Discussiones Mathematicae
Graph Theory 45 (2025) 1323–1329
https://doi.org/10.7151/dmgt.2588

HAMILTONIAN CYCLES THROUGH A LINEAR FOREST IN BIPARTITE GRAPHS

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Abstract

A graph is a linear forest if every component is a nontrivial path. Let G be a balanced bipartite graph with n vertices. Let F be a set of m edges of G that induces a linear forest. Zamani and West provided a sufficient condition for G to contain the linear forest in a Hamiltonian cycle, as presented in [*Spanning cycles through specified edges in bipartite graphs*, J. Graph Theory 71 (2012) 1–17]. The proof presented in this paper establishes the existence of Hamiltonian cycles C_i such that $|E(C_i) \cap F| = i$ $(0 \leqslant i \leqslant m)$ in G if any two nonadjacent vertices in opposite partite sets have degree-sum at least $n/2 + m + 1$.

Keywords: Hamiltonian cycle, bipartite graph, linear forest.

2020 Mathematics Subject Classification: 05C45, 05C70.

1. Introduction

Terminology and notation that are not defined in this paper are referenced from Bondy[2]. A *Hamiltonian path* (*cycle*) in a graph G is a path(cycle) containing all the vertices of G, and a graph with a *Hamiltonian cycle* is called *Hamiltonian*. The study of sufficient conditions for the existence of Hamiltonian cycles is a classical topic in graph theory. Ore's theorem [7] states that

[1]Supported by NSFC (12371356).
[2]Supported by the Major Research Project of Shanxi Province (No.202202020101006).
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every n-vertex ($n \geqslant 3$) graph G with $\sigma_2(G) \geqslant n$ is Hamiltonian, where $\sigma_2(G) = \min\{d(x) + d(y) : xy \notin E(G)\}$.

Bollobás and Brightwell investigated a similar sufficient condition to ensure the existence of cycles through prescribed vertices. They presented the following theorem.

Theorem 1 (Bollobás and Brightwell [1]). *Let G be a graph on n vertices, and let $W \subseteq V(G)$. If $|W| \geqslant 3$ and $d_G(x) + d_G(y) \geqslant n$ for every pair of non-adjacent vertices $x, y \in W$, then G has a cycle containing all the vertices of W.*

Let G be a graph and let $W \subseteq V(G)$, $|W| \geqslant 3$. The graph G is called *W-locally pancyclic* if for each i with $3 \leqslant i \leqslant |W|$, G contains cycles C_i such that C_i contains exactly i vertices from $|W|$. Let $G[W]$ be the graph induced on the set W. The following theorem shows the existence of cycles containing a specified number of vertices of W.

Theorem 2 (Stacho [4]). *Let G be a graph on n vertices, and let $W \subseteq V(G)$. If $|W| \geqslant 3$ and $d_G(x) + d_G(y) \geqslant n$ for every pair of non-adjacent vertices $x, y \in W$, then either G is W-locally pancyclic or $|W| = n$ and $G = K_{n/2,n/2}$ or else $|W| = 4$, $G[W] = K_{2,2}$.*

Let E' be a nonempty subset of $E(G)$. The subgraph of G whose vertex set is the set of ends of edges in E' and whose edge set is E' is called the subgraph of G *induced* by E' and is denoted by $G[E']$; $G[E']$ is an *edge-induced subgraph* of G. A graph is a linear forest if every component is a nontrivial path. Let F be a set of m edges of G and $G[F]$ is a linear forest. An edge uv is called an *F-edge*, if $uv \in F$. The *F-edge-length* of a cycle in G is defined as the number of *F-edges* that it contains.

Researchers explored the degree condition for the existence of Hamiltonian cycles through F, calling a graph *F-Hamiltonian* when such a cycle exists. For general graphs G, Pósa gave a sharp sufficient condition on $\sigma_2(G)$.

Theorem 3 (Pósa [3]). *Let G be a graph on n vertices. Let F be a set of m edges of G such that $G[F]$ is a linear forest. If $\sigma_2(G) \geqslant n + m$, then G is F-Hamiltonian.*

Takeshi Sugiyama proved the following theorem, which shows the existence of a Hamiltonian cycle containing a specified number of edges of a linear forest.

