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THE PLANAR TURÁN NUMBER OF $\{C_6, C_7\}$

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Abstract

Let $\mathcal{H}$ be a set of graphs. A graph is $\mathcal{H}$-free if it does not contain any copy of H as a subgraph where $H \in \mathcal{H}$. The planar Turán number of $\mathcal{H}$, denoted by $ex_p(n, \mathcal{H})$, is the maximum number of edges in an $\mathcal{H}$-free planar graph on n vertices. The upper bounds of $ex_p(n, \{C_k, C_{k+1}\})$ are known when $3 \leq k \leq 5$, and these bounds are tight. In this paper, we give the upper bound of $ex_p(n, \{C_6, C_7\})$ for all integers $n \geq 76$, and this bound is sharp.

Keywords: Turán number, planar graph, cycle.

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1. Introduction

All graphs considered in this paper are finite, undirected and simple. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. For any $v \in V(G)$, we use $N_G(v)$ to denote the set of neighbours of v in G. We define $N_G[v] := N_G(v) \cup \{v\}$. Let $d_G(v)$ be the degree of vertex v in G. We use $\delta(G)$ to denote the minimum degree

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of G. Let $V_i(G)$ denote the set of vertices of degree i in G. We say a cycle is a Hamilton cycle of G if it contains all vertices of G. Let $g(G)$ denote the length of the shortest cycle of G. For $V' \subseteq V(G)$ ($E' \subseteq E(G)$), $G[V']$ ($G[E']$) denotes a subgraph which is induced by V' (E') of G. We use C_t and K_t to denote the cycle and complete graph on t vertices, respectively. We use K_t^- to denote the graph obtained from K_t with one edge removed. A graph is k-degenerate if it can be reduced to K_1 by repeatedly deleting vertices of degree at most k. A graph is 2-connected if it is a connected graph on at least 3 vertices and without any cut vertex. For any integer k, let $[k] := \{1, 2, \ldots, k\}$.

A graph G is planar if it can be drawn in the plane so that its edges intersect only at their ends, and such a planar embedding of G is called a plane graph. The unbounded face of a plane graph is called the outer face. For any plane graph G, a vertex of G is called an internal vertex if it is not on the boundary of the outer face of G. We use $F(G)$ to denote the set of all faces of G. We denote the face of degree k and more than k by k-face and k^+-face, respectively. We denote the face by $(v_1, v_2, \ldots, v_k)$-face, if its boundary is a cycle with vertices $v_1, v_2, \ldots, v_k$ in order. If there exists a sequence of 3-faces $F_1, F_2, \ldots, F_k$ so that F_i and F_{i+1} exactly share one edge where $i = 1, 2, \ldots, k-1$ and $k \geq 2$, the 3-faces F_1 and F_k are equivalent. The terminology and notation used in this paper can refer to [1].

Let $\mathcal{H}$ be a set of graphs. A graph is $\mathcal{H}$-free if it does not contain any copy of H as a subgraph where $H \in \mathcal{H}$. The Turán number of $\mathcal{H}$, denoted by $ex(n, \mathcal{H})$, is the maximum number of edges in an $\mathcal{H}$-free graph on n vertices. When $\mathcal{H} = \{H\}$, we write $ex(n, \mathcal{H})$ as $ex(n, H)$. In 1941, Turán [18] proved a classical result in the field of extremal graph theory. He determined the exact value of $ex(n, K_k)$ for any integer n and all $k \geq 3$, and he also proved that the balanced complete $(k-1)$-partite graph on n vertices is the unique extremal graph. This has led to a considerable amount of research work including the classical Turán-type problem and the Turán-type problems when the host graphs are hypergraph, hypercube and random graph [10, 14, 15].

In recent years, the Turán-type problem when the host graph is planar have received much attention. In 2016, Dowden [3] considered the Turán-type problems when the host graph is a planar graph, i.e., how many edges can an $\mathcal{H}$-free planar graph on n vertices have? And the maximum number of edges is called the planar Turán number of $\mathcal{H}$, denoted by $ex_p(n, \mathcal{H})$. For $|\mathcal{H}| = 1$, let $\mathcal{H} = \{H\}$, we write $ex_p(n, \mathcal{H})$ as $ex_p(n, H)$. Early in 2007, Wang *et al.* [19] gave the upper bounds of $ex_p(n, C_k)$ when $3 \leq k \leq 7$, but they did not prove whether these bounds were tight. Dowden [3] determined the exact value of $ex_p(n, C_3)$ and its unique extremal graph, he also gave the tight upper bounds of $ex_p(n, C_k)$ when $k \in \{4, 5\}$. Lan *et al.* [12] obtained the upper bound of $ex_p(n, C_6)$. Ghosh *et al.* [6] improved the upper bound of $ex_p(n, C_6)$ given in [12] and obtained the tight upper bound of $ex_p(n, C_6)$. In 2023, Shi *et al.* [16] and Győri *et al.* [7]

independently obtained the tight upper bound of $ex_p(n, C_7)$.

Theorem 1. *Let n be an integer.*

(1) [3] *For all $n \geq 3$, $ex_p(n, C_3) = 2n - 4$.*

(2) [3] *For all $n \geq 4$, $ex_p(n, C_4) \leq \frac{15}{7}(n-2)$, the equality holds when $n \equiv 30 \pmod{70}$.*

(3) [3] *For all $n \geq 11$, $ex_p(n, C_5) \leq \frac{12n-33}{5}$, the equality holds for infinitely many n.*

(4) [6] *For all $n \geq 18$, $ex_p(n, C_6) \leq \frac{5}{2}n - 7$, the equality holds when $n \equiv 10 \pmod{18}$.*

(5) [7, 16] *For all $n > 38$, $ex_p(n, C_7) \leq \frac{18}{7}n - \frac{48}{7}$, the equality holds for infinitely many n.*

Regarding the lower bound of $ex_p(n, C_k)$, Cranston *et al.* [2] obtained that $ex_p(n, C_k) \geq 3n - 6 - \frac{3n+6}{k}$, when n is a function of k and sufficiently large for all $k \geq 11$. Lan *et al.* [11] obtained that $ex_p(n, C_k) \geq 3n - \frac{3-\frac{2}{k-1}}{k-6+\lfloor\frac{k-1}{2}\rfloor}n + \frac{12+3r-\frac{8+2r}{k-2}}{k-6+\lfloor\frac{k-1}{2}\rfloor} + \frac{4}{k-1} - \min\{r+10, 11\}$, for all n and k with $n \geq k \geq 11$ and for r being the remainder of $n-4$ when divided by $k - 6 + \lfloor\frac{k-1}{2}\rfloor$. Győri *et al.* [8] obtained that $ex_p(n, C_k) \geq 3n - 6 - \frac{6\cdot 3^{\log_2 3} n}{k^{\log_2 3}}$ for n sufficiently large and for all k. Regarding the upper bound of $ex_p(n, C_k)$, Cranston *et al.* [2] conjectured that $ex_p(n, C_k) \leq 3n - 6 - \frac{Dn}{k^{\log_2 3}}$ for n sufficiently large and for all k where D is a constant. Shi *et al.* [17] verified the above conjecture with $D = \frac{1}{4}$.

Theorem 2 [17]**.** *For all integers $n, k \geq 4$, $ex_p(n, C_k) \leq 3n - 6 - \frac{n}{4k^{\log_2 3}}$.*

For $|\mathcal{H}| \geq 2$, Du *et al.* [4] gave the tight upper bounds of $ex_p(n, \{C_k, C_{k+1}\})$ when $k \in \{3, 4\}$. Du *et al.* [5] obtained the tight upper bound of $ex_p(n, \{C_5, C_6\})$. By Theorem 1(1), we see that $ex_p(n, C_3) = 2n-4$ for all $n \geq 3$, so $ex_p(n, \{C_3, C_k\}) \leq 2n - 4$ for all $k \geq 5$. Since $K_{2,n-2}$ must be C_k-free for all $k \neq 4$, we have $ex_p(n, \{C_3, C_k\}) \geq 2n - 4$. Thus $ex_p(n, \{C_3, C_k\}) = 2n - 4$ when $k \geq 5$. The minimum degree of $K_{2,n-2}$ is 2, so Győri *et al.* [9] studied the maximum number of edges in a $\{C_3, C_{2k}\}$-free planar graph on n vertices with the minimum degree more than 2, and obtained the upper bounds when $k \in \{3, 4\}$.

