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ON TOTAL DOMINATION SUBDIVISION NUMBERS OF TREES

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Abstract

A set S of vertices in a graph G is a total dominating set of G if every vertex is adjacent to a vertex in S. The total domination number $\gamma_t(G)$ is the minimum cardinality of a total dominating set of G. The total domination subdivision number $\mathrm{sd}_{\gamma_t}(G)$ of a graph G is the minimum number of edges that must be subdivided (where each edge in G can be subdivided at most once) in order to increase the total domination number. Haynes *et al.* [*Total domination subdivision numbers of trees*, Discrete Math. 286 (2004) 195–202] have given a constructive characterization of trees whose total domination subdivision number is 3. In this paper, we give new characterizations of trees whose total domination subdivision number is 3.

Keywords: trees, total domination number, total domination subdivision number.

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1. Introduction

For graph theory notation and terminology, we generally follow [3]. Specifically, let G be a graph with vertex set $V(G)$ and edge set $E(G)$, and of order $n(G) = |V(G)|$ and size $m(G) = |E(G)|$. Two vertices u and v of G are *adjacent* if $uv \in E(G)$, and are called *neighbors*. The *open neighborhood* $N_G(v)$ of a vertex v in G is the set of neighbors of v, while the *closed neighborhood* of v is the set $N_G[v] = \{v\} \cup N_G(v)$. In general, for a subset $X \subseteq V(G)$, its *open neighborhood* is the set $N_G(X) = \bigcup_{v \in X} N_G(v)$, and its *closed neighborhood* is the set $N_G[X] = N_G(X) \cup X$.

The *degree* of a vertex v in G is the number of neighbors of v in G, and is denoted by $\deg_G(v)$, and so $\deg_G(v) = |N_G(v)|$. An *isolated vertex* in G is a vertex of degree zero. A graph without any isolated vertex is called an *isolate-free graph*. A vertex of degree 1 is called a *leaf*, and its (unique) neighbor is called a *support vertex*. The edge incident with a support vertex and a leaf neighbor of the support vertex is called a *pendant edge*. A *strong support vertex* is a vertex with at least two leaf neighbors, and a *weak support vertex* is a vertex with exactly one leaf neighbor. The set of leaves and the set of support vertices of G are denoted by $L(G)$ and $S(G)$, respectively.

A graph G is *connected* if there is a (u, v)-path in G joining every two vertices u and v in G. The *distance* between two vertices u and v in a connected graph G, denoted by $d_G(u, v)$, is the minimum length among all (u, v)-paths in G. If X and Y are subsets of vertices of G, then the *distance* $d_G(X, Y)$ between X and Y in G is the minimum distance $d_G(x, y)$ among all pairs of vertices where $x \in X$ and $y \in Y$. The *distance* $d_G(e_1, e_2)$ between two edges $e_1 = u_1v_1$ and $e_2 = u_2v_2$ of G is the distance between the sets $\{u_1, v_1\}$ and $\{u_2, v_2\}$. The *distance* $d_G(e, F)$ between an edge e and a subset F of edges in G is the minimum distance $d_G(e, f)$ between the edge e and all edges $f \in F$. If k is a positive integer and u is a vertex in G, then the *k-neighborhood* of u, denoted by $N_G^k(u)$, is the set of vertices at distance k from u, that is, $N_G^k(u) = \{x \in V(G) \colon d_G(u, x) = k\}$.

We use P_n, C_n, and K_n to denote a *path*, a *cycle*, and a *complete graph*, respectively, on n vertices. The *complete bipartite graph* $K_{r,s}$ is a bipartite graph with partite sets X and Y, where $|X| = r$, $|Y| = s$, and every vertex in X is adjacent to every vertex in Y. A *star* is a tree with at most one vertex that is not a leaf; that is, stars consist of complete bipartite graphs $K_{1,s}$ for $s \geq 1$ along with the trivial graph K_1. For $k \geq 1$ an integer, we let $[k]$ denote the set $\{1, \ldots, k\}$ and we let $[k]_0 = [k] \cup \{0\} = \{0, 1, \ldots, k\}$.

A *total dominating set*, abbreviated TD-set, of a graph G is a set S of vertices of G such that every vertex has a neighbor in S, and so $N_G(S) = V(G)$. The *total domination number* of G, denoted by $\gamma_t(G)$, is the minimum cardinality of a TD-set in G. For fundamentals on total domination theory in graphs we refer the reader to the authors' book [7], and to the so-called "domination books" [1, 2, 3].

The *total domination subdivision number* $\mathrm{sd}_{\gamma_t}(G)$ of a graph G is the minimum number of edges that must be subdivided (where each edge in G can be subdivided at most once) in order to increase the total domination number. The total domination subdivision number was introduced by Haynes *et al.* [4] and is now well studied in the literature, see [4, 5, 6], to mention but a few papers on the topic. Haynes *et al.* [4] have shown that the total domination subdivision number of a graph can be arbitrarily large, but the total domination subdivision number of a tree is either 1, 2, or 3, and so trees can be classified as Class 1, Class 2, or Class 3 depending on whether their total domination subdivision number is 1, 2, or 3, respectively. A constructive characterization of all trees in Class 3 has been provided by Haynes, Henning, and Hopkins in [6]. In this paper, inspired by results of Haynes, Henning, and Hopkins [6], we continue their studies of trees in Class 3 and, in particular, we provide a different new characterizations of the trees that belong to this class.

1.1. Known results

In this section, we present the constructive characterization of trees in Class 3 presented in [6]. For this purpose the authors of [6] describe a procedure to build a family $\mathcal{T}$ of labeled trees that are of Class 3 as follows. We assign to each vertex v a *label*, also called its *status*, and denoted by $\mathrm{sta}(v)$.

Definition 1. Let $\mathcal{T}$ be the family of labeled trees that

(1) contains a path P_6 where the two leaves have status C, the two support vertices have status B, and the two central vertices have status A; and

(2) is closed under the two operations $\mathcal{T}_1$ and $\mathcal{T}_2$, which extend the labeled tree T belonging to $\mathcal{T}$ by attaching a tree to the vertex $y \in V(T)$.