Theorem 4 (Takeshi Sugiyama [5]). *Let G be a graph on n vertices, where $n \geqslant 5$, and let F be a set of m edges of G such that $G[F]$ is a linear forest. If $\sigma_2(G) \geqslant n + m$, then G contains Hamiltonian cycles of all the F-edge-lengths from 0 to m.*

We explore the analogues of these results in the context of bipartite graphs. The X,Y-*bigraph* denotes to a bipartite graph with partite sets X and Y. When $|X| = |Y|$, the graph is a *balanced bipartite graph*. For an X,Y-bigraph, let $\sigma(G) = \min\{d(x) + d(y) : x \in X, y \in Y, xy \notin E(G)\}$. Las Vergnas [6] established that if $\sigma(G) > n/2+2$, then every perfect matching in G is contained within some Hamiltonian cycle. Regarding the general linear forest, Zamani and West gave a sufficiency threshold for $\sigma(G)$ in terms of n and m. They also demonstrated that this threshold on the degree-sum is sharp when $n > 3m$.

Theorem 5 (Zamani and West [8])**.** *Let G be an n-vertex balanced X,Y-bigraph. Let F be a set of m edges of G such that $G[F]$ is a linear forest and $G[F]$ has t_1 paths of odd length and t_2 paths of positive even length. If $\sigma(G) \geqslant n/2 + \lceil m/2 \rceil + \varepsilon(t_1, t_2)$, where*

$$\varepsilon(t_1,t_2) = \begin{cases} 1 & t_1 = 0, \\ 1 & (t_1,t_2) \in \{(1,0),(2,0)\}, \\ 0 & \textit{otherwise}, \end{cases}$$

then G is F-Hamiltonian.

This paper presents a theorem that demonstrates the existence of balanced bipartite graphs with Hamiltonian cycles, where the F-edge-lengths range from 0 to m.

Theorem 6. *Let G be an n-vertex balanced X,Y-bigraph. Let F be a set of m edges of G such that $G[F]$ is a linear forest. If*

(i) *G is not a complete bipartite graph and $\sigma(G) \geqslant n/2 + m + 1$,*

(ii) *G is a complete bipartite graph with $n \geqslant 20$,*

then G contains Hamiltonian cycles of all the F-edge-lengths from 0 to m.

2. Proof of Theorem 6

Let G be an n-vertex balanced X,Y-bigraph, and let F be a set of m edges of G that induces a linear forest. A cycle C with a specified orientation is denoted by $\overrightarrow{C}$. The successor and the predecessor of x on C are denoted by x^+ and x^-, respectively. For $x, y \in V(C)$, a path from x to y on $\overrightarrow{C}$ is denoted by $x\overrightarrow{C}y$. The inverse path of $x\overrightarrow{C}y$ is denoted by $y\overleftarrow{C}x$. A path with endvertices x and y is called an *x, y-path*. When X and Y represent sets of vertices, an *(X, Y)-path* is denoted as a path P where one end is in X and the other is in Y. The distance $d_G(x, y)$ is the length of the shortest x, y-path in G, if no such path exists, let $d_G(x, y) = \infty$.

Let P denote an x, y-path of odd length. An edge of P is an *odd edge* or *even edge* (with respect to P) when it has an odd position or an even position in a listing of the edges in order from one end of P. The set of all odd edges on P is denoted as $E_{odd}(P)$, while the set of all even edges on P is denoted as $E_{even}(P)$. An edge that lies on an x, y-path is $full$ (with respect to P) if one endpoint is adjacent to x and the other is adjacent to y.

Remark 7. Let P be an x, y-path through all vertices in G. Since each endpoint of an edge on P has at most one neighbor in x, y, the pigeonhole principle implies that if $xy \notin E(G)$ and $d(x) + d(y) \geqslant n/2 + p$, then there are at least p full odd edges and $p + 1$ full even edges along P.

