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THE EXCLUDED MINOR THEOREM FOR THE PETERSEN GRAPH CONTRACTING EXACTLY TWO EDGES OF A PERFECT MATCHING AND ONE OTHER EDGE

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Abstract

In order to characterize graphs which do not contain the Petersen graph as a minor, several authors explore characterizations of graphs which do not contain some minors of the Petersen graph. By symmetry, there are three minors that can be obtained from the Petersen graph by contracting exactly two edges of a perfect matching and one other edge. Let P_1, P_2 and P_3 be the three graphs. In this article, we characterize all 4-connected P_i-minor-free graphs for $i = 1, 2, 3$.

Keywords: forbidden minor, 4-connected graph, Petersen graph.

2020 Mathematics Subject Classification: 05C83.

1. Introduction

All graphs in this paper are simple. Let G and H be two graphs. The graph H is called a *minor* of G if it can be generated by deleting or contracting edges from G. And G is called H-*minor-free* if no minor of G is isomorphic to H. In graph theory, many important problems are about the property of H-minor-free graphs. For instance, Tutte's 4-flow conjecture asserts that every bridgeless Petersen-minor-free graph admits a 4-flow.

Define the *contraction* of an edge e to be identifying two ends of e and then deleting all but one edge from each parallel family. We denote by G/e the graph

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obtained from G by contracting e and $G\backslash e$ the graph obtained from G by deleting e. Two edges $e \neq f$ are adjacent if they have an end in common. Pairwise non-adjacent vertices or edges are called *independent*. More formally, a set of vertices or edges is independent if no two of its elements are adjacent. Independent sets of vertices are also called stable. Suppose that V' is a nonempty subset of V. The subgraph of G whose vertex set is V' and whose edge set is the set of those edges of G that have both ends in V' is called the subgraph of G induced by V' and is denoted by $G[V']$, we say that $G[V']$ is an *induced subgraph* of G.

To characterize graphs which do not contain the Petersen graph as a minor, several papers explore to characterize graphs which do not contain some minors of the Petersen graph. One of the most interesting work is due to Ferguson [2] that characterized the P-minor-free graphs, where P is isomorphic to the graph obtained by contracting three edges of a perfect matching of the Petersen graph.

Let v be a vertex of a 3-connected graph G such that $d(v) \geq 4$. Given two sets $A, B \subseteq N_G(v)$, where $N_G(v)$ is the set of vertices adjacent to v in G and $A \cap B = \emptyset$, $\min\{|A|, |B|\} \geq 2$. We mean a 3-*split* of v is the operation of first deleting v from G and adding two new adjacent vertices a, b, then joining a to vertices in A and b to vertices in B. It is clearly that a graph obtained by 3-splitting a vertex of a 3-connected graph will also be 3-connected.

Up to symmetry, there are three minors can be obtained from the Petersen graph by contracting exactly two edges of a perfect matching and one other edge. Let P_1, P_2 and P_3 be the three graphs (see Figures 1–3). In this article, we characterize 4-connected graphs that do not contain P_i as a minor, $i = 1, 2, 3$. In addition, we obtain two graphs by 3-splitting a vertex of the Octahedron. We denote the planar one by Oct_1^+ and the non-planar one by Oct_2^+ (as shown in Figure 4).

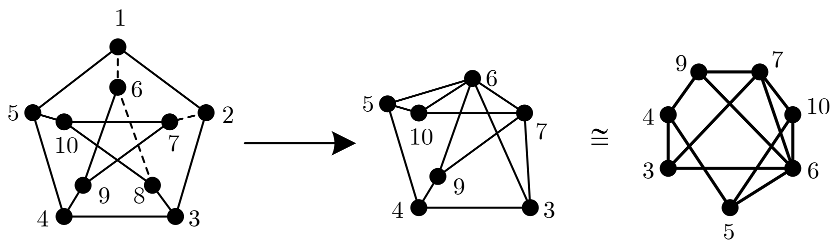

Figure 1. Graph P_1.

For each integer $n \geq 3$, let DW_n denote a *double-wheel*, which is a graph on $n+2$ vertices obtained from a cycle C_n by adding two adjacent vertices and connecting them to all vertices on the cycle. Let $\mathcal{DW} = \{DW_n : n \geq 3\}$. For each integer $n \geq 5$, let C_n^2 be a graph obtained from a cycle C_n by joining all pairs of vertices of distance two on the cycle. Let $\mathcal{C}_0 = \{C_{2n}^2 : n \geq 3\}$,

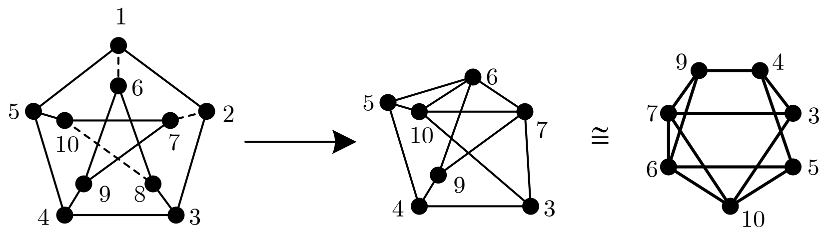

Figure 2. Graph P_2.