Theorem 3. *Let n be an integer.*

(1) [4] *For all $n \geq 4$, $ex_p(n, \{C_3, C_4\}) \leq \frac{5}{3}(n-2)$, the equality holds when $n \equiv 5 \pmod{15}$.*

(2) [4] *For all $n \geq 8$, $ex_p(n, \{C_4, C_5\}) \leq 2n - 6$, the equality holds when $n \equiv 3 \pmod{9}$.*

(3) [5] *For all* $n \geq 14$, $ex_p(n, \{C_5, C_6\}) \leq \frac{30n-84}{13}$, *the equality holds when* $n \equiv 7 \pmod{10}$.

In this paper, we obtain the sharp upper bound of the planar Turán number of $\{C_6, C_7\}$; the results are as follows.

Theorem 4. *For all integers* $n \geq 76$, *we have* $ex_p(n, \{C_6, C_7\}) \leq \frac{27}{11}n - \frac{72}{11}$.

Theorem 5. *If* $n \equiv 10 \pmod{22}$, *then* $ex_p(n, \{C_6, C_7\}) = \frac{27}{11}n - \frac{72}{11}$.

2. Proof of Theorem 4

To prove Theorem 4, we give some definitions and lemmas. Given a plane graph G, a triangular block of G is a subgraph induced by the edge set consisting of all edges on the boundaries of a 3-face and all 3-faces equivalent to it. If an edge of G is not on the boundary of any 3-face, then the subgraph induced by the edge is called a trivial triangular block of G. Let $\mathcal{T}(G)$ denote the set of all triangular blocks of G.

A triangular block on at most 5 vertices must be $\{C_6, C_7\}$-free. According to the definition of the triangular block, the triangular blocks on 5 vertices are T_5^1, T_5^2, T_5^3 and T_5^4, the triangular blocks on 4 vertices are T_4^1 and T_4^2, the triangular block on i vertices is T_i for $i = 2, 3$, as shown in Figure 1.

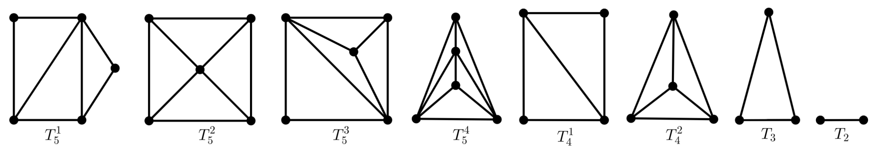


Figure 1. The triangular blocks on at most 5 vertices.

Lemma 6. *Any triangular block on* k *vertices is* C_6*-free if and only if* $k \leq 5$.

Proof. The sufficiency clearly holds for $k \leq 5$. Now we prove the necessity. A triangular block on at most 5 vertices must be C_6-free. So we only prove that any triangular block on k vertices must contain C_6 when $k \geq 6$. According to the definition of the triangular block, a triangular block on at least 6 vertices must contain a triangular block on 6 vertices as a subgraph. Therefore, we only need to prove that any triangular block on 6 vertices must contain C_6. Let T be any triangular block on 6 vertices. Notice that T must contain a triangular block H_1 on 5 vertices as a subgraph. Let $V(T) \setminus V(H_1) = \{x_0\}$. According to the definition of the triangular block, we see that x_0 must be adjacent to the ends x_1 and x_2 of an edge on the boundary of the outer face of H_1. All triangular blocks on 5 vertices are depicted in Figure 1. Clearly, T_5^i has a Hamilton cycle containing e for any edge e on the boundary of the outer face of T_5^i when $i = 1, 2, 3, 4$. Thus H_1 has a Hamilton cycle containing edge x_1x_2, we denote this cycle by $x_1Px_2x_1$. So T contains $C_6 = x_0x_1Px_2x_0$. Thus any triangular block on 6 vertices must contain a C_6. This completes the proof of the necessity. ■

Corollary 1. *Any $\{C_6, C_7\}$-free triangular block has at most 5 vertices.*

Proof. Any $\{C_6, C_7\}$-free triangular block must be C_6-free. By Lemma 1, we see that any $\{C_6, C_7\}$-free triangular block has at most 5 vertices. ■

Let G be a plane graph and T be a triangular block of G. An edge of T is called an outer edge of T if it is on the boundary of a 3^+-face of G. The ends of an outer edge of T are called the outer vertices of T. We use $t_G(v)$ to denote the number of the triangular blocks sharing v in G where $v \in V(G)$. For any $v \in V(G)$, when $t_G(v) \geq 2$, v is called a junction vertex; when $t_G(v) = 1$, v is called a non-junction vertex. We denote the set of all junction vertices of T by $\mathcal{J}(T)$. For any triangular block T of G, we define $n_G(T) := \sum_{v \in V(T)} \frac{1}{t_G(v)}$. For any $v \in V(G)$, since the number of the triangular blocks sharing v in G is $t_G(v)$, we have

$$(1) \qquad \sum_{T \in \mathcal{T}(G)} n_G(T) = \sum_{T \in \mathcal{T}(G)} \sum_{v \in V(T)} \frac{1}{t_G(v)} = \sum_{v \in V(G)} t_G(v) \cdot \frac{1}{t_G(v)} = |V(G)|.$$

Observation 1. *Let T be a triangular block of a plane graph G. Then $n_G(T) \leq |V(T)| - \frac{1}{2}|\mathcal{J}(T)|$.*

Proof. According to the definition of the junction vertex, for any $v \in V(G)$, we have $t_G(v) \geq 2$ when v is a junction vertex and $t_G(v) = 1$ when v is a non-junction vertex. So by the definition of $n_G(T)$, then $n_G(T) \leq \frac{1}{2} \cdot |\mathcal{J}(T)| + 1 \cdot (|V(T)| - |\mathcal{J}(T)|) = |V(T)| - \frac{1}{2}|\mathcal{J}(T)|$. ■

Let G be a 2-connected plane graph and F be a face of G. Since G is 2-connected, we know that the boundary of each face of G is a cycle, we denote the boundary of F by C_F. If two adjacent edges on the boundary of F are the outer edges of a T_5^4, then we call that this T_5^4 is closely related to F, as shown in Figure 2 (the gray area is F). For any $F \in F(G)$, let $k(F)$ be the number of T_5^4s which are closely related to F, we define $E_1(C_F) := \{e \in E(C_F) | e \in E(T_5^4), T_5^4$ is closely related to $F\}$ and $E_2(C_F) := E(C_F) \backslash E_1(C_F)$. For any $F \in F(G)$ and $e \in E(C_F)$, let $\ell(F) = |E(C_F)| - k(F)$, and we define

$$(2) \qquad f_F(e) := \begin{cases} \frac{1}{2\ell(F)}, & \text{if } e \in E_1(C_F); \\ \frac{1}{\ell(F)}, & \text{if } e \in E_2(C_F). \end{cases}$$

Since $|E(C_F)| = |E_1(C_F)| + |E_2(C_F)| = 2k(F) + |E_2(C_F)|$, we have $k(F) = \ell(F) - |E_2(C_F)|$.

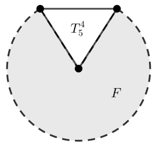


Figure 2. T_5^4 is closely related to F.

If T_5^4 is closely related to F, then we have $|E(T_5^4) \cap E(C_F)| = 2$. Since the triangular blocks are edge-disjoint, we have $|E_1(C_F)| = 2k(F)$ and $|E_2(C_F)| = |E(C_F)| - 2k(F)$. So for all $F \in F(G)$, we have $\sum_{e\in E(C_F)} f_F(e) = \sum_{e\in E_1(C_F)} f_F(e) + \sum_{e\in E_2(C_F)} f_F(e) = \sum_{e\in E_1(C_F)} \frac{1}{2\ell(F)} + \sum_{e\in E_2(C_F)} \frac{1}{\ell(F)} = 2k(F) \cdot \frac{1}{2\ell(F)} + (|E(C_F)| - 2k(F)) \cdot \frac{1}{\ell(F)} = \ell(F) \cdot \frac{1}{\ell(F)} = 1$.

