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THE STRONG PATH PARTITION CONJECTURE HOLDS FOR $a = 9$

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Abstract

The *detour order* of a graph G, denoted by $\tau(G)$, is the order of a longest path in G. If a and b are positive integers and the vertex set of G can be partitioned into two subsets A and B such that $\tau(\langle A\rangle) \le a$ and $\tau(\langle B\rangle) \le b$, we say that (A, B) is an (a, b)*-partition* of G. If equality holds in both instances, i.e., if $\tau(\langle A\rangle) = a$ and $\tau(\langle B\rangle) = b$, we call (A, B) an *exact* (a, b)*-partition*. The Path Partition Conjecture asserts that if G is any graph and a, b any pair of positive integers such that $\tau(G) = a + b$, then G has an (a, b)-partition. The Strong Path Partition Conjecture asserts that, under the same conditions, G has an exact (a, b)-partition. The Path Partition Conjecture is now more than 40 years old. It first appeared in the literature in a paper by Laborde, Payan and Xuong (1982). It is known that the Path Partition Conjecture holds for all $a \le 8$. The case $a \le 5$ was first proved by Vronka (1986), the case $a = 6$ by Dunbar and Frick (1999) and the cases $a = 7$ and $a = 8$ by Melnikov and Petrenko (2002 and 2005). Using a new partition strategy involving a recursive procedure, De Wet, Dunbar, Frick and Oellermann (2024) improved these results by showing that the Strong Path Partition Conjecture holds for $a \le 8$. By expanding and refining the recursive procedure, we prove that the Strong Partition Conjecture also holds for $a = 9$.

Keywords: Path Partition Conjecture, Strong Path Partition Conjecture, vertex partitions, path kernels, longest path.

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1. Introduction and Background

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. If S is a subset of either $V(G)$ or $E(G)$, then $\langle S \rangle$ denotes the subgraph of G induced by S. The number of vertices in G is called the *order* of G and denoted by $n(G)$. A longest path in G is called a *detour* of G. The number of vertices in a detour of G is called the *detour order* of G and denoted by $\tau(G)$. By a *k-path* in a graph G we mean a subgraph of G (not necessarily induced) that is isomorphic to P_k, the path on k vertices.

Throughout the paper, a and b will denote positive integers.

If the vertex set $V(G)$ of a graph G can be partitioned into two sets A and B such that

$$\tau(\langle A \rangle) \leq a \text{ and } \tau(\langle B \rangle) \leq b,$$

we say that (A, B) is an *(a, b)-partition* of G.

If equality holds in both instances, i.e., if

$$\tau(\langle A \rangle) = a \text{ and } \tau(\langle B \rangle) = b,$$

we call (A, B) an *exact (a, b)-partition.* If equality holds in the first instance (but not necessarily in the second) we call (A, B) a *semi-exact (a, b)-partition* of G.

The following conjecture, which first appeared in the paper [14] by Laborde, Payan and Xuong, has become known as the *Path Partition Conjecture* (PPC for short).

Conjecture 1. ***The* PPC**. *If G is any graph with $\tau(G) \leq a + b$, then G has an (a, b)-partition.*

Hedetniemi [11] listed the PPC as one of his top 10 favourite conjectures. Results supporting the PPC appear in [3, 5–12, 14–17]. For a survey of these results, the reader is referred to [7].

A set A of vertices in a graph G is called a *P_{a+1}-kernel* of G if $\tau(\langle A \rangle) \leq a$ and every vertex in $V(G) - A$ is adjacent to an end-vertex of a P_a in $\langle A \rangle$. We note that, if $\tau(G) < a$, then $V(G)$ is the only P_{a+1}-kernel of G, but if $\tau(G) \geq a$, then every P_{a+1}-kernel of G has detour order equal to a.

We observe the following.

Observation 1.1. *If A is a P_{a+1}-kernel of a graph G with $\tau(G) = a + b$ and $B = V(G) - A$, then (A, B) is a semi-exact (a, b)-partition of $V(G)$.*

Proof. Since $b \geq 1$ (by our earlier assumption), $\tau(G) > a$ and hence $\tau(\langle A \rangle) = a$ and $B \neq \emptyset$. Now, suppose x is an end-vertex of a path X in $\langle B \rangle$. Then x is adjacent to an end-vertex of an a-path in $\langle A \rangle$, and hence, since $\tau(G) = a + b$, it follows that X has at most b vertices. Thus $\tau(\langle B \rangle) \leq b$. ■

Broere, Hajnal and Mihók [2] conjectured that every connected graph has a P_{a+1}-kernel for every a. However, Aldred and Thomassen [1] constructed a connected graph with detour order 364 that has no P_{364}-kernel. Later, Katrenič and Semanišin [13] constructed a connected graph with no P_{155}-kernel and also showed that for each integer $r \geq 0$ there exists a connected graph G having no $P_{\tau(G)-r}$-kernel. However, they pointed out that in each of their examples $\tau(G)-r$ is still greater than $\tau(G)/2$ and hence the following conjecture, which is stronger than the PPC, has not yet been disproved.

Conjecture 2. ***The Revised Path Kernel Conjecture.*** *If G is a connected graph with detour order τ, then G has a P_{a+1}-kernel for every positive integer $a \leq \tau/2$.*

It is known that every graph has a P_{a+1}-kernel for each $a \leq 8$. The case $a \leq 5$ was proved by Vronka [17]. Later, Dunbar and Frick [5] proved the case $a = 6$ by developing a recursive procedure, which was subsequently extended and refined by Melnikov and Petrenko [15, 16] to prove the cases $a = 7$ and $a = 8$.

A corollary of the results above is that the PPC holds for all $a \leq 8$. In fact, in view of Observation 1.1, it follows that if $a \leq 8$ and $\tau(G) = a + b$, then G has a semi-exact (a, b)-partition. Recently, De Wet, Dunbar, Frick and Oellermann [4] improved the latter result, by showing that the following conjecture holds for each $a \leq 8$.

Conjecture 3. ***The Strong path Partition Conjecture.*** *If G is any graph such that $\tau(G) = a + b$, then G has an exact (a, b)-partition.*

The proof of the Strong Path Partition Conjecture for $a \leq 8$ relies on a recursive procedure developed in [4]. This procedure may be summed up roughly as follows.

Let G be a graph with $\tau(G) = a + b$. We begin by letting A consist of the first a vertices of some $(a + b)$-path in G and putting $B = V(G) - A$. Then $\tau(\langle A\rangle) = a$ and $\tau(\langle B\rangle) \geq b$. Now we apply the following recursive procedure.

Step 1. If $\tau(B) = b$, we STOP. If $\tau(B) > b$, we let X be a $(b + 1)$-path in $\langle B\rangle$ and proceed to Step 2.

Step 2. If we can move an end-vertex of X to A without creating an $(a+1)$-path in $\langle A\rangle$, we do so and then return to Step 1. Otherwise, we proceed to Step 3.

Step 3. We move one end-vertex of X to A, thus creating at least one $(a + 1)$-path in $\langle A\rangle$. We then select an a-path to be retained in $\langle A\rangle$ and we destroy all $(a + 1)$-paths in $\langle A\rangle$ by moving vertices to B that are not on the selected a-path. Then we return to Step 1.

We note that, upon completion of any step, $\tau(\langle A\rangle) = a$ and there is still a b-path in $\langle B\rangle$. During the implementation of Steps 2 or 3, at least one $(b+1)$-path in $\langle B\rangle$ is destroyed. Thus, after each implementation of Step 2, there is at least one less $(b+1)$-path in $\langle B\rangle$ than in the previous step. However, if Step 3 is implemented, the vertices from A that are returned to B may create other $(b+1)$-paths in $\langle B\rangle$, and hence performing Step 3 need not necessarily decrease the number of $(b+1)$-paths in $\langle B\rangle$. Thus, the main problem is to show that our procedure will terminate, so that we will end up with an exact (a, b)-partition of G.

It is shown in [4] that, if $a \leq 6$, only Steps 1 and 2 will be performed, and hence the number of $(b+1)$-paths will decrease with each step until none remain. Thus the recursive procedure will terminate.

However, if $a \geq 7$, we may encounter "problematic configurations", which will make it necessary to perform Step 3. As shown in [4], there are "forbidden edges" associated with each problematic configuration, and if a is 7 or 8, the edges that are added to $\langle A\rangle$ when Step 3 is performed, will eventually contribute to the complexity of the structure of $\langle A\rangle$ to such an extent as to prohibit any further occurrence of problematic configurations. Thus, after a finite number of steps, only Steps 1 and 2 will be performed, and hence the recursive procedure will eventually terminate.

In this paper, we study the structure of problematic configurations for the case $a = 9$ and refine the recursive procedure described above to prove that the Strong PPC holds for $a = 9$.

2. Preliminaries

In this section we provide some auxiliary results that will be used in the next section to prove the Strong PPC for $a = 9$. We state these results for arbitrary a, in anticipation that they might prove useful for extending our result beyond $a = 9$.

We first provide some notation. If $v \in V(G)$ and U and W are subsets of $V(G)$, then $N_U(v) = \{u \in U : uv \in E(G)\}$ and $N_U(W) = \bigcup_{w\in W} N_U(w)$. If the context is clear, the subscript U will be omitted.

Let T be an a-path in a graph G. If we let $T = t_1t_2\cdots t_a$, this labelling of the vertices of T imposes an orientation on T. We denote the same path with the opposite orientation by $\overleftarrow{T}$. Thus, the i-th vertex of T is the $(a+1-i)$-th vertex of $\overleftarrow{T}$.

We use the notation $i \sim j$ to indicate that the i-th vertex of T is adjacent (in G) to the j-th vertex of T. If $t_it_j \in E(G)$ for some $i, j \in \{1, \ldots, a\}$ such that $|i - j| \geq 2$, we call t_it_j an *external edge* of T (since t_it_j is an edge in the induced subgraph $\langle V(T)\rangle$, but $t_i t_j \notin E(T)$) and we call $i \sim j$ an *external adjacency* of T. The number of external edges of T in $E(G)$ is denoted by $ext(T)$.

If Z is a path in $G - V(T)$ such that Zt_i is a path in G for some $i \in \{2, \ldots, a-1\}$, we say that Zt_i is a *path attached to the i-th vertex of* T.

Next, we state an obvious but useful proposition.

Proposition 2.1. *Suppose $L_1, \ldots, L_m$ are vertex disjoint segments of a path L, of which k are end-segments of L ($k \in \{0,1,2\}$). Then $L - \bigcup_{i=1}^{m} V(L_i)$ consists of at most $m + 1 - k$ segments of L.*

Now let G be a graph with $\tau(G) = a+b$ and suppose we wish to prove that G has an exact (a, b)-partition by implementing the recursive procedure discussed in Section 1. Then we need to consider the possibility that at some step in our procedure there is a $(b+1)$-path $X = x_1 \cdots x_{b+1}$ in B such that

$$\tau(\langle\{x_1\} \cup A\rangle) > a \text{ and } \tau(\langle\{x_{b+1}\} \cup A\rangle) > a.$$

If this is the case, as observed in Lemma 2.2 of [4], $\langle A\rangle$ contains four paths

$$R = w_1 \cdots w_r,\ S = w_{r+1} \cdots w_{r+s},\ P = v_1 \cdots v_p,\ Q = v_{p+1} \cdots v_{p+q},$$

of order r, s, p, q, respectively, such that Rx_1S and $Px_{b+1}Q$ are $(a+1)$-paths. Thus, $r + s = p + q = a$. (Our assumption that Rx_1S and $Px_{b+1}Q$ are paths implies that P and Q are vertex disjoint, and so are R and S.)

We assume, without loss of generality, that $r \geq s$ and $p \geq q$. Then $r \geq a/2$ and $p \geq a/2$, which implies that the paths R and P intersect (since otherwise RXP would be a path with at least $(a + b + 1)$ vertices).

We denote by (X, P, Q, R, S) the subgraph of G induced by the edges of the $(b+1)$-path X and the edges of the two $(a+1)$-paths Rx_1S and $Px_{b+1}Q$, i.e.,

$$(X, P, Q, R, S) = \langle E(X) \cup E(Px_{b+1}Q) \cup E(Rx_1S)\rangle.$$

We call (X, P, Q, R, S) a *problematic configuration* and we call the component H of $\langle A\rangle$ containing the path P a *problematic component*.

Throughout the paper, we shall use the notation given above to describe a problematic configuration (X, P, Q, R, S) and the associated problematic component H of $\langle A\rangle$. The case $a = 9$ is illustrated in Figure 5.

As mentioned earlier, it is the occurrence of problematic configurations that may prevent the general recursive procedure described in Section 1 from terminating. Our next lemma provides useful results on the structure of problematic configurations in general.

Lemma 2.2. *Let G be a graph with $\tau(G) = a + b$ and let (A, B) be a partition of $V(G)$ such that $\tau(\langle A\rangle) = a$ and $\tau(\langle B\rangle) > b$. Suppose there is a $(b+1)$-path X in $\langle B\rangle$ and four paths P, Q, R, S defined and labelled as above, such that*

(X, P, Q, R, S) is a problematic configuration. Let H be the problematic component of $\langle A \rangle$ containing the path P (and hence also the path R). Then the following hold.

1. (a) *If $m \in \{1, p, p+1, a\}$, then every $x_1 v_m$-path in $\langle \{x_1\} \cup A \rangle$ has an internal vertex in $V(P) \cup V(Q)$.*

   (b) *No neighbour of x_1 is in $\{v_1, v_p, v_{p+1}, v_a\}$. In particular, $w_r, w_{r+1} \notin \{v_1, v_p, v_{p+1}, v_a\}$.*

   (c) *If Y is a $w_r w_{r+1}$-path of order at least 3 in $\langle A \rangle$, then at least one internal vertex of Y has a neighbour in $H - V(Y)$.*

   (d) *If $\{w_r, w_{r+1}\} = \{v_{p-2}, v_{p-1}\}$, then v_p is an internal vertex of either the path R or the path S and on that path both the predecessor and successor of v_p are in $H - \{v_{p-1}, v_{p-2}\}$.*

2. (a) *If $m \in \{1, r, r+1, a\}$, then every $x_{b+1} v_m$-path in $\langle \{x_{b+1}\} \cup A \rangle$ has an internal vertex in $V(R) \cup V(S)$.*

   (b) *No neighbour of x_{b+1} is in $\{w_1, w_r, w_{r+1}, w_a\}$. In particular, $v_p, v_{p+1} \notin \{w_1, w_r, w_{r+1}, w_a\}$.*

   (c) *If Y is a $v_p v_{p+1}$-path of order at least 3 in $\langle A \rangle$, then at least one internal vertex of Y has a neighbour in $H - V(Y)$.*

   (d) *If $\{v_p, v_{p+1}\} = \{w_{r-2}, w_{r-1}\}$, then w_r is an internal vertex of either the path P or the path Q and on that path both the predecessor and successor of w_r are in $H - \{w_{r-2}, w_{r-1}\}$.*

Proof. 1. (a) If v_m is an end-vertex of either P or Q and there is an $x_1 v_m$ path in $\langle \{x_1\} \cup A \rangle$ with no internal vertex in $V(P) \cup V(Q)$, then there is a path in G that contains all the vertices in $V(P) \cup V(X) \cup V(Q)$ and hence has order at least $p + (b+1) + q = a + b + 1$.

(b) It follows from (a) that $x_1 v_m$ is not an edge in G for any $m \in \{1, p, p+1, a\}$. In particular, since $\{w_r, w_{r+1}\} \subseteq N(x_1)$, it follows that $w_r, w_{r+1} \notin \{v_1, v_p, v_{p+1}, v_1\}$.

(c) Let Y' be the interior of the path Y (i.e., Y minus its two end-vertices). Suppose neither R nor S intersects Y'. Then $RY'S$ is a path of order at least $a+1$ in H, contradicting that $\tau(H) \leq a$. Thus, at least one of R and S intersects Y'.

Suppose R, but not S, intersects Y'. Then $V(R) \not\subseteq V(Y)$, since otherwise $\overleftarrow{X} w_r Y' S$ would be a path with at least $a+b+1$ vertices. This implies that some vertex of R in Y' has a neighbour that is not in Y.

By symmetry, if S but not R intersects Y', some vertex of S in Y' has a neighbour that is not in Y.

Now suppose each of R and S intersects Y'. If $V(R) \cup V(S) \subseteq V(Y)$, then, since $V(R) \cap V(S) = \emptyset$, it follows that $n(Y) \geq n(R) + n(S) = a$, but then YX is a path in G of order at least $a+b+1$. Thus at least one of R and S has a vertex in $H - V(Y)$ and hence some vertex of Y' has a neighbour in $H - V(Y)$.

(d) Suppose $w_r = v_{p-2}$ and $w_{r+1} = v_{p-1}$. Then $v_p \in V(R) \cup V(S)$ (since otherwise RXv_pS is a path of order $a+b+1$). If $v_p = w_{r-1}$, then $w_1 \cdots w_{r-1}\overleftarrow{X}w_rw_{r+1} \cdots w_a$ is a path of order $a+b+1$. If $v_p = w_{r+2}$, then $w_1 \cdots w_{r+1}Xw_{r+2}$ is an $(a+b+1)$-path. Thus $v_p \notin \{w_{r-1}, w_{r+2}\}$. By 2(b), v_p is not an end-vertex of either R or S and hence v_p is an internal vertex of either the path R or the path S and both the predecessor and the successor of v_p on that path is in $H - \{v_{p-2}, v_{p-1}\}$. The proof of the case where $w_{r+1} = v_{p-1}$ and $w_r = v_{p-2}$ is similar.

By symmetry, the proof of 2 is similar to that of 1. ■

As shown in [4], there are only two problematic configurations for the case $a = 7$, and in each case the associated problematic component of A contains a 7-path. These are illustrated in Figure 1.