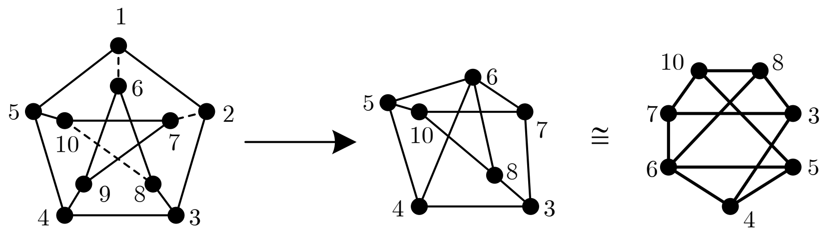

Figure 3. Graph P_3.

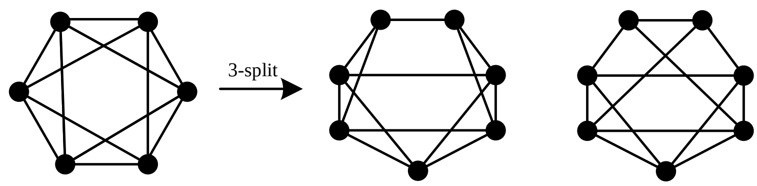

Figure 4. Graphs Oct_1^+ and Oct_2^+.

$\mathcal{C}_1 = \{C_{2n+1}^2 : n \geq 2\}$, and $\mathcal{C} = \mathcal{C}_0 \cup \mathcal{C}_1$. The graph $L(G)$ is called the *line graph* of G if $V(L(G)) = E(G)$, and for any two vertices e, f in $V(L(G))$, e and f are adjacent if and only if they are adjacent edges in G. Let $\mathcal{K}$ be the set of graphs that are 4-connected nonplanar minors of some $K_{4,n}$. In other words, these are 4-connected nonplanar graphs obtained from some $K_{4,n}$ ($n \geq 1$) by adding edges to the partite set of size four. It is routine to check that $\mathcal{K}$ contains exactly one graph (K_5) of five vertices, two graphs ($K_6 \backslash e$, DW_4) of six vertices, six graphs ($K_{4,3}^1, K_{4,3}^2, K_{4,3}^3, K_{4,3}^4, K_{4,3}^5, K_{4,3}^6$) of seven vertices, and eleven graphs of n ($n \geq 8$) vertices.

The following are the main results of this paper.

Theorem 1. *A* 4*-connected graph G is Oct_1^+-minor-free if and only if G belongs to $\mathcal{K} \cup \mathcal{C}_1 \cup \{K_6, C_6^2\}$.*

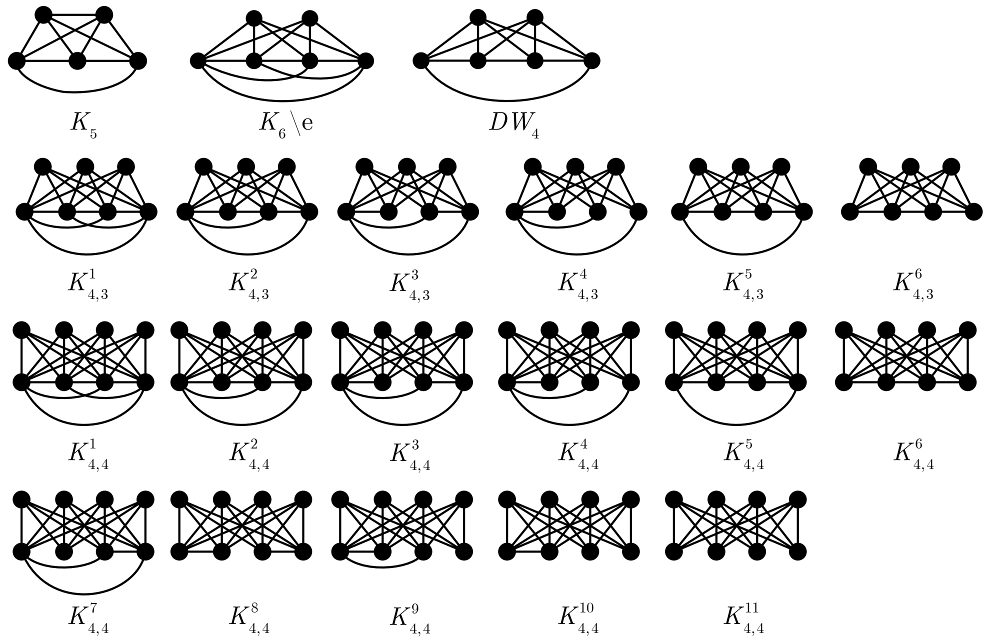

Figure 5. $\mathcal{K}$.