By Theorem 5, G contains a Hamiltonian cycle C of G such that $F \subseteq E(C)$. The proof can be completed by proving that if G contains a Hamiltonian cycle C of G such that $|E(C) \cap F| = l$ ($l \geqslant 1$), then there exists a Hamiltonian cycle C' of G such that $|E(C') \cap F| = l - 1$. So assume that G contains a Hamiltonian cycle C of G such that $|E(C) \cap F| = l$. Let $C = x_1 x_2 \cdots x_n$ and consider the subscripts as modulo n. Let $A = \{x_i : x_i x_{i+1} \in F\}$, $B = \{x_i : x_i x_{i+1} \notin F\}$ and $q = |F \setminus E(C)|$. Note that $q = m - l$. Let $G' = (V(G), E(G) \setminus \{F \setminus E(C)\})$.

Lemma 8. *If there exist $x_i \in A$ and $x_j \in B$ such that $d_C(x_i, x_j) \geqslant 2$, $x_i x_j \in E(G) \setminus F$ and $x_{i+1} x_{j+1} \notin F$, then there exists a Hamiltonian cycle C' such that $|E(C') \cap F| = l - 1$ and $x_i x_{i+1} \notin E(C')$.*

Proof. Let $x_i x_j \in E(G) \setminus F$ such that $x_i \in A$, $x_j \in B$ and $d_C(x_i, x_j) \geqslant 2$. Without loss of generality, we assume that $x_i \in X$, $x_j \in Y$. If $x_{i+1} x_{j+1} \in E(G) \setminus F$, then G contains a Hamiltonian cycle $C' = x_{i+1} \overrightarrow{C} x_j x_i \overleftarrow{C} x_{j+1}$ such that $|E(C') \cap F| = l - 1$. So we assume that $x_{i+1} x_{j+1} \notin E(G)$, then $d_G(x_{i+1}) + d_G(x_{j+1}) \geqslant \sigma(G) \geqslant n/2 + m + 1$. We have $d_{G'}(x_{i+1}) + d_{G'}(x_{j+1}) \geqslant n/2 + m + 1 - q$. There exists a Hamiltonian path $P = x_{j+1} \overrightarrow{C} x_i x_j \overleftarrow{C} x_{i+1}$ in G' such that $|E(P) \cap F| = l - 1 = m - q - 1$. According to Remark 7, P contains at least $m + 1 - q$ full odd edges. Consequently, there exists an edge $x'y' \in E(P) \setminus F$ such that $x_{j+1} y', x_{i+1} x' \in E(G) \setminus F$. Then there exists a Hamiltonian cycle $C' = x_{j+1} \overrightarrow{P} x' x_{i+1} \overleftarrow{P} y'$ such that $|E(C') \cap F| = l - 1$. Given that G' is a subgraph of G, G contains C'. ■

Lemma 9. *If there exist $b_1, b_2, b_3 \in B$ and $a \in A$ such that $d_C(a, b_i) \geqslant 2$ and $ab_i \in E(G)$ for every $i, 1 \leqslant i \leqslant 3$, then there exists a Hamiltonian cycle C' such that $|E(C') \cap F| = l - 1$.*

Proof. Let $b_1, b_2, b_3 \in B$ and $a \in A$ such that $d_C(a, b_i) \geqslant 2$ for every $i, 1 \leqslant i \leqslant 3$. Given that each edge of F induces a linear forest, without loss of generality, we can assume that $ab_1, ab_2 \notin F$ and $a^+ b_1^+ \notin F$. By Lemma 8, G contains a Hamiltonian cycle C' such that $|E(C') \cap F| = l - 1$. ■

The remainder of the proof is now partitioned into two cases.