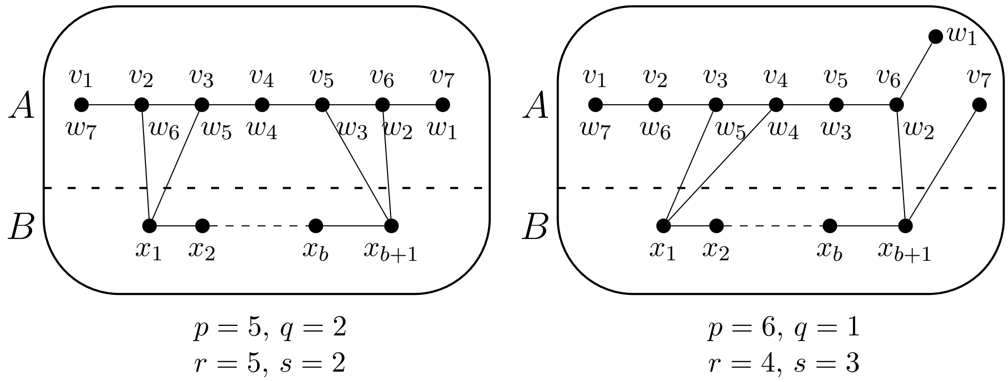


Figure 1. The two problematic configurations for $a = 7$.

However, for $a \geq 8$ there are problematic components with detour order less than a. For example, Figure 2 illustrates a problematic configuration for $a = 8$, where the associated problematic component H has detour order 7.

A a grows, the number as well as the complexity of the problematic configurations increase. Fortunately, our recursive procedure easily eliminates any problematic component with detour order less than a that we may encounter, since after applying Step 3, the resulting component of $\langle A \rangle$ will contain an a-path. Thus we can restrict our attention to problematic configurations where the associated problematic component H contains an a-path T. The study of these cases is facilitated by using Lemma 2.2 in conjunction with the following elementary lemma.

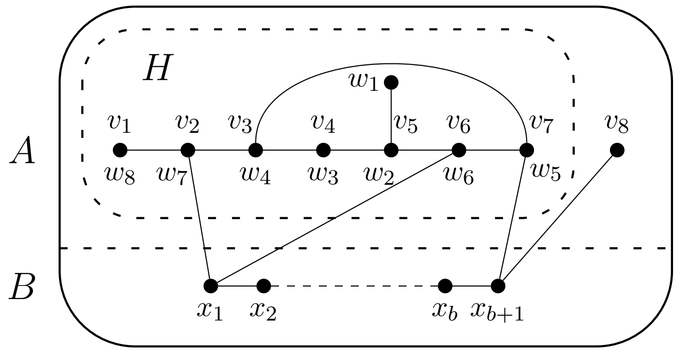


Figure 2. A problematic configuration for $a = 8$ with $\tau(H) = 7$.

Lemma 2.3. *Let G be a graph with $\tau(G) = a + b$, let $T = t_1 \cdots t_a$ be an a-path in G and suppose $X = x_1 \cdots x_{b+1}$ is a $(b + 1)$-path in $G - V(T)$. Now suppose that for some pair of distinct vertices $t_h, t_k \in V(T)$ there is an $x_1 t_h$-path F_1 and an $x_{b+1} t_k$-path F_2 such that F_1 and F_2 are vertex disjoint and all their internal vertices are in $G - (V(T) \cup V(X))$. Then each of the following holds.*

(1) $h, k \notin \{1, a\}$.

(2) $k \notin \{h - 1, h + 1\}$.

(3) $t_{h-1}t_{k-1}, t_{h+1}t_{k+1} \notin E(G)$.

(4) *If $h < k$, then*

  (a) *t_{h+1} is not adjacent to an end-vertex of any $(a-k)$-path in $G-(\{t_1, \ldots, t_k\}) \cup V(F_1) \cup V(F_2))$;*

  (b) *t_{k-1} is not adjacent to an end-vertex of any $(h-1)$-path in $G-(\{t_h, \ldots, t_a\} \cup V(F_1) \cup V(F_2))$.*

(5) *If $t_1 t_c \in E(G)$ for some $c \in \{3, \ldots, a - 1\}$ and $t_d t_a \in E(G)$ for some $d \in \{2, \ldots, a - 2\}$, then the following hold.*

  (a) $d \neq c - 1$.

  (b) *If $d = c + 1$, then either $h \leq c$ and $k \leq c$, or $h \geq d$ and $k \geq d$.*

  (c) $h, k \notin \{c - 1, d + 1\}$.

  (d) *If $h < k$, then $k \neq c + 1$ and $h \neq d - 1$.*

  (e) *If $d \leq c$, then $h, k \notin \{c + 1, d - 1\}$.*

The proofs of items (1), (2) and (3) of Lemma 2.3 are obvious. Indirect proofs of the statements in (4) and (5) are illustrated in Figures 3 and 4.

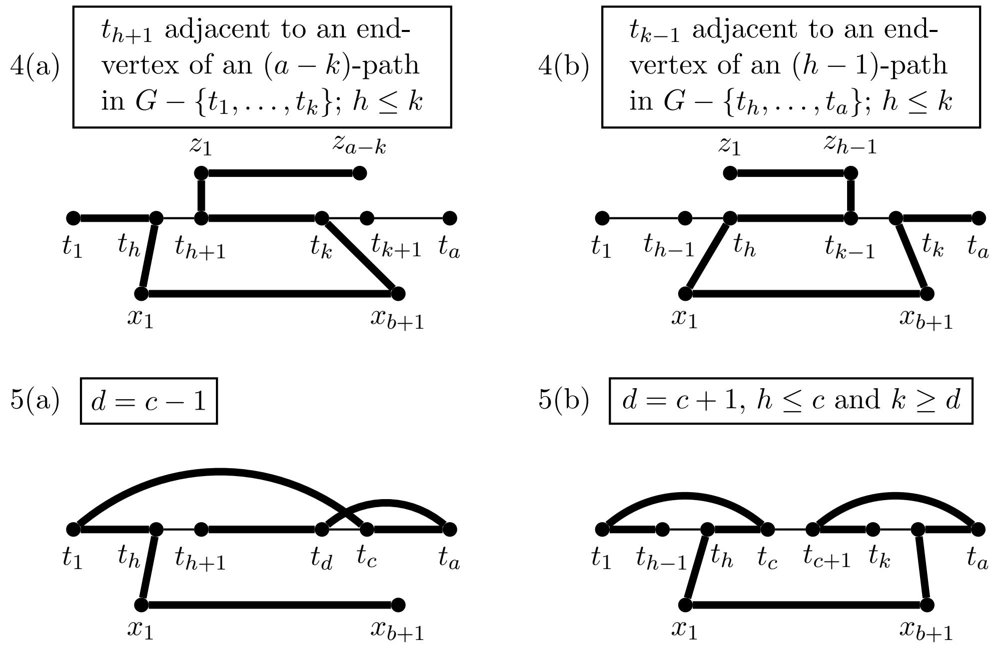


Figure 3. Illustrations of indirect proofs of Lemma 2.3(4(a)–(b)) and 2.3(5(a)–(b)). In each case, the heavy lines indicate a path of order greater than $a + b$ that would be in G if a condition in the corresponding item was violated.

3. PROOF OF THE STRONG PPC FOR $a = 9$

Throughout this section we let G be a graph with $\tau(G) = 9 + b$ and we let (A, B) be a partition of $V(G)$ such that $\tau(\langle A \rangle) = 9$.

For easy reference, we state the definitions of a problematic configuration and a problematic component for the specific case $a = 9$.

Definition 3.1. Suppose there is a $(b+1)$-path $X = x_1 \cdots x_{b+1}$ in $\langle B \rangle$ and that $\langle A \rangle$ contains four paths

$$P = v_1 \cdots v_p,\ Q = v_{p+1} \cdots v_{p+q},\ R = w_1 \cdots w_r,\ S = w_{r+1} \cdots w_{r+s}$$

$$\text{with } p \ge q,\ r \ge s \text{ and } p + q = r + s = 9,$$

such that Rx_1S and $Px_{b+1}Q$ are 10-paths. Let (X, P, Q, R, S) be the subgraph of G induced by the edges of the $(b+1)$-path X and the two 10-paths Rx_1S and $Px_{b+1}Q$, i.e.,

$$(X, P, Q, R, S) = \langle E(X) \cup E(Px_{b+1}Q) \cup E(Rx_1S) \rangle.$$

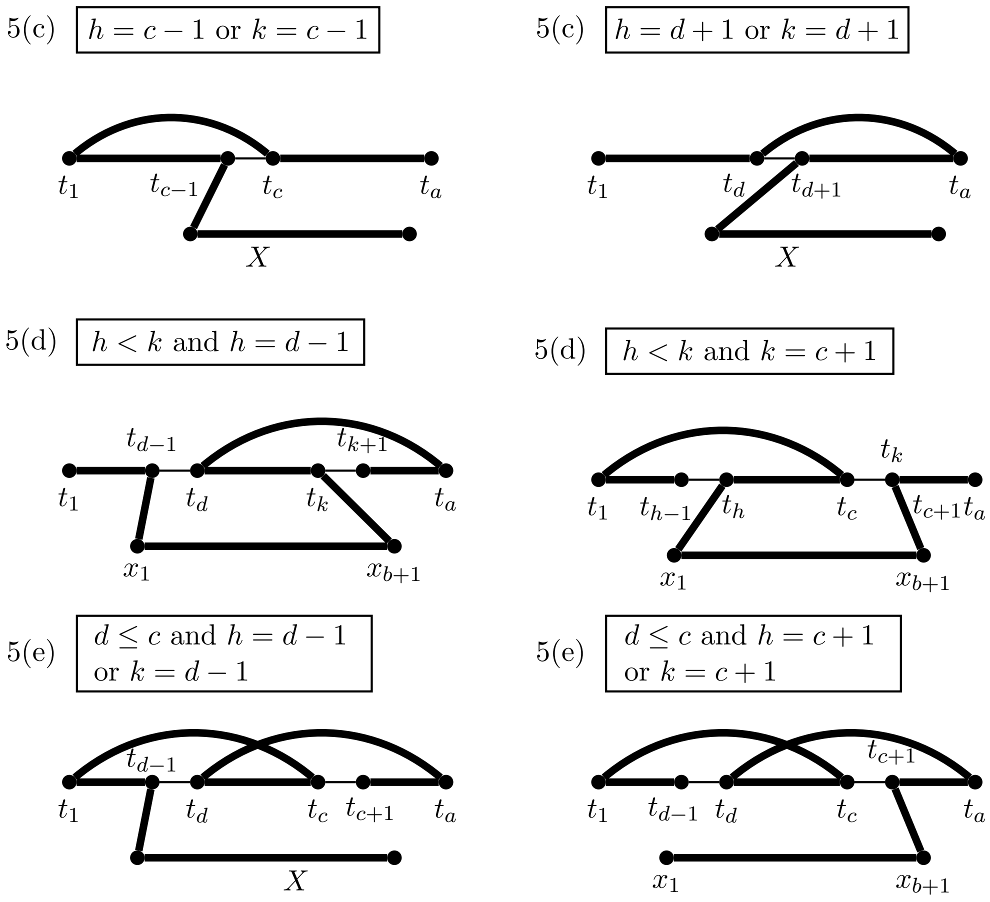


Figure 4. Illustrations of indirect proofs of Lemma 2.3 (5(c)–(e)). In each case, the heavy lines indicate a path of order greater than $a + b$ that would be in G if a condition in the corresponding item was violated.

Then we say (X, P, Q, R, S) is a **problematic configuration** in G and the component H of $\langle A \rangle$ that contains P is a **problematic component**.

Remark 3.2. We note the following concerning the paths P, Q, R, S defined in Definition 3.1.

(1) The vertices $w_r, w_{r+1}, v_p, v_{p+1}$ are four distinct vertices. (The fact that $w_r \neq w_{r+1}$ and $v_p \neq v_{p+1}$ follows from our assumption that Rx_1S and $Px_{b+1}Q$ are paths, and the fact that that $\{w_r, w_{r+1}\} \cap \{v_p, v_{p+1}\} = \emptyset$ follows from Lemma 2.2(1b).)

(2) Our assumption that $p \geq q$ and $r \geq s$ implies that $p \geq 5$ and $r \geq 5$, and hence the paths R and P have one or more vertices in common (since otherwise

$RX\overleftarrow{P}$ would be a path with at least $11 + b$ vertices).

(3) If $p \geq r$, then $s \geq q$ and hence, in this case, S also intersects P (since otherwise $P\overleftarrow{X}S$ would be a path with more than $9 + b$ vertices).

(4) If $r \geq p$, then $q \geq s$ and hence, in this case, Q also intersects R (since otherwise RXQ would be a path with more than $9 + b$ vertices).

Figure 5 illustrates a problematic configuration for $a = 9$ in the case where $p \geq r$.

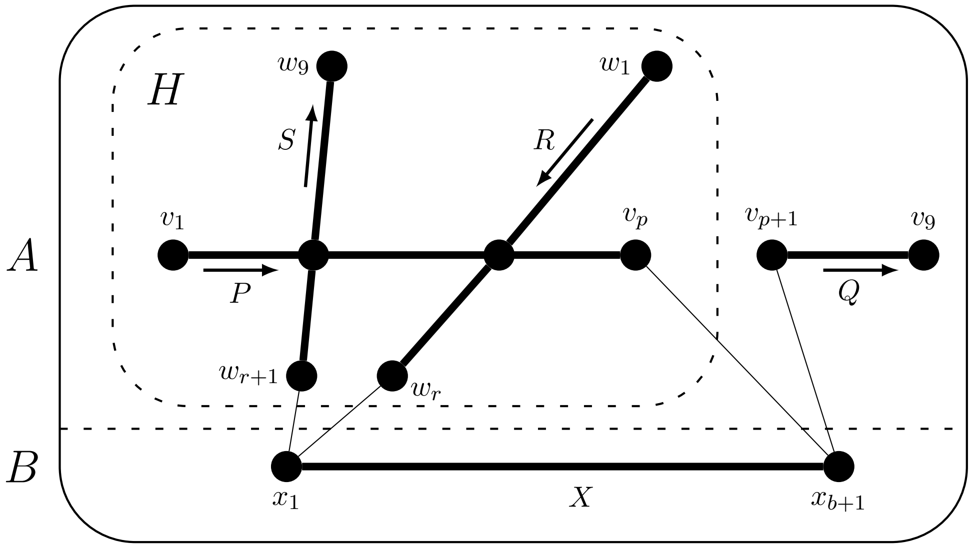


Figure 5. Illustrating a problematic configuration (X, P, Q, R, S) and the associated problematic component H, as defined in Definition 3.1, for the case $p \geq r$. Note that Q may be in H and each of R and S may intersect P and Q in several vertices.

By Definition 3.1 and Remark 3.2(2), (3) and (4), a component H of $\langle A \rangle$ is a problematic component if there is a problematic configuration (X, P, Q, R, S) in G such that H contains the paths P and R as well as at least one of the paths S and Q.

A component K of $\langle A \rangle$ is *non-problematic* if there is no problematic configuration (X, P, Q, R, S) in G such that the paths P and R are in K. This means that if K is a non-problematic component of $\langle A \rangle$ and X is any $(b+1)$-path in $\langle B \rangle$, we can move at least one of x_1 and x_{b+1} to A without creating a 10-path in $\langle A \rangle$ that intersects K.

To prove the Strong PPC for $a = 9$, we shall design a recursive procedure based on the general procedure described in Section 1. We shall show that the procedure will transform problematic components into non-problematic components until, after a finite number of steps, no more problematic components will be encountered, thus ensuring that the procedure will terminate. Since any problematic component with detour order less than 9 will be transformed into a component of $\langle A \rangle$ with detour order equal to 9, we restrict our attention to problematic components with detour order equal to 9. This leads to the following definition.

Definition 3.3. Suppose (X, P, Q, R, S) is a problematic configuration, defined and labelled as in Definition 3.1 and suppose the associated problematic component H contains a 9-path $T = t_1 \cdots t_9$. Then we let

$$(X, P, Q, R, S, T) = \langle E((X, P, Q, R, S)) \cup E(T) \rangle.$$

and we say (X, P, Q, R, S, T) is a **complex configuration** in G.

To avoid having to consider isomorphic copies of complex configurations which result from reversing the orientation of T or X, we restrict our investigation to the three types of complex configurations defined below.

Definition 3.4. Suppose (X, P, Q, R, S, T) is a complex configuration, defined and labelled as in Definition 3.3. Then we say

(a) (X, P, Q, R, S, T) is an **A-configuration** if $p \geq r$ and neither $\{w_r, w_{r+1}\}$ nor $\{v_p, v_{p+1}\}$ is a subset of $V(T)$.

(b) (X, P, Q, R, S, T) is a **B-configuration** if neither $\{w_r, w_{r+1}\}$ nor $\{v_p, v_{p+1}\}$ is a pair of consecutive vertices of T and each of the following holds.

  (1) $\{w_r, w_{r+1}\} = \{t_g, t_h\}$, for some pair g, h such that $2 \leq g \leq h - 2 \leq 6$.

  (2) If $t_k \in \{v_p, v_{p+1}\}$, then $k > g$.

(c) (X, P, Q, R, S, T) is a **C-configuration** if $\{w_r, w_{r+1}\} = \{t_h, t_{h+1}\}$ for some $h \in \{2, 3, 4\}$.

We now show that if T is any 9-path in a problematic component, then T can be oriented so that it is in an A-configuration, a B-configuration or a C-configuration in G.

Lemma 3.5. *Suppose (X, P, Q, R, S) is a problematic configuration, defined and labelled as in Definition 3.1 and let H be the associated problematic component of $\langle A \rangle$. Suppose H contains a 9-path $T = t_1 \cdots t_9$. Then at least one of T and $\overleftarrow{T}$ is in an A- B- or C-configuration with X or $\overleftarrow{X}$.*

Proof. We consider three possibilities regarding the intersections of the sets $\{w_r, w_{r+1}\}$ and $\{v_p, v_{p+1}\}$ with $V(T)$.

(a) *Suppose neither $\{w_r, w_{r+1}\}$ nor $\{v_p, v_{p+1}\}$ is a subset of $V(T)$.* Then (X, P, Q, R, S, T) is an A-configuration if $p \geq r$, and $(\overleftarrow{X}, P, Q, R, S, T)$ is an A-configuration if $r \geq p$.