Theorem 2. *A 4-connected graph G is P_1-minor-free if and only if G is planar or G belongs to $\{K_5, K_6, K_6\backslash e, DW_4, C_5^2, C_7^2, K_{4,3}^4, K_{4,4}^{11}\}$.*

Theorem 3. *A 4-connected graph G is P_2-minor-free if and only if G belongs to $\mathcal{C}_1 \cup \{K_5, K_6, C_6^2, DW_4, K_6\backslash e, K_{4,3}^4, K_{4,3}^5, K_{4,3}^6, K_{4,4}^{11}\}$.*

Theorem 4. *A 4-connected graph G is P_3-minor-free if and only if G is planar or G belongs to $\{K_5, K_6, K_6\backslash e, DW_4, C_5^2\}$.*

2. Preliminaries

A sequence of 4-connected graphs $G_0, G_1, \ldots, G_n$ is called a (G_0, G_n)-*chain* if each G_i $(i < n)$ has an edge e_i such that $G_i/e_i = G_{i+1}$. A graph G with at least six vertices is called *cyclically k-edge-connected* if the deletion of fewer than k edges from G does not create two components which both contain at least one cycle. Let $\mathcal{L} = \{L(G) : G$ be a cyclically 4-edge-connected cubic graph$\}$.

Let v be a vertex of a 4-connected graph G. A 4-*split* of v produces a new G' as follows. Given two sets, $A, B \subseteq N_G(v)$, where $A \cup B = N_G(v)$ and $\min\{|A|, |B|\} \geq 3$, the graph G' is obtained by adding to $G - v$ two adjacent vertices a and b such that $N_{G'}(a) = A \cup \{b\}$ and $N_{G'}(b) = B \cup \{a\}$. Clearly, G' is also 4-connected.

Martinov characterizes all 4-connected graphs in the following theorem.

Theorem 5 [5]**.** *Let G be a 4-connected graph. There exists a sequence of 4-connected graphs $H_0, \ldots, H_t$ $t \geq 0$ such that G is isomorphic to H_0 and H_i is obtained from H_{i-1} by contracting some edges $e \in E(H_{i-1})$ and deleting any resulting parallel edges. Moreover, H_t is either isomorphic to C_n^2 for some $n \geq 5$ or isomorphic to the line graph of a cubic cyclically 4-connected graph.*

The following strengthened result due to Qin and Ding is an important tool to characterize all 4-connected Oct_1^+-minor-free and P_i-minor-free graphs.

Theorem 6 [6]**.** *Let G be a 4-connected graph not in $\mathcal{C} \cup \mathcal{L}$. If G is planar, then there exists a (G, C_6^2)-chain; if G is non-planar, then there exists a (G, K_5)-chain.*

Lemma 7 [3]**.** *Let G be a 4-connected graph in $\mathcal{C} \cup \mathcal{L}$. Then G is Oct_1^+-minor-free if and only if $G \in \{C_6^2\} \cup \{C_{2k+1}^2 : k \geq 2\}$.*

Theorem 8 [3]**.** *If a 4-connected graph G is Oct_1^+-minor-free, then G is C_6^2, C_{2k+1}^2 $(k \geq 2)$ or it is obtained from C_5^2 by repeatedly 4-splitting vertices. And C_6^2 is the unique 4-connected planar Oct_1^+-minor-free graph.*

The following result due to Ding for 4-connected $\bar{P}_7$-minor-free graph is important to prove the main results, where $\bar{P}_7$ is the complement of a path on seven vertices.

Theorem 9 [1]**.** *A 4-connected graph G is $\bar{P}_7$-minor-free if and only if either G is planar or G belongs to $\mathcal{DW} \cup \mathcal{C}_1 \cup \mathcal{K} \cup \{K_6,\ L(K_{3,3}), \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5\}$, where $\Gamma_1, \ldots, \Gamma_5$ are the five graphs shown in Figure 6.*

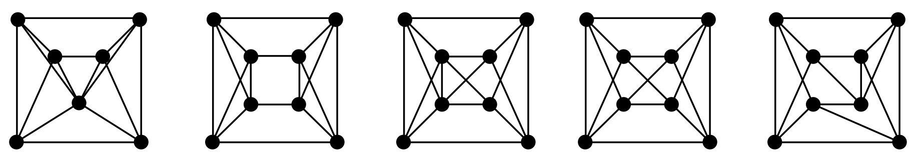

Figure 6. Graphs $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5$.

3. 4-Connected Oct_1^+-Minor-Free Graphs

In this section, we prove Theorem 1.

Lemma 10. *Every graph in $\mathcal{K}$ is Oct_1^+-minor-free.*

Proof. Note that $\max\{|V(K_5)|, |V(DW_4)|, |V(K_6\backslash e)|\} < 7$, which implies that K_5, DW_4, and $K_6\backslash e$ are Oct_1^+-minor-free.

For each integer $1 \leq i \leq 6$, if $K_{4,3}^i$ contains a Oct_1^+-minor, then Oct_1^+ can be obtained from $K_{4,3}^i$ by deleting some edges. Observe that every graph $K_{4,3}^i$ contains three independent vertices, while Oct_1^+ does not contain three independent vertices, a contradiction.

For each integer $1 \leq j \leq 11$, if $K_{4,4}^j$ contains a Oct_1^+-minor, then Oct_1^+ can be obtained from $K_{4,4}^j$ by contracting an edge and deleting some edges. Similarly, every resulting graph contains three independent vertices, a contradiction. ■

Lemma 11. *All graphs in* $\{DW_n : n \geq 5\} \cup \{\Gamma_n : 1 \leq n \leq 5\}$ *contain* Oct_1^+ *as a minor.*

Proof. It can be seen in Figure 7 that DW_5 and Γ_1 contain Oct_1^+ as a minor. Note that DW_n $(n \geq 6)$ contains DW_5 as a minor and Γ_1 is a minor of Γ_n $(n = 2, \ldots, 5)$. Thus they all contain Oct_1^+ as a minor.