For any triangular block T of G, we define $f_G(T) := \sum_{e\in E(T)}(f_{F_e^1}(e) + f_{F_e^2}(e))$, where two faces F_e^1 and F_e^2 are incident with e. The triangular blocks are edge-disjoint, so

$$\begin{aligned}\sum_{T\in\mathcal{T}(G)} f_G(T) &= \sum_{T\in\mathcal{T}(G)} \sum_{e\in E(T)} (f_{F_e^1}(e) + f_{F_e^2}(e)) = \sum_{e\in E(G)} (f_{F_e^1}(e) + f_{F_e^2}(e)) \\ &= \sum_{F\in F(G)} \sum_{e\in E(C_F)} f_F(e) = \sum_{F\in F(G)} 1 = |F(G)|\,. \end{aligned} \tag{3}$$

We denote the set of all faces of G incident with edges of T by $\mathcal{F}(T)$. According to the definition of $f_G(T)$, we have $f_G(T) = \sum_{F\in\mathcal{F}(T)} \sum_{e\in E_T(C_F)} f_F(e)$ where $E_T(C_F) = E(C_F) \cap E(T)$. We denote the set of all 3-faces of G incident with edges of T by $\mathcal{F}_3(T)$. If $F \in \mathcal{F}_3(T)$, then $E_1(C_F) = \emptyset$, $k(F) = 0$ and $\ell(F) = |E(C_F)|$. Thus $\sum_{e\in E(C_F)} f_F(e) = \sum_{e\in E(C_F)} \frac{1}{|E(C_F)|} = 1$. For any $F \in \mathcal{F}_3(T)$, we have $E_T(C_F) = E(C_F)$, so $\sum_{F\in\mathcal{F}_3(T)} \sum_{e\in E_T(C_F)} f_F(e) = \sum_{F\in\mathcal{F}_3(T)} \sum_{e\in E(C_F)} f_F(e) = \sum_{F\in\mathcal{F}_3(T)} 1 = |\mathcal{F}_3(T)|$. Thus

$$\begin{aligned} f_G(T) &= \sum_{F\in\mathcal{F}_3(T)} \sum_{e\in E_T(C_F)} f_F(e) + \sum_{F\in\mathcal{F}(T)\setminus\mathcal{F}_3(T)} \sum_{e\in E_T(C_F)} f_F(e) \\ &= |\mathcal{F}_3(T)| + \sum_{F\in\mathcal{F}(T)\setminus\mathcal{F}_3(T)} \sum_{e\in E_T(C_F)} f_F(e). \end{aligned} \tag{4}$$

Let G be a 2-connected planar graph. So for any $T \in \mathcal{T}(G)$, we must have $|\mathcal{J}(T)| \geq 2$ and $|\mathcal{F}(T) \setminus \mathcal{F}_3(T)| = |\mathcal{J}(T)|$. Thus for any trivial triangular block $H \in \mathcal{T}(G)$, there are only two faces in $\mathcal{F}(H)$, both faces are 3^+-faces, since two vertices of H must be junction vertices. Let $w(T) = 36f_G(T) + 9n_G(T) - 25|E(T)|$ for any $T \in \mathcal{T}(G)$.

Observation 2. *Let G be a 2-connected planar graph, T be a triangular block of G on at most 5 vertices with $\mathcal{F}(T) \setminus \mathcal{F}_3(T) = \{F_1, F_2, \ldots, F_{|\mathcal{J}(T)|}\}$ and $T \neq T_5^4$. Then $w(T) = 36\left(|\mathcal{F}_3(T)| + \sum_{i=1}^{|\mathcal{J}(T)|} \frac{|E_T(C_{F_i})|}{\ell(F_i)}\right) + 9n_G(T) - 25|E(T)|$.*

Proof. Since $T \neq T_5^4$, we have $E_T(C_{F_i}) \subseteq E_2(C_{F_i})$ for $i = 1, 2, \ldots, |\mathcal{J}(T)|$. By equation (2), we have $f_{F_i}(e) = \frac{1}{\ell(F_i)}$ for $e \in E_T(C_{F_i})$ and $i \in [|\mathcal{J}(T)|]$. By equation (4), we have $w(T) = 36(|\mathcal{F}_3(T)| + \sum_{i=1}^{|\mathcal{J}(T)|} \sum_{e \in E_T(C_{F_i})} f_{F_i}(e)) + 9n_G(T) - 25|E(T)| = 36(|\mathcal{F}_3(T)| + \sum_{i=1}^{|\mathcal{J}(T)|} \frac{|E_T(C_{F_i})|}{\ell(F_i)}) + 9n_G(T) - 25|E(T)|$. ■

Observation 3. *Let G be a 2-connected planar graph and T be a triangular block of G with $\mathcal{F}(T) \setminus \mathcal{F}_3(T) = \{F_1, F_2, \ldots, F_{|\mathcal{J}(T)|}\}$. Then $w(T) \leq 0$ if one of the following holds.* (1) *$T \cong T_5^j$ and $\ell(F_k) \geq 8$ for $j \in [3]$ and $k = 1, 2, \ldots, |\mathcal{J}(T)|$;* (2) *$T \cong T_4^j$ and $\ell(F_k) \geq 8$ for $j \in [2]$ and $k = 1, 2, \ldots, |\mathcal{J}(T)|$;* (3) *$T \cong T_5^1$, $|\mathcal{J}(T)| = 3$, $\ell(F_1) \geq 8$ and $\ell(F_k) \geq 5$ for $k = 2, 3$.*

Proof. By Observation 1, we have $n_G(T) \leq |V(T)| - \frac{1}{2}|\mathcal{J}(T)| \leq |V(T)| - 1$ since $|\mathcal{J}(T)| \geq 2$. If (1) holds, then $n_G(T) \leq 4$. When $T \cong T_5^1$, since $\sum_{i=1}^{|\mathcal{J}(T)|} |E_T(C_{F_i})| = 5$, we have $\sum_{i=1}^{|\mathcal{J}(T)|} \frac{|E_T(C_{F_i})|}{\ell(F_i)} \leq \frac{5}{8}$, combining $|\mathcal{F}_3(T)| = 3$, $|E(T)| = 7$ and Observation 2, we have $w(T) \leq 0$. When $T \cong T_5^j$, $j \in \{2, 3\}$, since $\sum_{i=1}^{|\mathcal{J}(T)|} |E_T(C_{F_i})| = 4$, we have $\sum_{i=1}^{|\mathcal{J}(T)|} \frac{|E_T(C_{F_i})|}{\ell(F_i)} \leq \frac{4}{8}$, combining $|\mathcal{F}_3(T)| = 4$, $|E(T)| = 8$ and Observation 2, we have $w(T) \leq 0$. If (2) holds, then $n_G(T) \leq 3$. When $T \cong T_4^j$, $j \in [2]$, since $\sum_{i=1}^{|\mathcal{J}(T)|} |E_T(C_{F_i})| = 5 - j$, we have $\sum_{i=1}^{|\mathcal{J}(T)|} \frac{|E_T(C_{F_i})|}{\ell(F_i)} \leq \frac{5-j}{8}$, combining $|\mathcal{F}_3(T)| = j + 1$, $|E(T)| = j + 4$ and Observation 2, we have $w(T) \leq 0$. If (3) holds, then by $|\mathcal{J}(T)| = 3$, we have $n_G(T) \leq 5 - \frac{1}{2}|\mathcal{J}(T)| \leq \frac{7}{2}$. Notice that $\sum_{i=1}^{|\mathcal{J}(T)|} \frac{|E_T(C_{F_i})|}{\ell(F_i)} = \frac{|E_T(C_{F_1})|}{\ell(F_1)} + \sum_{i=2}^{|\mathcal{J}(T)|} \frac{|E_T(C_{F_i})|}{\ell(F_i)} \leq \frac{|E_T(C_{F_1})|}{8} + \frac{5 - |E_T(C_{F_1})|}{5} = \frac{40 - 3|E_T(C_{F_1})|}{40} \leq \frac{37}{40}$. Combining $|\mathcal{F}_3(T)| = 3$, $|E(T)| = 7$ and Observation 2, we have $w(T) \leq 0$. ■