- **Operation $\mathcal{T}_1$.** Assume $\mathrm{sta}(y) = A$. Then add a path xwv and the edge xy. Let $\mathrm{sta}(x) = A$, $\mathrm{sta}(w) = B$, and $\mathrm{sta}(v) = C$.
- **Operation $\mathcal{T}_2$.** Assume $\mathrm{sta}(y) \in \{B, C\}$. Then add a path $xwvu$ and the edge xy. Let $\mathrm{sta}(x) = \mathrm{sta}(w) = A$, $\mathrm{sta}(v) = B$, and $\mathrm{sta}(u) = C$.

The operations $\mathcal{T}_1$ and $\mathcal{T}_2$, and a labeled tree belonging to the family $\mathcal{T}$ are illustrated in Figure 1. The *underlying tree* of a labeled tree T is the tree obtained from T by removing the vertex labels.

If $T \in \mathcal{T}$, we let $S_A(T)$, $S_B(T)$, and $S_C(T)$ be the sets of vertices of status A, B, and C, respectively, in T. The following observation is immediate from the way in which each tree in the family $\mathcal{T}$ is constructed.

Observation 1.1 [6]**.** *If $T \in \mathcal{T}$, then the following properties hold.*

1. *If $v \in S_A(T)$, then $|N_T(v) \cap (S_B(T) \cup S_C(T))| = 1$ and $N_T(v) \cap S_A(T) \neq \emptyset$;*

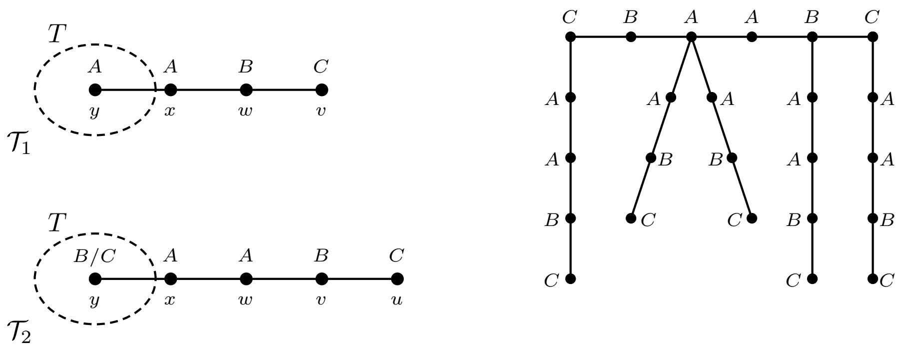

Figure 1. Operations $\mathcal{T}_1$ and $\mathcal{T}_2$, and a tree belonging to the family $\mathcal{T}$.

2. *If* $v \in S_B(T)$ $(v \in S_C(T)$, *respectively*), *then* $|N_T(v) \cap S_C(T)| = 1$ $(|N_T(v) \cap S_B(T)| = 1$, *respectively*), *and* $N_T(v) \setminus (S_B(T) \cup S_C(T)) \subseteq S_A(T)$;
3. $L(T) \subseteq S_C(T)$ *and* $S(T) \subseteq S_B(T)$;
4. $\{v \in V(T)\colon \max\{d_T(v, S_B(T)), d_T(v, S_C(T))\} = 2\} = S_A(T)$;
5. $|S_B(T)| = |S_C(T)|$.

Haynes, Henning and Hopkins [6] gave the following characterization of trees in Class 3.

Theorem 1.2 [6]**.** A tree is in Class 3 if and only if it is the underlying tree of a labeled tree that belongs to the family $\mathcal{T}$.

2. Two New Characterizations

We are interested in structural properties of trees belonging to Class 3. We first present another constructive characterization of labeled trees belonging to the family $\mathcal{T}$ that is a modification of the characterization given in [6].

Definition 2. Let $\mathcal{O}$ be the family of labeled trees that

(1) contains a path P_6 in which the two central vertices have status A, and all other vertices have status B; and

(2) is closed under the two operations $\mathcal{O}_1$ and $\mathcal{O}_2$, which extend the labeled tree T belonging to $\mathcal{O}$ by adding a labeled P_6, and then

- **Operation** $\mathcal{O}_1$. Identifying one B-B-A-path of T with one B-B-A-path of P_6;
- **Operation** $\mathcal{O}_2$. Identifying one B-B-edge of T with one B-B-edge of P_6.

The two operations $\mathcal{O}_1$ and $\mathcal{O}_2$, and an example of a labeled tree belonging to the family $\mathcal{O}$ are given in Figure 2.

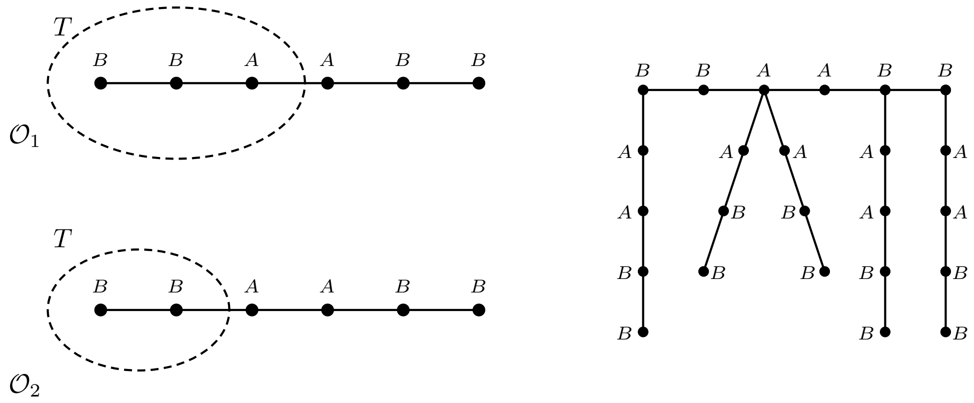

Figure 2. Operations $\mathcal{O}_1$ and $\mathcal{O}_2$, and a tree belonging to the family $\mathcal{O}$.

If $T \in \mathcal{O}$, we let $S_A(T)$ and $S_B(T)$ be the sets of vertices of status A and B, respectively, in T. The following observation follows immediately from the way in which each tree in the family $\mathcal{O}$ is constructed.