(b) *Suppose at least one of the sets* $\{w_r, w_{r+1}\}$ *and* $\{v_p, v_{p+1}\}$ *is contained in* $V(T)$*, but neither set is a pair of consecutive vertices of* T*.*

– Suppose $\{w_r, w_{r+1}\} = \{t_g, t_h\} \subset V(T)$, with $2 \le g \le h-2 \le 6$.
  - ⋆ If $\{v_p, v_{p+1}\} = \{t_k, t_m\} \subset V(T)$, with $2 \le k \le m-2 \le 6$, then (X, P, Q, R, S, T) is a B-configuration if $g < k$, and if $k < g$ then $(\overleftarrow{X}, P, Q, R, S, T)$ is a B-configuration.
  - ⋆ If $\{v_p, v_{p+1}\} \cap V(T) = \emptyset$, then (X, P, Q, R, S, T) is a B-configuration
  - ⋆ If $\{v_p, v_{p+1}\} \cap V(T) = \{t_k\}$, then (X, P, Q, R, S, T) is a B-configuration if $g < k$, and $(X, P, Q, R, S, \overleftarrow{T})$ is a B-configuration if $k > g$.

– Suppose $\{w_r, w_{r+1}\} \not\subset V(T)$. Then we may assume that $\{v_p, v_{p+1}\} = \{t_k, t_m\}$, with $2 \le k \le m-2 \le 6$.
  - ⋆ If $\{w_r, w_{r+1}\} \cap V(T) = \emptyset$, then $(\overleftarrow{X}, P, Q, R, S, T)$ is a B-configuration.
  - ⋆ If $\{w_r, w_{r+1}\} \cap V(T) = \{t_h\}$, then $(\overleftarrow{X}, P, Q, R, S, T)$ is a B-configuration if $k < h$, and $(\overleftarrow{X}, P, Q, R, S, \overleftarrow{T})$ is a B-configuration if $k > h$.

(c) *Suppose at least one of the sets* $\{w_r, w_{r+1}\}$ *and* $\{v_p, v_{p+1}\}$ *is a pair of consecutive vertices of* T*.*

– If $\{w_r, w_{r+1}\} = \{t_h, t_{h+1}\}$ for some $h \in \{2, \ldots, 7\}$, then (X, P, Q, R, S, T) is a C-configuration if $h \le 4$, and $(X, P, Q, R, S, \overleftarrow{T})$ is a C-configuration if $h \ge 5$ (because t_{h+1} is the $(9-h)$-th vertex of $\overleftarrow{T}$).

– If $\{v_p, v_{p+1}\} = \{t_k, t_{k+1}\}$ for some $k \in \{2, \ldots, 7\}$, then $(\overleftarrow{X}, P, Q, R, S, T)$ is a C-configuration if $k \le 4$, and $(\overleftarrow{X}, P, Q, R, S\overleftarrow{T})$ is a C-configuration if $k > 4$.

■

Corollary 3.6. *Let* $T = t_1 \cdots t_9$ *be a* 9*-path in* $\langle A \rangle$*. If neither* T *nor* $\overleftarrow{T}$ *is in an A-configuration, a B-configuration or a C-configuration with any* $(b+1)$*-path in* $\langle B \rangle$*, then the component* K *of* $\langle A \rangle$ *containing* T *is non-problematic.*

If (X, P, Q, R, S, T) is a complex configuration in G, the 9-path T may have external edges in G which are not necessarily in $E(R) \cup E(S) \cup E(P) \cup E(Q))$. In the case of C-configurations, external edges incident with the end-points of T require special consideration, as will become clear in the proof of our main theorem.

We call a C-configuration (X, P, Q, R, S, T) in G a **nice** C**-configuration** if at least one of the end-vertices of T is not incident with an external edge of T.

In the case of C-configurations that are not nice, we will need to consider *expanded C-configurations*, defined as follows.

Definition 3.7. Suppose (X, P, Q, R, S, T) is a C-configuration in G, defined and labelled as in Definition 3.4(c), and each of t_1 and t_9 is incident with an external edge of T. Let c be the smallest number in $\{3, \ldots, 8\}$ such that $t_1 t_c \in E(G)$ and let d be the smallest number in $\{2, \ldots, 7\}$ such that $t_9 t_d \in E(G)$. Let

$$(X, P, Q, R, S, T, c, d) = \langle E((X, P, Q, R, S, T)) \cup \{t_1 t_c, t_9 t_d\} \rangle.$$

Then we call (X, P, Q, R, S, T, c, d) an **expanded C-configuration**.

The following lemma regarding the structure of A-, B- and C-configurations will play a key role in the proof of our main theorem.

Lemma 3.8. *Suppose (X, P, Q, R, S, T) is a complex configuration in G, defined and labelled as in Definition* 3.4.

(A) *If (X, P, Q, R, S, T) is an* A-*configuration, then it is one of the three configurations* A1, A2, A3, *described and illustrated in Figure* 8.

(B) *If (X, P, Q, R, S, T) is a* B-*configuration, then it is one of the nine configurations* B1, ..., B9, *described and illustrated in Figures* 9, 10 *and* 11.

(C) *If (X, P, Q, R, S, T) is a C-configuration, then either (X, P, Q, R, S, T) is a nice C-configuration, or the expanded C-configuration (X, P, Q, R, S, T, c, d) is one of the seven configurations* C1, ..., C7 *described and illustrated in Figures* 12, 13 *and* 14.

Proof. (A) Suppose (X, P, Q, R, S, T) is an A-configuration. Then, according to Definition 3.4(a), $p \geq r$ and there is a $w \in \{w_r, w_{r+1}\}$ and a $v \in \{v_p, v_{p+1}\}$ such that $w, v \notin V(T)$.

Since $p \geq r$, it follows from Remark 3.2(2) and (3) that the paths R, S, P and T all lie in H. Thus, for some $h \in \{2, \ldots, 8\}$ there is a wt_h path F in H with no internal vertex in $V(T)$.

It follows from Remark 3.2(1) that $w \neq v$ and hence, if $v \notin V(F)$, then $v\overleftarrow{X}F$ is a path of order at least $4 + b$. On the other hand, if $v \in V(F)$, then $\overleftarrow{X}F$ is a path of order at least $4 + b$. In either case, there is a path of order $4 + b$ ending at t_h that contains no vertex in $T - \{t_h\}$. If $h \notin \{4, 5, 6\}$, then either the path $t_1 \cdots t_h$ or the path $t_h \cdots t_9$ has at least 7 vertices, and hence there is a path in G having at least $10 + b$ vertices, contradicting that $\tau(G) = 9 + b$. Thus $h \in \{4, 5, 6\}$.

Case 1. $h = 4$. In this case, $\overleftarrow{X}Ft_5 t_6 t_7 t_8 t_9$ is a path of order $b + 6 + n(F)$, and hence $n(F) \leq 3$.

By Lemma 2.2(1b), $w \neq v_p$ and hence wXv_p is a $(3 + b)$-path, which implies that $v_p \notin \{t_1, t_2, t_8, t_9\}$. It follows from Lemma 2.3(2) that $v_p \notin \{t_3, t_5\}$. Also, $v_p \neq t_6$, since otherwise $t_1 t_2 t_3 F X t_6 t_7 t_8 t_9$ would be a path with at least $10 + b$ vertices.

Thus v_p is either t_4 or t_7 or $v_p \notin V(T)$. We consider these three cases separately.

1.1. $v_p = t_4$. By the definition of an A-configuration, $\{v_p, v_{p+1}\} \not\subset V(T)$, and therefore $v_{p+1} \notin V(T)$.

Now suppose $v_{p+1} \in V(F)$. Then, since $w \neq v_{p+1}$ (by Lemma 2.2(1b)) and $n(F) \leq 3$ (as shown earlier), $F = wv_{p+1}v_p$. Since $wXv_{p+1}v_pt_5t_6t_7t_8t_9$ is a $(9+b)$-path, v_{p+1} is the only neighbour of w in $H - V(T)$, and hence $N_H(w) \subseteq \{t_4, v_{p+1}\} = \{v_p, v_{p+1}\}$. Thus w is not an internal vertex of either P or Q, and hence $x_1wv_{p+1}v_p$ is an x_1v_p-path containing no internal vertex of either P or Q, contradicting Lemma 2.2(1a).

Thus $v_{p+1} \notin V(F)$. Now $v_{p+1}\overleftarrow{X}Ft_5t_6t_7t_8t_9$ is a path of order $n(F) + 7 + b$, which implies that $n(F) = 2$, i.e., $F = wv_p$. It therefore follows from Lemma 2.2(1a) that w is an internal vertex of either P or Q.

Suppose w is an internal vertex of P, i.e., $w \in \{v_2 \cdots v_{p-1}\}$. If v_p is the only vertex of P on T, then, since $p \geq 5$, it follows that $v_1 \cdots v_pt_5t_6t_7t_8t_9$ is a path of order at least $p + 5 \geq 10$ in H, contradicting that $\tau(\langle A \rangle) = 9$. Thus $v_k \in V(T)$ for some $k \in \{1, \ldots, p-1\}$. But then v_p is not on the wv_k-subpath of P, contradicting that t_4 is on every subpath from w to T. This contradiction shows that w is not an internal vertex of P.

Thus w is an internal vertex of Q, and hence $q \geq 3$ and $w = v_{p+d}$, for some $d \geq 2$. Now, $v_{p+1} \cdots v_{p+d}Xt_4t_5t_6t_7t_8t_9$ is a path of order $d + b + 7$, and hence $d = 2$ and $q \geq 3$. Thus, $v_{p+3}v_{p+2}v_{p+1}X$ is a $(4+b)$-path. Now, $t_5 \notin N(x_1)$ by Lemma 2.3(2), but every vertex in T except for t_5 is an end-vertex of a 6-path in H, and hence x_1 has no neighbour in $V(T)$. Also, $t_9t_8t_8t_6t_5t_4v_{p+1}v_{p+2}\overleftarrow{X}$ is a $(9+b)$-path, and hence v_{p+2} is the only neighbour of x_1 in A, contradicting that both w_r and w_{r+1} are neighbours of x_1.

1.2. $v_p = t_7$. As in the previous case, $v_{p+1} \notin V(T)$ and $v_{p+1}\overleftarrow{X}w$ is a $(3+b)$-path, which implies that neither v_{p+1} nor w has a neighbour in $\{t_1, t_2, t_3, t_6, t_7, t_8, t_9\}$. Since $t_9t_8t_7\overleftarrow{X}w$ is a $(5+b)$-path, neither t_5 nor t_6 is in $N(w)$, and since $t_1t_2t_3t_3t_4t_5t_6t_7\overleftarrow{X}w$ is a $(9+b)$-path, w has no neighbour in $A - V(T)$. Thus t_4 is the only neighbour of w in A, and hence w is not an internal vertex of either P or Q. By Lemma 2.2(1b), w is also not an end-vertex of either P or Q. Thus $w \notin V(P) \cup V(Q)$.

If $p = 8$, then PXw is a $(10+b)$-path. Thus $p \leq 7$, and hence $q \geq 2$. Since $t_1t_2t_3t_4wXv_{p+1}$ is a $(7+b)$-path, $t_5, t_6 \notin N(v_{p+1})$ and hence t_4 is the only possible neighbour of v_{p+1} in T. We have already shown that w is not a neighbour of v_{p+1} and hence, since $t_9t_8t_7t_6t_5t_4wXv_{p+1}$ is a $(9+b)$-path, it follows that t_4 is the only neighbour of v_{p+1} in A. This implies that $v_{p+2} = t_4$, and hence $v_1 \cdots v_pwXv_{p+2} \cdots v_9$ is a $(10+b)$-path.

1.3 $v_p \notin V(T)$. Suppose $v_p \notin V(F)$. Then $v_p\overleftarrow{X}Ft_5t_6t_7t_8t_9$ is a path of order $7+n(F)+b$. Thus $n(F)=2$, i.e., $F=wt_4$ and $N_A(v_p) \subseteq \{w, t_4, t_5, t_6, t_7, t_8, t_9\}$. However, since $t_1t_2t_3t_4wXv_p$ is a $(7+b)$-path, $t_5, t_6, t_7, t_8, t_9 \notin N(v_p)$. Thus $N_A(v_p) \subseteq \{t_4, w\}$, and hence v_{p-1} is either w or t_4.

If $v_{p-1} = t_4$, then $t_9t_8t_7t_6t_5t_4v_p\overleftarrow{X}$ is a $(9+b)$-path and hence $N_A(w) \subseteq \{v_{p-1}, v_p\}$, which implies that $w \notin V(P) \cup V(Q)$. Thus $v_1 \cdots v_{p-1}wXQ$ is an $(a+b+1)$-path.

Thus $v_{p-1} = w$. Now $t_9t_8t_7t_6t_5t_4v_{p-1}v_p\overleftarrow{X}$ is a $(9+b)$-path, and $t_1t_2t_3t_4wv_p\overleftarrow{X}$ is a $(7+b)$-path, and hence $N(x_1) \subseteq \{v_{p-1}, t_4\}$. Thus $t_9t_8t_7t_6t_5t_4Xv_pv_{p-1}$ is a $(9+b)$-path, and hence $N_A(v_{p-1}) \subseteq \{t_4, v_p\}$, which implies that $v_{p-2} = t_4$. This implies that $\{w_r, w_{r+1}\} = \{v_{p-1}, v_{p-2}\}$ and $N(v_p) \subseteq \{v_{p-1}, v_{p-2}\}$, contradicting Lemma 2.2(1d).

Thus $v_p \in V(F)$, and hence $F = wv_pt_4$. Now, $t_9t_8t_7t_6t_5t_4v_p\overleftarrow{X}w$ is a $(9+b)$-path and $t_1t_2t_3t_4v_p\overleftarrow{X}w$ is a $(7+b)$-path, and hence $N(w) \subseteq \{v_p, t_4\}$. Since x_1wv_p is an x_1v_p-path, it follows from Lemma 2.2(1a) that w is an internal vertex of either P or Q, and hence $w = v_{p-1}$ and $t_4 = v_{p-2}$. Thus $v_1 \cdots v_{p-2}v_pv_{p-1}Xv_{p+1} \cdots v_9$ is a $(10+b)$-path.

Thus, the case $h=4$ does not occur.

Case 2. $h=5$. Since we are assuming that $p \geq r$, it follows from Remark 3.10 that both the paths R and S intersect the path P in one or more vertices.

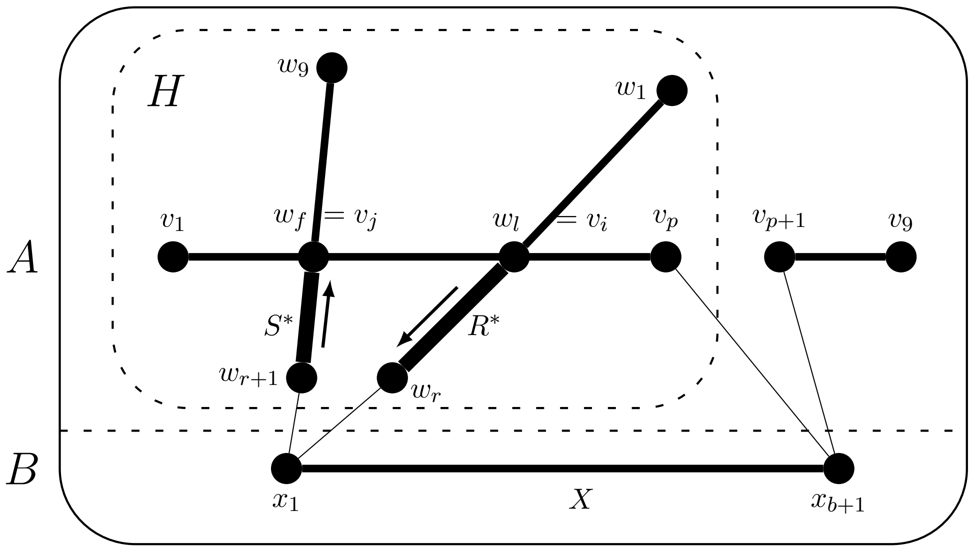


Figure 6. Ilustrating the subpaths R^* and S^* in a problematic configuration (X, P, Q, R, S) with $p \geq r$.

Let w_l be the last vertex of R that lies on P. Let $w_l = v_i$ and let R^* be the $w_l w_r$-subpath of R, i.e.,

$$R^* = w_l \cdots w_r.$$

Let w_f be the first vertex of S that lies on P. Let $w_f = v_j$ and let S^* be the $w_{r+1} w_f$-subpath of S, i.e.,

$$S^* = w_{r+1} \cdots w_f.$$

Note that, if $w_r = w_l$, then $R^* = w_r = v_i$, and if $w_{r+1} = w_f$, then $S^* = w_{r+1} = v_j$.

Since Case 1 does not occur, any path in H from w to $V(T)$ contains the vertex t_5.

By Lemma 2.3(2) and the fact that $\overleftarrow{X} F t_6 t_7 t_8 t_9$ and $\overleftarrow{X} F t_4 t_3 t_2 t_1$ are $(7+b)$-paths, t_5 lies on every path in $\langle \{x_{b+1}\} \cup V(H) \rangle$ from x_{b+1} to T. In particular, $N_{V(T)}(w) \cup N_{V(T)}(x_{b+1}) \subseteq \{t_5\}$.

2.1. $v_p \in V(T)$. In this case, $v_p = t_5$, and hence $V(Q) \cap V(T) = \emptyset$. If $w \in V(P)$, then $w = v_m$ for some $m \in \{2, \ldots, p-1\}$ and there is a path from w to each vertex in $\{v_1, v_2, \ldots, v_{p-1}\}$ that does not contain the vertex $v_p = t_5$. This implies that $v_1, \ldots, v_{p-1} \notin V(T)$. Thus $v_1 \cdots v_p t_6 t_7 t_8 t_9$ is a path of order $p+4$, and hence, since $\tau(H) = 9$, it follows that $p = 5$ and $q = 4$. This implies that $Q \overleftarrow{X} v_m \cdots v_p t_6 t_7 t_8 t_9$ is a path with at least $11+b$ vertices, contradicting that $\tau(G) = 9 + b$.

Thus $w \notin V(P)$. Since F is a wv_p-path in $\langle A \rangle$, it follows from Lemma 2.2(1b) that F contains an internal vertex of Q, and hence $q \geq 3$. Thus XQ is a path of order at least $4+b$ in $G - V(T)$. Since every vertex in $V(T)$ except t_5 is an end-vertex of a 6-path, it follows that $N_{V(T)}(x_1) \subseteq \{t_5\}$. However, $t_5 = v_p \notin N(x_1)$ by Lemma 2.2(1b). Thus $N_{V(T)}(x_1) = \emptyset$. Thus $w_r, w_{r+1} \notin V(T)$, and hence $w_r, w_{r+1} \notin V(P)$ (since we have shown that $w \notin V(P)$)).