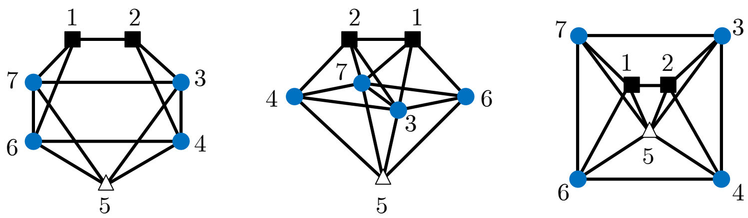

Figure 7. Graph Oct_1^+, and Oct_1^+-minor of DW_5 and Γ_1.

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Proof of Theorem 1. Since Oct_1^+ is a minor of $\bar{P}_7$, the result follows from Theorem 8, Theorem 9, Lemma 10 and Lemma 11. ■

4. 4-Connected P_1-Minor-Free Graphs

In this section, we characterize all 4-connected P_1-minor-free graphs.

Lemma 12. *If a* 4*-connected graph* $G \in \mathcal{C}_1$ *is* P_1*-minor-free, then* G *is* C_5^2 *or* C_7^2*.*

Proof. Since $|V(C_5^2)| < |V(P_1)|$, C_5^2 is P_1-minor-free. If C_7^2 contains a P_1-minor, we can assume that P_1 is obtained from C_7^2 by deleting some edges. Note that P_1 contains a vertex of degree 5. However, the maximum degree of C_7^2 is 4, a contradiction.

Let $V(C_9^2) = \{1, 2, \ldots, 9\}$ be labeled as shown in Figure 8. The graph obtained by contracting the edges 29 and 35, contains a P_1-minor. It is easy to verify that each graph in $\mathcal{C}_1$ with order more than 11 contains C_9^2 as a minor, thus contains a P_1-minor.

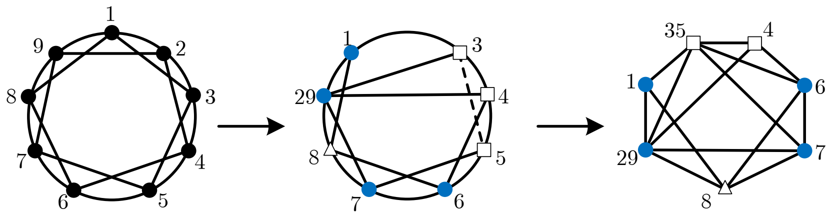

Figure 8. P_1-minor of C_9^2.

■

Lemma 13. *If a 4-connected graph $G \in \mathcal{K}$ and G is P_1-minor-free, then G belongs to $\{K_5, DW_4, K_6\backslash e, K_{4,3}^4, K_{4,4}^{11}\}$.*

Proof. Clearly, K_5, DW_4, and $K_6\backslash e$ are P_1-minor-free. If $K_{4,3}^4$ contains a P_1-minor, we can assume that some edges of $K_{4,3}^4$ are deleted. Let $V(P_1) = \{1, 2, \ldots, 7\}$. Let $X = \{x_1, x_2, x_3; y_1, y_2, y_3, y_4\}$ be the vertex set of $K_{4,3}^4$. The $K_{4,3}^4$ has three independent vertices, x_1, x_2, x_3 or y_2, y_3, y_4 (see Figure 9). Up to symmetry, we consider x_1, x_2, x_3. Note that P_1 contains exactly two sets of three independent vertices, $\{1, 5, 7\}$ and $\{1, 4, 5\}$.

Case 1. $\{x_1, x_2, x_3\} = \{1, 5, 7\}$, $\{y_1, y_2, y_3, y_4\} = \{2, 3, 4, 6\}$. Without loss of generality, we assume that $x_1 = 1$, $x_2 = 5$ and $x_3 = 7$. Since $P_1[\{2, 4, 6\}]$ is isomorphic to K_3, $K_{4,3}^4[\{y_1, y_2, y_3, y_4\}]$ contains no subgraph isomorphic to K_3, $K_{4,3}^4$ does not contain P_1 as a minor.

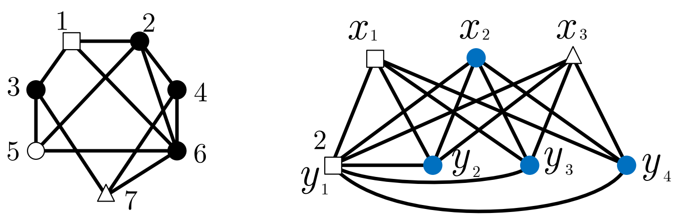

Figure 9. Graphs P_1 and $K_{4,3}^4$ in *Case* 1.