Observation 2.1. *If $T \in \mathcal{O}$, then the following two properties hold.*

1. $|N_T(v) \cap S_B(T)| = 1$ *for each* $v \in V(T)$;
2. $L(T) \cup S(T) \subseteq S_B(T)$.

We are now in a position to prove the following characterization of trees that belong to Class 3.

Theorem 2.2. *If T is a tree of order at least 6, then the following statements are equivalent.*

(1) *T is in Class 3;*

(2) $T \in \mathcal{T}$;

(3) $T \in \mathcal{O}$.

(4) *There is a uniquely determined subset F of $E(T)$ such that*

  (a) *each pendant edge of T belongs to F;*

  (b) $d_T(e, F \setminus \{e\}) = 3$ *for each* $e \in F$;

  (c) *if e and f are distinct elements of F, then there is a unique sequence $(e_0, e_1, \ldots, e_k)$ of elements of F such that $e_0 = e$, $e_k = f$, and $d_T(e_{i-1}, e_i) = 3$ for $i \in [k]$.*

For example, if T is the tree illustrated in Figure 3, then the subset F of broad edges of T indicated in Figure 3 satisfies property (4) in the statement of Theorem 2.2. Moreover if $e = e_0$ and $f = e_4$, then the sequence $(e_0, e_1, e_2, e_3, e_4)$ satisfies property (4c) in the statement of Theorem 2.2.

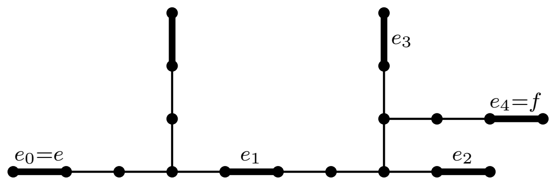

Figure 3. An example to the statement (d) of Theorem 2.2.

Proof. The equivalence of the statements (1) and (2) has been proved in [6], see Theorem 1.2. The proof of the equivalence of the statements (2) and (3) is straightforward by Observations 1.1 and 2.1, and we omit the details. We shall prove the equivalence of the statements (3) and (4).

Assume first that $T \in \mathcal{O}$. Thus the tree T can be obtained from a sequence $T_1, \ldots, T_p$ of trees, where $T_1 = P_6$ and $T = T_p$, and, if $p \geq 2$, T_{i+1} can be obtained from T_i by operation $\mathcal{O}_1$ or $\mathcal{O}_2$ for $i \in [p-1]$. Let F be the set of all B-B edges in T. By induction on the number p we shall prove that F has the desired properties.

If $p = 1$, then $T = P_6$, and the set F (consisting of the two pendant edges of P_6) has the desired properties. This establishes the base case. Assume, then, that the result holds for all trees that can be constructed from a sequence of fewer than p trees, where $p \geq 2$. Let $T \in \mathcal{O}$ be obtained from a sequence $T_1, T_2, \ldots, T_p$ of p trees. By our inductive hypothesis, the set F' of the *B-B* edges of T_{p-1} has the desired properties in T_{p-1}. We now consider two possibilities depending on whether T is obtained from T_{p-1} by operation $\mathcal{O}_1$ or $\mathcal{O}_2$.

Assume first that T is obtained from T_{p-1} by operation $\mathcal{O}_1$. Suppose, without loss of generality, that T is obtained from T_{p-1} by identifying a B-B-A path xyz in T_{p-1} with a B-B-A path in the labeled P_6, say with xyz in P_6: $xyzabc$, where z and a have status A, while x, y, b, c have status B. Now from the properties of F' in T_{p-1}, we infer that the set $F' \cup \{bc\}$ (of all B-B edges in T) has properties (4a)–(4c) in T. Similarly, if T is obtained from T_{p-1} by operation $\mathcal{O}_2$, then again the set of all B-B edges of T has properties (4a)–(4c) in T.

Assume now that T is a tree of order at least 6 in which there is a unique nonempty subset F of $E(T)$ having properties (4a)–(4c). By induction on the order $n \geq 6$ of T, we shall prove that $T \in \mathcal{O}$. The only tree of order 6 having the desired properties is the path P_6, and so in this case $T \in \mathcal{O}$. This establishes the base step. Let $n > 6$, and assume that if T' is a tree of order n' with $6 \leq n' < n$

that has a unique subset of edges having properties (4a)–(4c), then $T' \in \mathcal{O}$. Let T be a tree of order n with a unique subset F of $E(T)$ having properties (4a)–(4c). Let $P\colon v_0v_1\dots v_k$ be a longest path in T. For convenience, we root T at the leaf v_k. From the properties (4a) and (4b) of edges belonging to F and by the maximality of the path P, it follows that $k \geq 5$ and $d_T(v_1) = d_T(v_2) = 2$. We consider two cases depending on the degree of v_3 in T.

Case 1. $d_T(v_3) = 2$. In this case, we let T' denote the subtree $T - \{v_0, v_1, v_2, v_3\}$ of T. Since v_0v_1 is a pendant edge in T, it follows from the properties (4a) and (4b) that there is exactly one vertex $v_4' \in N_T(v_4) \setminus \{v_3\}$ such that $v_4v_4' \in F$. Now, from the properties (4a)–(4c) of F in T it follows that $F' = F \setminus \{v_0v_1\}$ has the properties (4a)–(4c) in T'. Applying the inductive hypothesis we infer that $T' \in \mathcal{O}$. Thus, T can be obtained from T' by operation $\mathcal{O}_2$ (identifying the B-B edge v_4v_4' of T' with the B-B edge v_4v_4' of the labeled $P_6\colon v_0v_1v_2v_3v_4v_4'$), and so, $T \in \mathcal{O}$.

Case 2. $d_T(v_3) \geq 3$. In this case, we let T' denote the subtree $T - \{v_0, v_1, v_2\}$ of T. Since v_0v_1 is a pendant edge in T, it follows from the properties (4a) and (4b) that there are vertices $x \in N_T(v_3) \setminus \{v_2\}$ and $x' \in N_T(x) \setminus \{v_3\}$ such that $xx' \in F$. As in Case 1, from the properties (4a)–(4c) of F in T it follows that $F' = F - \{v_0v_1\}$ has the properties (4a)–(4c) in T'. Hence, an application of the inductive hypothesis implies that $T' \in \mathcal{O}$. In this case, T can be obtained from T' by applying operation $\mathcal{O}_1$ (identifying the B-B-A path $x'xv_3$ of T' with the B-B-A path $x'xv_3$ of the labeled $P_6\colon v_0v_1v_2v_3xx'$). This proves that $T \in \mathcal{O}$, and completes the proof of the equivalence of the statements (3) and (4). ■

As an immediate consequence of Theorem 2.2, we characterize the paths that are in Class 3.