Observation 4. *Let G be a 2-connected planar graph and T be a triangular block of G. For all $F \in \mathcal{F}(T) \setminus \mathcal{F}_3(T)$, if $k(F) \geq 1$, then G contains two cycles $C_{|E(C_F)|+1}$ and $C_{|E(C_F)|+2}$.*

Proof. Let $u_1, u_2, \ldots, u_{|E(C_F)|}$ denote vertices of C_F in order. Since $k(F) \geq 1$, there must exist a T_5^4 closely related to C_F, i.e., there must exist two adjacent edges of C_F that are two outer edges of a T_5^4, without loss of generality, let these two edges be u_1u_2 and u_2u_3. Notice that there exists a path P_i of length $i + 2$ between u_1 and u_3 in T_5^4, $i = 1, 2$, so G contains $C_{|E(C_F)|+i} = u_1P_iu_3u_4 \cdots u_{|E(C_F)|}u_1$. ■

Observation 5. *Let G be a 2-connected $\{C_6, C_7\}$-free planar graph and T be a triangular block of G. For all $F \in \mathcal{F}(T) \setminus \mathcal{F}_3(T)$, we have $\ell(F) \geq 4$ and $\ell(F) \notin \{6, 7\}$.*

Proof. By the definition of $\ell(F)$, we obtain that $\ell(F) = |E(C_F)| - k(F) \geq 3$ and G contains cycles of length $\ell(F)$ and $|E(C_F)|$. Since G is $\{C_6, C_7\}$-free, we have $\ell(F) \notin \{6, 7\}$. Now we prove that $\ell(F) \geq 4$. Assume $\ell(F) = 3$. Recall that $k(F) = \ell(F) - |E_2(C_F)| \leq \ell(F)$ and $|E(C_F)| = k(F) + 3$. If $k(F) = 0$, then $|E(C_F)| = 3$, this contradicts the fact that $F \in \mathcal{F}(T) \setminus \mathcal{F}_3(T)$. If $k(F) = i$, $i \in \{1, 2\}$, then $|E(C_F)| = i+3$, by Observation 4, we know that G contains C_{i+5}, a contradiction. If $k(F) = 3$, then $|E(C_F)| = 6$, a contradiction. So $\ell(F) \geq 4$. ■

Let G be a 2-connected $\{C_6, C_7\}$-free planar graph and T be a triangular block of G with $T \neq T_5^4$. For all $F \in \mathcal{F}(T) \setminus \mathcal{F}_3(T)$, let $V(C_F) \cap \mathcal{J}(T) = \{u_F^T, v_F^T\}$. Let $C_{\ell(F)}$ be a cycle of length $\ell(F)$ containing all edges of $E_2(C_F)$. Let P_F^T be a path in $C_{\ell(F)}$ with ends u_F^T and v_F^T and containing no edges of $E_T(C_F)$. Since $T \neq T_5^4$, we have $E_T(C_F) \subseteq E_2(C_F)$, so $|E(P_F^T)| = |E(C_{\ell(F)})| - |E_T(C_F)| = \ell(F) - |E_T(C_F)|$. Let H be a subgraph of G. For $\{u, v\} \subseteq V(H)$, let $\mathcal{L}_H(u, v)$ be the set of lengths of all paths between u and v in H. When H is a triangular block of G with $|V(H)| \in \{4, 5\}$, it is easy to verify that for $\{u, v\} \subseteq V(H)$, we have $\{2, 3, \ldots, |V(H)| - 2\} \subseteq \mathcal{L}_H(u, v)$; moreover, we have $|V(H)| - 1 \in \mathcal{L}_H(u, v)$ if uv is an outer edge of H or $uv \notin E(H)$.

Observation 6. *Let G be a 2-connected $\{C_6, C_7\}$-free planar graph and T be a triangular block of G with $T \neq T_5^4$. For all $F \in \mathcal{F}(T) \setminus \mathcal{F}_3(T)$, we have $\ell(F) \geq 8$, if one of the following holds.* (1) $|V(T)| \in \{4, 5\}$ *and* $|E_T(C_F)| = 1$; (2) $|V(T)| = 5$, $|E_T(C_F)| = 2$ *and* $u_F^T v_F^T \notin E(T)$; (3) $T \in \{T_5^2, T_5^3\}$ *and* $|E_T(C_F)| = 3$.

Proof. Assume (1) holds. If $\ell(F) = i$, then $|E(P_F^T)| = \ell(F) - |E_T(C_F)| = i - 1$, $i \in \{4, 5\}$; we obtain that G contains C_{i+2} since $3 \in \mathcal{L}_T(u_F^T, v_F^T)$, a contradiction. So $\ell(F) \notin \{4, 5\}$. Assume (2) holds. If $\ell(F) = i$, then $|E(P_F^T)| = i - 2$, $i \in \{4, 5\}$; we obtain that G contains C_{i+2} since $4 \in \mathcal{L}_T(u_F^T, v_F^T)$, a contradiction. So $\ell(F) \notin \{4, 5\}$. Assume (3) holds. Since $|E_T(C_F)| = 3$, we have $\ell(F) \neq 4$. If $\ell(F) = 5$, then $|E(P_F^T)| = 2$; we obtain that G contains C_6 since $4 \in \mathcal{L}_T(u_F^T, v_F^T)$, a contradiction. So $\ell(F) \notin \{4, 5\}$. By Observation 5, if (1), (2) or (3) holds, then $\ell(F) \geq 8$. ■

Observation 7. *Let G be a 2-connected $\{C_6, C_7\}$-free planar graph and T be a triangular block of G. For all $F \in \mathcal{F}(T) \setminus \mathcal{F}_3(T)$, we have $\ell(F) = 4$ or $\ell(F) \geq 8$, if one of the following holds.* (1) $T \cong T_i^1$, $|E_T(C_F)| = i - 2$ *for* $i \in \{4, 5\}$ *and* $u_F^T v_F^T \notin E(T)$; (2) $T \cong T_4^2$ *and* $|E_T(C_F)| = 2$.

Proof. Assume (1) or (2) holds. If $\ell(F) = 5$, then $|E(P_F^T)| = \ell(F) - |E_T(C_F)| = 5 - (|V(T)| - 2) = 7 - |V(T)|$, with $|V(T)| - 1 \in \mathcal{L}_T(u_F^T, v_F^T)$; we obtain that G

contains C_6, a contradiction. So $\ell(F) \neq 5$. By Observation 5, we have $\ell(F) = 4$ or $\ell(F) \geq 8$. ■

Observation 8. *Let G be a 2-connected $\{C_6, C_7\}$-free planar graph and T be a triangular block of G. For all $F \in \mathcal{F}(T) \setminus \mathcal{F}_3(T)$, if $T \cong T_5^3$, $|E_T(C_F)| = 2$ and $u_F^T v_F^T \in E(T)$, then $\ell(F) = 4$ or $\ell(F) \geq 9$.*

Proof. If $\ell(F) = i$, then $|E(P_F^T)| = \ell(F) - |E_T(C_F)| = i - 2$, $i \in \{5, 8\}$, with $\{1, 3\} \subseteq \mathcal{L}_T(u_F^T, v_F^T)$, and we obtain that G contains C_{i-1} and C_{i+1}, a contradiction. So $\ell(F) \notin \{5, 8\}$. By Observation 5, we have $\ell(F) = 4$ or $\ell(F) \geq 9$. ■

