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TOTAL {2}-DOMINATION IN A GRAPH AND ITS COMPLEMENT

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Abstract

Let G be a graph with no isolated vertex. A function $f : V(G) \rightarrow \{0, 1, 2\}$ is a total $\{2\}$-dominating function on G if $\sum_{u \in N_G(v)} f(u) \geq 2$ for every vertex $v \in V(G)$. The total $\{2\}$-domination number of G is the minimum weight $\omega(f) = \sum_{v \in V(G)} f(v)$ among all total $\{2\}$-dominating functions f on G. In this paper, we study some relationships among some parameters of a graph and the total $\{2\}$-domination number of its complement, emphasizing in results of the Nordhaus-Gaddum type.

Keywords: total $\{2\}$-domination number, total domination number, complement, Nordhaus-Gaddum bounds.

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1. Introduction

For notation and graph theory terminology, we in general follow [10]. Let G be a graph with vertex set $V(G)$, edge set $E(G)$ and order $n = |V(G)|$. Given a vertex $v \in V(G)$, $N_G(v)$ and $N_G[v]$ represents the *open neighborhood* and the *closed neighborhood* of v in G, respectively. The minimum and maximum degrees among all vertices of G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. We denote the *complement* of G by $\overline{G}$ and let $\delta^*(G) = \min\{\delta(G), \delta(\overline{G})\}$. As usual, given a set $D \subseteq V(G)$, $G[D]$ denotes the subgraph of G induced by D. If G is the disjoint union of k copies of a graph H, then we write $G = kH$. Moreover, we use the notation K_n and C_n for complete graphs and cycle graphs of order n, respectively.

Total domination in graphs was introduced in [6] by Cockayne, Dawes and Hedetniemi, and is the most studied variant of domination, with more than 600 published papers. Given a graph G with no isolated vertex, a set $D \subseteq V(G)$ is a *total dominating set* of G if every vertex in G is adjacent to at least one vertex in D. The *total domination number* of G, denoted by $\gamma_t(G)$, is the minimum cardinality among all total dominating sets of G. For more information on total domination and its many variations, we suggest the books [9, 10].

A well-studied variant of total domination in graphs is total $\{2\}$-domination. A function $g : V(G) \to \{0, 1, 2\}$ on a graph G with no isolated vertex is a *total $\{2\}$-dominating function* (T{2}DF) if $\sum_{x \in N_G(v)} g(x) \geq 2$ for every vertex $v \in V(G)$. The *total $\{2\}$-domination number* of G, denoted by $\gamma_{\{2\},t}(G)$, is the minimum weight $\omega(g) = \sum_{x \in V(G)} g(x)$ among all T{2}DFs g on G. A $\gamma_{\{2\},t}(G)$-function is a T{2}DF of weight $\gamma_{\{2\},t}(G)$. For more information on this parameter, see [2–5, 12].

In [2], the authors showed the following relationships between the two aforementioned parameters for any graph with no isolated vertex.

Theorem 1 [2]. *For any graph G with no isolated vertex of order n,*

$$3 \leq \gamma_t(G) + 1 \leq \gamma_{\{2\},t}(G) \leq 2\gamma_t(G) \leq 2n.$$

To illustrate the previous definitions and relationships, consider the two graphs shown in Figure 1. As can be seen, on the left is shown a graph G that satisfies $\gamma_{\{2\},t}(G) = \gamma_t(G) + 2 = 5$, and on the right is shown the graph $\overline{G}$ that satisfies $\gamma_{\{2\},t}(\overline{G}) = 2\gamma_t(\overline{G}) = 6$. In both cases, the black vertices and the labels describe a total dominating set of minimum cardinality and the positive weights of a T{2}DF of minimum weight, respectively.

In this paper, we study several relationships between some parameters of a graph G and the total $\{2\}$-domination number of its complement $\overline{G}$, emphasizing in results of the Nordhaus-Gaddum type. In Section 2 we provide some bounds

Figure 1. A graph G with $\gamma_{\{2\},t}(G) = \gamma_t(G) + 2 = 5$ and its complement $\overline{G}$ with $\gamma_{\{2\},t}(\overline{G}) = 2\gamma_t(\overline{G}) = 6$.

on $\gamma_{\{2\},t}(\overline{G})$ involving the 2-packing number, the total domination number and the girth of G. Finally, in Section 3 we present Nordhaus-Gaddum type results for the total $\{2\}$-domination number.

2. Bounding the Total {2}-Domination Number of the Complement of a Graph

We begin with some results bounding the total $\{2\}$-domination number of $\overline{G}$ involving the 2-*packing number* $\rho(G) = \max\{|P| : P$ is a 2-packing in $G\}$, considering that a 2-*packing* in G is a subset of $V(G)$ that satisfies that any two vertices in that subset have distance at least three.

Theorem 2. *The following statements hold for any nontrivial graph G.*

(i) $\rho(G) \geq 3$ *if and only if* $\gamma_{\{2\},t}(\overline{G}) = 3$.

(ii) *If* $\rho(G) = 2$, *then* $\gamma_{\{2\},t}(\overline{G}) = 4$.

(iii) *If* $\rho(G) = 1$ *and* $\overline{G}$ *is connected, then* $4 \leq \gamma_{\{2\},t}(\overline{G}) \leq 2(\delta(G) + 1)$.