Corollary 1. *A path P_n is in Class 3 if and only if $n \equiv 2 \pmod 4$.*

Let v be a weak support vertex of degree at least 3 in a tree T. If sets A and B form a partition of the set $N_T(v) \setminus L(T)$, then by T_A (T_B, respectively) we denote the component of $T - B$ ($T - A$, respectively) that contains the vertex v, as illustrated in Figure 4.

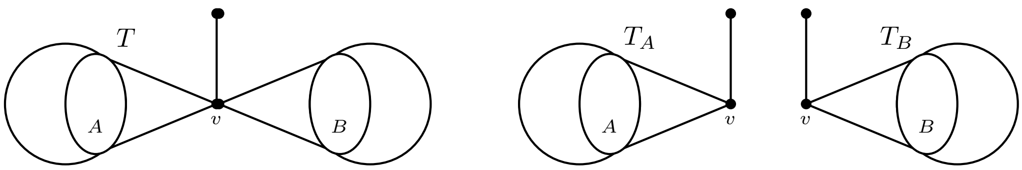

Figure 4. A tree T and the components T_A and T_B.

Lemma 2.3. *Let v be a weak support vertex of degree at least* 3 *in a tree T, and let sets A and B form a partition of the set $N_T(v) \setminus L(T)$. Then the tree T is in Class* 3 *if and only if both subtrees T_A and T_B are in Class* 3*.*

Proof. The proof follows readily from the fact that a subset F of $E(T)$ has properties (4a)–(4c) of Theorem 2.2 in T if and only if each of the sets $F \cap E(T_A)$ and $F \cap E(T_B)$ has properties (4a)–(4c) of Theorem 2.2 in T_A and T_B, respectively. ∎

A *caterpillar* is a tree of order at least 3 with the property that the removal of its leaves results in a path, called the *spine* of the caterpillar. The code C of a caterpillar T with spine $v_0v_1\ldots v_s$ is the sequence of nonnegative integers $(t_0, t_1, \ldots, t_s)$, where t_i is the number of leaves adjacent to v_i in T. We say that two leaves of a caterpillar T are *consecutive* if no inner vertex of the path joining their neighbors is a support vertex in T. Haynes *et al.* [4] characterized caterpillars in Class 3.

Theorem 2.4 [4]**.** *A caterpillar T with code $C = (t_0, \ldots, t_s)$ is in Class* 3 *if and only if $t_i \in \{0, 1\}$ for $i \in [s]_0$, and any two consecutive nonzero entries in C are at distance* $3 \pmod 4$*.*

Let T be a caterpillar with spine $v_0v_1\ldots v_s$ and with code $C = (t_0, \ldots, t_s)$ that is in Class 3. If t_i and t_j are two consecutive nonzero entries in C, then by Theorem 2.4 we have $t_i = t_j = 1$ and the vertices v_i and v_j on the spine are at distance $3 \pmod 4$, that is, $d_T(v_i, v_j) \equiv 3 \pmod 4$. Let l_i and l_j be the leaf neighbors of the vertices v_i and v_j, respectively. Then, l_i and l_j are consecutive leaves in T and $d_T(l_i, l_j) \equiv 1 \pmod 4$. Conversely, if T is a caterpillar with code $C = (t_0, \ldots, t_s)$ where $t_i \in \{0, 1\}$ and where any two consecutive leaves of T are at distance $1 \pmod 4$, then the neighbors of these leaves (that belong to the spine of T) are at distance $3 \pmod 4$, and so by Theorem 2.4, the caterpillar T is in Class 3. This yields the following equivalent statement of Theorem 2.4.

Theorem 2.5 [4]**.** *A caterpillar T of order at least* 6 *is in Class* 3 *if and only if every two consecutive leaves of T are at distance* $1 \pmod 4$*.*

We present next a short proof of Theorem 2.5.

Proof. Let T be a caterpillar of order $n \geq 6$. We proceed by induction on the number $l \geq 2$ of leaves in T to show that T is in Class 3 if and only if every two consecutive leaves of T are at distance $1 \pmod 4$. If $l = 2$, then T is a path P_n, and so by Corollary 1, $n \equiv 2 \pmod 4$, and so the two leaves of T are at distance $1 \pmod 4$. This establishes the base case. Let $l \geq 3$ and assume that every caterpillar T' of order at least 6 with l' leaves is in Class 3 if and only if every two consecutive leaves of T' are at distance $1 \pmod 4$. Let T be a caterpillar of order $n \geq 6$ with l leaves.

If T has a strong support vertex v and if v_1 and v_2 are two leaf neighbors of v, then subdividing two edges vv_1 and vv_2 increases the total domination number, implying that T is in Class 1 or 2, a contradiction. Hence, every support vertex of T is a weak support vertex. Since T has at least three leaves, the caterpillar T has a leaf, say d, whose (unique) neighbor, say c, is of degree 3. Let a and b be the two neighbors of c on the spine of T. Let T_a (T_b, respectively) be the component of $T - b$ ($T - a$, respectively) that contains a (b, respectively). We note that both T_a and T_b are caterpillars. By Corollary 2.3, T is in Class 3 if and only if both T_a and T_b are in Class 3. By the inductive hypothesis, T_a and T_b are in Class 3 if and only if every two consecutive leaves in T_a are at distance $1 \pmod 4$ and every two consecutive leaves in T_b are at distance $1 \pmod 4$. From these properties of the caterpillars T_a and T_b, we infer that every two consecutive leaves of T are at distance $1 \pmod 4$. This proves Theorem 2.5. ■

3. Another Characterization

We show in this section that trees belonging to the family $\mathcal{O}$ are 2-subdivisions of trees or can be obtained from 2-subdivisions of trees, where a 2-subdivision of a graph is defined as follows. Recall that the *corona* $H \circ K_1$ of a graph H is the graph obtained from H by adding for each vertex $v \in V(H)$ a new vertex v' and the edge vv'.

