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HAMILTONIAN CYCLES THROUGH A LINEAR FOREST IN BIPARTITE GRAPHS

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Abstract

A graph is a linear forest if every component is a nontrivial path. Let G be a balanced bipartite graph with n vertices. Let F be a set of m edges of G that induces a linear forest. Zamani and West provided a sufficient condition for G to contain the linear forest in a Hamiltonian cycle, as presented in [Spanning cycles through specified edges in bipartite graphs, J. Graph Theory 71 (2012) 1–17]. The proof presented in this paper establishes the existence of Hamiltonian cycles C_i such that $|E(C_i) \cap F| = i$ ($0 \le i \le m$) in G if any two nonadjacent vertices in opposite partite sets have degree-sum at least n/2 + m + 1.

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1. INTRODUCTION

Terminology and notation that are not defined in this paper are referenced from Bondy[2]. A Hamiltonian path (cycle) in a graph G is a path(cycle) containing all the vertices of G, and a graph with a Hamiltonian cycle is called Hamiltonian. The study of sufficient conditions for the existence of Hamiltonian cycles is a classical topic in graph theory. Ore's theorem [7] states that

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every *n*-vertex $(n \ge 3)$ graph G with $\sigma_2(G) \ge n$ is Hamiltonian, where $\sigma_2(G) = \min\{d(x) + d(y) : xy \notin E(G)\}.$

Bollobás and Brightwell investigated a similar sufficient condition to ensure the existence of cycles through prescribed vertices. They presented the following theorem.

Theorem 1 (Bollobás and Brightwell [1]). Let G be a graph on n vertices, and let $W \subseteq V(G)$. If $|W| \ge 3$ and $d_G(x) + d_G(y) \ge n$ for every pair of non-adjacent vertices $x, y \in W$, then G has a cycle containing all the vertices of W.

Let G be a graph and let $W \subseteq V(G)$, $|W| \ge 3$. The graph G is called Wlocally pancyclic if for each i with $3 \le i \le |W|$, G contains cycles C_i such that C_i contains exactly i vertices from |W|. Let G[W] be the graph induced on the set W. The following theorem shows the existence of cycles containing a specified number of vertices of W.

Theorem 2 (Stacho [4]). Let G be a graph on n vertices, and let $W \subseteq V(G)$. If $|W| \ge 3$ and $d_G(x) + d_G(y) \ge n$ for every pair of non-adjacent vertices $x, y \in W$, then either G is W-locally pancyclic or |W| = n and $G = K_{n/2,n/2}$ or else |W| = 4, $G[W] = K_{2,2}$.

Let E' be a nonempty subset of E(G). The subgraph of G whose vertex set is the set of ends of edges in E' and whose edge set is E' is called the subgraph of G induced by E' and is denoted by G[E']; G[E'] is an edge-induced subgraph of G. A graph is a linear forest if every component is a nontrivial path. Let F be a set of m edges of G and G[F] is a linear forest. An edge uv is called an F-edge, if $uv \in F$. The F-edge-length of a cycle in G is defined as the number of F-edges that it contains.

Researchers explored the degree condition for the existence of Hamiltonian cycles through F, calling a graph F-Hamiltonian when such a cycle exists. For general graphs G, Pósa gave a sharp sufficient condition on $\sigma_2(G)$.

Theorem 3 (Pósa [3]). Let G be a graph on n vertices. Let F be a set of m edges of G such that G[F] is a linear forest. If $\sigma_2(G) \ge n + m$, then G is F-Hamiltonian.

Takeshi Sugiyama proved the following theorem, which shows the existence of a Hamiltonian cycle containing a specified number of edges of a linear forest.

Theorem 4 (Takeshi Sugiyama [5]). Let G be a graph on n vertices, where $n \ge 5$, and let F be a set of m edges of G such that G[F] is a linear forest. If $\sigma_2(G) \ge n + m$, then G contains Hamiltonian cycles of all the F-edge-lengths from 0 to m.

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We explore the analogues of these results in the context of bipartite graphs. The X,Y-bigraph denotes to a bipartite graph with partite sets X and Y. When |X| = |Y|, the graph is a balanced bipartite graph. For an X,Y-bigraph, let $\sigma(G) = \min\{d(x) + d(y) : x \in X, y \in Y, xy \notin E(G)\}$. Las Vergnas [6] established that if $\sigma(G) > n/2+2$, then every perfect matching in G is contained within some Hamiltonian cycle. Regarding the general linear forest, Zamani and West gave a sufficiency threshold for $\sigma(G)$ in terms of n and m. They also demonstrated that this threshold on the degree-sum is sharp when n > 3m.