Since the paths R and S do not intersect, $w_l \neq w_f$. Thus, if $w_l = v_i$ and $w_f = v_j$, then $i \neq j$ and hence at least one of i and j is less than p. If $i < p$, there is a path from w_r to every vertex on the path $v_1 \cdots v_{p-1}$ that does not contain v_p, and hence $v_1 \cdots v_{p-1} \notin V(T)$, which implies that $p = 5$. Now, if $i \in \{3, 4\}$ then $v_1 \cdots v_{i-1} R^* X t_5 \cdots t_9$ is a path of order greater than $9 + b$. The case $i = 3$ is illustrated in Figure 7. If $i \leq 2$, then $X R^* v_{i+1} \cdots v_5 t_6 \cdots t_9$ is a path of order greater than 9. If $j < p$ we get a similar contradiction. This proves that Case 2.1 does not occur.

2.2. $v_p \notin V(T)$. In this case, t_5 is on every path from x_1 to T in $\langle V(H) \cup \{x_1\} \rangle$ as well as on every path from x_{b+1} to T in $\langle V(H) \cup \{x_{b+1}\} \rangle$.

2.2.1. $t_5 \notin V(P)$. Then $V(P) \cap V(T) = \emptyset$. Thus $t_5 \notin N(x_1)$ (since otherwise $t_1 \cdots t_5 X \overleftarrow{P}$ would be a path with at least $11+b$ vertices). Thus $N_{V(T)}(x_1) = \emptyset$, and hence $w_r, w_{r+1} \notin V(T)$. Since both P and T are in the component H of $\langle A \rangle$, there is a $t_5 v_c$-path D in H for some vertex $v_c \in V(P)$, with all internal

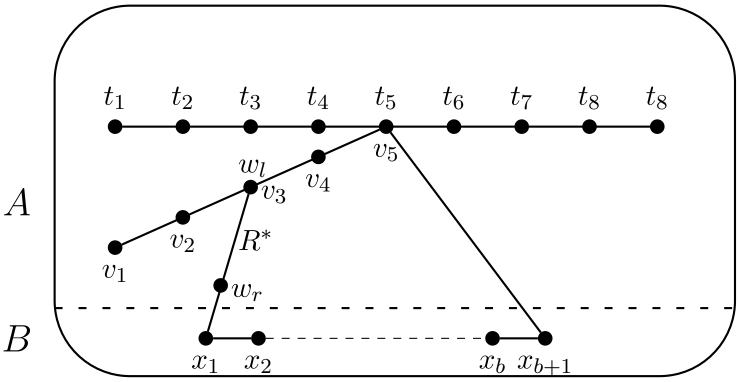


Figure 7. A step in the proof of Lemma 3.8 (A2.1).

vertices in $H-(V(T)\cup V(P))$. Since $\tau(\langle A\rangle)=9$, it follows that $c\leq 4$, and since $\tau(G)=9+b$, it follows that $c\geq p-2$. Thus $p=5$ or 6, and in either case, $D=t_5v_c$.

If $c=p-2$, then $t_1\cdots t_5v_{p-2}v_{p-1}v_p\overleftarrow{X}$ is a $(9+b)$-path, and hence $\{w_r,w_{r+1}\}=\{v_{p-1},v_{p-2}\}$. However, $t_1\cdots t_5v_{p-2}v_{p-1}\,Xv_p$ is a $(9+b)$-path, and hence $N_A(v_p)=\{v_{p-1},v_{p-2}\}$, contradicting Lemma 2.2(1d).

If $c=p-1$, then $c=4$ and $p=5$. Thus $t_1\cdots t_5v_4v_5\overleftarrow{X}$ is an $(8+b)$-path. It follows that $\{w_l,w_f\}=\{v_4,v_5\}$. If $w_l=v_p$, then it follows from Lemma 2.2 (1a and b) that R^* contains an internal vertex of Q. If $n(R^*)\geq 3$, then the path $t_1\cdots t_5v_4R^*X$ has at least $10+b$ vertices. Thus $n(R^*)=2$ and hence w_r is an internal vertex of Q, i.e., $w_r=v_{p+d}$ for some $d\geq 2$. It follows that $Xv_{p+1}\cdots v_{p+d}v_5v_4t_5\cdots t_9$ is a path with at least $10+b$ vertices. If $w_f=v_p$, we obtain a similar contradiction. Case 2.2.1 does therefore not occur.

2.2.2 $t_5=v_{p-1}$. As in the previous case, $t_5\notin V(Q)$, and hence $V(Q)\cap V(T)=\emptyset$.

If $w\in V(P)$, then $w=v_m$ for some $m\in\{2,\ldots,p-2\}$ (since $w\notin V(T)$ and $w\neq v_p$). Since every path from w to T contains the vertex $t_5=v_{p-1}$, it follows that v_{p-1} is the only vertex of P in $V(T)$. This implies that $v_1\cdots v_{p-1}t_6t_7t_8t_9$ is a path in H of order $p+3$, and hence $p\leq 6$ and $q\geq 3$. But then $\overleftarrow{Q}\overleftarrow{X}v_m\cdots v_{p-1}t_6t_7t_8t_9$ is a path with at least $10+b$ vertices.

Thus $w\notin V(P)$.

If $w_r\in V(T)$, then $w_r=t_5=v_{p-1}$, and hence in this case $w_l=v_{p-1}$. On the other hand, if $w_r\notin V(T)$, then it follows from the above that $w_r\notin V(P)$, which implies that $w_r\neq w_l$ and hence $n(R^*)\geq 2$. We let $w_l=v_i$. Now, if $1<i<p-1$, then $v_1\cdots v_{i-1}R^*Xv_pt_5\cdots t_9$ is a path of order greater than $9+b$, and if $i=1$, then $t_1\cdots t_4v_{p-1}v_{p-2}\cdots v_2R^*Xv_p$ is a path of order greater than

$9+b$. Thus $v_i \in \{v_{p-1}, v_p\}$. A similar argument shows that $v_j \in \{v_{p-1}, v_p\}$. Thus $\{w_l, w_f\} = \{v_i, v_j\} = \{v_{p-1}, v_p\}$.

Suppose $w_l = v_{p-1}$ and $w_f = v_p$. Then $t_5 \in V(R)$ (since $t_5 = v_{p-1} = w_l$) and $V(S) \cap V(T) = \emptyset$ (since any path from w_{r+1} to T contains the vertex t_5 which is in $V(R)$, and $V(S) \cap V(R) = \emptyset$). Recall that S^* is the $w_{r+1}w_f$-subpath of S. Thus x_1S^* is an x_1v_p-path in $\langle V(H) \cup \{x_1\}$. It therefore follows from Lemma 2.2(1a) that S^* contains an internal vertex of Q, and hence $q \geq 3$. Since $t_1 \cdots t_5 S^* X$ is a path of order $6 + n(S^*) + b$, it follows that $n(S^*) \leq 3$.

Now suppose $v_{p+1} \in V(S^*)$. Then $S^* = v_p v_{p+1} w_{r+1}$ and hence w_{r+1} is an internal vertex of Q. Since $t_1t_2t_3t_4t_5v_pv_{p+1}w_r$ is an 8-path in H, it follows that $w_{r+1} = v_{p+2}$. Clearly, $v_{p+3} \notin V(T)$ (since $t_5 = v_{p-1}$ and $t_i \notin N(w_{r+1})$ if $i \neq 5$). Thus $t_1t_2t_3t_4t_5v_pv_{p+1}Xv_{p+2}v_{p+3}$ is a $(10+b)$-path.

Thus $v_{p+1} \notin V(S^*)$, and hence $v_1v_2v_3v_4t_5S^*v_{p+1}\overleftarrow{X}$ is a $(9+b)$-path, and hence $w_r = t_5$. Also, $v_1v_2v_3v_4v_5S^*Xv_{p+1}$ is a path of order $7+n(S^*)+b$, which implies that $n(S^*) = 2$, and hence $v_p = w_{r+2}$. However, since $q \geq 3$ (as shown earlier) it follows that $s \geq 3$ (since $r + s = p + q$ and $p \geq r$). Since $t_1t_2t_3t_4t_5Xw_{r+1}v_{p+1}v_p$ is a $(9+b)$-path, $N_A(v_p) \subseteq \{w_{r+1}, v_{p+1}, t_5\}$. However, $t_5 \notin V(S)$, since $t_5 \in V(R)$, and also $v_{p+1} \notin N(v_p)$ (since otherwise $t_1t_2t_3t_4t_5v_pv_{p+1}\overleftarrow{X}v_{p+2}v_{p+3}$ would be a path of order $10+b$). Thus, w_{r+2} has no successor on the path S, and hence $S = S^* = w_{r+1}w_{r+2}$, contradicting that $s \geq 3$.

If $w_l = v_p$, then $w_f = v_{p-1}$ we get a similar contradiction. Thus Case 2.2.2 does not occur.

2.2.3. $t_5 = v_{p-2}$. Suppose $w_r = v_i \in V(P)$. If $i < p-2$, then $w_r \notin V(T)$ and t_5 is the only vertex of P in $V(T)$. But then $v_1 \cdots v_i X v_p v_{p-1} v_{p-2}\ t_6 \cdots t_9$ is a path with at least $b+10$ vertices. Thus $v_i \in \{v_{p-2}, v_{p-1}\}$.

If $w_r \notin V(P)$, then $t_1 \cdots t_4 v_{p-2} v_{p-1} v_p \overleftarrow{X} w_r$ is a $(b+9)$-path and hence $N_A(w_r) \subseteq \{v_{p-2}, v_{p-1}, v_p\}$. This implies that w_r is not an internal vertex of Q and hence, by Lemma 2.2(1a), $v_i \neq v_p$. Thus we again have $v_i \in \{v_{p-1}, v_{p-2}\}$. In this case $R^* = v_i w_r$.

Thus we have shown that $v_i \in \{v_{p-1}, v_{p-2}\}$ and either $w_r = v_i$ or $R^* = v_i w_r$. Similarly, $v_j \in \{v_{p-1}, v_{p-2}\}$ and either $w_{r+1} = v_j$ or $S^* = w_{r+1}v_j$.

If $w_r = v_{p-1}$, then $v_j = v_{p-2} = t_5$ and hence $V(R) \cap V(T) = \emptyset$. But then $t_1 \cdots t_5 R$ is a path of order at least 10 in H. Thus $w_r \neq v_{p-1}$.

If $R^* = w_r v_{p-1}$, then $t_1 \cdots t_5 v_{p-1} w_r X$ is an $(8+b)$-path and hence either $q = 1$ or $v_{p+2} = w_r$. If the former, then $p = 8$ and hence $P\overleftarrow{X}w_r$ is a $(10+b)$-path. If the latter, then $t_1 \cdots t_4 S^* v_{p+1} w_r v_{p-1} v_p$ is a $(10+b)$-path in G. Thus $R^* \neq w_r v_{p-1}$ and, similarly, $S^* \neq w_{r+1} v_{p-1}$.

It follows that $w_{r+1} = v_{p-1}$, and either $w_r = v_{p-2}$ or $R^* = w_r v_{p-2}$.

If $R^* = w_r v_{p-2}$, then (X, P, Q, R, S, T) is the configuration A1 in Figure 4.

If $w_r = v_{p-2}$, then, by Lemma 2.2(1d), v_p is an internal vertex of either R or

S. In this case (X, P, Q, R, S, T) is the configuration A2.

2.2.4. $t_5 = v_{p-3}$. In this case $t_1 \cdots t_5 v_{p-2} v_{p-1} v_p \overleftarrow{X}$ is a $(9+b)$-path, and hence $w_r, w_{r+1} \in \{v_{p-1}, v_{p-2}, v_{p-3}\}$.

If $\{w_r, w_{r+1}\} = \{v_{p-2}, v_{p-1}\}$, then $t_1 \cdots t_5 v_{p-2} v_{p-1} X v_p$ is a $(9+b)$-path and hence $N_A(v_p) \subset \{v_{p-1}, v_{p-2}\}$, contradicting Lemma 2.2(1d).

If $\{w_r, w_{r+1}\} = \{v_{p-3}, v_{p-1}\}$, then, by Lemma 2.2(1c), v_{p-2} has a neighbour z in $A - \{v_{p-3}, v_{p-1}\}$. If $z = v_p$, then $t_1 \cdots t_5\ v_{p-2} v_p v_{p-1} X v_{p+1}$ is a $(10+b)$-path in G, and if $z \neq v_p$, then $t_1 \cdots t_5 X v_p v_{p-1}\ v_{p-2} z$ is a $(10+b)$-path in G.

Thus the only possibility is that $\{w_r, w_{r+1}\} = \{v_{p-3}, v_{p-2}\}$, and then (X, P, Q, R, S, T) is the configuration A3.

Case 3. $h = 6$. This case does not occur. Due to symmetry, the proof is similar to the proof that the case $h = 4$ does not occur.

(B) Suppose (X, P, Q, R, S, T) is a B-configuration. Then, by Definition 3.4, neither $\{w_r, w_{r+1}\}$ nor $\{v_p, v_{p+1}\}$ is a pair of consecutive vertices of T and $\{w_r, w_{r+1}\} = \{t_g, t_h\}$ for some pair g, h such that $2 \leq g \leq h - 2 \leq 6$.

It follows from Remark 3.2 and Lemma 2.3(2) that no two vertices in the set $\{t_g, t_h, v_p, v_{p+1}\}$ is a pair of consecutive vertices of T.

If $g = 2$, then $t_9 t_8 \cdots t_2 X$ is a $(9+b)$-path, and hence, in this case, $\{v_p, v_{p+1}\} \subset T$. If $g \geq 3$, then it follows from the condition (2) of Definition 3.4(b) that at least one of v_p and v_{p+1} is not in $V(T)$. We therefore only need to consider the following cases.

Case 1. $\{g, h\} = \{2, 4\}$ *and* $\{v_p, v_{p+1}\} = \{t_6, t_8\}$. By Lemma 2.3(3) and (5c), the set $\{t_1, t_3, t_5, t_7, t_9\}$ is an independent set, and by Lemma 2.3(4a and b), neither t_3 nor t_5 nor t_7 has a neighbour in $H - V(T)$. Also, it is easily seen that neither t_2 nor t_4 nor t_8 is adjacent to a vertex of a 2-path in $H - V(T)$. Thus $H - \{t_2, t_4, t_6, t_8\}$ is an independent set, and hence each component of $L - \{t_2, x, t_4, t_6, t_8\}$ consists of a single vertex. However, since $t_2 x t_4$ is a segment of the path L, Proposition 2.1 implies that $L - \{t_2, x, t_4, t_6, t_8\}$ has at most 4 components, and hence $n(L) \leq 9$. This contradiction shows that this case does not occur.

Case 2. $\{g, h\} = \{2, 6\}$ *and* $\{v_p, v_{p+1}\} = \{t_4, t_8\}$. If t_5 has no neighbour in $H - V(T)$ and $t_3 t_7 \notin E(H)$, then $H - \{t_2, t_4, t_6, t_8\}$ is an independent set and we obtain a similar contradiction as in Case 1.

If t_5 has a neighbour in $H - V(T)$, then (X, P, Q, R, S, T) is the configuration B1. If $t_3 t_7 \in E(H)$, then (X, P, Q, R, S, T) is the configuration B2.

Case 3. $\{g, h\} = \{2, 8\}$ *and* $\{v_p, v_{p+1}\}) = \{t_4, t_6\}$. In this case, $H - \{t_2, t_4, t_6, t_8\}$ is an independent set. Now consider the 10-path $L' = P x_{b+1} Q$. Then $t_4 x_{b+1} t_6$ is a segment of L' and hence, by Lemma 2.1, $L' - \{t_2, t_4, x_{b+1}, t_6, t_8\}$

has at most four components, each consisting of a single vertex. But then $n(L') \leq 9$. This case does therefore not occur.

Case 4. $\{g, h\} = \{3, 5\}$, $v_p = t_7$ *and* $v_{p+1} \notin V(T)$. Since $t_3x_1t_5$ is a segment of L, it follows from Proposition 2.1 that $L - \{t_3, x_1, t_5, t_7\}$ has at most 3 components.

By Lemma 2.3(4a), neither t_4 nor t_6 is adjacent to an end-vertex of a 2-path in $H - \{t_1 \cdots t_7\}$. In particular, $t_8, t_9 \notin N(t_4, t_6)$. By Lemma 2.3(4b), $t_1, t_2 \notin N(t_6)$. Also, $t_1, t_2 \notin N(t_4)$, since otherwise $t_9t_8 \cdots t_4v_2t_3Xv_{p+1}$ or $t_9t_8 \cdots t_4v_1v_2t_3Xv_{p+1}$ would be a path of order greater than $9 + b$ in G. Thus, each component of $L - \{t_3, x_1, t_5, t_7\}$ has at most 2 vertices. But each such component of order 2 has at least one vertex of degree 1 in H. Since L has at most two end-vertices, it follows that $L - \{t_3, x_1, t_5, t_7\}$ has at most two components of order two. Thus $n(L) \leq 4 + 2(2) + 1 = 9$. This contradiction show that this case does not occur.

Case 5. $\{g, h\} = \{3, 5\}$, $v_p = t_8$ *and* $v_{p+1} \notin V(T)$. By Lemma 2.2(1c), t_4 has a neighbour in $H - \{t_3, t_5\}$. It is easily checked that t_6 is the only possible neighbour of t_4 in $H - \{t_3, t_5\}$. Thus $t_6t_4 \in E(H)$, and hence (X, P, Q, R, S, T) is the configuration B3.

Case 6. $\{g, h\} = \{3, 6\}$, $v_p = t_8$ *and* $v_{p+1} \notin V(T)$. Lemma 2.2(1c) implies that t_4 or t_5 has a neighbour in $H - \{t_3, t_6\}$.

If $t_7 \in N(t_4)$, then (X, P, Q, R, S, T) is the configuration B4.

If $t_2 \in N(t_5)$, then (X, P, Q, R, S, T) is the configuration B5.

If $t_7 \notin N(t_4)$ and $t_2 \notin N(t_5)$, then inspection show that, since $n(L) = 10$, it is necessary for L to contain the edge t_8t_4 as well as an edge v_5z for some $z \in H - V(T)$. So in this case (X, P, Q, R, S, T) is the configuration B6.