Definition 3. Let G be a connected graph of order at least 2, and let $\mathcal{P} = \{\mathcal{P}(v) \colon v \in V(G)\}$ be a family in which $\mathcal{P}(v)$ is a partition of the set $N_G(v)$ for each $v \in V(G)$. The 2-subdivision of G with respect to $\mathcal{P}$ is the graph $G(\mathcal{P})$ with vertex set

$$V(G(\mathcal{P})) = V(G) \cup (V(G) \times \{1\}) \cup \bigcup_{v \in V(G)} (\{v\} \times \mathcal{P}(v))$$

and edge set $E(G(\mathcal{P})) = E_1 \cup E_2 \cup E_3$ where

$$\begin{aligned} E_1 &= \{v(v,1) \colon v \in V(G)\}, \\ E_2 &= \bigcup_{v \in V(G)} \{v(v,A) \colon A \in \mathcal{P}(v)\}, \\ E_3 &= \bigcup_{uv \in E(G)} \{(u,A)(v,B) \colon A \in \mathcal{P}(u), B \in \mathcal{P}(v), u \in B, v \in A\}. \end{aligned}$$

More intuitively, $G(\mathcal{P})$ is the graph obtained from the corona $G \circ K_1$ by inserting two new vertices into each inner edge of $G \circ K_1$, and then identifying newly inserted vertices according to the partition $\mathcal{P}(v)$ of $N_G(v)$, that is, if $A = \{w_1, \ldots, w_k\} \in \mathcal{P}(v)$, then we contract all neighbors of v on the edges $vw_1, \ldots, vw_k$ into a single vertex (v, A) and replace all multiple edges in the resulting graph by single edges, for each $v \in V(G)$ and $A \in \mathcal{P}(v)$.

Example 3.1. If G is the tree shown in Figure 5(a), and $\mathcal{P} = \{\mathcal{P}(a), \ldots, \mathcal{P}(f)\}$ is a family of partitions of the sets $N_G(a), \ldots, N_G(f)$, respectively, where $\mathcal{P}(a) = \{\{d, e\}\}$, $\mathcal{P}(b) = \{\{e\}\}$, $\mathcal{P}(c) = \{\{e, f\}\}$, $\mathcal{P}(d) = \{\{a\}\}$, $\mathcal{P}(e) = \{\{a, b\}, \{c\}\}$, and $\mathcal{P}(f) = \{\{c\}\}$, then $G(\mathcal{P})$ is the tree shown in Figure 5(b).

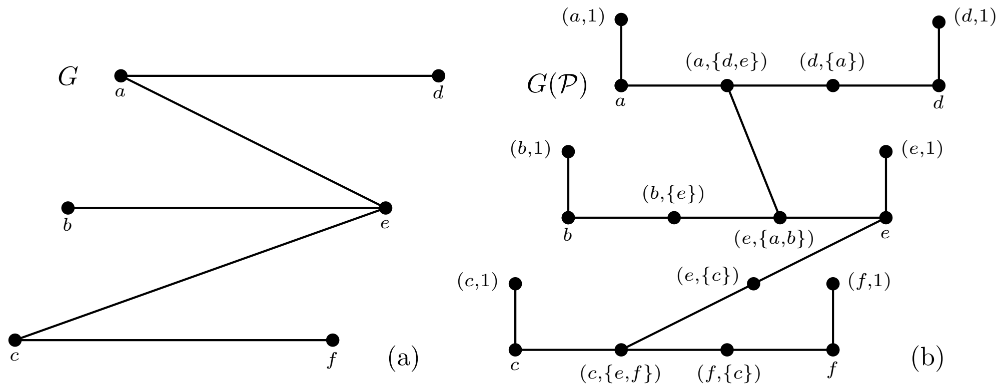

Figure 5. The trees G and $G(\mathcal{P})$.

Let $\mathcal{O}'$ be the family of all trees $T(\mathcal{P})$, where T is any tree of order at least 2, and $\mathcal{P} = \{\mathcal{P}(v) \colon v \in V(T)\}$ is a family in which $\mathcal{P}(v)$ is a partition of the set $N_T(v)$ for each $v \in V(T)$. If $T(\mathcal{P}) \in \mathcal{O}'$, then we observe that the set $F = \{v(v, 1) \colon v \in V(T)\}$ of all pendant edges of $T(\mathcal{P})$ has properties (4a)–(4c) of Theorem 2.2. Therefore, $T(\mathcal{P}) \in \mathcal{O}$, and so $\mathcal{O}' \subseteq \mathcal{O}$. The path P_{10} proves that $\mathcal{O}'$ is a proper subfamily of the family $\mathcal{O}$. We state these observations formally as follows.

Observation 3.2. *The family $\mathcal{O}'$ is a proper subfamily of the family $\mathcal{O}$.*

In the next theorem we characterize trees belonging to the family $\mathcal{O}'$ in terms of 2-packings. A set S of vertices of a graph G is a 2-*packing* in G if the vertices in S are pairwise at distance at least 3 in G, that is, $N_G[u] \cap N_G[v] = \emptyset$ for every pair of distinct vertices $u, v \in S$. The 2-*packing number* of G, denoted by $\rho(G)$, is the maximum cardinality of a 2-packing in G.

Theorem 3.3. *If T is a tree of order at least 6, then T is in the family $\mathcal{O}'$ if and only if the set of weak support vertices of T is a maximum 2-packing in T.*

Proof. Assume first that $T \in \mathcal{O}'$, say $T = R(\mathcal{P})$ for some tree R and some family $\mathcal{P}$ of partitions of the sets $N_R(v)$, $v \in V(R)$. From the definition of $R(\mathcal{P})$ it follows that $V(R)$ is the set of weak supports in $R(\mathcal{P})$. In addition, the set $V(R)$ is also a 2-packing in $R(\mathcal{P})$ (as $d_{R(\mathcal{P})}(u, v) \geq 3$ for each pair of vertices $u, v \in V(R)$), and therefore $|V(R)| \leq \rho(R(\mathcal{P}))$. On the other hand since $\{N_{R(\mathcal{P})}[v] \colon v \in V(R)\}$ is a partition of the set $V(R(\mathcal{P}))$ and every 2-packing has at most one vertex in $N_R[v]$ for every $v \in V(R)$ (as $d_{R(\mathcal{P})}(x, y) \leq 2$ for each pair of vertices $x, y \in N_{R(\mathcal{P})}[v]$) it follows that every 2-packing in $R(\mathcal{P})$ has at most $|V(R)|$ vertices, and therefore $\rho(R(\mathcal{P})) \leq |V(R)|$. This implies that $V(R)$ is a maximum 2-packing in $R(\mathcal{P})$.