Observation 9. *Let G be a 2-connected $\{C_6, C_7\}$-free planar graph and T be a triangular block of G. For all $F \in \mathcal{F}(T) \setminus \mathcal{F}_3(T)$, we have $|E(C_F)| = \ell(F)$ and each edge of $E(C_F) \setminus E_T(C_F)$ is in a trivial triangular block, if one of the following holds.* (1) *$T = T_5^1$, $|E_T(C_F)| = 3$, $\ell(F) = 4$ and $u_F^T v_F^T \notin E(T)$;* (2) *$T \in \{T_5^3, T_4^2\}$, $|E_T(C_F)| = 2$, $\ell(F) = 4$ and $u_F^T v_F^T \in E(T)$;* (3) *$T = T_4^1$, $|E_T(C_F)| = 2$, $\ell(F) = 4$ and $u_F^T v_F^T \notin E(T)$;* (4) *$T = T_4^1$, $|E_T(C_F)| = 3$, $\ell(F) = 5$ and $u_F^T v_F^T \in E(T)$.*

Proof. Firstly, we prove that $|E(C_F)| = \ell(F)$. Suppose $|E(C_F)| \neq \ell(F)$. Since $k(F) = |E(C_F)| - \ell(F)$ and $|E(C_F)| \geq \ell(F)$, we have $k(F) \geq 1$. Notice that $T \neq T_5^4$, since $k(F) = \ell(F) - |E_2(C_F)|$ and $|E_2(C_F)| \geq |E_T(C_F)|$, we have $k(F) \leq \ell(F) - |E_T(C_F)| \leq 2$. According to $|E(C_F)| = \ell(F) + k(F)$, we have $|E(C_F)| \in \{5, 6, 7\}$. By Observation 4, we obtain that G contains C_6 or C_7, a contradiction. So $|E(C_F)| = \ell(F)$.

Secondly, we prove that each edge of $E(C_F) \setminus E_T(C_F)$ is in a trivial triangular block. If (1) holds, then $|E(C_F) \setminus E_T(C_F)| = |E(C_F)| - |E_T(C_F)| = \ell(F) - |E_T(C_F)| = 1$, so $E(C_F) \setminus E_T(C_F) = \{u_F^T v_F^T\}$. Let H be a triangular block containing $u_F^T v_F^T$. If $H \neq G[u_F^T v_F^T]$, then $u_F^T v_F^T$ must be incident with a 3-face of H, then $2 \in \mathcal{L}_H(u_F^T, v_F^T)$, we know that G contains C_6 since $4 \in \mathcal{L}_T(u_F^T, v_F^T)$, a contradiction. So $H = G[u_F^T v_F^T]$. If (2), (3) or (4) holds, then $|E(C_F) \setminus E_T(C_F)| = 2$. Let $E(C_F) \setminus E_T(C_F) = \{u_F^T w, v_F^T w\}$. Now we prove that $u_F^T w$ and $v_F^T w$ are in different triangular blocks. Suppose $u_F^T w$ and $v_F^T w$ belong to a triangular block T'. Clearly, $T' \neq T_2$. Since $|E(C_F)| = \ell(F)$, we have $k(F) = 0$, i.e., $T' \neq T_5^4$. If $T' \in \{T_3, T_4^1\}$ and $u_F^T v_F^T \in E(T')$, then $d_G(w) = 2$, a contradiction. If $T' \in \{T_5^1, T_5^2, T_5^3, T_4^2\}$, or $T' = T_4^1$ and $u_F^T v_F^T \notin E(T')$, then $3 \in \mathcal{L}_{T'}(u_F^T, v_F^T)$, so G contains C_6 since $3 \in \mathcal{L}_T(u_F^T, v_F^T)$, a contradiction. So $u_F^T w$ and $v_F^T w$ are in different triangular blocks. Let H_u and H_v be triangular blocks containing $u_F^T w$ and $v_F^T w$, respectively. If $H_u \neq G[u_F^T w]$, then $u_F^T w$ must be incident with a 3-face of H_u, i.e., $3 \in \mathcal{L}_{H_u \cup H_v}(u_F^T, v_F^T)$, we know that G contains C_6 since $3 \in \mathcal{L}_T(u_F^T, v_F^T)$, a contradiction. So $H_u = G[u_F^T w]$. Similarly, $H_v = G[v_F^T w]$. ■

Observation 10. *Let G be a 2-connected $\{C_6, C_7\}$-free planar graph, H be a triangular block of G with $E(H) = \{v_1 v_2\}$, $\mathcal{F}(H) = \{F_1, F_2\}$ and $\ell(F_1) = 4$.*

Then (1) $w(H) \leq \frac{18}{r} - \frac{23}{2}$ *if* $t_G(v_i) \geq r$, $i = 1, 2$, *where* $r \in \{2, 3\}$; (2) $w(H) \leq -4$ *if* $t_G(v_1) \geq 2$ *and* $t_G(v_2) \geq 3$.

Proof. Since $\ell(F_1) = 4$, we have $|E(P^H_{F_1})| = \ell(F_1) - |E_H(C_{F_1})| = 3$. If $\ell(F_2) = i$, $i \in \{4, 5\}$, then $|E(P^H_{F_2})| = \ell(F_2) - |E_H(C_{F_2})| = i - 1$, so G contains $C_{i+2} = v_1 P^H_{F_1} v_2 P^H_{F_2} v_1$, a contradiction. By Observation 5, we have $\ell(F_2) \geq 8$. Since $|E(H)| = 1$, we have $|\mathcal{F}_3(H)| = 0$ and $|E_H(C_{F_i})| = 1$ for $i = 1, 2$. By Observation 2, we have $w(H) \leq 36(0 + \frac{1}{4} + \frac{1}{8}) + 9n_G(H) - 25|E(H)| = 9n_G(H) - \frac{23}{2}$. Recall that $n_G(H) = \sum_{v \in V(H)} \frac{1}{t_G(v)} = \frac{1}{t_G(v_1)} + \frac{1}{t_G(v_2)}$. If $t_G(v_i) \geq r$, $i = 1, 2$, then $n_G(H) \leq \frac{2}{r}$, so $w(H) \leq \frac{18}{r} - \frac{23}{2}$ where $r \in \{2, 3\}$; if $t_G(v_1) \geq 2$ and $t_G(v_2) \geq 3$, then $n_G(H) \leq \frac{5}{6}$, so $w(H) \leq -4$. ■

Lemma 7 [13]**.** *If G is a 2-degenerate graph on n vertices with $n \geq 2$, then $|E(G)| \leq 2n - 3$.*

Lemma 8 [20]**.** *If a connected graph on n vertices is composed of blocks $G_1, G_2, \ldots, G_s$, then $n = \sum_{i=1}^{s} |V(G_i)| - s + 1$.*

Lemma 9. *Let G be a 2-connected and $\{C_6, C_7\}$-free planar graph on n vertices with $n \geq 7$. If the degree of any internal vertex of one plane graph of G is at least 3, then $|E(G)| \leq \frac{27}{11}n - \frac{72}{11}$.*

We will give the proof of Lemma 9 in Section 3. Now we prove Theorem 4.