Proof. We first proceed to prove (i). Assume that $\gamma_{\{2\},t}(\overline{G}) = 3$. Let g be a $\gamma_{\{2\},t}(\overline{G})$-function. Observe that $g(x) \leq 1$ for every vertex $x \in V(G)$. Let $D = \{x \in V(G) : g(x) = 1\}$. So, $|D| = 3$ and as a consequence, it follows that $\overline{G}[D] = C_3$, which implies that D is an independent set of G. In addition, if there exists a vertex $x \in V(G) \setminus D$ such that $|N_G(x) \cap D| \geq 2$, then $|N_{\overline{G}}(x) \cap D| \leq 1$, which contradicts the fact that g is a T{2}DF on $\overline{G}$. Hence, $|N_G(x) \cap D| \leq 1$ for every $x \in V(G) \setminus D$, which implies that D is a 2-packing in G. Therefore, $\rho(G) \geq |D| = 3$, as desired. Conversely, assume that $\rho(G) \geq 3$. Let $P = \{u, v, w\}$ be any 2-packing in G. Since any two vertices in P have no neighbors in common, it follows that the function f, defined by $f(u) = f(v) = f(w) = 1$ and $f(x) = 0$

otherwise, is a T{2}DF on $\overline{G}$. Hence, $\gamma_{\{2\},t}(\overline{G}) \leq \omega(f) = 3$, and by Theorem 1 we conclude that $\gamma_{\{2\},t}(\overline{G}) = 3$, which completes the proof of (i).

As an immediate consequence of the previous equivalence, we have that if $\rho(G) \leq 2$, then $\gamma_{\{2\},t}(\overline{G}) \neq 3$, which implies by Theorem 1 that $\gamma_{\{2\},t}(\overline{G}) \geq 4$. Now, assume that $\rho(G) = 2$. Let P be a 2-packing in G such that $|P| = \rho(G)$. Observe that the function f, defined by $f(x) = 2$ if $x \in P$ and $f(x) = 0$ otherwise, is a T{2}DF on $\overline{G}$. Hence, $\gamma_{\{2\},t}(\overline{G}) \leq \omega(f) = 4$. Therefore $\gamma_{\{2\},t}(\overline{G}) = 4$, which completes the proof of (ii).

Finally, assume that $\rho(G) = 1$ and that $\overline{G}$ is connected. To complete the proof of (iii), we only need to prove that $\gamma_{\{2\},t}(\overline{G}) \leq 2(\delta(G) + 1)$. For this purpose, we fix a vertex $v \in V(G)$ of minimum degree. Let $D = N_G(v)$, $D' = \{x \in D : D \subseteq N_G[x]\}$ and $X = V(G) \setminus N_G[v]$. Observe that every vertex in X has a neighbor in D. Now, we consider the following two complementary cases.

Case 1. $D' \neq \emptyset$. Observe that $\Delta(G) < |V(G)| - 1$ because $\overline{G}$ is connected. This implies that for every vertex $x \in D'$ there exists a vertex $y_x \in X \setminus N_G(x)$. Let $X' = \cup_{x \in D'}\{y_x\}$. By the definition of X', it is straightforward that $|X'| \leq |D'|$. It is easy to check that the function f, defined by $f(x) = 2$ if $x \in (D \setminus D') \cup X' \cup \{v\}$ and $f(x) = 0$ otherwise, is a T{2}DF on $\overline{G}$. Therefore, $\gamma_{\{2\},t}(\overline{G}) \leq \omega(f) \leq 2|(D \setminus D') \cup X' \cup \{v\}| \leq 2(|D| - |D'| + |X'| + 1) \leq 2(|D| + 1) = 2(\delta(G) + 1)$, as desired.

Case 2. $D' = \emptyset$. In this case, it follows that $\overline{G}[D]$ is a subgraph of $\overline{G}$ with no isolated vertex. Since $\overline{G}$ is a connected graph, there exist two vertices $u \in D$ and $w \in X$ such that $uw \notin E(G)$. Observe that $D \setminus N_G(u) \neq \emptyset$ due to the fact that $D' = \emptyset$. Let $z \in D \setminus N_G(u)$ be a vertex such that $|N_G(z) \cap D|$ is maximum. If $\overline{G}[D \setminus \{u\}]$ has no isolated vertex, then the function f', defined by $f'(x) = 2$ if $x \in (D \setminus \{u\}) \cup \{v, w\}$ and $f'(x) = 0$ otherwise, is a T{2}DF on $\overline{G}$. Therefore, $\gamma_{\{2\},t}(\overline{G}) \leq \omega(f') \leq 2|(D \setminus \{u\}) \cup \{v, w\}| = 2(\delta(G) + 1)$, as desired. Finally, if $\overline{G}[D \setminus \{u\}]$ has an isolated vertex, then vertex $z \in D \setminus N_G(u)$ has degree one in the subgraph $\overline{G}[D]$ because $|N_G(z) \cap D|$ is maximum. As a consequence, it is easy to check that the function f'', defined by $f''(x) = 2$ if $x \in (D \setminus \{z\}) \cup \{v, w\}$ and $f''(x) = 0$ otherwise, is a T{2}DF on $\overline{G}$. Therefore, $\gamma_{\{2\},t}(\overline{G}) \leq \omega(f'') \leq 2|(D \setminus \{z\}) \cup \{v, w\}| = 2(\delta(G) + 1)$, as desired. ■

The following result is a consequence of the previous theorem.