Theorem 5 (Zamani and West [8]). Let G be an n-vertex balanced X,Y-bigraph. Let F be a set of m edges of G such that G[F] is a linear forest and G[F] has t_1 paths of odd length and t_2 paths of positive even length. If $\sigma(G) \ge n/2 + \lceil m/2 \rceil + \varepsilon(t_1, t_2)$, where

$$\varepsilon(t_1, t_2) = \begin{cases} 1 & t_1 = 0, \\ 1 & (t_1, t_2) \in \{(1, 0), (2, 0)\}, \\ 0 & otherwise, \end{cases}$$

then G is F-Hamiltonian.

This paper presents a theorem that demonstrates the existence of balanced bipartite graphs with Hamiltonian cycles, where the F-edge-lengths range from 0 to m.

Theorem 6. Let G be an n-vertex balanced X,Y-bigraph. Let F be a set of m edges of G such that G[F] is a linear forest. If

(i) G is not a complete bipartite graph and $\sigma(G) \ge n/2 + m + 1$,

(ii) G is a complete bipartite graph with $n \ge 20$,

then G contains Hamiltonian cycles of all the F-edge-lengths from 0 to m.

2. Proof of Theorem 6

Let G be an n-vertex balanced X,Y-bigraph, and let F be a set of m edges of G that induces a linear forest. A cycle C with a specified orientation is denoted by \overrightarrow{C} . The successor and the predecessor of x on C are denoted by x^+ and x^- , respectively. For $x, y \in V(C)$, a path from x to y on \overrightarrow{C} is denoted by $x\overrightarrow{C}y$. The inverse path of $x\overrightarrow{C}y$ is denoted by $y\overleftarrow{C}x$. A path with endvertices x and y is called an x, y-path. When X and Y represent sets of vertices, an (X, Y)-path is denoted as a path P where one end is in X and the other is in Y. The distance $d_G(x, y)$ is the length of the shortest x, y-path in G, if no such path exists, let $d_G(x, y) = \infty$.

Let P denote an x, y-path of odd length. An edge of P is an *odd edge* or *even edge* (with respect to P) when it has an odd position or an even position in a listing of the edges in order from one end of P. The set of all odd edges on P is denoted as $E_{odd}(P)$, while the set of all even edges on P is denoted as $E_{even}(P)$. An edge that lies on an x, y-path is *full* (with respect to P) if one endpoint is adjacent to x and the other is adjacent to y.

Remark 7. Let P be an x, y-path through all vertices in G. Since each endpoint of an edge on P has at most one neighbor in x, y, the pigeonhole principle implies that if $xy \notin E(G)$ and $d(x) + d(y) \ge n/2 + p$, then there are at least p full odd edges and p + 1 full even edges along P.

By Theorem 5, G contains a Hamiltonian cycle C of G such that $F \subseteq E(C)$. The proof can be completed by proving that if G contains a Hamiltonian cycle C of G such that $|E(C) \cap F| = l$ $(l \ge 1)$, then there exists a Hamiltonian cycle C' of G such that $|E(C') \cap F| = l - 1$. So assume that G contains a Hamiltonian cycle C of G such that $|E(C) \cap F| = l - 1$. Let $C = x_1 x_2 \cdots x_n$ and consider the subscripts as modulo n. Let $A = \{x_i : x_i x_{i+1} \in F\}, B = \{x_i : x_i x_{i+1} \notin F\}$ and $q = |F \setminus E(C)|$. Note that q = m - l. Let $G' = (V(G), E(G) \setminus \{F \setminus E(C)\})$.

Lemma 8. If there exist $x_i \in A$ and $x_j \in B$ such that $d_C(x_i, x_j) \ge 2$, $x_i x_j \in E(G) \setminus F$ and $x_{i+1}x_{j+1} \notin F$, then there exists a Hamiltonian cycle C' such that $|E(C') \cap F| = l - 1$ and $x_i x_{i+1} \notin E(C')$.

Proof. Let $x_i x_j \in E(G) \setminus F$ such that $x_i \in A, x_j \in B$ and $d_C(x_i, x_j) \ge 2$. Without loss of generality, we assume that $x_i \in X, x_j \in Y$. If $x_{i+1}x_{j+1} \in E(G) \setminus F$, then G contains a Hamiltonian cycle $C' = x_{i+1}Cx_j x_iCx_{j+1}$ such that $|E(C') \cap F| = l - 1$. So we assume that $x_{i+1}x_{j+1} \notin E(G)$, then $d_G(x_{i+1}) + d_G(x_{j+1}) \ge \sigma(G) \ge n/2 + m + 1$. We have $d_{G'}(x_{i+1}) + d_{G'}(x_{j+1}) \ge n/2 + m + 1 - q$. There exists a Hamiltonian path $P = x_{j+1}Cx_i x_jCx_{i+1}$ in G' such that $|E(P) \cap F| = l - 1 = m - q - 1$. According to Remark 7, P contains at least m + 1 - q full odd edges. Consequently, there exists an edge $x'y' \in E(P) \setminus F$ such that $x_{j+1}y', x_{i+1}x' \in E(G) \setminus F$. Then there exists a Hamiltonian cycle $C' = x_{j+1}Px'x_{i+1}Py'$ such that $|E(C') \cap F| = l - 1$. Given that G' is a subgraph of G, G contains C'.