Case 1. $m \leqslant n/2 - 3$. Since $|A| = l \geqslant 1$, without loss of generality, we may assume that $a \in X \cap A$, then $|B \cap X| \leqslant n/2 - 1$. If $q = 0$, since $|B| = |E(C) \setminus F| \geqslant n - l \geqslant n/2 + 3 \geqslant |X| + 3$, there are at least three vertices in $B \cap Y$. Consequently, there exists a $b \in B \cap Y$ such that $d_C(a, b) \geqslant 2$. If $ab \in E(G)$, then by Lemma 8, G contains a Hamiltonian cycle C' such that $|E(C') \cap F| = l - 1$. Hence we may only consider the case $ab \notin E(G)$. If $q \geqslant 1$, we have $|B| = |E(C) \setminus F| \geqslant n/2 + 4 \geqslant |B \cap X| + 5$. There are at least five vertices in $B \cap Y$, hence there exist $b_1, b_2, b_3 \in B \cap Y$ such that $d_C(a, b_i) \geqslant 2$, $1 \leqslant i \leqslant 3$. If $ab_1, ab_2, ab_3 \in E(G)$, by Lemma 9, G contains a Hamiltonian cycle C' such that $|E(C') \cap F| = l - 1$. Hence we may only consider the case where at least ab_1, ab_2, ab_3 is not in $E(G)$. Therefore, in both cases $q = 0$ and $q \geqslant 1$, we can assume that there exist $a \in A \cap X$ and $b \in B \cap Y$ such that $ab \notin E(G)$ and $d_C(a, b) \geqslant 2$. We have $d_G(a) + d_G(b) \geqslant n/2 + m + 1$.

If $a^+b^+ \in E(G)$, we have a Hamiltonian path $P = a\overleftarrow{C}b^+a^+\overrightarrow{C}b$ such that $|E(P) \cap F| = l - 1 = m - q - 1$. Since $d_{G'}(a) + d_{G'}(b) \geqslant n/2 + m + 1 - q$, according to Remark 7, P contains at least $m + 1 - q$ full odd edges. Consequently, there exists an edge $x'y' \in E(P) \setminus F$ such that $ay', bx' \in E(G) \setminus F$. Then there exists a Hamiltonian cycle $C' = a\overrightarrow{P}x'b\overleftarrow{P}y'$ such that $|E(C') \cap F| = l - 1$ in G'. Since G' is subgraph of G, G contains C'.

If $a^+b^+ \notin E(G)$, then $d_G(a^+) + d_G(b^+) \geqslant n/2 + m + 1$. Let $P_1 = \overleftarrow{C}[b^-, a^+]$, $P_2 = \overrightarrow{C}[b^+, a^-]$. We have both P_1 and P_2 are (X, Y)-path. Let $E' = E_{even}(P_1) \cup E_{even}(P_2)$. Since $|V(P_1)| + |V(P_2)| = n - 2$, we have $|E'| = n/2 - 3$. Now we consider the graph G', $d_{G'[E']}(a) + d_{G'[E']}(b) \geqslant n/2 + m + 1 - q - 4 = n/2 - 3 + m - q$. By pigeonhole principle and $|E' \cap F| \leqslant l - 1 = m - q - 1$, there exists an edge uv in $E' \setminus F$ such that one endpoint is adjacent to a and the other is adjacent to b. Without loss of generality, we may assume that $uv \in E_{even}(P_1)$. Then $P = b^+\overrightarrow{C}au\overleftarrow{C}bv\overleftarrow{C}a^+$ is a (X, Y)-Hamiltonian path such that $|E(P) \cap F| = l - 1$. Since $d_{G'}(a^+) + d_{G'}(b^+) \geqslant n/2 + m + 1 - q$, by Remark 7, P contains at least $m + 1 - q$ full odd edges. Hence there exists an edge $x'y' \in E(P) \setminus F$. Then there exists a Hamiltonian cycle $C' = b^+\overrightarrow{P}x'a^+\overleftarrow{P}y'$ such that $|E(C') \cap F| = l - 1$ in G'. Since G' is subgraph of G, G contains C'.

Case 2. $m \geqslant n/2 - 2$. By the degree condition, G is complete bipartite graph. Since $n \geqslant 20$, we have $m \geqslant 8$. Let uv be an edge of $F \cap E(C)$. Let $G^* = (V(G'), E(G') \setminus \{uv\})$, and let $F' = F \cap E(C) - uv$ and $m' = m - q - 1$.