Theorem 3. *The next statements hold for any graph G with no isolated vertex.*

(i) *If $\overline{G}$ is connected, then $\gamma_{\{2\},t}(\overline{G}) \leq 2(\delta(G) + 1)$.*

(ii) *If $\gamma_{\{2\},t}(G) = 3$, then $\gamma_{\{2\},t}(\overline{G}) \geq 6$.*

(iii) *If $\gamma_{\{2\},t}(G) = 3$, $\delta(G) = 2$ and $\overline{G}$ is connected, then $\gamma_{\{2\},t}(\overline{G}) = 6$.*

Proof. If $\overline{G}$ is connected, then $\gamma_{\{2\},t}(\overline{G}) \leq 2(\delta(G)+1)$ is an immediate consequence of Theorem 2. Hence, (i) follows. From now on, we assume that $\gamma_{\{2\},t}(G) = 3$. By Theorem 2(i) we have that $\rho(\overline{G}) \geq 3$. Let f' be a $\gamma_{\{2\},t}(\overline{G})$-function and let P be a 2-packing in $\overline{G}$ such that $|P| = \rho(\overline{G})$. Observe that $\gamma_{\{2\},t}(\overline{G}) = \omega(f') \geq \sum_{v\in P} f'(N_{\overline{G}}[v]) \geq 2|P| \geq 6$, which completes the proof of (ii). In addition, if $\delta(G) = 2$ and $\overline{G}$ is connected, then by statements (i) and (ii) it follows that $\gamma_{\{2\},t}(\overline{G}) = 6$, which completes the proof of (iii). ■

The following two theorems provide new tight bounds on $\gamma_{\{2\},t}(\overline{G})$ for the case where $\rho(G) = 1$. The first one improves the upper bound given in Theorem 2(iii) whenever $\gamma_t(G) \geq 3$. Before, we need to introduce the next known results.

Theorem 4. *Let G and $\overline{G}$ be two nontrivial connected graphs.*

(i) [7] *If $\rho(G) = 1$, then $\gamma_t(G) \leq \delta(G) + 1$.*

(ii) [11] $(\gamma_t(G) - 2)(\gamma_t(\overline{G}) - 2) \leq \delta^*(G) - 1$.

Theorem 5. *Let G be a graph with $\rho(G) = 1$ such that $\gamma_t(G) \geq 3$. If $\overline{G}$ is connected, then*

$$\gamma_{\{2\},t}(\overline{G}) \leq 4 + \frac{2(\delta^*(G)-1)}{\gamma_t(G)-2}.$$

Proof. By Theorem 1 and Theorem 4(ii) we have that $\gamma_{\{2\},t}(\overline{G}) \leq 2\gamma_t(\overline{G})$ and $(\gamma_t(G)-2)(\gamma_t(\overline{G})-2) \leq \delta^*(G) - 1$, respectively. From the previous inequalities, we deduce that $\gamma_{\{2\},t}(\overline{G}) \leq 4 + 2(\gamma_t(\overline{G})-2) \leq 4 + 2(\delta^*(G)-1)/(\gamma_t(G)-2)$, which completes the proof. ■

For $n \geq 3$, let us consider the graph G_n defined as follows. Let $V(G_n) = S \cup C \cup \{w\}$, where $S = \{s_1, \ldots, s_n\}$ is an independent set and $C = \{c_1, \ldots, c_n\}$ is a clique. To complete the edges of G_n, we have that $ws_i \in E(G_n)$ for every $i \in \{1, \ldots, n\}$, and $s_i c_j \in E(G_n)$ for every $i, j \in \{1, \ldots, n\}$ such that $i \neq j$. For instance, the graph G_3 is the graph G given in Figure 1. For any $n \geq 3$, the bound given in Theorem 5 is achieved for the graph $\overline{G_n}$. It is easy to check that $\rho(G_n) = 1$, $\delta^*(G_n) = 2$ and $\gamma_t(G_n) = 3$. Hence, $\gamma_{\{2\},t}(\overline{G_n}) = 6 = 4 + 2(\delta^*(G_n) - 1)/(\gamma_t(G_n) - 2)$, as required.

Theorem 6. *The following statements hold for any nontrivial graph G of order n with $\rho(G) = 1$.*

(i) $\gamma_{\{2\},t}(G) \leq 2(\delta(G) + 1)$.

(ii) *If $\gamma_{\{2\},t}(G) = 2(\delta(G)+1)$ and $\Delta(G) < n - 1$, then $\gamma_{\{2\},t}(\overline{G}) \leq 6$.*

Proof. By Theorem 1 and Theorem 4(i) we have that $\gamma_{\{2\},t}(G) \leq 2\gamma_t(G) \leq 2(\delta(G)+1)$, which completes the proof of (i). Now, assume that $\gamma_{\{2\},t}(G) = 2(\delta(G)+1)$ and that $\Delta(G) < n-1$. Hence, we have equalities in the previous inequality chain. In particular, we obtain that $\gamma_t(G) = \delta(G) + 1$, which implies that for any vertex $v \in V(G)$ of minimum degree, $D = N_G[v]$ is a total dominating set of G of cardinality $\gamma_t(G)$. By the minimality of D, there exists a vertex $u \in N_G(v)$ such that $N_G(u) \cap D = \{v\}$. If $D = \{u, v\}$, then $|N_G(u)| = n-1$, which contradicts the fact that $\Delta(G) < n - 1$. So, there exists a vertex $u' \in D \setminus \{u, v\}$. Now, we observe that by the minimality of D, there exists a vertex $w \in V(G) \setminus D$ such that $N_G(w) \cap D = \{u'\}$. This implies that $uw \notin E(G)$. So, it is easy to check that the function f, defined by $f(v) = f(u) = f(w) = 2$ and $f(x) = 0$ otherwise, is a T{2}DF on $\overline{G}$. Therefore, $\gamma_{\{2\},t}(\overline{G}) \leq \omega(f) = 6$, which completes the proof of (ii). ■

Now, we bound the total {2}-domination number of $\overline{G}$ involving the *girth* $g(G)$ of a graph G, that is, the length of a shortest cycle in the graph G. Before, we need to establish the following useful lemma.