Case 7. $\{g, h\} = \{3, 7\}$, $v_p = t_5$ *and* $v_{p+1} \notin V(T)$. If $t_4t_8 \in E(H)$, then (X, P, Q, R, S, T) is the configuration B7. If $t_2t_6 \in E(H)$, then (X, P, Q, R, S, T) is the configuration B8.

Now suppose that neither t_2t_6 nor t_4t_8 is in $E(H)$. Then $N_{V(T)}(t_2) \subseteq \{t_1, t_3, t_5, t_7\}$, $N_{V(T)}(\{t_4, t_6\}) \subseteq \{t_3, t_5, t_7\}$ and $N_{V(T)}(t_8) \subseteq \{t_3, t_5, t_7, t_9, \}$.

If Z is a path in $H - V(T)$ having an end-vertex adjacent to a vertex $v_i \in V(T)$, then, since $t_9t_8t_7t_6t_5\overleftarrow{X}t_3$ and $t_1t_2t_3t_4t_5\overleftarrow{X}t_7$ are $(7+b)$-paths, and $t_1t_2t_3Xt_5$ is a $(5 + b)$-path, it follows that

$$n(Z) \leq \begin{cases} 3 & \text{if } i = 5, \\ 2 & \text{if } i \in \{3, 7\}, \\ 1 & \text{if } i \in \{2, 4, 6, 8\}. \end{cases}$$

Clearly, $q = 1$ and hence $p = 8$. Since $P\overleftarrow{X}$ is a $(9 + b)$-path, it follows that $t_3, t_7 \in V(P)$. Since $t_5 = v_p$, it follows from Proposition 2.1 that $P - \{t_3, t_5, t_7\}$ has at most three segments. Now suppose $P - \{t_3, t_5, t_7\}$ has a segment with more than 2 vertices. Then that segment is a 3-path Z in $\langle A \rangle$, and neither end-vertex of Z has a neighbour in T other than t_5, which implies that $p \leq 4$. This contradiction implies that no segment of $P-\{t_3,t_3,t_7\}$ has more than 2 vertices. Moreover, each segment of $P-\{t_3,t_5,t_7\}$ with two vertices is an end-segment of P, since it contains a vertex of degree 1 in H. Thus, since t_5 is an end-vertex of P, it follows that $P-\{t_3,t_3,t_7\}$ has at most one segment of order 2. Thus $n(P)\leq 2+1+1+3=7$, contradicting that $n(P)=8$. This case does therefore not occur.

Case 8. $\{g,h\}=\{4,6\}$, $v_p=t_8$ *and* $v_{p+1}\notin V(T)$. Lemma 2.2 implies that t_5 has a neighbour in $A-\{t_4,t_6\}$. Thus, since G has no $(10+b)$-path, $t_3t_5\in E(H)$ and (X,P,Q,R,S,T) is the configuration B9.

(C) Suppose (X,P,Q,R,S,T) is a C-configuration. Then $\{w_r,w_{r+1}\}=\{t_h, t_{h+1}\}$ for some $h\in\{2,3,4\}$.

Now suppose (X,P,Q,R,S,T) is not a nice C-configuration and consider the expanded C-configuration (X,P,Q,R,S,T,c,d), defined and labelled as in Definition 3.7.

Since the paths P and T are both in the component H of $\langle A\rangle$, there is an $x_{b+1}t_k$-path in G with all its internal vertices in $A-V(T)$, for some $k\in\{2,\ldots,8\}$. The following claims follow from Lemma 2.3 and our assumption that $h\leq 4$.

Claim 1. (a) $h\notin\{1,c-2,c-1,d,d+1,5,6,7,8,9\}$.

(b) $k\notin\{1,c-1,d+1,h-1,h,h+1,h+2,9\}$.

Claim 2. (a) *If $k>h$, then $h\notin\{d-2,d-1\}$ and $k\neq c+1$.*

(b) *If $k<h$, then $h\notin\{c,c+1\}$ and $k\neq d-1$.*

Claim 3. *If $d\leq c$, then $h\notin\{c,c+1,d-2,d-1\}$ and $k\notin\{c+1,d-1\}$.*

Claim 4. $d\neq c-1$.

Claim 5. *If $d=c+1$, then either $h<c$ and $k\leq c$, or $h\geq d$ and $k\geq d$.*

We consider the following possibilities.

$c=3, d\leq 5$. This case does not occur, because of the following. Claims 1 and 2 imply that $k\neq 2$ and $h\neq 2$. Thus $h\in\{3,4\}$ and $k>h$. It therefore follows from Claims 1 and 2(a) that $d\neq\{h-1,h,h+1,h+2,\}$. Thus $d\geq 6$ if $h=3$ and $d\geq 7$ if $h=4$.

$c=3, d=6$. In this case $h=3$, and $k\in\{6,8\}$. If both v_p and v_{p+1} are in $V(T)$, then $\{v_{p+1},v_p\}=\{t_6,t_8\}$. If $z\in N(t_7)$ for some $z\in A-V(T)$, then $zt_7t_8t_9t_4\overleftarrow{X}t_4t_3t_2t_1$ is a $(10+b)$-path. Thus $N_A(t_7)\subset V(T)$. However, $t_2,t_3,t_4,t_5\notin N(t_7)$ by Lemma 2.3(3), and $t_1,t_9\notin N(t_7)$ by Lemma 2.3(5c). Thus $N_A(p_7)=\{t_6,t_8\}$, contradicting Lemma 2.2(2c).

Thus v_p is either t_6 or t_8 and $v_{p+1}\notin V(T)$. Thus $q=1$ and $p=8$.

If $v_p=t_8$, then $(X,P,Q,R,S,T,3,6$ is the configuration C1.

If $v_p = t_6$, then neither t_7 nor t_9 has a neighbour in $H - \{t_6, t_7, t_8, t_9\}$. Thus, if $t_5t_8 \notin E(G)$, then any path in H that ends at v_p and contains one or more vertices in $\{t_7, t_8, t_9\}$ has at most 4 vertices. Thus $t_7, t_8, t_9 \notin V(P)$, and hence Pt_7t_8 is a 10-path in H. This contradiction shows that $t_5t_8 \in E(G)$ and then $(X, P, Q, R, S, T, 3, 6)$ is the configuration C2.

$c = 3, d = 7$. In this case, either $h = 3$ and $k \in \{6, 7\}$, or $h = 4$ and $k = 7$.

• Suppose $h = 3$ and $\{v_p, v_{p+1}\} = \{t_6, t_7\}$. In this case $(X, P, Q, R, S, T, 3, 7)$ is the configuration C3.

• Suppose $h = 3$, $v_p = t_6$ and $v_{p+1} \notin V(T)$. If $V(Q) \cap V(T) = \emptyset$, then $q = 1$ and $p = 8$. Since G has no $(10 + b)$-path, t_3 is the only neighbour of t_1 or t_2 in $H - \{t_1, t_2\}$. Also, t_7 is the only possible neighbour of t_8 or t_9 in $H - \{t_8, t_9\}$.

Now $t_3, t_4 \in V(P)$, since otherwise $P\overleftarrow{X}t_3$ or $P\overrightarrow{X}t_4$ would be a $(10 + b)$-path in G. This implies that there is a t_3t_7-path in $H - v_p$ and hence $t_3t_7 \in E(P)$.

Let P' be the t_3t_6-subpath of P. If $t_7 \in V(P')$, then $P' = t_3t_7t_6 = v_6v_7v_8$, and hence $v_1 \cdots v_6$ is a 6-path in $H - \{t_6, t_7\}$, with $t_3 = v_6$. But it follows from Lemma 2.3 that neither t_4 nor t_5 is adjacent to an end-vertex of a 3-path in $H - \{t_1, \ldots t_6\}$. But then any path in $H - \{v_7, v_8\}$ ending at v_6 has at most 5 vertices. This contradiction shows that $t_7 \notin V(P')$, Thus, since $t_7t_3 \in V(P)$ and any path in $H - \{t_3, t_6\}$ ending at t_7 has at most 3 vertices, it follows that $t_3 = v_4$, and hence P' is a t_3t_6-path of order 5 in $H - \{t_1, t_2, t_7, t_8, t_9\}$. But then $t_1t_2P't_7t_8t_9$ is a 10-path in H.

This contradiction implies that $V(Q) \cap V(T) \neq \emptyset$, and hence $v_{p+2} = t_7$. Thus, in this case, $(X, P, Q, R, S, T, 3, 7)$ is the configuration C3, with the edge $x_{b+1}t_7$ subdivided. (It is unnecessary for us to consider this case separately, as will become clear in the proof of our main theorem.)

• Suppose $h = 3$, $v_p = t_7$ and $v_{p+1} \notin V(T)$. Then $q = 1$ and hence $p = 8$, and it is easily seen that $N_A(\{t_8, t_9\}) = \{v_p, t_8, t_9\}$. Thus, if t_8 or t_9 is in $V(P)$, then P would have at most 3 vertices. Thus $t_8, t_9 \notin V(P)$ and hence Pt_8t_9 is a 10-path in H. This situation does therefore not occur.

• Suppose $h = 3$ and $v_p, v_{p+1} \notin V(T)$. Then $v_{p-1} = t_7$ and $(X, P, Q, R, S, T, 3, 7)$ is the configuration C4.

• Suppose $h = 4$. Then $v_p = t_7$ and $V(Q) \cap V(T) = \emptyset$. It is easily seen that $N_A(\{t_8, t_9\}) = \{v_p, t_8, t_9\}$, which implies that $t_8, t_9 \notin V(P)$. Thus, $p \leq 7$ (since otherwise Pt_8t_9 would be a 10-path in H), and hence $q = 2$ and $(X, P, Q, R, S, T, 3, 7)$ is the configuration C5.

$c = 4, d \leq 6$. By Claim 1, $h = 4 = c$ and hence $d > 4$ by Claim 3. Thus, it follows from Claim 5 that $d \neq c + 1$. If $d = 6$, then $h = d - 2$ and hence Claim 2(a) implies that $k < h$, contradicting Claim 2(b). This case does therefore not occur.

$c = 4, d = 7$. By Claim 1, $h = 4$, and hence it follows from Claim 2(b) that $k > h$, and hence $k = 7$. Thus, in this case, $(X, P, Q, R, S, T, 4, 7)$ is the configuration C5. (The proof is similar to that of the case $c = 3, d = 7, h = 4$.)

$c = 5, d \leq 4$. This case does not occur, since it follows from Claim 1(a) that $h = 2$ and hence $d \neq 2$, by Claim 1, but $d \notin \{3, 4\}$, by Claim 3.

$c = 5, d = 5$. Since $h = 2$, both v_p and v_{p+1} are in $V(T)$. By Claim 1(b), $k \in \{5, 7, 8\}$. If $\{v_p, v_{p+1}\} = \{t_5, t_8\}$, then $N_A\{t_6, t_7\} = \{t_5, t_6, t_7, t_8\}$, contradicting Lemma 2.2(2c). If $\{v_p, v_{p+1}\} = \{t_5, t_7\}$, then $N_A(t_6) = \{t_5, t_7\}$, which also contradicts Lemma 2.2(2c). Thus $\{v_p, v_{p+1}\} = \{t_7, t_8\}$ and hence $(X, P, Q, R, S, T, 5, 5)$ is the configuration C6.

$c = 5, d = 6$. It follows from Claim 1(b) that $h = 2$, and hence $v_{p+1}, v_p \in V(T)$. But, by Claim 5, the only possible neighbour of x_{b+1} in $V(T)$ is t_5. This case can therefore not occur.

$c = 5, d = 7$. Since $h = 2$, it follows from Claim 1(b) and 2(a) that $\{v_p, v_{p+1}\} = \{t_5, t_7\}$. But $N_A(t_6) = \{t_5, t_7\}$, contradicting Lemma 2.2(2c). This case does therefore not occur.

$c = 6$. In this case $h = 2$ or 3. In either case it follows from Claim 1(b) that $k \geq 6$ and from Claim 2(a) that $k \neq 7$. Hence $N_{V(T)}(x_{b+1}) \subseteq \{t_6, t_8\}$.

Furthermore, it follows from Claims 1(a), 3 and 4 that $d \geq 6$.

If $\{v_p, v_{p+1}\} = \{t_6, t_8\}$, then $d = 6$ and $N_A(t_7) = \{t_6, t_8\}$, contradicting Lemma 2.2(2c).

If $v_p \notin V(T)$, then $h = 3$ and $t_9t_8t_7t_6t_1t_2t_3t_4Xv_{p+1}$ is a $(10 + b)$-path. If $v_{p+1} \notin V(T)$, we obtain a similar contradiction. The case $c = 6$ does therefore not occur.

$c = 7$. If $d = 7$, then Claim 1(b) implies that $k \neq 8$, and if $d < 7$, then Claim 3 implies that $k \neq 8$.

• Suppose $h = 2$. Then both v_p and v_{p+1} are in $V(T)$, and hence $\{v_p, v_{p+1}\} = \{t_5, t_7\}$. Thus $N_A(t_6) = \{t_5, t_7\}$, contradicting Lemma 2.2(2c).

• Suppose $h = 3$. Then it follows from Claims 1 and 3 that $k = 7$ and $d = 7$. Thus t_7 is either v_p or v_{p-1} and $v_{p+1} \notin V(T)$. In either case, $q = 1$ and $p = 8$.

If $v_p = t_7$, then $t_4 \in V(P)$ (otherwise $P\overleftarrow{X}t_4$ would be a $(10 + b)$-path). Clearly, $N_A(\{t_8, t_9\}) = \{t_7, t_8, t_9\}$ and hence any path in H containing t_4 as well as t_8 or t_9 also contains $t_7 = v_p$. This implies that $t_8, t_9 \notin V(P)$. But then Pt_8t_9 is a 10-path in H.

Thus $v \notin V(T)$ and $v_{p-1} = t_7$, and hence $(X, P, Q, R, S, T, 7, 7)$ is the configuration C7.

• Suppose $h = 4$. Then it follows from Claims 1(b) and 2(b) that $k \in \{2, 7\}$ and $d \in \{2, 7\}$.

Since $t_9t_8t_7t_1t_2t_3t_4t_5X$ is a $(9+b)$-path, both v_p and v_{p+1} are in $V(T)$. Thus $\{v_p, v_{p+1}\} = \{t_2, t_7\}$. Now, $N_A(t_1) = \{t_2, t_7\} = \{v_p, v_{p+1}\}$. Thus $t_1 \notin V(P)$, otherwise p would be 2.

Now suppose $t_1 \in V(Q)$. Then $Q = t_2t_1$ if $v_p = t_7$, and $Q = t_7t_1$ if $v_p = t_2$. In either case, the end-vertex t_1 of Q is adjacent to v_p, and hence $P\overleftarrow{Q}X$ is an $(a+b+1)$-path. On the other hand, if $t_1 \notin V(Q)$, then Pt_1Q is a 10-path in H. Thus case does therefore not occur.

$c = 8$. In this case, $v_1 \cdots v_8v_1$ is an $(a-1)$-cycle. Now, if $v_p \notin V(T)$, then Xv_p is a $(b+2)$-path, attached to this $(a-1)$-cycle at t_h. But then G contains an $(a+b+1)$-path. Thus $v_p \in V(T)$ and, similarly, $v_{p+1} \in V(T)$.

• Suppose $h = 2$. Then $v_p, v_{p+1} \in \{t_5, t_6, t_8\}$. If $\{v_p, v_{p+1}\} = \{v_5, v_6\}$, then it follows from Claims 1, 2 and 3 that the end-vertex t_9 of T is not incident with an external edge, contradicting our assumption.

If $\{v_p, v_{p+1}\} = \{t_5, t_8\}$, it follows from Claims 1, 2 and 3 that $d = 5$. Since $t_5x_{b+1}t_8$ is a segment of the 10-path $M = Px_{b+1}Q$, it follows that $M - \{t_2, t_5, x_{b+1}, t_8\}$ has at most 3 components. It is easily checked that there are no edges between the two sets $\{t_3, t_4\}$ and $\{t_6, t_7\}$ and no vertex in either of these two sets has a neighbour in $H - \{t_2, \dots, t_8\}$. Furthermore, $H - \{t_2, \dots, t_8\}$ is an independent set. Thus $M - \{t_2, t_5, x_{b+1}, t_8\}$ has at most two components of order 2, and hence $n(M) \le 4 + 2(2) + 1 = 9$, contradicting that $n(M) = 10$.

If $\{v_p, v_{p+1}\} = \{t_6, t_8\}$, then it follows from Claims 1, 2 and 3 that $d = 6$. But then $N_A(t_7) = \{t_6, t_8\}$, contradicting Lemma 2.2(2c).

• Suppose $h = 3$. Then $\{v_p, v_{p+1}\} = \{t_6, t_8\}$, and we obtain a similar contradiction as in the case where $h = 2$.

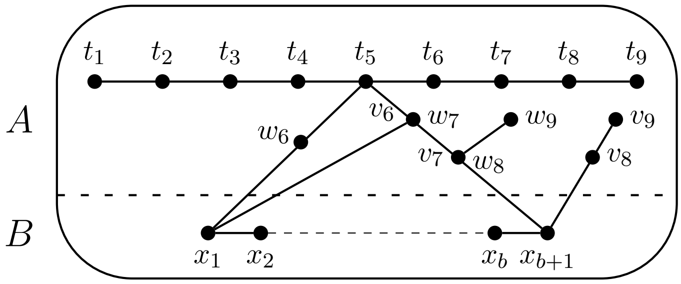
• Suppose $h = 4$. Then $\{v_p, v_{p+1}\} = \{t_2, t_8\}$ and it follows from Claims 1, 2 and 3 that $d = 2$. If $t_1 \notin V(P) \cup V(Q)$, then Pt_1Q is a 10-path in H, and if $t_9 \notin V(P) \cup V(Q)$, then Pt_9Q is a 10-path in H. Hence t_1 as well as t_9 are in $V(P) \cup V(Q)$. But $N_A(t_1) = N_A(t_9) = \{v_p, v_{p+1}\}$, which implies that $p = q = 2$.

These contradictions show that the case $c = 8$ does not occur. ■

Suppose (X, P, Q, R, S, T) is a complex configuration, defined and labelled as in Definition 3.3. If adding an external edge t_it_j to T creates a 10-path in $\langle A \rangle$ or a $(10+b)$-path in G, we say that t_it_j is a *forbidden edge*, and $i \sim j$ a *forbidden adjacency* for the given complex configuration.