Assume now that T is a tree of order at least 6 in which the set $S'(T)$ of weak support vertices is a maximum 2-packing. We shall prove that T is in the family $\mathcal{O}'$. We first prove three claims.

Claim 1. *The tree T has no strong support vertex.*

Proof. Suppose, to the contrary, that v is a strong support vertex in T. Let v' be any leaf adjacent to v in T. Certainly, neither v nor v' is in $S'(T)$. Since $S'(T)$ is a maximum 2-packing in T, the set $S'(T) \cup \{v'\}$ is not a 2-packing in T and therefore there exists a vertex (in fact, exactly one vertex), say u, in $N_T(v) \cap S'(T)$. Now, if u' is the leaf adjacent to u, then $S = (S'(T) \setminus \{u\}) \cup \{u', v'\}$ is a 2-packing in T, and so $\rho(T) \geq |S| > |S'(T)| = \rho(T)$, a contradiction. □

Claim 2. *$d_T(x, S'(T) \setminus \{x\}) = 3$ for each $x \in S'(T)$.*

Proof. Since $S'(T)$ is a 2-packing in T, $d_T(x, y) \geq 3$ for each pair of vertices $x, y \in S'(T)$, and therefore $d_T(x, S'(T) \setminus \{x\}) \geq 3$ for each $x \in S'(T)$. Suppose that $d_T(v, S'(T) \setminus \{v\}) = k \geq 4$ for some vertex $v \in S'(T)$. Let $u \in S'(T)$ be a vertex such that $d_T(u, v) = d_T(v, S'(T) \setminus \{v\}) = k$, and let $v_0v_1 \ldots v_k$ be the (v, u)-path in T where $v = v_0$ and $u = v_k$. If $k \geq 5$ and if u' and v' are the leaves adjacent to u and v, respectively, then the set $S = (S'(T) \setminus \{u, v\}) \cup \{v', u', v_2\}$ is a 2-packing in T, and so $\rho(T) \geq |S| > |S'(T)| = \rho(T)$, a contradiction. Hence, $k = 4$. In this case, let $A = \{w \in S'(T) \colon d_T(v, w) = 4\}$. Since $u \in A$, we note that $|A| \geq 1$. For each $w \in A \cup \{v\}$, let w' be the (unique) leaf neighbor of w in T. Now, the set

$$S = (S'(T) \setminus (A \cup \{v\})) \cup (\{v', v_2\} \cup \{w' \colon w \in A\})$$

is a 2-packing in T, and so $\rho(T) \geq |S| > |S'(T)| = \rho(T)$, a contradiction. □

Claim 3. *If x and y are distinct vertices in $S'(T)$, then there is exactly one sequence of vertices, say $(x_0, x_1, \ldots, x_k)$ where $x = x_0$ and $y = x_k$, of distinct vertices in $S'(T)$ such that $d_T(x_{i-1}, x_i) = 3$ for $i \in [k]$.*

Before we present a proof of Claim 3, as an illustration of the claim, if $T = G(\mathcal{P})$ is the tree shown in Figure 5(b), then the set $S'(T) = \{a, b, c, d, e, f\}$.

Moreover, if $x = a$ and $y = f$, for example, then x and y are distinct vertices in $S'(T)$ and there is exactly one sequence (x_0, x_1, x_2, x_3) where $x_0 = x$, $x_1 = e$, $x_2 = c$, and $x_3 = y$ of distinct vertices in $S'(T)$ such that $d_T(x_{i-1}, x_i) = 3$ for $i \in [3]$.

As a further illustration of Claim 3, consider the tree T shown in Figure 6. If $S'(T)$ is the set of weak support vertices in T and if $x = a$ and $y = x_6$, for example, then x and y are distinct vertices in $S'(T)$ and there is exactly one sequence $(x_0, x_1, x_2, x_3, x_4, x_5, x_6)$ where $x_0 = x$ and $x_6 = y$ of distinct vertices in $S'(T)$ such that $d_T(x_{i-1}, x_i) = 3$ for $i \in [6]$.

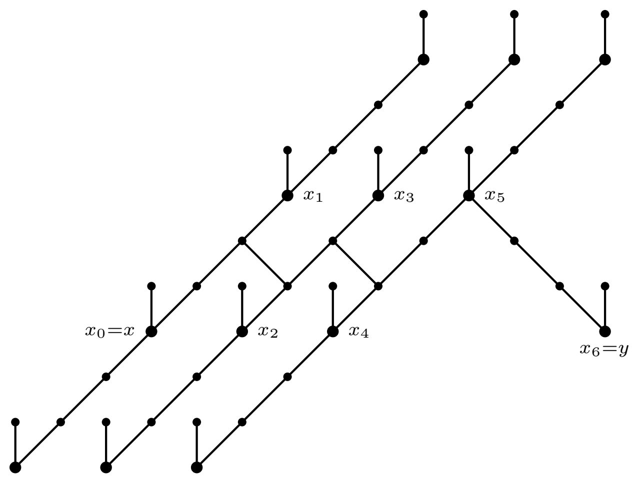

Figure 6. A tree T illustrating Claim 3.

***Proof of Claim* 3.** It follows from Claim 2 that $d_T(x, y) \geq 3$. To prove the desired result, we proceed by induction on $d_T(x, y)$. If $d_T(x, y) = 3$, then the sequence (x, y) has the desired property and this establishes the base case. Assume, then, that the result holds for all pairs $x', y' \in S'(T)$ such that $3 \leq d_T(x', y') < q$, where $q \geq 4$. Assume that $x, y \in S'(T)$, $d_T(x, y) = q$, and let $y_0y_1 \ldots y_q$ be the (x, y)-path in T where $x = y_0$ and $y = y_q$. Thus, $y_0 \in S'(T)$, and $y_1, y_2 \notin S'(T)$ (as $S'(T)$ is a 2-packing).