Lemma 7. *Let G be a graph. If $g(G) \geq 7$ and $G \notin \{C_7, C_8\}$, then $\rho(G) \geq 3$.*

Proof. If $g(G) \geq 9$ or G is a disconnected graph with $g(G) \geq 7$, then $\rho(G) \geq 3$, as desired. Assume that G is a connected graph such that $g(G) \in \{7, 8\}$ and $G \notin \{C_7, C_8\}$. Let C be a cycle in G such that $|V(C)| = g(G)$. By the assumptions, it follows that $V(G) \setminus V(C) \neq \emptyset$. Let $v \in V(C)$ such that $N_G(v) \setminus V(C) \neq \emptyset$. Let v_1 and v_2 be the vertices of C at distance two from vertex v and let $u \in N_G(v) \setminus V(C)$. It is easy to check that $\{u, v_1, v_2\}$ is a 2-packing in G. Therefore, $\rho(G) \geq |\{u, v_1, v_2\}| = 3$, which completes the proof. ■

Theorem 8. *Let G be a graph of order $n \geq 6$ such that G and its complement $\overline{G}$ have no isolated vertex.*

(i) *If G is a triangle-free graph different from a complete bipartite graph, then $\gamma_{\{2\},t}(\overline{G}) \leq 5$.*
(ii) *If $g(G) = 6$, then $\gamma_{\{2\},t}(\overline{G}) \leq 4$.*
(iii) *If $g(G) \geq 7$ and $G \notin \{C_7, C_8\}$, then $\gamma_{\{2\},t}(\overline{G}) = 3$.*

Proof. If $g(G) = 6$, then $\rho(G) \geq 2$. Hence, by Theorem 2(i)–(ii) we have that $\gamma_{\{2\},t}(\overline{G}) \leq 4$. Thus, (ii) follows. Now, if $g(G) \geq 7$ and $G \notin \{C_7, C_8\}$, then by Lemma 7 we have that $\rho(G) \geq 3$. So, Theorem 2(i) leads to $\gamma_{\{2\},t}(\overline{G}) = 3$. Therefore, (iii) follows. Finally, we proceed to prove (i). Assume that G is a traingle-free graph different from a complete bipartite graph. If $\rho(G) \geq 2$, then by Theorem 2(i)–(ii) we have that $\gamma_{\{2\},t}(\overline{G}) \leq 4$. From now on, we assume that $\rho(G) = 1$. Let $v \in V(G)$ and let $X = V(G) \setminus N_G[v]$. By the assumptions, it follows that $X \neq \emptyset$ and that $N_G(v)$ is an independent set of G. Observe that $N_G(x) \cap N_G(v) \neq \emptyset$ for every $x \in X$. Next, we analyze the following two complementary scenarios.

Case 1. X is an independent set of G. As G is different from a complete bipartite graph, there exist two vertices $x \in X$ and $y \in N_G(v) \setminus N_G(x)$. Observe that $\{x, y\}$ is a $\gamma_t(\overline{G})$-set because $\rho(G) = 1$. So, Theorem 1 leads to $\gamma_{\{2\},t}(\overline{G}) \leq 2\gamma_t(\overline{G}) < 5$.

Case 2. There exist $u, w \in X$ such that $uw \in E(G)$. As G is a triangle-free graph, there exist $u' \in N_G(u) \cap N_G(v)$ and $w' \in N_G(w) \cap (N_G(v) \setminus \{u'\})$. We claim that the function f, defined by $f(x) = 1$ if $x \in D = \{v, w', w, u, u'\}$ and $f(x) = 0$ otherwise, is a T{2}DF on $\overline{G}$. It is easy to check that $f(N_{\overline{G}}(x)) = |N_{\overline{G}}(x) \cap D| \geq 2$ for every $x \in D \cup N_G(v)$. Now, let $x \in X \setminus \{u, w\}$. It is straightforward that $x \in N_{\overline{G}}(v)$ and as G is a triangle-free graph; it follows that $N_{\overline{G}}(x) \cap (D \setminus \{v\}) \neq \emptyset$. So $f(N_{\overline{G}}(x)) = |N_{\overline{G}}(x) \cap D| \geq 2$. Hence, f is a T{2}DF on $\overline{G}$, as desired. Therefore, $\gamma_{\{2\},t}(\overline{G}) \leq \omega(f) = |D| = 5$.

From the previous cases, (i) follows. Therefore, the proof is completed. ■

3. Nordhaus-Gaddum Type Inequalities

A Nordhaus-Gaddum type result is a lower or an upper bound on the sum or the product of a parameter of a graph and its complement. The excellent survey by Aouchiche and Hansen [1] provides several Nordhaus-Gaddum relations on several parameters in graphs, including various domination parameters. In this section, we initiate the study of Nordhaus-Gaddum type inequalities for the total {2}-domination number. Before, we need to introduce the following well-known relations, which will be useful for our study.