Lemma 9. If there exist $b_1, b_2, b_3 \in B$ and $a \in A$ such that $d_C(a, b_i) \ge 2$ and $ab_i \in E(G)$ for every $i, 1 \le i \le 3$, then there exists a Hamiltonian cycle C' such that $|E(C') \cap F| = l - 1$.

Proof. Let $b_1, b_2, b_3 \in B$ and $a \in A$ such that $d_C(a, b_i) \ge 2$ for every $i, 1 \le i \le 3$. Given that each edge of F induces a linear forest, without loss of generality, we can assume that $ab_1, ab_2 \notin F$ and $a^+b_1^+ \notin F$. By Lemma 8, G contains a Hamiltonian cycle C' such that $|E(C') \cap F| = l - 1$.

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The remainder of the proof is now partitioned into two cases.

Case 1. $m \leq n/2 - 3$. Since $|A| = l \geq 1$, without loss of generality, we may assume that $a \in X \cap A$, then $|B \cap X| \leq n/2 - 1$. If q = 0, since $|B| = |E(C) \setminus F| \geq n - l \geq n/2 + 3 \geq |X| + 3$, there are at least three vertices in $B \cap Y$. Consequently, there exists a $b \in B \cap Y$ such that $d_C(a, b) \geq 2$. If $ab \in E(G)$, then by Lemma 8, G contains a Hamiltonian cycle C' such that $|E(C') \cap F| = l - 1$. Hence we may only consider the case $ab \notin E(G)$. If $q \geq 1$, we have $|B| = |E(C) \setminus F| \geq n/2 + 4 \geq |B \cap X| + 5$. There are at least five vertices in $B \cap Y$, hence there exist $b_1, b_2, b_3 \in B \cap Y$ such that $d_C(a, b_i) \geq 2$, $1 \leq i \leq 3$. If $ab_1, ab_2, ab_3 \in E(G)$, by Lemma 9, G contains a Hamiltonian cycle C' such that $|E(C') \cap F| = l - 1$. Hence we may only consider the case where at least ab_1, ab_2, ab_3 is not in E(G). Therefore, in both cases q = 0 and $q \geq 1$, we can assume that there exist $a \in A \cap X$ and $b \in B \cap Y$ such that $ab \notin E(G)$ and $d_C(a, b) \geq 2$. We have $d_G(a) + d_G(b) \geq n/2 + m + 1$.

If $a^+b^+ \in E(G)$, we have a Hamiltonian path $P = a\overleftarrow{C}b^+a^+\overrightarrow{C}b$ such that $|E(P)\cap F| = l-1 = m-q-1$. Since $d_{G'}(a) + d_{G'}(b) \ge n/2 + m+1-q$, according to Remark 7, P contains at least m+1-q full odd edges. Consequently, there exists an edge $x'y' \in E(P) \setminus F$ such that $ay', bx' \in E(G) \setminus F$. Then there exists a Hamiltonian cycle $C' = a\overrightarrow{P}x'b\overrightarrow{P}y'$ such that $|E(C')\cap F| = l-1$ in G'. Since G' is subgraph of G, G contains C'.

If $a^+b^+ \notin E(G)$, then $d_G(a^+) + d_G(b^+) \ge n/2 + m + 1$. Let $P_1 = \overleftarrow{C}[b^-, a^+]$, $P_2 = \overrightarrow{C}[b^+, a^-]$. We have both P_1 and P_2 are (X, Y)-path. Let $E' = E_{even}(P_1) \cup E_{even}(P_2)$. Since $|V(P_1)| + |V(P_2)| = n - 2$, we have |E'| = n/2 - 3. Now we consider the graph G', $d_{G'[E']}(a) + d_{G'[E']}(b) \ge n/2 + m + 1 - q - 4 = n/2 - 3 + m - q$. By pigeonhole principle and $|E' \cap F| \le l - 1 = m - q - 1$, there exists an edge uv in $E' \setminus F$ such that one endpoint is adjacent to a and the other is adjacent to b. Without loss of generality, we may assume that $uv \in E_{even}(P_1)$. Then $P = b^+ \overrightarrow{C} au \overrightarrow{C} bv \overleftarrow{C} a^+$ is a (X, Y)-Hamiltonian path such that $|E(P) \cap F| = l - 1$. Since $d_{G'}(a^+) + d_{G'}(b^+) \ge n/2 + m + 1 - q$, by Remark 7, P contains at least m + 1 - q full odd edges. Hence there exists an edge $x'y' \in E(P) \setminus F$. Then there exists a Hamiltonian cycle $C' = b^+ \overrightarrow{P} x' a^+ \overleftarrow{P} y'$ such that $|E(C') \cap F| = l - 1$ in G'. Since G' is subgraph of G, G contains C'.