Case 2. $\{x_1, x_2, x_3\} = \{1, 4, 5\}$, $\{y_1, y_2, y_3, y_4\} = \{2, 3, 6, 7\}$. Without loss of generality, we assume that $x_1 = 1$, $x_2 = 4$, $x_3 = 5$. Since $P_1[\{2, 3, 6, 7\}]$ is a 4-path, $K_{4,3}^4[\{y_1, y_2, y_3, y_4\}]$ contains no subgraph isomorphic to a 4-path, $K_{4,3}^4$ contains no P_1-minor.

Therefore, $K_{4,3}^4$ is P_1-minor-free.

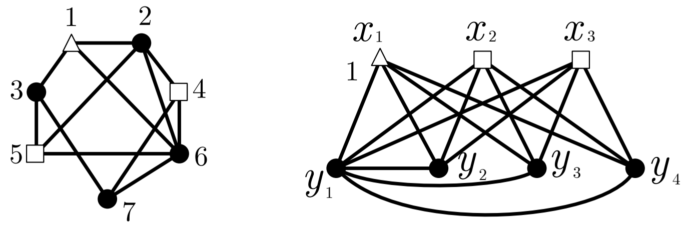

Figure 10. Graphs P_1 and $K^4_{4,3}$ in *Case* 2.

It is obvious that both $K^6_{4,3}$ and $K^3_{4,3}$ contain P_1 as a minor (see Figure 11). What is more, $K^1_{4,3}$ and $K^2_{4,3}$ contain $K^3_{4,3}$ as a minor. Since $K^1_{4,4}$, $K^2_{4,4}$, $K^5_{4,4}$, $K^5_{4,3}$ and $K^6_{4,4}$ contain $K^6_{4,3}$ as a minor, they all contain P_1 as a minor.

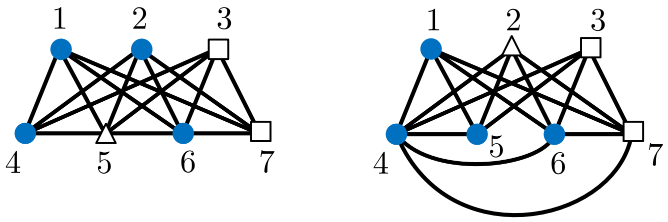

Figure 11. P_1-minor of $K^6_{4,3}$ and $K^3_{4,3}$.

Note that every graph obtained by contracting an edge in $K^j_{4,4}$ for $j = 3, 4, 7, 8, 9, 10$, contains $K^6_{4,3}$ as a minor (see Figure 12). Thus $K^j_{4,4}$ contains P_1 as a minor.

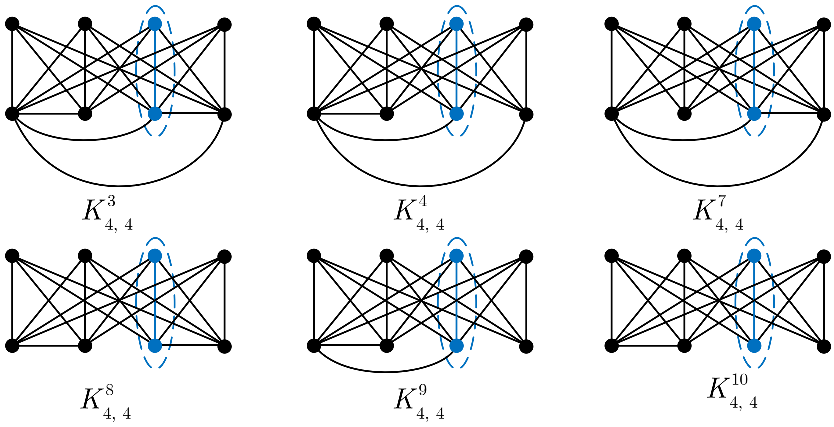

Figure 12. $K^6_{4,3}$-minor of $K^3_{4,4}, K^4_{4,4}, K^7_{4,4}, K^8_{4,4}, K^9_{4,4}, K^{10}_{4,4}$.

If $K_{4,4}^{11}$ contains a P_1-minor, we can assume that one of its edges is contracted. Note that $K_{4,4}^{11}/e \cong K_{4,3}^{4}$ (see Figure 13). Up to the discussion above, no edges can be deleted in $K_{4,4}^{11}/e$ to obtain a P_1-minor. Therefore, $K_{4,4}^{11}$ contains no P_1-minor. ■

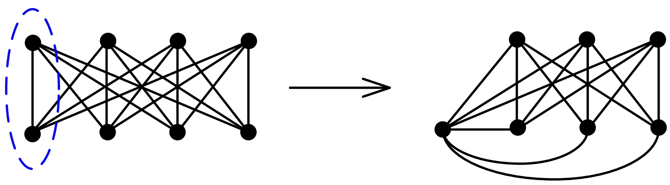

Figure 13. Graphs $K_{4,4}^{11}$ and $K_{4,4}^{11}/e$.