Theorem 9. *Let G be a graph of order n. If G and its complement $\overline{G}$ have no isolated vertex, then the following holds.*

(i) [6] $\gamma_t(G) + \gamma_t(\overline{G}) \leq n + 2$, *with equality if and only if* $G \in \left\{\frac{n}{2}K_2, \overline{\frac{n}{2}K_2}\right\}$.

(ii) [8] *If* $\min\{\gamma_t(G), \gamma_t(\overline{G})\} \geq 4$, *then* $\gamma_t(G) + \gamma_t(\overline{G}) \leq \delta^*(G) + 3$.

We first establish lower and upper bounds on the sum of the total {2}-domination numbers of a graph and its complement.

Theorem 10. *Let G be a graph of order n. If G and its complement $\overline{G}$ have no isolated vertex, then*

$$8 \leq \gamma_{\{2\},t}(G) + \gamma_{\{2\},t}(\overline{G}) \leq 2n + 4.$$

Furthermore,

(i) $\gamma_{\{2\},t}(G) + \gamma_{\{2\},t}(\overline{G}) = 8$ *if and only if* $\gamma_{\{2\},t}(G) = \gamma_{\{2\},t}(\overline{G}) = 4$.

(ii) $\gamma_{\{2\},t}(G) + \gamma_{\{2\},t}(\overline{G}) = 2n + 4$ *if and only if* $G \in \{2K_2, \overline{2K_2}\}$.

(iii) $\gamma_{\{2\},t}(G)+\gamma_{\{2\},t}(\overline{G})=2n+3$ *if and only if* $G\in\left\{\frac{n}{2}K_2,\overline{\frac{n}{2}K_2}\right\}$, *with* $n\geq 6$.

Proof. If $\gamma_{\{2\},t}(G)=3$ or $\gamma_{\{2\},t}(\overline{G})=3$, then Theorem 3(ii) leads to $\gamma_{\{2\},t}(\overline{G})\geq 6$ or $\gamma_{\{2\},t}(G)\geq 6$, respectively. This implies that $\gamma_{\{2\},t}(G)+\gamma_{\{2\},t}(\overline{G})>8$. Conversely, if $\gamma_{\{2\},t}(G)\geq 4$ and $\gamma_{\{2\},t}(\overline{G})\geq 4$, then $\gamma_{\{2\},t}(G)+\gamma_{\{2\},t}(\overline{G})\geq 8$ and it is easy to check that the previous equality holds if and only if $\gamma_{\{2\},t}(G)=\gamma_{\{2\},t}(\overline{G})=4$. Hence, the lower bound and the equivalence given in (i) follow.

Moreover, from Theorem 1 and Theorem 9(i) we deduce that

$$\gamma_{\{2\},t}(G)+\gamma_{\{2\},t}(\overline{G})\leq 2(\gamma_t(G)+\gamma_t(\overline{G}))\leq 2(n+2)=2n+4. \tag{1}$$

Hence, the upper bound follows.

Now, we proceed to prove the equivalence given in (ii). If $G\in\{2K_2,\overline{2K_2}\}$, then we are done. Conversely, assume that $\gamma_{\{2\},t}(G)+\gamma_{\{2\},t}(\overline{G})=2n+4$. Observe that we have equalities in the inequality chain (1). In particular, we obtain that $\gamma_t(G)+\gamma_t(\overline{G})=n+2$. Hence, Theorem 9(i) leads to $G\in\left\{\frac{n}{2}K_2,\overline{\frac{n}{2}K_2}\right\}$. If $n\geq 6$, then $\rho\left(\frac{n}{2}K_2\right)\geq 3$, and by Theorem 2(i) we have that $\gamma_{\{2\},t}\left(\overline{\frac{n}{2}K_2}\right)=3$. As $\gamma_{\{2\},t}\left(\frac{n}{2}K_2\right)=2n$ we obtain that $\gamma_{\{2\},t}(G)+\gamma_{\{2\},t}(\overline{G})=2n+3$, a contradiction. Therefore, $n=4$, which implies that $G\in\{2K_2,\overline{2K_2}\}$, as desired.

Finally, we proceed to prove the equivalence given in (iii). As previously shown, if $n\geq 6$ and $G\in\left\{\frac{n}{2}K_2,\overline{\frac{n}{2}K_2}\right\}$, then we are done. Conversely, assume that $\gamma_{\{2\},t}(G)+\gamma_{\{2\},t}(\overline{G})=2n+3$. If $\delta^*(G)\geq 2$, then it is straightforward that $\gamma_{\{2\},t}(G)+\gamma_{\{2\},t}(\overline{G})\leq 2n$, a contradiction. Hence $\delta^*(G)=1$. First, we assume that $\delta(G)=1$. This implies that $\overline{G}$ is connected and by Theorem 3(i) we obtain that $\gamma_{\{2\},t}(\overline{G})\leq 4$. Now, we fix a vertex $v\in V(G)$ of degree one and let $u\in V(G)$ its unique neighbor. If $|N_G(u)|\geq 2$, then $V(G)\setminus\{v\}$ is a total dominating set of G, which implies that $\gamma_t(G)\leq|V(G)\setminus\{v\}|=n-1$. By using the previous bounds and the upper bound given in Theorem 1 it follows that $\gamma_{\{2\},t}(G)+\gamma_{\{2\},t}(\overline{G})\leq 2\gamma_t(G)+4\leq 2(n-1)+4\leq 2n+2$, a contradiction. Therefore $|N_G(u)|=1$, which implies that $G=\frac{n}{2}K_2$ or $G=rK_2\cup H$, for some graph H with $\delta(H)\geq 2$ and some integer $r\geq 1$. If $G=rK_2\cup H$, then it is easy to check that $\gamma_{\{2\},t}(G)\leq|V(H)|+4r$. In addition, previously it was shown that $\gamma_{\{2\},t}(\overline{G})\leq 4$. Therefore,