***Proof of Theorem* 4.** Let G be any $\{C_6, C_7\}$-free planar graph. We need to prove that $|E(G)| \leq \frac{27}{11}|V(G)| - \frac{72}{11}$, i.e., $27\,|V(G)| - 11\,|E(G)| \geq 72$ when $|V(G)| \geq 76$. If G is 2-degenerate, by Lemma 7, we have $|E(G)| \leq 2|V(G)| - 3 \leq \frac{27}{11}|V(G)| - \frac{72}{11}$ when $|V(G)| \geq 76$. If G is not 2-degenerate, repeatedly delete vertices of degree at most 2 from G until the degree of any vertex of the remaining graph is at least 3, the remaining graph is denoted by G^-. Clearly, we have $\delta(G^-) \geq 3$ and $|E(G)| \leq |E(G^-)| + 2(|V(G)| - |V(G^-)|)$. So $27|V(G)| - 11|E(G)| \geq 27|V(G)| - 11(|E(G^-)| + 2(|V(G)| - |V(G^-)|)) = 5(|V(G)| - |V(G^-)|) + 27|V(G^-)| - 11|E(G^-)|$, i.e.,

$$27|V(G)| - 11|E(G)| \geq 5(|V(G)| - |V(G^-)|) + 27|V(G^-)| - 11|E(G^-)|. \tag{5}$$

Let $T_1, T_2, \ldots, T_t$ be components of G^- and $H_{c_{i-1}+1}, H_{c_{i-1}+2}, \ldots, H_{c_i}$ be the blocks which belong to T_i of G^- where $c_0 = 0$. Clearly, the number of the blocks in G^- is c_t. Since the blocks are edge-disjoint, we have $|E(G^-)| = \sum_{i=1}^{c_t} |E(H_i)|$. By Lemma 8, we have $|V(T_i)| = \sum_{j=c_{i-1}+1}^{c_i} |V(H_j)| - (c_i - c_{i-1}) + 1$. So

$$\big|V(G^-)\big| = \sum_{i=1}^{t} |V(T_i)| = \sum_{i=1}^{c_t} |V(H_i)| - c_t + t. \tag{6}$$

Let b_j and $b_{\geq 7}$ be the number of the blocks on j vertices and at least 7 vertices in G^-, respectively, when $j = 2, 3, \ldots, 6$. So $c_t = \sum_{i=2}^{6} b_i + b_{\geq 7}$ and $\sum_{i=1}^{c_t} |V(H_i)| = \sum_{i\geq 2} ib_i$. For any block H_i of G^-, by Lemma 9, we have $27|V(H_i)| - 11|E(H_i)| - 27 \geq 72 - 27 = 45$ when $|V(H_i)| \geq 7$. Clearly, $|E(H_i)| = 1$ when $|V(H_i)| = 2$ and $|E(H_i)| \leq 3|V(H_i)| - 6$ when $3 \leq |V(H_i)| \leq 6$, then $27|V(H_i)| - 11|E(H_i)| - 27 \geq 3$ when $2 \leq |V(H_i)| \leq 6$. By inequality (5), we have $27|V(G)| - 11|E(G)| \geq 5(|V(G)| - |V(G^-)|) + 27|V(G^-)| - 11|E(G^-)| = 5(|V(G)| - |V(G^-)|) + 27(\sum_{i=1}^{c_t} |V(H_i)| - c_t + t) - 11 \sum_{i=1}^{c_t} |E(H_i)| = 5(|V(G)| - |V(G^-)|) + \sum_{i=1}^{c_t}(27|V(H_i)| - 11|E(H_i)| - 27) + 27t \geq \frac{3}{5}(|V(G)| - |V(G^-)|) + 45b_{\geq 7} + 3\sum_{i=2}^{6} b_i + 27t$, i.e.,

$$27|V(G)| - 11|E(G)| \geq \frac{3}{5}(|V(G)| - |V(G^-)|) + 45b_{\geq 7} + 3\sum_{i=2}^{6} b_i + 27t.$$

If $b_{\geq 7} \geq 1$, then by $t \geq 1$, we have $27\,|V(G)| - 11\,|E(G)| \geq 72$. If $b_{\geq 7} = 0$, then $c_t = \sum_{i=2}^{6} b_i$ and $\sum_{i=1}^{c_t} |V(H_i)| = \sum_{i=2}^{6} ib_i$. By equation (6), we have $|V(G^-)| = \sum_{i=2}^{6}(i-1)b_i + t \leq 5\sum_{i=2}^{6} b_i + t$, so $27|V(G)| - 11|E(G)| \geq \frac{3}{5}(|V(G)| - (5\sum_{i=2}^{6} b_i + t)) + 3\sum_{i=2}^{6} b_i + 27t = \frac{3}{5}|V(G)| + \frac{132}{5}t$. Thus $27\,|V(G)| - 11\,|E(G)| \geq 72$ when $|V(G)| \geq 76$.

In summary, for all $n \geq 76$, we have $ex_p(n, \{C_6, C_7\}) \leq \frac{27}{11}n - \frac{72}{11}$. ■

3. Proof of Lemma 9

Let G be a plane graph satisfying the condition of Lemma 9. To prove that $|E(G)| \leq \frac{27}{11}|V(G)| - \frac{72}{11}$, by Euler formula, we prove that $11|E(G)| \leq 27|V(G)| - 36(|V(G)| - |E(G)| + |F(G)|)$, i.e., $36\,|F(G)| + 9\,|V(G)| - 25\,|E(G)| \leq 0$. Since the triangular blocks are edge-disjoint, we have $|E(G)| = \sum_{T\in\mathcal{T}(G)} |E(T)|$. By equations (1) and (3), we have $36\,|F(G)| + 9\,|V(G)| - 25\,|E(G)| = \sum_{T\in\mathcal{T}(G)}(36f_G(T) + 9n_G(T) - 25|E(T)|)$. Thus we only need to prove that $\sum_{T\in\mathcal{T}(G)} w(T) \leq 0$. By Corollary 1, we see that T is isomorphic to a triangular block in Figure 1 for all $T \in \mathcal{T}(G)$. We consider eight cases based on the structure of the triangular blocks. For any $T \in \mathcal{T}(G)$, since G is 2-connected, we have $|\mathcal{J}(T)| \geq 2$ and $|\mathcal{F}(T) \setminus \mathcal{F}_3(T)| = |\mathcal{J}(T)|$, let $\mathcal{F}(T) \setminus \mathcal{F}_3(T) = \{F_1, F_2, \ldots, F_{|\mathcal{J}(T)|}\}$.

Case 1. $T \cong T_5^1$. Let $V(T) = \{v_1, v_2, v_3, v_4, v_5\}$. The triangular block T is depicted in Figure 3(a). Clearly, $|E(T)| = 7$, $|\mathcal{F}_3(T)| = 3$ and $|\mathcal{J}(T)| \in \{2, 3, 4, 5\}$.

Subcase 1.1. $|\mathcal{J}(T)| = 2$.

(1) Assume two junction vertices are the ends of an edge e of T. If e is not an outer edge of T, then $v_i \notin \mathcal{J}(T)$, so $d_G(v_i) = 2$, $i = 1, 4$. Since G is 2-connected, we obtain that v_1 or v_4 must be an internal vertex of G, a contradiction. So e

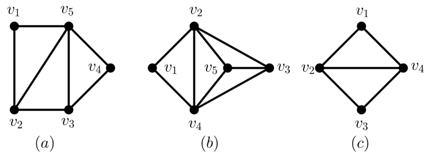


Figure 3. Three cases of T.

is an outer edge of T. The 3^+-faces in $\mathcal{F}(T)$ are either only incident with e or with other four outer edges of T. Notice that $\mathcal{F}(T) \setminus \mathcal{F}_3(T) = \{F_1, F_2\}$, without loss of generality, let $|E_T(C_{F_1})| = 1$ and $|E_T(C_{F_2})| = 4$. Since $e \in E(T)$, we have $\ell(F_2) \notin \{4, 5\}$. By Observations 5 and 6(1), we have $\ell(F_i) \geq 8$, $i = 1, 2$. By Observation 3(1), we have $w(T) \leq 0$.

(2) Assume two junction vertices are not the ends of any edge of T. Let $\mathcal{J}(T) = \{u, w\}$. Notice that $\mathcal{F}(T) \setminus \mathcal{F}_3(T) = \{F_1, F_2\}$, without loss of generality, let $|E_T(C_{F_1})| = 2$ and $|E_T(C_{F_2})| = 3$. By Observations 6(2) and 7(1), we have $\ell(F_1) \geq 8$, and $\ell(F_2) = 4$ or $\ell(F_2) \geq 8$.