Lemma 14. *Every graph in* $\{DW_5, L(K_{3,3})\} \cup \{\Gamma_n : 1 \leq n \leq 5\}$ *contains* P_1 *as a minor.*

Proof. As shown in Figure 14, both DW_5 and Γ_1 contain P_1 as a minor. Thus, for $i \geq 2$, every Γ_i also contains a P_1-minor. In addition, the graph obtained by contracting the edges 28 and 56 in $L(K_{3,3})$ contains P_1 as a minor. ■

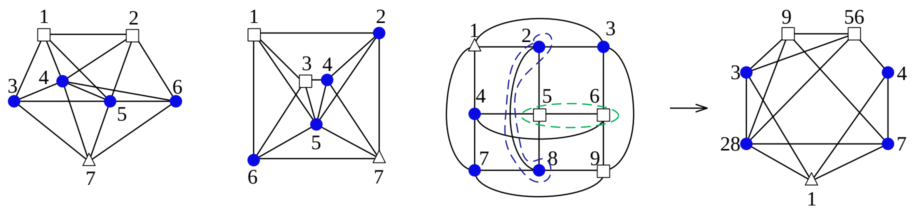

Figure 14. P_1-minor of DW_5, Γ_1 and $L(K_{3,3})$.

Proof of Theorem 2. Note that P_1 is a minor of $\bar{P}_7$ and is nonplanar, then Theorem 2 follows from Theorem 9, Lemmas 12, 13 and 14. ■

5. 4-CONNECTED P_2-MINOR-FREE GRAPHS

In this section, we characterize all 4-connected P_2-minor-free graphs.

Recall the definition that C_n^2 $(n \geq 5)$ is a graph obtained from a cycle C_n by joining all pairs of vertices of distance two on the cycle. The edges of the cycle C_n are called *rung edges*, while the remaining edges are called *rim edges*.

Let v be a vertex of a graph G adjacent to exactly three vertices, a, b and c. A $Y\triangle$-*exchange* on v of G is obtained by removing the vertex v, then adding a triangle on the vertices a, b and c, and removing any parallel edges created in the process. The resulting graph will be denoted by $G_v^{Y\triangle}$.

Lemma 15 [4]**.** *Let H be a 4-connected graph and G a graph with a cubic vertex v. If H is a minor of G, then H is a minor of $G_v^{Y\triangle}$.*

Lemma 16. *For every integer $n \geq 2$, C_{2n+1}^2 is P_2-minor-free.*

Proof. The result is valid for $n = 2$. If C_7^2 contains a P_2-minor, then two edges of C_7^2 must be deleted, which results in seven graphs H_i $(1 \leq i \leq 7)$ as shown in Figure 15. Note that $\{1, 3, 4\}$ is a unique set of three independent vertices in P_2. What is more, $V' = \{5, 6, 7\} \subseteq V(P_2)$ and $G[V'] \cong K_3$. However, the subgraph of H_1 induced by the set $\{3, 4, 6, 7\}$ cannot be isomorphic to K_3. Similarly, for every $j \geq 2$, H_j is P_2-minor-free, where neither H_5 nor H_6 contains three independent vertices.

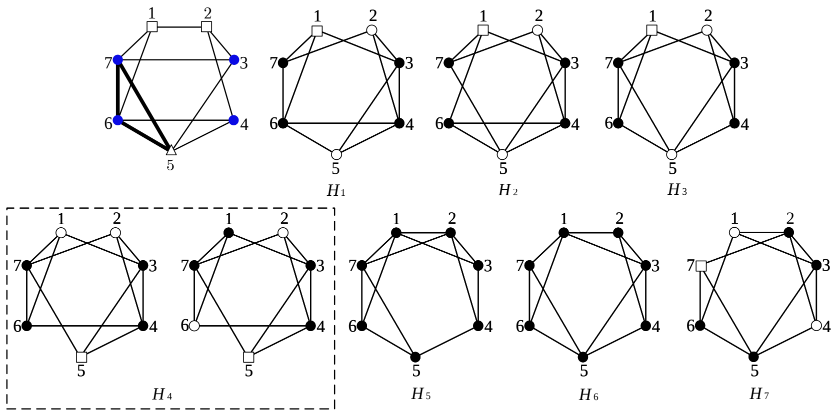

Figure 15. Graphs P_2, H_i $(1 \leq i \leq 7)$.

Let $k \geq 4$ be the smallest integer such that C_{2k+1}^2 contains a P_2-minor. Note that since P_2 has 7 vertices and C_{2k+1}^2 has an odd number of vertices, it can be assumed that at least two edges are contracted.

Assume that at least two rim edges of C_{2k+1}^2 are contracted. Without loss of generality, first contract the edge 13 to obtain a graph N with a new vertex v (see Figure 16). By Lemma 15, $N_2^{Y\triangle}$ must also contain a P_2-minor. Note that $N_2^{Y\triangle}$ is isomorphic to C_{2k-1}^2, a smaller graph that also contains P_2. Therefore, no rim edge of C_{2k+1}^2 can be contracted.

Assume that at least two rung edges of C_{2k+1}^2 are contracted. Let G be the graph obtained by contracting a rung edge of C_{2k+1}^2. Without loss of generality, contract the edge 12 to create a new vertex v (see Figure 17). By Lemma 15, the graph $H = G_3^{Y\triangle}$ must also contain a P_2-minor. Notice that H is a minor of C_{2k-1}^2. This contradicts the minimality of k. ■

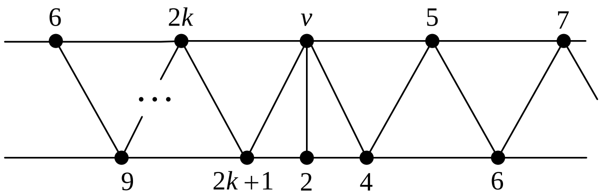

Figure 16. The graph is obtained by contracting a rim edge of C^2_{2k+1}.