$$2n+3=\gamma_{\{2\},t}(G)+\gamma_{\{2\},t}(\overline{G})\leq(|V(H)|+4r)+4=(2n-|V(H)|)+4. \tag{2}$$

From inequality chain (2) we obtain that $|V(H)|\leq 1$, a contradiction. Thus $G=\frac{n}{2}K_2$, and by the equivalence given in (ii) it follows that $n\geq 6$. By symmetry, if $\delta(\overline{G})=1$, then we obtain that $\overline{G}=\frac{n}{2}K_2$ with $n\geq 6$. Therefore, $G\in\left\{\frac{n}{2}K_2,\overline{\frac{n}{2}K_2}\right\}$, with $n\geq 6$, which completes the proof. ■

The following lemma will be a useful tool to prove the next two theorems.

Lemma 11. *There is no graph G of order six with $\rho(G) = \rho(\overline{G}) = 1$ such that $\gamma_{\{2\},t}(G) = \gamma_{\{2\},t}(\overline{G}) = 6$.*

Proof. Suppose that there exists a graph G with $\rho(G) = \rho(\overline{G}) = 1$ such that $\gamma_{\{2\},t}(G) = \gamma_{\{2\},t}(\overline{G}) = |V(G)| = 6$. By Theorem 3(i), Theorem 6(i) and the fact that $\delta(\overline{G}) = 5 - \Delta(G)$ we obtain that $6 = \gamma_{\{2\},t}(G) \leq 2(\delta(G) + 1)$ and $6 = \gamma_{\{2\},t}(G) \leq 2(\delta(\overline{G})+1) = 2(6-\Delta(G))$, which implies that $2 \leq \delta(G) \leq \Delta(G) \leq 3$. Hence, G contains a cycle. Since G is different from a complete bipartite graph, it follows by Theorem 8 that $g(G) = 3$. Let $C = vv_1v_2$ be a triangle in G, where $|N_G(v)| \leq \min\{|N_G(v_1)|, |N_G(v_2)|\}$. If $|N_G(v)| = 2$, then the function f, defined by $f(x) = 0$ if $x \in V(G) \setminus \{v_1, v_2\}$ and $f(v_1) = f(v_2) = 2$, is a T{2}DF on G due to the fact that $\rho(G) = 1$. Hence, $\gamma_{\{2\},t}(G) \leq \omega(f) = 4$, a contradiction. Thus $|N_G(v)| = |N_G(v_1)| = |N_G(v_2)| = 3$, which implies that there exists a vertex $w \in V(G) \setminus V(C)$ such that $|N_G(w)| = 3$. By the fact that $n = 6$, there exists a vertex $u \in V(C)$ such that $N_G(u) \subseteq (V(C) \setminus \{u\}) \cup \{w\}$. Now, we observe that the function f', defined by $f'(x) = 1$ if $x \in V(G) \setminus \{u\}$ and $f'(u) = 0$, is a T{2}DF on G. So, $\gamma_{\{2\},t}(G) \leq \omega(f') = 5$, a contradiction too. Therefore, the initial assumption is false, which completes the proof. ∎

Next, we show that the upper bound given in Theorem 10 can be improved if we restrict the minimum degree on both G and $\overline{G}$ to be at least two.

Theorem 12. *Let G be a graph of order n such that $\delta^*(G) \geq 2$. Then*

$$\gamma_{\{2\},t}(G) + \gamma_{\{2\},t}(\overline{G}) \leq n + 5.$$

Proof. It is easy to check that if $\delta(G) = 2$ or $\delta(G) \geq 3$, then $\gamma_{\{2\},t}(G) \leq n$ or $\gamma_{\{2\},t}(G) \leq n - 1$, respectively. If $\rho(G) \geq 2$ or $\rho(\overline{G}) \geq 2$, then by Theorem 2(i)–(ii) we have that $\gamma_{\{2\},t}(\overline{G}) \leq 4$ or $\gamma_{\{2\},t}(G) \leq 4$, respectively. As a consequence, it follows that $\gamma_{\{2\},t}(G) + \gamma_{\{2\},t}(\overline{G}) \leq n + 4 < n + 5$, as desired. From now on, let us consider that $\rho(G) = \rho(\overline{G}) = 1$. Since $\delta^*(G) \geq 2$, we note that $n \geq 5$. If $n = 5$ then $G = C_5$, which satisfies the required inequality. If $n = 6$, then Lemma 11 leads to $\min\{\gamma_{\{2\},t}(G), \gamma_{\{2\},t}(\overline{G})\} \leq 5$. As a consequence, it follows that $\gamma_{\{2\},t}(G) + \gamma_{\{2\},t}(\overline{G}) \leq 11 = n + 5$, as desired. Suppose that $n \geq 7$. Without loss of generality, assume that $\gamma_t(G) \geq \gamma_t(\overline{G})$. If $\gamma_t(\overline{G}) = 2$, then it is straightforward that $\gamma_{\{2\},t}(G)+\gamma_{\{2\},t}(\overline{G}) \leq n+2\gamma_t(\overline{G}) = n+4 < n+5$, as required. Now, suppose that $\gamma_t(\overline{G}) = 3$. If $\delta(G) = 2$, then by Theorem 6(i) it follows that $\gamma_{\{2\},t}(G) + \gamma_{\{2\},t}(\overline{G}) \leq 2(\delta(G) + 1) + 2\gamma_t(\overline{G}) = 12 \leq n + 5$, as required. On the other hand, if $\delta(G) \geq 3$, then $\gamma_{\{2\},t}(G) + \gamma_{\{2\},t}(\overline{G}) \leq n - 1 + 2\gamma_t(\overline{G}) = n + 5$, as required. Finally, if $\gamma_t(\overline{G}) \geq 4$, then from Theorem 9(ii) and the fact that $\delta^*(G) \leq (n - 1)/2$ we deduce that