For example, $2 \sim 6$ is a forbidden adjacency for an A1-configuration, since adding the edge t_2t_6 to the complex configuration in Figure 8(A1) creates the 10-path $t_9t_8t_7t_6t_2t_3t_4t_5v_6v_7$ in $\langle A \rangle$. In fact, it is easily checked that if $i \in \{1, 2, 3, 4\}$ and $j \in \{6, 7, 8, 9\}$, then $i \sim j$ is forbidden for each of the three A-configurations in Figure 8.

A1

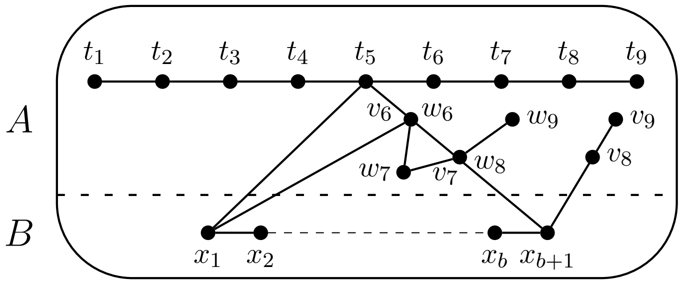


Forbidden adjacencies

$1 \sim i,\ i = 6, \ldots, 9$
$2 \sim i,\ i = 6, \ldots, 9$
$3 \sim i,\ i = 6, \ldots, 9$
$4 \sim i,\ i = 6, \ldots, 9$

$Rx_1S = t_1t_2t_3t_4t_5w_6x_1w_7w_8w_9$
$Px_{b+1}Q = t_1t_2t_3t_4t_5v_6v_7x_{b+1}v_8v_9$, with $v_6 = w_7$, $v_7 = w_8$

A2

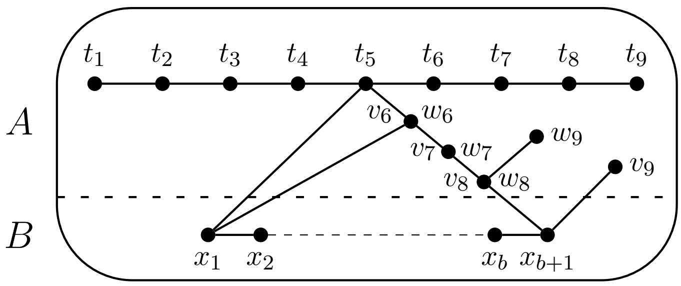


Forbidden adjacencies

$1 \sim i,\ i = 6, \ldots, 9$
$2 \sim i,\ i = 6, \ldots, 9$
$3 \sim i,\ i = 6, \ldots, 9$
$4 \sim i,\ i = 6, \ldots, 9$

$Rx_1S = t_1t_2t_3t_4t_5x_1w_6w_7w_8w_9$
$Px_{b+1}Q = t_1t_2t_3t_4t_5v_6v_7x_{b+1}v_8v_9$, with $v_6 = w_6$, $v_7 = w_8$

A3



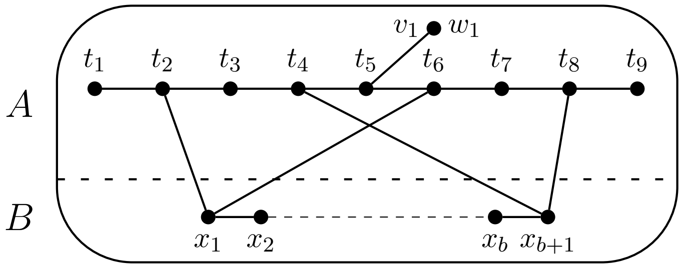
Forbidden adjacencies

$1 \sim i,\ i = 6, \ldots, 9$
$2 \sim i,\ i = 6, \ldots, 9$
$3 \sim i,\ i = 6, \ldots, 9$
$4 \sim i,\ i = 6, \ldots, 9$

$Rx_1S = t_1t_2t_3t_4t_5x_1w_6w_7w_8w_9$
$Px_{b+1}Q = t_1t_2t_3t_4t_5v_6v_7v_8x_{b+1}v_9$, with $v_i = w_i$, for $i = 6, 7, 8$

Figure 8. Configurations of Type A consisting of $(X, P, Q, R, S, T) = \langle E(X) \cup E(Rx_1S) \cup E(Px_{b+1}Q) \cup E(T) \rangle$.

B1

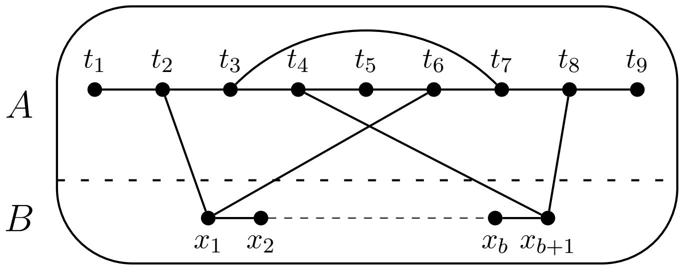


Forbidden adjacencies

$1 \sim i,\ i = 3, \ldots, 9$
$2 \sim i,\ i = 5, 7, 9$
$3 \sim i,\ i = 5, 6, 8, 9$
$4 \sim i,\ i = 7, 9$
$5 \sim i,\ i = 7, 8, 9$
$6 \sim 9$
$7 \sim 9$

$Rx_1S = w_1t_5t_4t_3t_2x_1t_6t_7t_8t_9$
$Px_{b+1}Q = v_1t_5t_6t_7t_8x_{b+1}t_4t_3t_2t_1$, $v_1 = w_1 \in A - V(T)$

B2

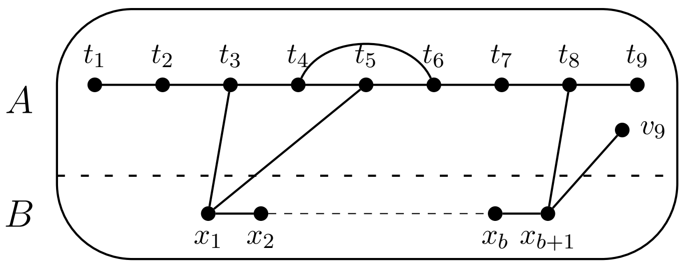


Forbidden adjacencies

$1 \sim i,\ i = 3, \ldots, 9$
$2 \sim i,\ i = 5, 7, 9$
$3 \sim i,\ i = 5, 6, 8, 9$
$4 \sim i,\ i = 7, 9$
$5 \sim i,\ i = 7, 8, 9$
$6 \sim 9$
$7 \sim 9$

$Rx_1S = t_9t_8t_7t_3t_4t_5t_6x_1t_2t_1$
$Px_{b+1}Q = t_1t_2t_3t_7t_6t_5t_4x_{b+1}t_8t_9$

B3



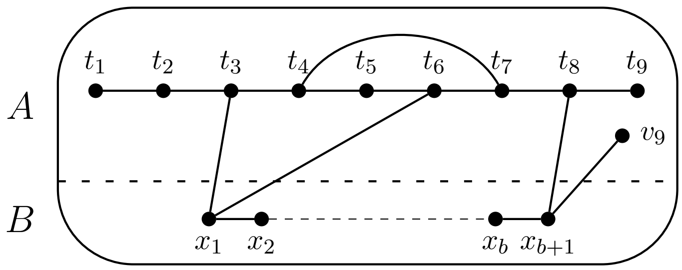
Forbidden adjacencies

$1 \sim i,\ i = 4, \ldots, 9$
$2 \sim i,\ i = 4, \ldots, 9$
$3 \sim i,\ i = 7, 8, 9$
$4 \sim i,\ i = 7, 8, 9$
$5 \sim i,\ i = 7, 8, 9$
$6 \sim 9$
$7 \sim 9$

$Rx_1S = t_9t_8t_7t_6t_4t_5x_1t_3t_2t_1$
$Px_{b+1}Q = t_1t_2t_3t_4t_5t_6t_7t_8x_{b+1}v_9$, $v_9 \in A - V(T)$

Figure 9. Configurations of Type B, consisting of $(X, P, Q, R, S, T) = \langle E(X) \cup E(Rx_1S) \cup E(Px_{b+1}Q) \cup E(T) \rangle$.

B4

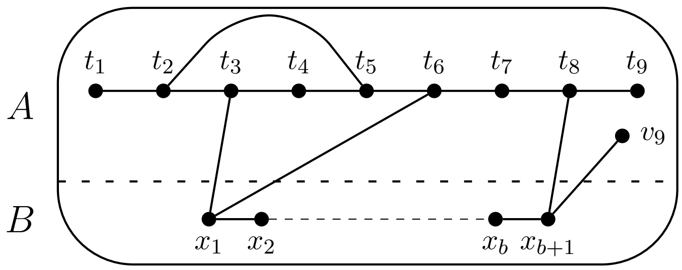


$Rx_1S = t_9t_8t_7t_4t_5t_6x_1t_3t_2t_1$
$Px_{b+1}Q = t_1t_2t_3t_4t_5t_6t_7t_8x_{b+1}v_9,\ v_9 \in A - V(T)$

Forbidden adjacencies

$1 \sim i,\ i = 4, \ldots, 9$
$2 \sim i,\ i = 4, \ldots, 9$
$3 \sim i,\ i = 5, 7, 8, 9$
$4 \sim 9$
$5 \sim i,\ i = 7, 8, 9$
$6 \sim 9$
$7 \sim 9$

B5

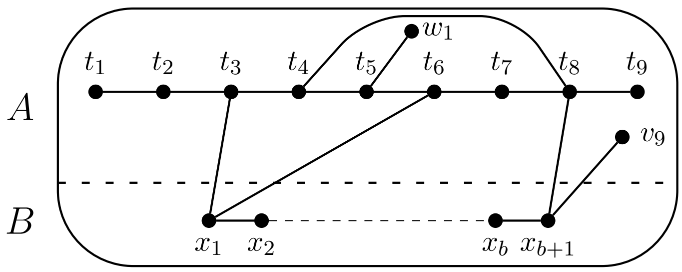


$Rx_1S = t_1t_2t_5t_4t_3x_1t_6t_7t_8t_9$
$Px_{b+1}Q = t_1t_2t_3t_4t_5t_6t_7t_8x_{b+1}v_9,\ v_9 \in A - V(T)$

Forbidden adjacencies

$1 \sim i,\ i = 3, \ldots, 9$
$2 \sim i,\ i = 4, 6, 7, 8, 9$
$3 \sim i,\ i = 7, 8, 9$
$4 \sim i,\ i = 6, 7, 8, 9$
$5 \sim i,\ i = 7, 8, 9$
$6 \sim 9$
$7 \sim 9$

B6



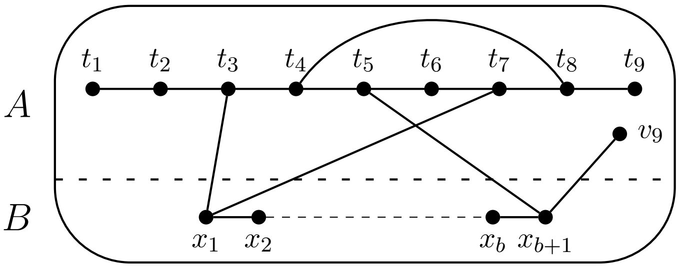
$Rx_1S = w_1t_5t_4t_8t_7t_6x_1t_3t_2t_1$
$Px_{b+1}Q = t_1t_2t_3t_4t_5t_6t_7t_8x_{b+1}v_9,\ v_9 \in A - V(T)$

Forbidden adjacencies

$1 \sim i,\ i = 4, \ldots, 9$
$2 \sim i,\ i = 4, \ldots, 9$
$3 \sim i,\ i = 5, 7, 9$
$4 \sim i,\ i = 6, 7, 9$
$5 \sim i,\ i = 7, 8, 9$
$7 \sim 9$

Figure 10. Configurations of Type B, consisting of $(X, P, Q, R, S, T) = \langle E(X) \cup E(Rx_1S) \cup E(Px_{b+1}Q) \cup E(T)\rangle$ — continued.

B7

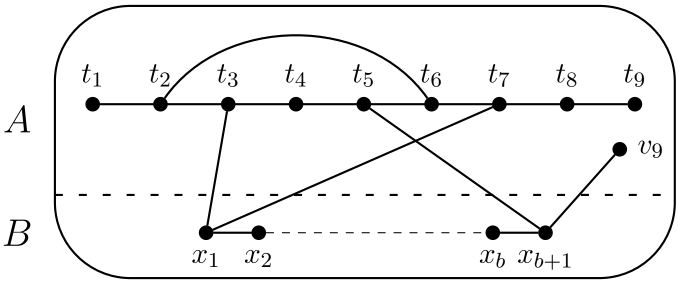


$Rx_1S = t_9t_8t_4t_5t_6t_7x_1t_3t_2t_1$
$Px_{b+1}Q = t_1t_2t_3t_4t_8t_7t_6t_5x_{b+1}v_9$, $v_9 \in A - V(T)$

Forbidden adjacencies

$1 \sim i$, $i = 4, \ldots, 9$
$2 \sim i$, $i = 4, 5, 7, 8, 9$
$3 \sim i$, $i = 6, 8, 9$
$4 \sim i$, $i = 6, 7, 9$
$5 \sim i$, $i = 8, 9$
$6 \sim i$, $i = 8, 9$
$7 \sim 9$

B8

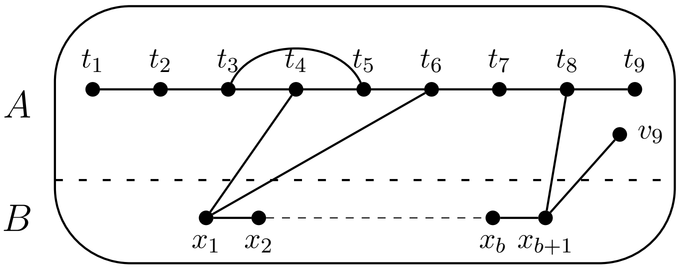


$Rx_1S = t_1t_2t_6t_5t_4t_3x_1t_7t_8t_9$
$Px_{b+1}Q = t_9t_8t_7t_6t_2t_3t_4t_5x_{b+1}v_9$, $v_9 \in A - V(T)$

Forbidden adjacencies

$1 \sim i$, $i = 3, \ldots, 9$
$2 \sim i$, $i = 4, 5, 7, 8, 9$
$3 \sim i$, $i = 6, 8, 9$
$4 \sim i$, $i = 6, 7, 9$
$5 \sim i$, $i = 8, 9$
$6 \sim i$, $i = 8, 9$

B9



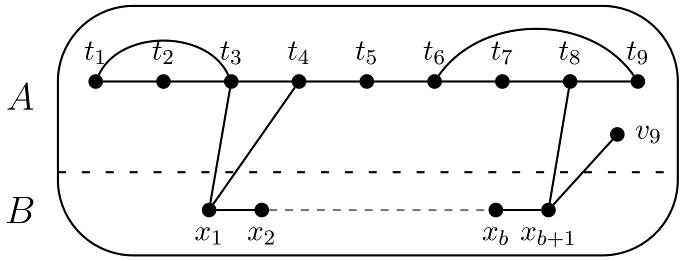
$Rx_1S = t_1t_2t_3t_5t_4x_1t_6t_7t_8t_9$
$Px_{b+1}Q = t_1t_2t_3t_4t_5t_6t_7t_8x_{b+1}v_9$, $v_9 \in A - V(T)$

Forbidden adjacencies

$1 \sim i$, $i = 4, \ldots, 9$
$2 \sim i$, $i = 4, \ldots, 9$
$3 \sim i$, $i = 7, 8, 9$
$4 \sim i$, $i = 7, 8, 9$
$5 \sim i$, $i = 7, 8, 9$
$6 \sim 9$
$7 \sim 9$

Figure 11. Configurations of Type B, consisting of $(X, P, Q, R, S, T) = \langle E(X) \cup E(Rx_1S) \cup E(Px_{b+1}Q) \cup E(T) \rangle$ — continued.

C1

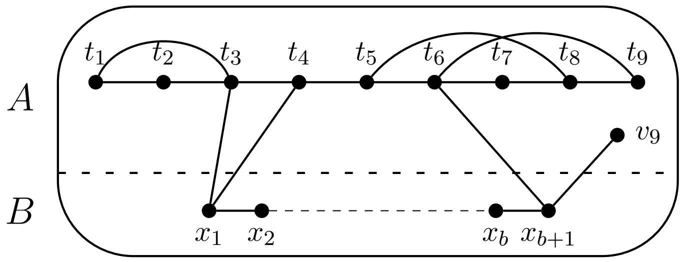


Forbidden adjacencies

$1 \sim i,\ i = 4, \ldots, 9$
$2 \sim i,\ i = 4, \ldots, 9$
$3 \sim i,\ i = 5, 7, 8, 9$
$4 \sim i,\ i = 7, 9$
$5 \sim i,\ i = 7, 9$
$7 \sim 9$

$Rx_1S = t_9t_8t_7t_6t_5t_4x_1t_3t_2t_1$, $Px_{b+1}Q = t_1t_2t_3t_4t_5t_6t_7t_8x_{b+1}v_9$
$v_9 \in A - V(T)$, external edges t_1t_3 and t_9t_6

C2

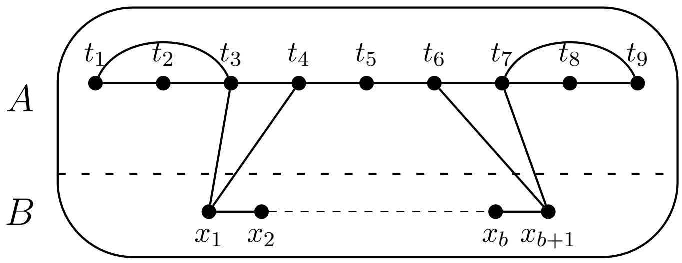


Forbidden adjacencies

$1 \sim i,\ i = 4, \ldots, 9$
$2 \sim i,\ i = 4, \ldots, 9$
$3 \sim i,\ i = 5, \ldots, 9$
$4 \sim i,\ i = 7, 9$
$5 \sim i,\ i = 7, 9$
$7 \sim 9$

$Rx_1S = t_9t_8t_7t_6t_5t_4x_1t_3t_2t_1$, $Px_{b+1}Q = t_1t_2t_3t_4t_5t_8t_7t_6x_{b+1}v_9$
$v_9 \in A - V(T)$, external edges t_1t_3 and t_9t_6

C3



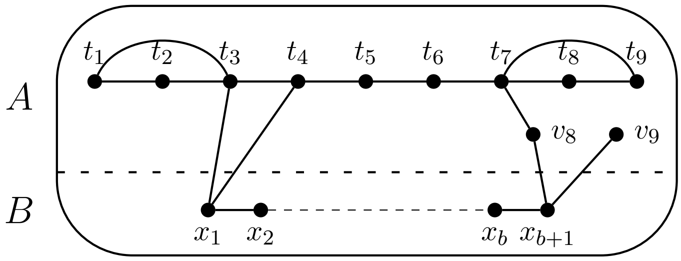
Forbidden adjacencies

$1 \sim i,\ i = 4, \ldots, 9$
$2 \sim i,\ i = 4, \ldots, 9$
$3 \sim i,\ i = 5, 6, 8, 9$
$4 \sim i,\ i = 7, 8, 9$
$5 \sim i,\ i = 7, 8, 9$
$6 \sim i,\ i = 8, 9$

$Rx_1S = t_9t_8t_7t_6t_5t_4x_1t_3t_2t_1$, $Px_{b+1}Q = t_1t_2t_3t_4t_5t_6x_{b+1}t_7t_8t_9$
external edges t_1t_3 and t_9t_7

Figure 12. Expanded C-configurations consisting of $(X, P, Q, R, S, T, c, d) = \langle E(X) \cup E(Rx_1S) \cup E(Px_{b+1}Q) \cup E(T) \cup \{t_1t_c, t_9t_d\}\rangle$.