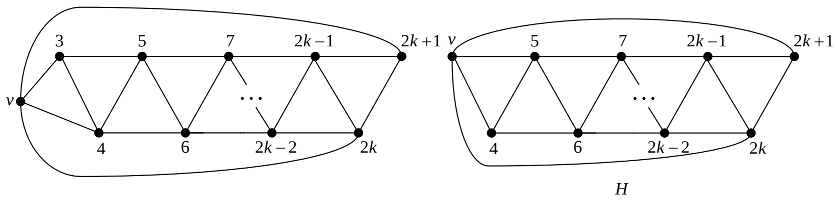

Figure 17. The graph is obtained by contracting a rung edge of C^2_{2k+1}.

Lemma 17. *If a* 4*-connected graph* $G \in \mathcal{K}$ *and* G *is* P_2*-minor-free, then* G *belongs to* $\{K_5, DW_4, K_6\backslash e, K^4_{4,3}, K^5_{4,3}, K^6_{4,3}, K^{11}_{4,4}\}$.

Proof. Clearly, K_5, DW_4, and $K_6\backslash e$ are P_2-minor-free. If $K^4_{4,3}$ contains P_2 as a minor, then P_2 can be obtained from $K^4_{4,3}$ by deleting some edges. Let $V(P_2) = \{1, 2, \ldots, 7\}$. Let $X = \{x_1, x_2, x_3; y_1, y_2, y_3, y_4\}$ be the vertex set of $K^4_{4,3}$. Note that the subgraph of $K^4_{4,3}$ induced by the set $\{y_1, y_2, y_3, y_4\}$ cannot be isomorphic to K_3. By a similar argument as that of Lemma 16, $K^4_{4,3}$ is P_2-minor-free. Similarly, $K^5_{4,3}$ and $K^6_{4,3}$ are P_2-minor-free (see Figure 18).

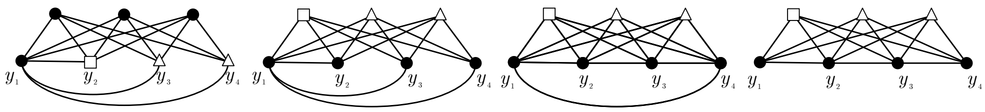

Figure 18. $K^4_{4,3}, K^5_{4,3}$ and $K^6_{4,3}$ are P_2-minor-free graphs.

Figure 19 shows that $K^3_{4,3}$ contains a P_2-minor. Note that both $K^1_{4,3}$ and $K^2_{4,3}$ contain $K^3_{4,3}$ as a minor. Thus $K^1_{4,3}$ and $K^2_{4,3}$ contain P_2 as a minor.

Similarly, $K^4_{4,4}$, $K^5_{4,4}$, $K^7_{4,4}$, $K^8_{4,4}$, $K^9_{4,4}$ and $K^{10}_{4,4}$ contain P_2 as a minor (see Figure 20). Note that $K^7_{4,4}$ is a minor of $K^1_{4,4}$, $K^2_{4,4}$ and $K^3_{4,4}$. Moreover, $K^8_{4,4}$ is a minor of $K^6_{4,4}$. Thus $K^1_{4,4}$, $K^2_{4,4}$, $K^3_{4,4}$ and $K^6_{4,4}$ contain P_2 as a minor.

If $K^{11}_{4,4}$ contains a P_2-minor, then P_2 can be obtained from $K^{11}_{4,4}$ by deleting some edges and contracting an edge, which is isomorphic to $K^4_{4,3}$. By an argument similar to that of $K^4_{4,3}$, $K^{11}_{4,4}$ is P_2-minor-free. ■

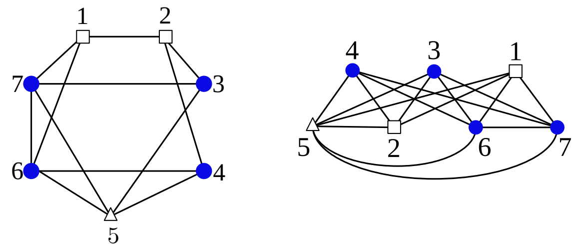

Figure 19. P_2-minor of $K^3_{4,3}$.