$$\gamma_{\{2\},t}(G)+\gamma_{\{2\},t}(\overline{G}) \leq 2(\gamma_t(G)+\gamma_t(\overline{G})) \leq 2(\delta^*(G)+3) \leq 2\left(\frac{n-1}{2}\right)+6 \leq n+5.$$

Therefore, the proof is completed. ∎

Now, we establish lower and upper bounds on the product of the total $\{2\}$-domination numbers of a graph and its complement.

Theorem 13. *Let G be a graph of order $n \geq 6$. If G and its complement $\overline{G}$ have no isolated vertex, then*

$$16 \leq \gamma_{\{2\},t}(G)\gamma_{\{2\},t}(\overline{G}) \leq 6n.$$

Furthermore,

(i) $\gamma_{\{2\},t}(G)\gamma_{\{2\},t}(\overline{G}) = 16$ *if and only if* $\gamma_{\{2\},t}(G) = \gamma_{\{2\},t}(\overline{G}) = 4$.

(ii) $\gamma_{\{2\},t}(G)\gamma_{\{2\},t}(\overline{G}) = 6n$ *if and only if* $G \in \left\{\frac{n}{2}K_2, \overline{\frac{n}{2}K_2}\right\}$.

Proof. If $\gamma_{\{2\},t}(G) = 3$ or $\gamma_{\{2\},t}(\overline{G}) = 3$, then Theorem 3(ii) leads to $\gamma_{\{2\},t}(\overline{G}) \geq 6$ or $\gamma_{\{2\},t}(G) \geq 6$, respectively. This implies that $\gamma_{\{2\},t}(G)\gamma_{\{2\},t}(\overline{G}) > 16$. Conversely, if $\gamma_{\{2\},t}(G) \geq 4$ and $\gamma_{\{2\},t}(\overline{G}) \geq 4$, then $\gamma_{\{2\},t}(G)\gamma_{\{2\},t}(\overline{G}) \geq 16$ and it is easy to check that the previous equality holds if and only if $\gamma_{\{2\},t}(G) = \gamma_{\{2\},t}(\overline{G}) = 4$. Hence, the lower bound and the equivalence given in (i) follow.

Now, we proceed to prove the upper bound and the equivalence given in (ii). First, we assume that $\rho(G) \geq \rho(\overline{G})$. Next, we analyze the following three complementary cases.

Case 1. $\rho(G) \geq 3$. By Theorem 2(i) and Theorem 1 we have that $\gamma_{\{2\},t}(\overline{G}) = 3$ and $\gamma_{\{2\},t}(G) \leq 2\gamma_t(G) \leq 2n$, respectively. Therefore, $\gamma_{\{2\},t}(G)\gamma_{\{2\},t}(\overline{G}) \leq 6n$, as desired. Observe that the previous equality holds if and only if $\gamma_{\{2\},t}(G) = 2n$, which only happens if $\gamma_t(G) = n$, that is, $G = \frac{n}{2}K_2$.

Case 2. $\rho(G) = 2$. By Theorem 2(i) we have that $\gamma_{\{2\},t}(\overline{G}) = 4$. In addition, G has at most two components. If K_2 is not a component of G, then it is well-known that $\gamma_t(G) \leq \frac{2n}{3}$. So, by Theorem 1 we deduce that $\gamma_{\{2\},t}(G)\gamma_{\{2\},t}(\overline{G}) \leq 16n/3 < 6n$, as desired. From now on, we assume that $G = K_2 \cup H$, with $|V(H)| \geq 4$ and $\rho(H) = 1$. If $\delta(H) = 1$, then by Theorem 6(i) we deduce that $\gamma_{\{2\},t}(H) \leq 2(\delta(H) + 1) = 4$. As a consequence, $\gamma_{\{2\},t}(G)\gamma_{\{2\},t}(\overline{G}) = 4(4 + \gamma_{\{2\},t}(H)) \leq 32 < 6n$, as desired. Finally, if $\delta(H) \geq 2$ then it is easy to check that $\gamma_{\{2\},t}(H) \leq |V(H)| = n-2$. This implies that $\gamma_{\{2\},t}(G) \leq n+2$, and as a consequence, $\gamma_{\{2\},t}(G)\gamma_{\{2\},t}(\overline{G}) \leq 4(n+2) < 6n$, as desired.