C4

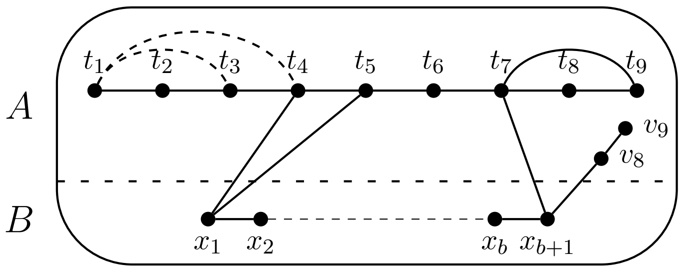


Forbidden adjacencies

$1 \sim i,\ i = 4, 5, 6, 8, 9$
$2 \sim i,\ i = 4, 5, 6, 8, 9$
$3 \sim i,\ i = 5, 6, 8, 9$
$4 \sim i,\ i = 8, 9$
$5 \sim i,\ i = 8, 9$
$6 \sim i,\ i = 8, 9$

$Rx_1S = t_9t_8t_7t_6t_5t_4x_1t_3t_2t_1$, $Px_{b+1}Q = t_1t_2t_3t_4t_5t_6t_7v_8x_{b+1}v_9$
$v_8, v_9 \in A - V(T)$, external edges t_1t_3 and t_9t_7

C5

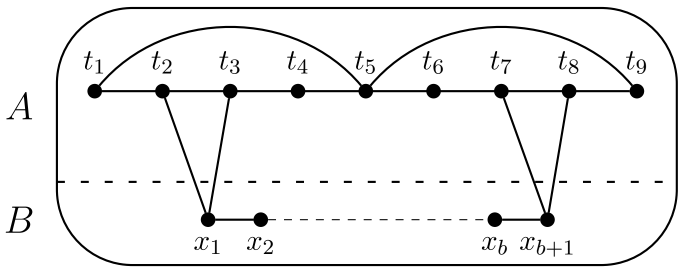


Forbidden adjacencies

$1 \sim i,\ i = 5, \ldots, 9$
$2 \sim i,\ i = 5, \ldots, 9$
$3 \sim i,\ i = 5, \ldots, 9$
$4 \sim i,\ i = 6, \ldots, 9$
$5 \sim i,\ i = 8, 9$
$6 \sim i,\ i = 8, 9$

$Rx_1S = t_9t_8t_7t_6t_5x_1t_4t_3t_2t_1$, $Px_{b+1}Q = t_1t_2t_3t_4t_5t_6t_7x_{b+1}v_8v_9$
$v_8, v_9 \in A - V(T)$, external edges (t_1t_3 or t_1t_4) and t_9t_7

C6

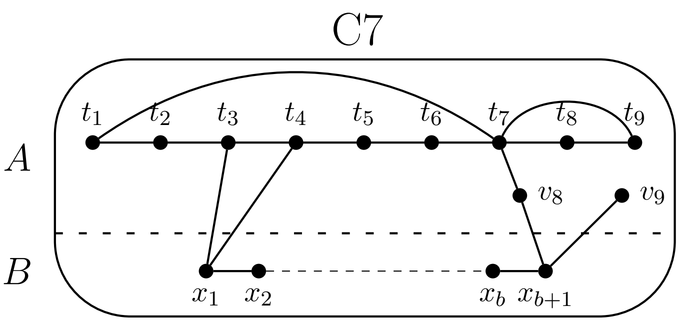


Forbidden adjacencies

$1 \sim i,\ i = 3, 4, 6, 7, 8, 9$
$2 \sim i,\ i = 4, 6, 7, 8, 9$
$3 \sim i,\ i = 6, \ldots, 9$
$4 \sim i,\ i = 6, \ldots, 9$
$6 \sim i,\ i = 8, 9$
$7 \sim i,\ i = 9$

$Rx_1S = t_9t_8t_7t_6t_5t_4t_3x_1t_2t_1$, $Px_{b+1}Q = t_1t_2t_3t_4t_5t_6t_7x_{b+1}t_8t_9$
external edges t_1t_5 and t_9t_5

Figure 13. Expanded C-configurations consisting of $(X, P, Q, R, S, T, c, d) = \langle E(X) \cup E(Rx_1S) \cup E(Px_{b+1}Q) \cup E(T) \cup \{t_1t_c, t_9t_d\}\rangle$ — continued.



Forbidden adjacencies

$1 \sim i,\ i = 4, 5, 6, 8, 9$
$2 \sim i,\ i = 4, 5, 6, 8, 9$
$3 \sim i,\ i = 5, 6, 8, 9$
$4 \sim i,\ i = 8, 9$
$5 \sim i,\ i = 8, 9$
$6 \sim i,\ i = 8, 9$

$Rx_1S = t_9t_8t_7t_6t_5t_4x_1t_3t_2t_1$, $Px_{b+1}Q = t_1t_2t_3t_4t_5t_6t_7v_8x_{b+1}v_9$
$v_8, v_9 \in A - V(T)$, external edges t_1t_7 and t_9t_7

Figure 14. Expanded C-configurations consisting of $(X, P, Q, R, S, T, c, d) = \langle E(X) \cup E(Rx_1S) \cup E(Px_{b+1}Q) \cup E(T) \cup \{t_1t_c, t_9t_d\}\rangle$ — continued.

For easy reference, forbidden adjacencies for each A- and B-configuration and each expanded C-configuration are listed in Figures 8, 9, 10, 11, 12 and 13.

If attaching a d-path to T at a vertex $t_i \in V(T)$ creates a 10-path in $\langle A\rangle$ or a $(10+b)$-path in G, we say that a d-path attached to t_i is a *forbidden attachment* for the given complex configuration. For example, a 4-path attached to T at t_4 is a forbidden attachment for a B9-configuration, because, if $z_1z_2z_3t_4$ is a path in the complex configuration in Figure 11(B9), with $z_1z_2, z_3 \in H - V(T)$, then $z_1z_2z_3t_4t_3t_5t_6t_7t_8\overleftarrow{X}$ is a $(10+b)$-path. In fact, it can easily be checked (by consulting Figures 9, 10 and 11) that a 4-path attached at either t_4 or t_6 is a forbidden attachment for every B-configuration.

We note that in the complex configurations in Figures 7–10, the labeling of the vertices of the 9-path T under consideration is important. Suppose, for example, that $T = t_1 \cdots t_9$ is a 9-path in $\langle A\rangle$ and t_7t_9 is an edge in $\langle A\rangle$. Then T has the external adjacency $7 \sim 9$, and $\overleftarrow{T}$ has the external adjacency $1 \sim 3$. Since $7 \sim 9$ is a forbidden adjacency in a B3-configuration, but $1 \sim 3$ is allowed, T cannot be in a B3-configuration, but $\overleftarrow{T}$ may well be.

If a 9-path T in $\langle A\rangle$ has external edges and/or attached paths in $\langle A\rangle$ that prohibit both T and $\overleftarrow{T}$ from being in any A-, B- or C-configuration in G, we say that T is an *ineligible* 9-path in $\langle A\rangle$.

Lemma 3.9. *Suppose $T = t_1 \cdots t_9$ is a 9-path in $\langle A\rangle$ such that $t_4t_7 \in E(G)$ and there is a 4-path $z_1z_2z_3t_4$ in $\langle A\rangle$ with $z_1, z_2, z_3 \in A - V(T)$. Then T is an ineligible 9-path.*

Proof. As mentioned earlier, $4 \sim 7$ as well as $3 \sim 6$ are forbidden adjacencies for each A-configuration. Since T has $4 \sim 7$ and $\overleftarrow{T}$ has $3 \sim 6$, it follows that neither T nor $\overleftarrow{T}$ can be in any A-configuration in G. Also, since T has a 4-path attached to its 4-th vertex (which is the 6-th vertex of $\overleftarrow{T}$), neither T nor $\overleftarrow{T}$ can be in any B-configuration in G, as can be deduced from Figures 9, 10 and 11.

Now suppose there is a $(b+1)$-path X in $\langle B \rangle$ and four paths P, Q, R, S in $\langle A \rangle$, such that (X, P, Q, R, S, T) is a C-configuration. Then $\{w_r, w_{r+1}\} = \{t_h, t_{h+1}\}$ for some $h \in \{2, 3, 4\}$. Since P and T lie in the same component H of $\langle A \rangle$, there is a $v_p t_k$-path M in G with all its internal vertices in $A - V(T)$, for some $k \in \{1, \dots, 9\}$. Also note that any vertex in $z_1 z_2 z_3$ lies on a 2-path in $A - V(T)$.

• Suppose $h = 2$ or 3. Then $t_9 t_8 t_7 t_6 t_5 t_4 t_3 X$ is an $(8+b)$-path, and hence M does not intersect the path $z_1 z_2 z_3$. Thus $z_1 z_2 z_3 t_4 t_3 X$ is a $(6+b)$-path, which implies that $k \notin \{5, 6, 8, 9\}$. Lemma 2.3(1) and (2) imply that $k \notin \{1, 2, 3, 4\}$. Also, $k \neq 7$, since otherwise $t_1 t_2 t_3 X M t_6 t_5 t_4 z_3 z_2$ would be a path of order greater than $9 + b$. This case can therefore not occur.

• Suppose $h = 4$. Then $t_1 t_2 t_3 t_4 t_7 t_6 t_5 X$ is an $(8+b)$-path, and hence M does not intersect the path $z_1 z_2 z_3$. Thus $k \neq 2$, since $z_1 z_2 z_3 t_4 t_7 t_6 t_5 X M t_3$ would be a path of order greater than $9 + b$. Also, Lemma 2.3(1) and (2) imply that $k \notin \{1, 3, 4, 5, 6, 9\}$, and Lermma 2.3(3) implies that $k \neq 8$. Hence $k = 7$ and $t_1 t_2 t_3 t_4 t_5 X M t_8 t_9$ is a path of order $8 + b + n(M)$. Thus $M = v_p = t_7$. Similar arguments show that any path from v_{p+1} to T contains the vertex v_7. Since $v_7 = v_p \notin V(Q)$, it follows that $V(Q) \cap V(T) = \emptyset$. Since $t_1 t_2 t_3 t_4 t_7 t_6 t_5 X Q$ is a path of order $8 + b + q$, it follows that $q = 1$, and hence $p = 8$. Thus, $v_8 = v_p = t_7$ and $v_9 = v_{p+1} \notin V(T)$. Now, if $V(P) \cap \{t_8, t_9\} = \emptyset$, then $P t_8 t_9$ would be a 10-path in $\langle A \rangle$, and if $t_4 \notin V(P)$, then $P \overleftarrow{X} t_4$ would be a $(10+b)$-path in G. Thus, P contains t_4 and at least one vertex in $\{t_8, t_9\}$, and hence there is a path in $\langle A \rangle - \{t_7\}$ from t_4 to a vertex in $\{t_8, t_9\}$. However, it is easily checked that this is not the case, since otherwise there would be a path of order greater than $9 + b$ in G.

These contradictions prove that T is not in any C-configuration.

Now suppose $\overleftarrow{T}$ is in some C-configuration in G. Then there are a $(b+1)$-path X and four paths P, Q, R, S in $\langle A \rangle$, defined and labelled as in Definition 3.1, such that $(X, P, Q, R, S, \overleftarrow{T})$ is a C-configuration. Let us relabel $\overleftarrow{T}$ as $\overleftarrow{T} = u_1 \cdots u_9$. Then $u_3 u_6 \in E(G)$ and $z_1 z_2 z_3 u_6$ is a 4-path in $\langle A \rangle$ and $\{w_r, w_{r+1}\} = \{u_h, u_{h+1}\}$ for some $h \in \{2, 3, 4\}$, and there is an $x_{b+1} t_k$- path M in G with all its internal vertices in $A - V(T)$, for some $k \in \{1, \dots, 9\}$.

• Suppose $h = 2$. Then $u_9 u_8 \cdots u_2 X$ is a $(9+b)$-path, and hence $N_A(x_{b+1}) \subset V(\overleftarrow{T})$. By Lemma 2.3(1), (2) and (3), $u_1, u_2, u_3, u_4, u_5, u_9 \notin N(x_{b+1})$. Also, $u_7, u_8 \notin N(x_{b+1})$, since $z_1 z_2 z_3 u_6 u_5 u_4 u_3 u_2 X$ is a $(9+b)$-path. Thus $N_A(x_{b+1}) \subset \{t_6\}$, contradicting that $\{v_p, v_{p+1}\} \subseteq N_T(x_{b+1})$.

• Suppose $h = 3$. In this case Lemma 2.3(1), (2) and (3) imply that $k \notin$

$\{1,2,3,4,5,7\}$. Since $z_1z_2z_3u_6u_5u_4u_3X$ is an $(8+b)$-path, $k \neq 8$ and any neighbour of x_{b+1} in $A-V(T)$ is an isolated vertex in $\langle A\rangle$. Thus $v_p = u_6$, $v_{p+1} \notin V(T)$, and $q=1$ and $p=8$. Since $\tau(\langle A\rangle)=9$, any path in$\langle A\rangle$ that contains vertices from both the sets $\{u_7,u_8,u_9\}$ and $\{z_1,z_2,z_3\}$ contains the vertex u_6. Thus, since $u_6 = v_p$, it follows that P contains vertices from at most one of the sets $\{u_7,u_8,u_9\}$ and $\{z_1,z_2,z_3\}$, and hence either Pu_7u_8 or Pz_3z_2 is a 10-path in $\langle A\rangle$.

• Suppose $h=4$. In this case, it follows from Lemma 2.3(1), (2) and (3) that $u_1,u_3,u_4,u_5,u_6, \notin N(x_{b+1})$. Also, it is easily seen that $u_7,u_8,u_9 \notin N(x_{b+1})$. Since $u_9u_8u_7u_6u_3u_4u_5X$ is an $(8+b)$-path, it follows that $u_2 \notin N(x_{b+1})$ and hence any neighbour of x_{b+1} in $A-V(T)$ is an isolated vertex in $\langle A\rangle$, contradicting that $v_p \in N(x_{b+1})$ and $v_pv_{p+1} \in E(\langle A\rangle$.

The above contradictions prove that $\overleftarrow{T}$ is also not in any C-configuration in G.

We conclude that T is an ineligible 9-path. ■

Remark 3.10. Lemma 3.8 implies that if a component K of $\langle A\rangle$ contains an ineligible 9-path, then K is not a problematic component.

We now prove our main result.

Theorem 3.11. *Let G be a graph with detour order $9+b$. Then G has an exact $(9,b)$-partition.*

Proof. We begin by choosing a path of order $9+b$ in G. We let A consist of the first nine vertices of this path and we let $B = V(G)-A$. Then $\tau\langle A\rangle = 9$ and $\tau\langle B\rangle \geq b$.

We now describe a recursive procedure for moving vertices back and forth between A and B until we have an exact $(9,b)$-partition of G.

Step 1. If $\tau(\langle B\rangle)=b$, then (A,B) is an exact $(9,b)$-partition of G, so then we stop. If $\tau(\langle B\rangle) > b$, let $X = x_1\cdots x_{b+1}$ be a $(b+1)$-path in $\langle B\rangle$ and proceed to Step 2.

Step 2. If $\tau(\langle A\cup\{x_i\}\rangle)=9$ for $i=1$ or $b+1$, we move x_1 to A if $i=1$; otherwise, we move x_{b+1} to A. Then we return to Step 1.

Step 3. If $\tau(\langle A\cup\{x_1\}\rangle) > 9$ and $\tau(\langle A\cup\{x_{b+1}\}\rangle) > 9$, then there are paths P,Q,R,S, defined and labelled as in Definition 3.1 such that (X,P,Q,R,S) is a problematic configuration. Let H be the component of $\langle A\rangle$ that contains the paths P and R.

(a) If $\tau(H) < 9$, we move x_1 to A. This creates at least one 10-path in H. In particular, Rx_1S becomes a 10-path in H. We now destroy all 10-paths in H

by moving end-vertices of 10-paths in H to B until $\tau(H) = 9$. Then we return to Step 1.

(b) If $\tau(H) = 9$, let T be a 9-path in H. By Lemma 3.5, we may assume that the paths X, P, Q, R, S and T are labelled such that (X, P, Q, R, S, T) is an A-configuration, a B-configuration or a C-configuration, as defined in Definition 3.4. We now move x_1 to H, thus creating at least one 10-path in H. We choose a 9-path T' that we wish to retain in $\langle H \rangle$ and then we destroy all 10-paths in $\langle H \rangle$ by moving certain vertices from $H - V(T')$ to B. (The way we choose the 9-path T' and select the vertices to be moved to B will be explained when we consider the different types of complex configurations that may occur.) Then we return to Step 1.

We note the following.