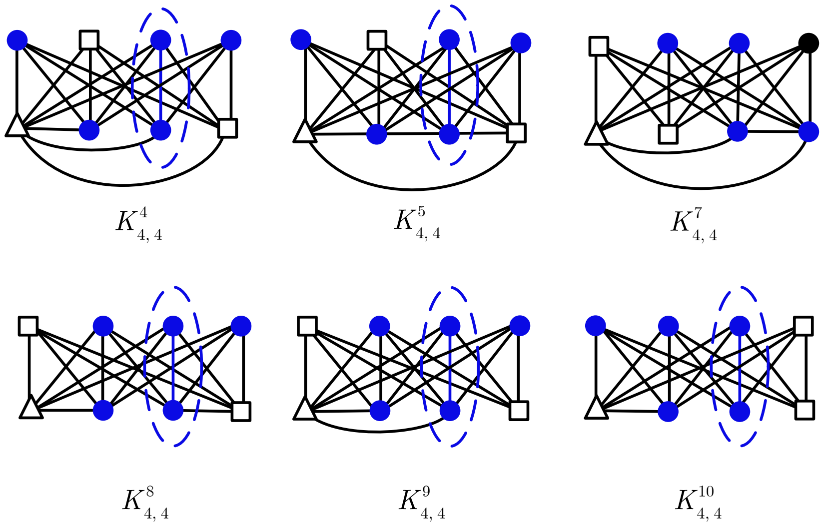

Figure 20. P_2-minor of $K^4_{4,4}$, $K^5_{4,4}$, $K^7_{4,4}$, $K^8_{4,4}$, $K^9_{4,4}$ and $K^{10}_{4,4}$.

***Proof of Theorem* 3.** It is easy to see that P_2 is a minor of the graph Oct_1^+. Hence Theorem 3 follows from Theorem 1, Lemmas 16 and 17. ■

6. 4-Connected P_3-Minor-Free Graphs

In this section, we characterize all 4-connected P_3-minor-free graphs.

Lemma 18. *If a 4-connected graph $G \in \mathcal{K}$ and G is P_3-minor-free, then G belongs to $\{K_5, DW_4, K_6\backslash e\}$.*

Proof. Clearly, K_5, DW_4, and $K_6\backslash e$ are P_3-minor-free. Note that both $K^4_{4,3}$ and $K^6_{4,3}$ contain P_3 as a minor (see Figure 21). In addition, $K^1_{4,3}$, $K^2_{4,3}$, $K^3_{4,3}$ contain $K^4_{4,3}$ as a minor, and $K^5_{4,3}$ contains a $K^6_{4,3}$-minor. For $i = 1, 2, \ldots, 11$, $K^i_{4,4}$ contains $K^4_{4,3}$ as a minor, thus they all contain P_3 as a minor. ■

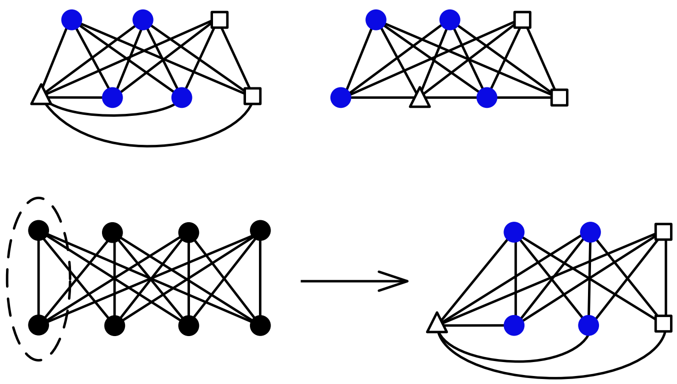

Figure 21. P_3-minor of $K^4_{4,3}$, $K^6_{4,3}$ and $K^{11}_{4,4}$.

Lemma 19. *If a 4-connected graph $G \in \mathcal{DW} \cup \mathcal{C}_1$ and G is P_3-minor-free, then G is DW_3, DW_4 and C_5^2.*

Proof. It can be seen in Figure 22 that DW_5 contains P_3 as a minor. Notice that DW_n $(n \geq 6)$ contains DW_5 as a minor, thus contains P_3 as a minor.

Observe that a P_3-minor can be obtained from C_7^2 by deleting edges 17, 34 and 56 (as shown in Figure 22). What is more, C_n^2 $(n \geq 9)$ contains C_7^2 as a minor, thus contains P_3 as a minor. ■

***Proof of Theorem* 4.** Note that P_3 is a minor of $\bar{P}_7$ and is a nonplanar graph. Both $L(K_{3,3})$ and Γ_1 contain P_3 as a minor as shown in Figure 23 and Figure 22, respectively. Then Theorem 4 follows from Theorem 9, Lemmas 18 and 19. ■

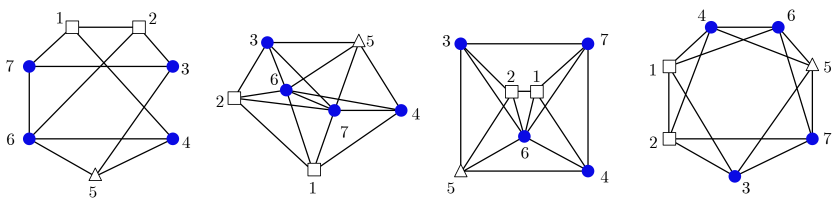

Figure 22. Graph P_3 and P_3-minor of DW_5, Γ_1 and C_7^2.

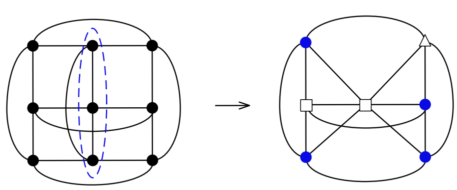

Figure 23. P_3-minor of $L(K_{3,3})$.

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