We now prove that a vertex (and then exactly one vertex) belonging to $N_T(y_2) \setminus \{y_1\}$ is in $S'(T)$. Suppose, to the contrary, that $(N_T(y_2) \setminus \{y_1\}) \cap S'(T) = \emptyset$. Thus, $N_T[y_2] \cap S'(T) = \emptyset$. In this case, the set $S = (N_T(S'(T)) \cap L(T)) \cup \{y_2\} = L(T) \cup \{y_2\}$ is a 2-packing in T, and so $\rho(T) \geq |S| = |S'(T)| + 1 > |S'(T)| = \rho(T)$, a contradiction. Therefore, $(N_T(y_2) \setminus \{y_1\}) \cap S'(T) \neq \emptyset$.

If $y_3 \in S'(T)$, then $3 \leq d_T(y_3, y) < q$ and, by our inductive hypothesis, there is exactly one sequence $(z_0, z_1, \ldots, z_k)$ of distinct vertices in $S'(T)$ where $z_0 = y_3$ and $y = z_k$ and such that $d_T(z_{i-1}, z_i) = 3$ for $i \in [k]$. Thus, $(x_0, x_1, \ldots, x_{k+1})$

where $x = x_0$, $y = x_{k+1}$ and where $x_i = z_{i+1}$ for $i \in [k]$, is the desired sequence. Hence, we may assume that $y_3 \notin S'(T)$, for otherwise the desired result follows.

Let y_2' be the (unique) element of $(N_T(y_2) \setminus \{y_1, y_3\}) \cap S'(T)$. Thus, $3 \le d_T(y_2', y) < n$ and the inductive hypothesis guarantees that there is exactly one sequence $(z_0, z_1, \ldots, z_k)$ of distinct vertices in $S'(T)$ where $z_0 = y_2'$ and $z_k = y$ and such that $d_T(z_{i-1}, z_i) = 3$ for $i \in [k]$. Thus, $(x_0, x_1, \ldots, x_{k+1})$ where $x = x_0$ and where $x_{i+1} = z_i$ for $i \in [k]$, yielding the desired sequence. □

We now return to the proof of Theorem 3.3, and are ready to prove that T is in the family $\mathcal{O}'$. We shall prove that T is isomorphic to $R(\mathcal{P})$ for some tree R and some family $\mathcal{P}$ of partitions of the sets $N_R(u)$ where $u \in V(R)$. Let $R = (V(R), E(R))$ be a graph with vertex set $V(R) = S'(T)$ and edge set $E(R) = \{uv\colon u, v \in V(R) \text{ and } d_T(u, v) = 3\}$. It follows from Claim 3 that R is a tree. Let $\mathcal{P} = \{\mathcal{P}(u)\colon u \in V(R)\}$ be a family, where for a vertex $u \in V(R)$, $\mathcal{P}(u)$ is the family $\{A_{ux}\colon x \in N_T(u) \setminus L(T)\}$ in which

$$A_{ux} = \{y \in S'(T)\colon d_T(x, y) = 2\} = S'(T) \cap N_T^3(u) \cap N_T^2(x)$$

for $x \in N_T(u) \setminus L(T)$. We note that if $u \in V(R)$, then

$$\begin{aligned}
\bigcup_{x \in N_T(u) \setminus L(T)} A_{ux} &= \bigcup_{x \in N_T(u) \setminus L(T)} (S'(T) \cap N_T^3(u) \cap N_T^2(x)) \\
&= S'(T) \cap N_T^3(u) \cap \left(\bigcup_{x \in N_T(u) \setminus L(T)} N_T^2(x) \right) \\
&= S'(T) \cap N_T^3(u) \\
&= N_R(u).
\end{aligned}$$

In addition, since

$$A_{ux} \cap A_{uy} = S'(T) \cap N_T^3(u) \cap N_T^2(x) \cap N_T^2(y)$$

and T is a tree, the sets A_{ux} and A_{uy} are disjoint if $x, y \in N_T(u) \setminus L(T)$ and $x \neq y$. The above implies that $\mathcal{P}(u) = \{A_{ux}\colon x \in N_T(u) \setminus L(T)\}$ is a partition of the set $N_R(u)$ for $u \in V(R)$, and so

$$\{u\} \times \mathcal{P}(u) = \big\{(u, A_{ux})\colon x \in N_T(u) \setminus L(T)\big\}$$

for every vertex $u \in V(R)$. Let us consider the graph $R(\mathcal{P})$ for the above defined set R and the partition $\mathcal{P}$. By construction, $R(\mathcal{P})$ is a graph with vertex set

$$V(R(\mathcal{P})) = V(R) \cup (V(R) \times \{1\}) \cup \left(\bigcup_{u \in V(R)} \{(u, A_{ux})\colon x \in N_T(u) \setminus L(T)\} \right)$$

and edge set $E(R(\mathcal{P})) = E_1 \cup E_2 \cup E_3$ where

$$\begin{aligned} E_1 &= \{u(u,1)\colon u \in V(R)\}, \\ E_2 &= \bigcup_{u \in V(R)} \{u(u,A)\colon A \in \mathcal{P}(u)\}, \\ E_3 &= \bigcup_{uv \in E(R)} \{(u,A)(v,B)\colon A \in \mathcal{P}(u), B \in \mathcal{P}(v), u \in B, v \in A\}. \end{aligned}$$

Equivalently, the edge sets E_2 and E_3 are the sets

$$E_2 = \bigcup_{u \in V(R)} \{u(u,A_{ux})\colon x \in N_T(u) \setminus L(T)\}$$

and

$$\begin{aligned} E_3 = \bigcup_{uv \in E(R)} \big\{&(u,A_{ux})(v,B_{vy})\colon x \in N_T(u) \setminus L(T), y \in N_T(v) \setminus L(T), \\ &u \in B_{vy}, v \in A_{ux}\big\}. \end{aligned}$$