Case 2. $m \ge n/2 - 2$. By the degree condition, G is complete bipartite graph. Since $n \ge 20$, we have $m \ge 8$. Let uv be an edge of $F \cap E(C)$. Let $G^* = (V(G'), E(G') \setminus \{uv\})$, and let $F' = F \cap E(C) - uv$ and m' = m - q - 1. If q = 0, we have

$$\sigma(G^*) \ge n/2 + m - 1 \ge n/2 + m/2 + 3.$$

Since $n/2 + \lceil m'/2 \rceil + 1 \leq n/2 + m'/2 + 2 \leq n/2 + m/2 + 3/2$, we have $\sigma(G^*) \geq n/2 + \lceil m'/2 \rceil + 1$.

If $q \ge 1$, we have

$$\sigma(G^*) \ge n/2 + m - 3 \ge n/2 + m/2 + 1.$$

Since $n/2 + \lceil m'/2 \rceil + 1 \leq n/2 + m'/2 + 2 \leq n/2 + m/2 + 1$, we have $\sigma(G^*) \ge n/2 + \lceil m'/2 \rceil + 1$.

Therefore, in both cases q = 0 and $q \ge 1$, we have $\sigma(G^*) \ge n/2 + \lceil m'/2 \rceil + 1$. By Theorem 5, then G^* has a Hamiltonian cycle C' containing F'. Therefore, C' is a Hamiltonian cycle in G such that $|E(C') \cap F| = l - 1$. The proof is complete.

3. A NOTE ON THEOREM 6

Notably, in the case where F is a matching and G is a complete balanced X, Y-bigraph, a better conclusion can be derived as follows.

Theorem 10. Let G be an n-vertex complete balanced X,Y-bigraph with $n \ge 8$, and let F be a matching with m edges. If $\sigma(G) \ge n/2 + m + 1$, then G contains Hamiltonian cycles of all the F-edge-lengths from 0 to m.

Proof. Let C be a Hamiltonian cycle of G such that $|E(C) \cap F| = l$. The terminology of Section 2 can now be applied to C. Through a proof similar to that of Lemma 8, the following lemma can be obtained.

Lemma 11. If there exist $a \in A$ and $b \in B$ such that $d_C(a,b) \ge 2$ and $ab \in E(G)$, then there exists a Hamiltonian cycle C' such that $|E(C') \cap F| = l - 1$.

By the degree condition, we have $n/2 - 2 \leq m \leq n/2$. Let $a \in A \cap X$. If $l \leq n/2 - 2$, then $|B| = |E(C) \setminus F| \geq n/2 + 2 \geq |B \cap X| + 3$. There are at least three vertices in $B \cap Y$. Consequently, we have $b \in B \cap Y$ such that $d_C(a,b) \geq 2$. By Lemma 11, G contains a Hamiltonian cycle such that $|E(C') \cap F| = l - 1$. If l = m = n/2, immediately G contains a Hamiltonian cycle C' such that $|E(C') \cap F| = l - 1$. If l = m = n/2, immediately G contains a Hamiltonian cycle C' such that $|E(C') \cap F| = l - 1$. Hence we can assume l = n/2 - 1. If there exists $b \in B \cap Y$ such that $d_C(a, b) \geq 2$, we apply Lemma 11. If any $b \in B \cap Y$ such that $d_C(a, b) = 1$, we obtain $A = \{a\} \cup (Y \setminus \{a^+, a^-\})$ and $B = \{a^+, a^-\} \cup (X \setminus \{a\})$. Since $n \geq 8$, there exist $a_1 \in A \cap Y$ and $b_1 \in B \cap X$ such that $d_C(a_1, b_1) \geq 2$. Then $C' = a_1^+ \overrightarrow{C} b_1 a_1 \overleftarrow{C} b_1^+$ such that $|E(C') \cap F| = l - 1$.

4. Conclusion

According to our thorough analysis, the lower bound of n/2 + m + 1 for $\sigma(G)$, as specified in Theorems 6 and 10, does not appear to be the most optimal value. The purpose of the lower bound n/2 + m + 1 for $\sigma(G)$ is to guarantee the existence HAMILTONIAN CYCLES THROUGH A LINEAR FOREST IN BIPARTITE GRAPHS 7

of a full odd edge e in P, such that $e \notin F$, thereby facilitating the acquisition of a Hamiltonian cycle C' in G that includes fewer edges from F. In a future study, it would be of considerable interest to determine the most appropriate lower bound.

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