When $\ell(F_2) = 4$, by Observation 9(1), we have $|E(C_{F_2})| = \ell(F_2) = 4$. Since $|E(C_{F_2}) \setminus E_T(C_{F_2})| = 1$, we have $uw \in E(C_{F_2})$. Let H_1 be a triangular block containing uw. By Observation 9(1), we have $H_1 = G[uw]$. We next calculate $w(T)$ and $w(H_1)$. Since $V(H_1) \subseteq V(T)$, we have $|V(T) \cup V(H_1)| = 5$. Since $n \geq 7$ and G is 2-connected, we have $t_G(u) \geq 3$ and $t_G(w) \geq 3$. Firstly, we calculate $w(T)$. For all $v \notin \mathcal{J}(T)$, we have $t_G(v) = 1$. Thus $n_G(T) = \sum_{v \in V(T)} \frac{1}{t_G(v)} \leq 2 \cdot \frac{1}{3} + 3 \cdot 1 = \frac{11}{3}$. Combining $|\mathcal{F}_3(T)| = 3$, $|E(T)| = 7$ and Observation 2, we have $w(T) \leq 36(3 + \frac{2}{8} + \frac{3}{4}) + 9n_G(T) - 25|E(T)| \leq 2$. Secondly, we calculate $w(H_1)$. Notice that $H_1 = G[uw]$, $F_2 \in \mathcal{F}(H_1)$ and $\ell(F_2) = 4$, by Observation 10(1), we have $w(H_1) \leq -\frac{11}{2}$. Let $\mathcal{T}_1' = \{T, H_1\}$. Then $\sum_{H \in \mathcal{T}_1'} w(H) \leq 0$.

When $\ell(F_2) \geq 8$, by Observation 3(1), we have $w(T) \leq 0$.

Subcase 1.2. $|\mathcal{J}(T)| = 3$. Notice that $\{v_1, v_4\} \cap \mathcal{J}(T) \neq \emptyset$, otherwise $v_i \notin \mathcal{J}(T)$ for $i = 1, 4$, we obtain that $d_G(v_i) = 2$ and v_1 or v_4 must be an internal vertex of G since G is 2-connected, a contradiction.

(1) Assume $|\{v_1, v_4\} \cap \mathcal{J}(T)| = 1$. Without loss of generality, let $v_1 \in \mathcal{J}(T)$ and $v_4 \notin \mathcal{J}(T)$.

(1.1) If $v_5 \notin \mathcal{J}(T)$, then $\mathcal{J}(T) = \{v_1, v_2, v_3\}$. Let $E_T(C_{F_1}) = \{v_1v_2\}$, $E_T(C_{F_2}) = \{v_2v_3\}$ and $E_T(C_{F_3}) = \{v_3v_4, v_4v_5, v_1v_5\}$. By Observations 6(1) and 7(1), we have $\ell(F_i) \geq 8$ for $i = 1, 2$ and $\ell(F_3) = 4$ or $\ell(F_3) \geq 8$.

When $\ell(F_3) = 4$, by Observation 9(1), we have $|E(C_{F_3})| = \ell(F_3) = 4$. Since $|E(C_{F_3}) \setminus E_T(C_{F_3})| = 1$, we have $v_1v_3 \in E(C_{F_3})$. Let H_1 be a triangular block containing v_1v_3. By Observation 9(1), we have $H_1 = G[v_1v_3]$. By $V(H_1) \subseteq V(T)$, we have $|V(T) \cup V(H_1)| = 5$. Since G is 2-connected, $n \geq 7$ and $H_1 = G[v_1v_3]$, there exists $v \in \mathcal{J}(T)$ satisfying $t_G(v) \geq 3$. For all $u \in \mathcal{J}(T) \setminus \{v\}$ and $w \notin \mathcal{J}(T)$, we have $t_G(u) \geq 2$ and $t_G(w) = 1$. So $n_G(T) = \sum_{v \in V(T)} \frac{1}{t_G(v)} \leq 1 \cdot \frac{1}{3} + 2 \cdot \frac{1}{2} + 2 \cdot 1 = \frac{10}{3}$. Combining $|\mathcal{F}_3(T)| = 3$, $|E(T)| = 7$ and Observation 2, we have $w(T) \leq 36(3 + \frac{1}{8} + \frac{1}{8} + \frac{3}{4}) + 9n_G(T) - 25|E(T)| \leq 0$.

When $\ell(F_3) \geq 8$, by Observation 3(1), we have $w(T) \leq 0$.

(1.2) If $v_5 \in \mathcal{J}(T)$, then $\mathcal{J}(T) = \{v_1, v_2, v_5\}$ or $\mathcal{J}(T) = \{v_1, v_3, v_5\}$. When $\mathcal{J}(T) = \{v_1, v_2, v_5\}$, without loss of generality, let $|E_T(C_{F_1})| = |E_T(C_{F_2})| = 1$ and $|E_T(C_{F_3})| = 3$. Since $v_2v_5 \in E(T)$, we have $\ell(F_3) \neq 4$. By Observations 5 and 6(1), we have $\ell(F_i) \geq 8$ for $i = 1, 2$ and $\ell(F_3) \geq 5$. By Observation 3(3), we have $w(T) \leq 0$.

When $\mathcal{J}(T) = \{v_1, v_3, v_5\}$, let $E_T(C_{F_1}) = \{v_1v_5\}$, $E_T(C_{F_2}) = \{v_1v_2, v_2v_3\}$ and $E_T(C_{F_3}) = \{v_3v_4, v_4v_5\}$. By Observations 5 and 6(1,2), we have $\ell(F_i) \geq 8$ for $i = 1, 2$ and $\ell(F_3) \geq 4$. By Observation 1, we have $n_G(T) \leq \frac{7}{2}$. Combining $|\mathcal{F}_3(T)| = 3$, $|E(T)| = 7$ and Observation 2, we have $w(T) \leq 36(3 + \frac{1}{8} + \frac{2}{8} + \frac{2}{4}) + 9n_G(T) - 25|E(T)| \leq 0$.

(2) Assume $|\{v_1, v_4\} \cap \mathcal{J}(T)| = 2$, i.e., $\{v_1, v_4\} \subseteq \mathcal{J}(T)$. According to the symmetry of T, there are two cases as follows.

When $\mathcal{J}(T) = \{v_1, v_2, v_4\}$, let $E_T(C_{F_1}) = \{v_1v_2\}$, $E_T(C_{F_2}) = \{v_1v_5, v_4v_5\}$ and $E_T(C_{F_3}) = \{v_2v_3, v_3v_4\}$. By Observation 6(1,2), we have $\ell(F_i) \geq 8$ for $i = 1, 2, 3$. By Observation 3(1), we have $w(T) \leq 0$.

When $\mathcal{J}(T) = \{v_1, v_4, v_5\}$, let $E_T(C_{F_1}) = \{v_1v_5\}$, $E_T(C_{F_2}) = \{v_4v_5\}$ and $E_T(C_{F_3}) = \{v_1v_2, v_2v_3, v_3v_4\}$. According to analysis similar to Subcase 1.2 (1.1), we have $w(T) \leq 0$.

Subcase 1.3. $|\mathcal{J}(T)| = k$ for $k \in \{4, 5\}$. Notice that $\mathcal{F}(T) \setminus \mathcal{F}_3(T) = \{F_1, F_2, \ldots, F_k\}$, then the number of 3^+-faces with exactly one edge of T is $k - 1$ when $k = 4$ and k when $k = 5$. Without loss of generality, let $|E_T(C_{F_i})| = 1$ for $i = 1, 2, \ldots, k-1$ and $|E_T(C_{F_k})| = 6 - k$. By Observations 5 and 6(1), we have $\ell(F_j) \geq 8$ for $j = 1, 2, \ldots, k-1$ and $\ell(F_k) \geq 4$. By Observation 1, we have $n_G(T) \leq 5 - \frac{k}{2}$. Combining $|\mathcal{F}_3(T)| = 3$, $|E(T)| = 7$ and Observation 2, we have $w(T) \leq 36(3 + \frac{k-1}{8} + \frac{6-k}{4}) + 9n_G(T) - 25|E(T)| \leq 0$.