If $q = 0$, we have

$$\sigma(G^*) \geqslant n/2 + m - 1 \geqslant n/2 + m/2 + 3.$$

Since $n/2 + \lceil m'/2 \rceil + 1 \leqslant n/2 + m'/2 + 2 \leqslant n/2 + m/2 + 3/2$, we have $\sigma(G^*) \geqslant n/2 + \lceil m'/2 \rceil + 1$.

If $q \geqslant 1$, we have

$$\sigma(G^*) \geqslant n/2 + m - 3 \geqslant n/2 + m/2 + 1.$$

Since $n/2 + \lceil m'/2 \rceil + 1 \leqslant n/2 + m'/2 + 2 \leqslant n/2 + m/2 + 1$, we have $\sigma(G^*) \geqslant n/2 + \lceil m'/2 \rceil + 1$.

Therefore, in both cases $q = 0$ and $q \geqslant 1$, we have $\sigma(G^*) \geqslant n/2 + \lceil m'/2 \rceil + 1$. By Theorem 5, then G^* has a Hamiltonian cycle C' containing F'. Therefore, C' is a Hamiltonian cycle in G such that $|E(C') \cap F| = l - 1$. The proof is complete.

3. A Note on Theorem 6

Notably, in the case where F is a matching and G is a complete balanced X, Y-bigraph, a better conclusion can be derived as follows.

Theorem 10. *Let G be an n-vertex complete balanced X,Y-bigraph with $n \geqslant 8$, and let F be a matching with m edges. If $\sigma(G) \geqslant n/2 + m + 1$, then G contains Hamiltonian cycles of all the F-edge-lengths from 0 to m.*

Proof. Let C be a Hamiltonian cycle of G such that $|E(C) \cap F| = l$. The terminology of Section 2 can now be applied to C. Through a proof similar to that of Lemma 8, the following lemma can be obtained.

Lemma 11. *If there exist $a \in A$ and $b \in B$ such that $d_C(a, b) \geqslant 2$ and $ab \in E(G)$, then there exists a Hamiltonian cycle C' such that $|E(C') \cap F| = l - 1$.*

By the degree condition, we have $n/2 - 2 \leqslant m \leqslant n/2$. Let $a \in A \cap X$. If $l \leqslant n/2 - 2$, then $|B| = |E(C) \setminus F| \geqslant n/2 + 2 \geqslant |B \cap X| + 3$. There are at least three vertices in $B \cap Y$. Consequently, we have $b \in B \cap Y$ such that $d_C(a, b) \geqslant 2$. By Lemma 11, G contains a Hamiltonian cycle such that $|E(C') \cap F| = l - 1$. If $l = m = n/2$, immediately G contains a Hamiltonian cycle C' such that $|E(C') \cap F| = l - 1$. Hence we can assume $l = n/2 - 1$. If there exists $b \in B \cap Y$ such that $d_C(a, b) \geqslant 2$, we apply Lemma 11. If any $b \in B \cap Y$ such that $d_C(a, b) = 1$, we obtain $A = \{a\} \cup (Y \setminus \{a^+, a^-\})$ and $B = \{a^+, a^-\} \cup (X \setminus \{a\})$. Since $n \geqslant 8$, there exist $a_1 \in A \cap Y$ and $b_1 \in B \cap X$ such that $d_C(a_1, b_1) \geqslant 2$. Then $C' = a_1^+ \overrightarrow{C} b_1 a_1 \overleftarrow{C} b_1^+$ such that $|E(C') \cap F| = l - 1$. ∎

4. Conclusion

According to our thorough analysis, the lower bound of $n/2 + m + 1$ for $\sigma(G)$, as specified in Theorems 6 and 10, does not appear to be the most optimal value. The purpose of the lower bound $n/2+m+1$ for $\sigma(G)$ is to guarantee the existence of a full odd edge e in P, such that $e \notin F$, thereby facilitating the acquisition of a Hamiltonian cycle C' in G that includes fewer edges from F. In a future study, it would be of considerable interest to determine the most appropriate lower bound.

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Received 15 September 2024
Revised 27 April 2025
Accepted 27 April 2025
Available online 29 May 2025