We now let

$$\varphi\colon V(R(\mathcal{P})) \to V(T)$$

be a function defined in such a way that $\varphi(x) = x$ for all $x \in V(R)$, $\varphi((x,1)) = l_x$ if $(x,1) \in V(R) \times \{1\}$ and l_x is the only leaf adjacent to x in T, and, finally, $\varphi((u,A_{ux})) = x$ if $u \in V(R)$, $x \in N_T(u) \setminus L(T)$, and $A_{ux} \in \mathcal{P}(u)$. It is immediate from the definitions of R, $\mathcal{P}$, and $R(\mathcal{P})$ that the function φ is an isomorphism of T and $R(\mathcal{P})$. This completes the proof. ■

In this final subsection, we give a simple construction that makes it possible to build trees belonging to the family $\mathcal{O}$ from smaller trees belonging to the family $\mathcal{O}'$. Let M be a matching in a graph G. By $G \mp M$ we denote the graph obtained from $G - M$ by adding exactly one new pendant edge to each vertex covered by M. On the other hand, let N be a matching in the complement $\overline{G}$ of G, and assume that every vertex covered by N is a weak support in G. Then by $G \pm N$ we denote the graph obtained from $G \cup N$ by removing the leaf neighbor of every vertex in N. It follows from these definitions that if M is a matching in G, then $(G \mp M) \pm M = G$. Similarly, if N is a matching in $\overline{G}$ and every vertex covered by N is a weak support in G, then $(G \pm N) \mp N = G$, see Figure 7 for an illustration.

As a consequence of Theorem 3.3, we have the following corollary.

Corollary 4. *Let u and v be weak support vertices in disjoint trees T_1 and T_2, respectively, and let T_{uv} be the tree obtained from the union $T_1 \cup T_2$ by adding the edge uv and removing the leaf neighbor adjacent to u and v, respectively. Then the tree T_{uv} is in Class 3 if both T_1 and T_2 are in Class 3.*

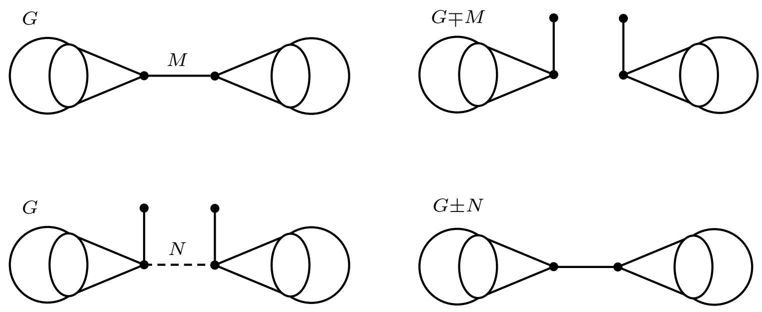

Figure 7. M and N are matchings in G and $\overline{G}$, respectively.

Proof. Assume that T_1 and T_2 are in Class 3, and let F_1 and F_2 be the subsets of $E(T_1)$ and $E(T_2)$ having properties (4a)–(4c) of Theorem 2.2 in T_1 and T_2, respectively. Let uu' and vv' be pendant edges incident with u and v in T_1 and T_2, respectively. Then the set $(F_1 \cup F_2 \cup \{uv\}) \setminus \{uu', vv'\}$ has properties (4a)–(4c) of Theorem 2.2 in T_{uv} and, therefore, T_{uv} is in Class 3. ■

We remark that the example of a path P_{10} illustrates that the converse implication of Corollary 4 does not hold without additional assumptions on the edge uv and vertices u and v in T_{uv}.

Theorem 3.4. *If T is a tree of order at least 6, then $T \in \mathcal{O}$ if and only if $T \in \mathcal{O}'$ or every component of $T \mp M$ belongs to $\mathcal{O}'$ for some matching M in T.*

Proof. Assume that $T \in \mathcal{O}$. Then in T there is a uniquely determined subset F of $E(T)$ that has properties (4a)–(4c) of Theorem 2.2. If every edge in F is a pendant edge in T, then it follows from (4b) that the set $S'(T)$ (of weak supports of T) is a 2-packing in T.

We claim that $S'(T)$ is a maximum 2-packing in T. Suppose, to the contrary, that there is a 2-packing S in T such that $|S| > |S'(T)|$. Since $|N_T[x] \cap S| \leq 1$ for each $x \in S'(T)$, the inequality $|S| > |S'(T)|$ implies that the set $S \setminus N_T[S'(T)]$ is nonempty, say $x_0 \in S \setminus N_T[S'(T)]$. Thus, $d_T(x_0, S'(T)) \geq 2$ and, therefore, if e and f are pendant edges in T belonging to distinct components of $T - x_0$, then no e–f sequence has property (4c), a contradiction. This proves that $S'(T)$ is a maximum 2-packing in T and $T \in \mathcal{O}'$, by Theorem 3.3.

We may therefore assume that not every edge belonging to F is a pendant edge in T. Let M be the set of non-pendant edges in T belonging to F. Now the graph $T \mp M$ is disconnected, it has $k = |M| + 1$ components, say $T_1, \ldots, T_k$, and $S'(T) \cup V(M)$ is the set of weak support vertices in $T \mp M$ (where $V(M)$ is the set of vertices covered by M). From the fact that F has properties (4a)–(4c) of Theorem 2.2 it follows that the set of pendant edges in T_i where $i \in [k]$ has properties (4a)–(4c) of Theorem 2.2. Then, as in the beginning of this proof, the set $S'(T_i)$ of weak support vertices of T_i is a maximum 2-packing in T_i, and, consequently, $T_i \in \mathcal{O}'$, by Theorem 3.3 for $i \in [k]$.

Assume now that M is a matching in T such that every component, say $T_1, \ldots, T_k$, of $T \mp M$ belongs to $\mathcal{O}'$. Then the set of weak support vertices of T_i is a maximum 2-packing in T_i for $i \in [k]$ (and the set of weak support vertices of $T \mp M$ is a maximum 2-packing in $T \mp M$). Consequently, the set F_i of pendant edges of T_i has properties (4a)–(4c) of Theorem 2.2 in T_i. This implies that the set of edges $M \cup \bigcup_{i=1}^{k} (F_i \cap E(T))$ has properties (4a)–(4c) of Theorem 2.2 in T. From these observations and from Theorem 2.2 we infer that $T \in \mathcal{O}$. ■

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