Case 3. $\rho(G) = 1$. In this case, it follows that $\rho(\overline{G}) = 1$. If $\max\{\gamma_t(G), \gamma_t(\overline{G})\} \leq 3$, then by Theorem 1 we have that $\gamma_{\{2\},t}(G)\gamma_{\{2\},t}(\overline{G}) \leq 4\gamma_t(G)\gamma_t(\overline{G}) \leq 36 \leq$

$6n$. In addition, if $\gamma_{\{2\},t}(G)\gamma_{\{2\},t}(\overline{G}) = 6n$, then we have equalities in the previous inequality chain. In particular, we obtain that $\gamma_{\{2\},t}(G) = \gamma_{\{2\},t}(\overline{G}) = n = 6$, which contradicts Lemma 11. Therefore, $\gamma_{\{2\},t}(G)\gamma_{\{2\},t}(\overline{G}) < 6n$. From now on, we assume that $\min\{\gamma_t(G), \gamma_t(\overline{G})\} \geq 4$. By Theorem 4(ii) we have that $(\gamma_t(G)-2)(\gamma_t(\overline{G})-2) \leq \delta^*(G)-1$. Rewriting this previous inequality to isolate $\gamma_t(G)\gamma_t(\overline{G})$ we obtain that $\gamma_t(G)\gamma_t(\overline{G}) \leq \delta^*(G) - 5 + 2(\gamma_t(G)+\gamma_t(\overline{G}))$. Now, by Theorem 9(ii) and the fact that $\delta^*(G) \leq (n-1)/2$, it follows that

$$\gamma_t(G)\gamma_t(\overline{G}) \leq \delta^*(G) - 5 + 2(\delta^*(G)+3) = 3\delta^*(G) + 1 \leq 3\left(\frac{n-1}{2}\right) + 1 = \frac{3n-1}{2}.$$

In addition, Theorem 1 and the previous bound lead to $\gamma_{\{2\},t}(G)\gamma_{\{2\},t}(\overline{G}) \leq 4\gamma_t(G)\gamma_t(\overline{G}) \leq 4\left(\frac{3n-1}{2}\right) = 6n-4 < 6n$, as desired.

As a consequence of the three cases above it follows that if $\rho(G) \geq \rho(\overline{G})$, then $\gamma_{\{2\},t}(G)\gamma_{\{2\},t}(\overline{G}) \leq 6n$ and equality holds if and only if $G = \frac{n}{2}K_2$. By symmetry, and proceeding in an analogous manner to the three previous cases, we deduce that if $\rho(\overline{G}) \geq \rho(G)$ then $\gamma_{\{2\},t}(G)\gamma_{\{2\},t}(\overline{G}) \leq 6n$ and equality holds if and only if $\overline{G} = \frac{n}{2}K_2$. Therefore, the upper bound and the equivalence given in (ii) follow, which completes the proof. ■

Finally, we show that the upper bound given in Theorem 13 can be improved if we restrict the minimum degree on both G and $\overline{G}$ to be at least two.

Theorem 14. *Let G be a graph of order n such that $\delta^*(G) \geq 2$. Then*

$$\gamma_{\{2\},t}(G)\gamma_{\{2\},t}(\overline{G}) \leq \begin{cases} 4(n+\delta^*(G)) & \text{if } \rho(G) = \rho(\overline{G}) = 1, \\ 4n & \text{otherwise.} \end{cases}$$

Proof. It is easy to check that if $\delta^*(G) \geq 2$, then $\max\{\gamma_{\{2\},t}(G), \gamma_{\{2\},t}(\overline{G})\} \leq n$. If $\rho(G) \geq 2$ or $\rho(\overline{G}) \geq 2$, then by Theorem 2(i)–(ii) we have that $\gamma_{\{2\},t}(\overline{G}) \leq 4$ or $\gamma_{\{2\},t}(G) \leq 4$, respectively. As a consequence, it follows that $\gamma_{\{2\},t}(G)\gamma_{\{2\},t}(\overline{G}) \leq 4n$, as desired. From now on, let us consider that $\rho(G) = \rho(\overline{G}) = 1$. This implies that $\min\{\gamma_t(G), \gamma_t(\overline{G})\} \geq 3$.

By Theorem 5 we have that $\gamma_{\{2\},t}(\overline{G}) \leq 4 + (2(\delta^*(G)-1))/(\gamma_t(G)-2)$, which implies that $(\gamma_{\{2\},t}(\overline{G})-4)(\gamma_t(G)-2) \leq 2(\delta^*(G)-1)$. As a consequence, it follows that $\big(\gamma_{\{2\},t}(\overline{G})-4\big)\big(\gamma_{\{2\},t}(G)-4\big) \leq \big(\gamma_{\{2\},t}(\overline{G})-4\big)\big(2\gamma_t(G)-4\big) \leq 4(\delta^*(G)-1)$. Moreover, expanding and collecting terms in the previous inequality we obtain $\gamma_{\{2\},t}(G)\gamma_{\{2\},t}(\overline{G}) \leq 4\delta^*(G) - 20 + 4(\gamma_{\{2\},t}(G) + \gamma_{\{2\},t}(\overline{G}))$. By assumption, $\delta^*(G) \geq 2$. Hence, Theorem 12 leads to

$$\gamma_{\{2\},t}(G)\gamma_{\{2\},t}(\overline{G}) \leq 4\delta^*(G) - 20 + 4(n+5) = 4(n+\delta^*(G)).$$

Therefore, the proof is completed. ■

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