Upon completion of any step, the detour order of $\langle A \rangle$ equals 9, and there is still a b-path in $\langle B \rangle$. After each execution of Step 2, there are fewer $(b+1)$-paths in $\langle B \rangle$ than before. However, executing Step 3(a) or (b) may result in $\langle B \rangle$ having at least as many $(b+1)$-paths as previously, since the vertices that were returned to B may now be in $(b+1)$-paths in $\langle B \rangle$. We shall show, however, that this will not prevent our recursive procedure from terminating, and hence we shall end up with an exact (a, b)-partition of G.

We note that, if at some stage in our procedure, $\langle A \rangle$ contains a non-problematic component K, then it may well happen that, at some later stage, K becomes a problematic component, due to vertices moved from B to K and/or vertices moved from other components of $\langle A \rangle$ to B. However, if K contains an ineligible 9-path, then it follows from Remark 3.10 that K is a non-problematic component of $\langle A \rangle$ and K will remain non-problematic throughout the procedure, because that ineligible 9-path will remain in K, since our procedure does not move any vertices out of a non-problematic component of $\langle A \rangle$.

We shall now show that, after a finite number of steps of our procedure, every component of $\langle A \rangle$ will contain an ineligible 9-path, unless the procedure terminates before that point is reached. Thus, eventually, problematic components will no longer be encountered, and hence, since the number of $(b+1)$-paths in $\langle B \rangle$ decreases with each application of Step 2, we will end up with an exact (a, b)-partition of G.

If H is a problematic component of $\langle A \rangle$ with detour order less than 9, then after an application of Step 3(a), H will contain a 9-path. Thus, for the remainder of the proof, we assume that every problematic component has detour order equal to 9.

Now we suppose that H is a problematic component containing a 9-path T. Then, as explained in Step 3(b,) we may assume that the complex configuration (X, P, Q, R, S, T) is an A, B, or C-configuration, as defined in Definition 3.4. We now consider the effect of applying Step 3(b) to the different types of complex configurations.

• Suppose (X, P, Q, R, S, T) is an A-configuration. We shall show that, in this case, after at most two applications of Step 3(b), H will become a non-problematic component and will then remain non-problematic throughout our procedure.

– Suppose (X, P, Q, R, S, T) is an A1-configuration, as depicted in Figure 8(A1).

We observe that, if $U_1 = \{t_2, t_3, t_4\}$, $U_2 = \{t_6, t_7, t_8\}$ and $U_3 = \{w_6, x_1, w_7, w_8, w_9\}$, there is no edge in $\langle A\rangle$ between any two of the three sets U_1, U_2, U_3 (otherwise there would be $(10 + b)$-path in G). Thus, if D is a path in G with $V(D) \subseteq \{t_5\} \cup \{U_1 \cup U_2 \cup U_3\}$, then D does not have vertices from more than two of the sets U_1, U_2, U_3 , and hence D has at most 9 vertices.

Now we move x_1 to A. Then $\langle A\rangle$ contains the 10-paths $t_1t_2t_3t_4t_5w_6\, x_1w_7w_8w_9$ and $t_9t_8t_7t_6t_5w_6x_1w_7w_8w_9$. We destroy these two 10-paths in $\langle A\rangle$ by moving t_1 and t_9 to B and we choose

$$T' = t_2t_3t_4t_5w_6x_1w_7w_8w_9.$$

Now suppose there is still a 10-path M in $\langle A\rangle$. Then, by our observation in the first paragraph, $V(M)$ has at least one vertex z that is not in $V(T') \cup \{t_6, t_7, t_8\}$. We now move z to B and we repeat the process with other 10-paths in $\langle A\rangle$, until there are no more 10-paths in $\langle A\rangle$, but the 9-path T' and the 4-path $t_5t_6t_7t_8$ remain in $\langle A\rangle$.

We relabel T' as $t'_1t'_2t'_3t'_4t'_5t'_6t'_7t'_8t'_9$ and note that $t'_4 = t_5$ and $t'_7 = w_7$ (see Figure 15(A1)). Thus, the 4-path $t_5t_6t_7t_8$ is attached to the 4-th vertex of T', and T' has the external adjacency $4 \sim 7$ (since $t_5w_7 \in E(H)$, as indicated in Figure 8(A1)). Thus, by Lemma 3.9, T' is an ineligible 9-path in H.

– Suppose (X, P, Q, R, S, T) is an A2- or A3-configuration, as depicted in Figure 8(A2 and A3).

In either case, we let

$$T' = t_2t_3t_4t_5x_1w_6w_7w_8w_9.$$

As in the previous case, we destroy all 10-paths in $\langle A\rangle$, without deleting any vertex in $V(T') \cup \{t_6, t_7, t_8\}$. We relabel T' as $T' = t'_1 \cdots t'_9$ and note that $t'_4 = t_5$ and $t'_6 = w_6$. Since $t_5w_6 \in E(G)$, it follows that T' as well as $\overleftarrow{T'}$ has the external adjacency $4 \sim 6$, which is a forbidden adjacency in each A-configuration. Also, the 4-path $t_8t_7t_6t'_4$ prohibits T as well as $\overleftarrow{T'}$ from being in any B-configuration in G. Using similar arguments to those used in the proof of Lemma 3.9, it is easily seen that $\overleftarrow{T'}$ is not in any C-configuration in G.

Now suppose H is still a problematic component. Then there are a $(b + 1)$-path $X' = x'_1 \cdots x'_{b+1}$ in $\langle B\rangle$ and four paths R', S', P', Q' in $\langle A\rangle$ such that

(X', R', S', P', Q', T') is a C-configuration in G. It is easily seen that $t'_3 \notin N(x'_1)$, and hence $\{w'_r, w'_{r+1}\} = \{t'_4, t'_5\}$. Now we perform Step 3(b) again, by moving x'_1 to H, choosing the 9-path

$$T'' = t'_1 t'_2 t'_3 t'_4 x'_1 t'_5 t'_6 t'_7 t'_8$$

and destroying all 10-paths in $\langle A \rangle$ without moving any vertex in $V(T'') \cup \{t_6, t_7, t_8\}$ to B. Then T'' has the external adjacency $4 \sim 7$ and a 4-path attached to its 4-th vertex, and hence it follows from Lemma 3.9 that T'' is an ineligible path in H.

Thus, by Remark 3.10, after at most two applications of Step 3(b), H will become a non-problematic component and will remain non-problematic throughout our procedure. This implies that, at some stage in our procedure, A-configurations will cease to occur.

For the remainder of the proof, we therefore assume that every complex configuration that we encounter will be a B- or C-configuration.

• Suppose (X, P, Q, R, S, T) is a B-configuration. Then $w_r = t_g$ and $w_{r+1} = t_h$ for some pair $g, h \in \{2, \ldots, 8\}$ such that $|g - h| \geq 2$.

If (X, P, Q, R, S, T) is a B1-configuration or a B6-configuration, we choose

$$T' = Rx_1S - \{w_1\}$$

and if (X, P, Q, R, S, T) is any other B-configuration, we let

$$T' = Rx_1S - \{t_9\}.$$

We now investigate the difference between the number of external edges of T and those of T'.

We first state some general observations regarding B-configurations.

(1) If both t_{g-1} and t_{g+1} are in $V(T')$, then at least one of the two edges $t_g t_{g-1}$ and $t_g t_{g+1}$ of T is an external edge of T' (because $xt_g \in E(T')$ and hence at least one of $t_g t_{g-1}$ and $t_g t_{g+1}$ is not in $E(T')$).

(2) If both t_{h-1} and t_{h+1} are in $V(T')$, then at least one of the two edges $t_h t_{h-1}$ and $t_h t_{h+1}$ of T is an external edge of T' (because $xt_h \in E(T')$ and hence at least one of $t_h t_{h-1}$ and $t_h t_{h+1}$ is not in $E(T')$).

(3) If $t_i t_j$ is an external edge of T that is not an external edge of T', then either $t_i t_j \in E(T')$, or at least one of t_i and t_j is not in $V(T')$.

We remind the reader that, in each case, T may have external edges in H that are not edges of the configuration (X, P, Q, R, S, T) and are therefore not shown in the sketch representing that configuration. Fortunately, we do not need

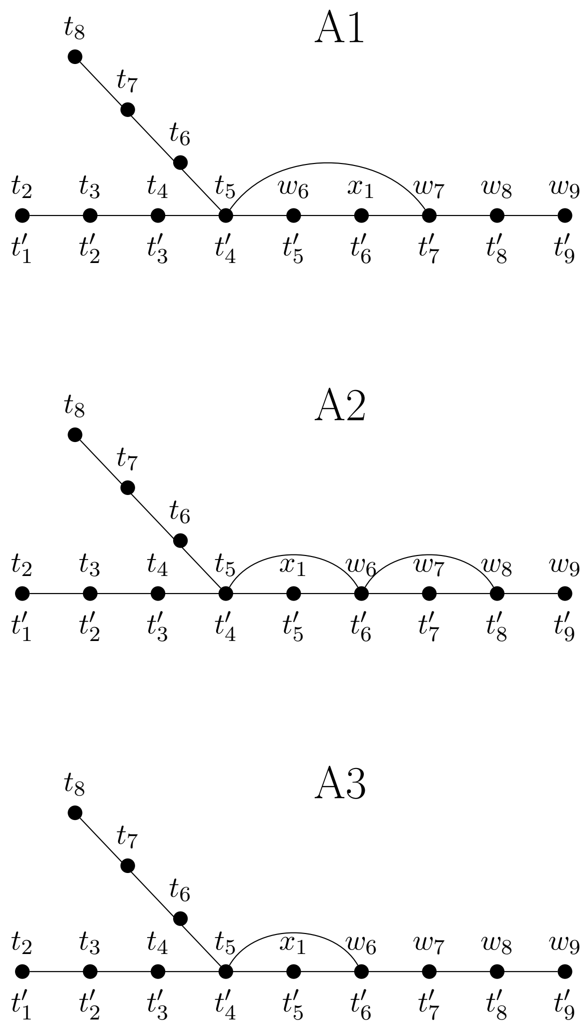


Figure 15. The path T' when step 3(b) is applied to type A-configurations.

to determine all possible external edges of T and T', since it follows from (3) that only external edges of T that are in $E(R) \cup E(S)$, or are incident with a vertex of T that is not in T', can affect the difference between $ext(T)$ and $ext(T')$.

– Suppose (X, P, Q, R, S, T) is any B-configuration other than B1 and B6.

Then $V(T') = \{x_1, t_1, t_2, \ldots, t_8)$ and $g, h \in \{2, \ldots, 7\}$. It therefore follows from (1) and (2) that at least two edges of T are external edges of T'.

From the representation of the configuration in Figures 9, 10 and 11, we note that T has exactly one external edge that is an edge of T', and all external edges of T incident with t_9 are forbidden. It therefore follows from (3) that at most one external edge of T is not an external edge of T'. Thus $ext(T') \geq ext(T) + 1$.

– Suppose (X, P, Q, R, S, T) is a B1-configuration.

Then $T' = t_5t_4t_3t_2x_1t_6t_7t_8t_9$, and hence the edge t_5t_6 of T is an external edge of T' . Thus, since the configuration does not contain an external edge of T, and all external edges of T incident with t_9 are forbidden, it follows from (3) that $ext(T') \geq ext(T) + 1$.

– Suppose (X, P, Q, R, S, T) is a B6-configuration. Then $T' = Rx_1S - \{w_1\} = t_5t_4t_8t_7t_6x_1t_3t_2t_1$, and hence $V(T') = \{x_1\} \cup (V(T) - \{t_9\})$.

Thus, (1) and (2) imply that at least two edges of T are external edges of T', and (3) implies that t_4t_8 is the only external edge of T that is an edge of T'. Moreover, every external edge of T incident with t_9 is forbidden, except for t_9t_6.

Thus, if $t_9t_6 \notin E(H)$, then $ext(T') = ext(T) + 1$.

On the other hand, if $t_9t_6 \in E(H)$, then $ext(T') = ext(T)$. In this case, T' has the external adjacencies $2 \sim 7$ and $1 \sim 5$, and $\overleftarrow{T}$ has the external adjacencies $3 \sim 8$ and $5 \sim 9$. However, $5 \sim 9$ as well as $1 \sim 5$ are forbidden in each B-configuration as well as in C1, C2,C3, C4, C5 and C7, and $2 \sim 7$ is forbidden in C6. Thus, if the component of $\langle A \rangle$ containing T' is a problematic component, then the only possibility is that T' or $\overleftarrow{T}$ is in a nice C-configuration in G.

Thus we have shown that if T is in any B-configuration in G, then $ext(T) \geq ext(T')$, and if equality holds, then T is in a B6-configuration and $t_9t_5 \in E(T)$. In this case, if the component of $\langle A \rangle$ containing T' is a problematic component, then $\overleftarrow{T'}$ is in a nice C-configuration in G.

• Suppose (X, P, Q, R, S, T) is a C-configuration. Then $w_{r+1} = t_h$ and $w_r = t_{h+1}$ for some $h \in \{2, 3, 4\}$. Moving x_1 to A creates the 10-path

$$L = t_1 \cdots t_h x_1 t_{h+1} \cdots t_9.$$

We choose $T' = L - \{t_1\}$ or $T' = L - \{t_9\}$, depending on the type of C-configuration (as will be explained below). For either choice, the following holds.

(a) The edge t_ht_{h+1} of T is an external edge of T'.

(b) All external edges of T are external edges of T', except for those that are incident with the vertex of T that is not in T'.

Suppose (X, P, Q, R, S, T) is a nice C-configuration.

Then, by Definition 3.3(c), at most one of t_1 and t_9 is incident with an external edge of T.

If t_1 is not incident with an external edge of T, we choose $T' = L - \{t_1\}$; and if t_1 is incident with an external edge of T, we choose $T' = L - \{t_9\}$. In either case it follows from (a) and (b) above that $ext(T') \geq ext(T) + 1$.

Suppose (X, P, Q, R, S, T) is a C1- or C2-configuration, as depicted in Figure 12(C1, C2). In either case, $t_1t_3 \in E(T)$, but no other external edge incident with

t_1 is allowed. We choose

$$T' = L - \{t_1\} = t_2t_3x_1t_4t_5t_6t_7t_8t_9.$$

Then we relabel T' so that $T' = t'_1t'_2t'_3t'_4t'_5t'_6t'_7t'_8t'_9$.

The edge t_3t_4 of T is now the external edge $t'_2t'_4$ of T', and every external edge of T except for t_1t_3 is also an external edge of T'. Thus $ext(T') = ext(T)$ and T' has the external adjacency $2 \sim 4$. It can easily be checked that in C1 as well as in C2, all external edges incident with t_2 are forbidden. Thus T' has no external edge incident with t'_1, and hence T' is not in any C-configuration other than a nice C-configuration.

We note that T' has the external adjacencies $2 \sim 4$ and $6 \sim 9$, and hence $\overleftarrow{T'}$ has the external adjacencies $6 \sim 8$ and $1 \sim 4$. Since $2 \sim 4$ as well as $1 \sim 4$ are forbidden adjacencies for a B6-configuration, neither T' nor $\overleftarrow{T'}$ can be in a B6-configuration in G.

Thus, if the component containing T' is still a problematic component, then T' is either in a nice C-configuration or in a B-configuration other than B6.

Let (X, P, Q, R, S, T) be a C_i-configuration for some $i \in \{3, 4, 5, 6, 7\}$. Then we choose

$$T' = t_1 \cdots t_hx_1t_{h+1} \cdots t_8$$

and we note that T has only one external edge incident with t_9. It therefore follows from (a) and (b) above that $ext(T') = ext(T)$.

Also, in each case, all external edges of T incident with t_8 are forbidden, and t_8 is an end-vertex of T'. Thus T' cannot be in any C-configuration in G that is not a nice C-configuration.

Since t_ht_{h+1} is an external edge of T' and $h \in \{2, 3, 4\}$, it follows that T' has the external adjacency $2 \sim 4$, $3 \sim 5$ or $4 \sim 6$, each if which is forbidden in a B6-configuration. Thus T' is not in a B6-configuration. Also, $\overleftarrow{T'}$ has one of the external adjacencies $8 \sim 6$, $7 \sim 5$ and $6 \sim 4$, and of these, only $6 \sim 8$ is allowed in a B6-configuration. However, $\overleftarrow{T'}$ has $6 \sim 8$ only if T was in a C6-configuration, and in that case, $\overleftarrow{T'}$ also has $9 \sim 4$, which is forbidden in a B6-configuration.

Thus, either the component containing T' is non-problematic, or at least one of T' and $\overleftarrow{T'}$ is in a nice C-configuration or a B-configuration other than B6. Thus, by applying Step 3(b) again, we obtain a derived path T'' such that $ext(T'') > ext(T') = ext(T)$.

From the above we conclude that after a finite number of steps of our recursive procedure, we will no longer encounter any problematic component associated with an A-component. Thereafter, if Step 3(b) is repeatedly applied to a problematic component of $\langle A \rangle$ due to recurring B- or C-configurations, the number of external edges of the derived 9-path will not decrease at any step, and will remain constant for at most two steps at a time. Thus, eventually, each component of $\langle A \rangle$ will contain a 9-path that has enough external edges so that it is an ineligible 9-path, unless our procedure terminates before. This proves that, after a finite number of steps, we will no longer encounter any problematic components, and hence our procedure is guaranteed to terminate. ∎

4. Concluding Remarks

As a increases beyond 9, there is a steep increase in the number of problematic configurations that need to be addressed by the recursive procedure used in this paper. Thus, if we wish to use the same basic recursive procedure to further our results on the Strong PPC, we need to step away from considering individual problematic configurations.

We have already observed that it is unnecessary to consider individual problematic configurations where the associated problematic component has detour order less than a, since our recursive procedure deals efficiently with such configurations. In our proof of the Strong PPC for $a = 9$, we considered three categories of configurations having an associated problematic component with detour order equal to 9, namely A-, B- and C-configurations. Unfortunately, our proof depended to some extent on considering individual members of these categories. However, it became apparent that the effect of repeatedly applying the recursive procedure is essentially the same for all members of the same category.

It seems that the way forward would be to categorize the configurations that may occur for larger values of a as generalized A-, B- and C-configurations and then try to refine our recursive procedure so as to ensure that repeated applications of the procedure will eventually complicate the structure of $\langle A \rangle$ to such an extent that, eventually, all the components of $\langle A \rangle$ will become non-problematic.

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