Case 2. $T \cong T_5^2$. Recall that $\mathcal{F}(T) \setminus \mathcal{F}_3(T) = \{F_1, F_2, \ldots, F_{|\mathcal{J}(T)|}\}$ and $|\mathcal{J}(T)| \in \{2, 3, 4\}$. According to the structure of T_5^2, we see that $u_{F_i}^T v_{F_i}^T \notin E(T)$ when $|E_T(F_i)| = 2$, $i \in \{1, 2, \ldots, |\mathcal{J}(T)|\}$. By Observation 6, we have $\ell(F_i) \geq 8$ for $i = 1, 2, \ldots, |\mathcal{J}(T)|$. By Observation 3(1), we have $w(T) \leq 0$.

Case 3. $T \cong T_5^3$. Let $V(T) = \{v_1, v_2, v_3, v_4, v_5\}$. The triangular block T is depicted in Figure 3(b). Clearly, $|E(T)| = 8$, $|\mathcal{F}_3(T)| = 4$ and $|\mathcal{J}(T)| \in \{2, 3, 4\}$.

Subcase 3.1. $|\mathcal{J}(T)| = 2$.

(1) Assume two junction vertices are the ends of an edge e of T. According to whether e is an outer edge, there are two cases as follows.

(1.1) If e is an outer edge of T, then the 3^+-faces in $\mathcal{F}(T)$ are either only incident with e or with other three outer edges of T. Notice that $\mathcal{F}(T) \setminus \mathcal{F}_3(T) = \{F_1, F_2\}$, without loss of generality, let $|E_T(C_{F_1})| = 1$ and $|E_T(C_{F_2})| = 3$. By Observation 6(1,3), we have $\ell(F_i) \geq 8$, $i = 1,2$. By Observation 3(1), we have $w(T) \leq 0$.

(1.2) If e is not an outer edge of T, then $\mathcal{J}(T) = \{v_2, v_4\}$ and $|E_T(C_{F_1})| = |E_T(C_{F_2})| = 2$. By Observation 8, we have $\ell(F_i) = 4$ or $\ell(F_i) \geq 9$ for $i = 1,2$. Without loss of generality, we assume that $\ell(F_1) \leq \ell(F_2)$.

When $\ell(F_i) = 4$, $i = 1,2$, by Observation 9(2), we have $|E(C_{F_i})| = \ell(F_i) = 4$, so $|E(C_{F_i}) \setminus E_T(C_{F_i})| = |E(C_{F_i})| - |E_T(C_{F_i})| = 2$. Let $E(C_{F_i}) \setminus E_T(C_{F_i}) = \{e_{i,1}, e_{i,2}\}$ and $H_{i,j}$ be a triangular block containing $e_{i,j}$, $i = 1,2$ and $j = 1,2$. By Observation 9(2), we have $H_{i,j} = G[e_{i,j}]$, $i = 1,2$ and $j = 1,2$. Let $V(H_{i,1}) \cap V(H_{i,2}) = \{v_{i+5}\}$, $i = 1,2$. Clearly, $t_G(v_j) \geq 2$, $j = 6,7$. Notice that $\{H_{1,1}, H_{1,2}\} \cap \{H_{2,1}, H_{2,2}\} = \emptyset$, otherwise $\{H_{1,1}, H_{1,2}\} = \{H_{2,1}, H_{2,2}\}$, then $v_6 = v_7$ and $|V(G)| = |V(T) \cup V(H_{1,1}) \cup V(H_{1,2})| = |V(T) \cup \{v_6\}| = |V(T)| + 1 < 7$, a contradiction. We next calculate $w(T)$ and $w(H_{i,j})$ for $i = 1,2$ and $j = 1,2$. Since $H_{i,j}$ is a triangular block, $i = 1,2$ and $j = 1,2$, we have $t_G(v_k) \geq 3$ for $k = 2,4$. Firstly, we calculate $w(T)$. For all $v \notin \mathcal{J}(T)$, we have $t_G(v) = 1$. Thus $n_G(T) = \sum_{v \in V(T)} \frac{1}{t_G(v)} \leq 2 \cdot \frac{1}{3} + 3 \cdot 1 = \frac{11}{3}$. Combining $|\mathcal{F}_3(T)| = 4$, $|E(T)| = 8$ and Observation 2, we have $w(T) \leq 36(4 + \frac{2}{4} + \frac{2}{4}) + 9n_G(T) - 25|E(T)| \leq 13$. Secondly, we calculate $w(H_{i,j})$ for $i = 1,2$ and $j = 1,2$. Without loss of generality, let $H_{1,1} = G[v_2v_6]$. Combining $F_1 \in \mathcal{F}(H_{1,1})$, $\ell(F_1) = 4$ and Observation 10(2), we have $w(H_{1,1}) \leq -4$. Similarly, $w(H_{i,j}) \leq -4$ for $i = 1,2$ and $j = 1,2$. Let $\mathcal{T}_2' = \{T, H_{1,1}, H_{1,2}, H_{2,1}, H_{2,2}\}$. Then $\sum_{H \in \mathcal{T}_2'} w_G(H) \leq 0$.

When $\ell(F_1) = 4$ and $\ell(F_2) \geq 9$, by Observation 9(2), we have $|E(C_{F_1})| = \ell(F_1) = 4$, so $|E(C_{F_1}) \setminus E_T(C_{F_1})| = 2$. Let $E(C_{F_1}) \setminus E_T(C_{F_1}) = \{e_1, e_2\}$ and H_i be a triangular block containing e_i, $i = 1,2$. By Observation 9(2), we have $H_i = G[e_i]$, $i = 1,2$. Let $V(H_1) \cap V(H_2) = \{v_6\}$. Clearly, $t_G(v_6) \geq 2$. We next calculate $w(T)$ and $w(H_i)$ for $i = 1,2$. Firstly, we calculate $w(T)$. Since $H_i = G[e_i]$ for $i = 1,2$, we have $|V(T) \cup V(H_1) \cup V(H_2)| = |V(T) \cup \{v_6\}| = 6$. Since $n \geq 7$, G is 2-connected and $H_i = G[e_i]$, $i = 1,2$, there exists $v \in \mathcal{J}(T)$ satisfying $t_G(v) \geq 3$. For all $u \in \mathcal{J}(T) \setminus \{v\}$ and $w \notin \mathcal{J}(T)$, we have $t_G(u) \geq 2$ and $t_G(w) = 1$. Thus $n_G(T) = \sum_{v \in V(T)} \frac{1}{t_G(v)} \leq \frac{1}{3} + \frac{1}{2} + 3 \times 1 = \frac{23}{6}$. Combining $|\mathcal{F}_3(T)| = 4$, $|E(T)| = 8$ and Observation 2, we have $w(T) \leq 36(4 + \frac{2}{4} + \frac{2}{9}) + 9n_G(T) - 25|E(T)| \leq \frac{9}{2}$. Secondly, we calculate $w(H_i)$ for $i = 1,2$. Without loss of generality, let $H_1 = G[v_2v_6]$. Since $v_2 \in \mathcal{J}(T)$, we have $t_G(v_2) \geq 2$. Combining $F_1 \in \mathcal{F}(H_1)$, $\ell(F_1) = 4$ and Observation 10(1), we have $w(H_1) \leq -\frac{5}{2}$. Similarly, $w(H_2) \leq -\frac{5}{2}$. Let $\mathcal{T}_3' = \{T, H_1, H_2\}$. Then $\sum_{H \in \mathcal{T}_3'} w_G(H) \leq 0$.

When $\ell(F_i) \geq 9$, $i = 1,2$, by Observation 3(1), we have $w(T) \leq 0$.

(2) Assume two junction vertices are not the ends of any edge of T. Clearly,

$\mathcal{J}(T) = \{v_1, v_3\}$ and $|E_T(C_{F_1})| = |E_T(C_{F_2})| = 2$. By Observation 6(2), we have $\ell(F_i) \geq 8$, $i = 1, 2$. By Observation 3(1), we have $w(T) \leq 0$.

Subcase 3.2. $|\mathcal{J}(T)| = 3$. Notice that $\mathcal{F}(T) \setminus \mathcal{F}_3(T) = \{F_1, F_2, F_3\}$, without loss of generality, let $|E_T(C_{F_i})| = 1$ for $i = 1, 2$ and $|E_T(C_{F_3})| = 2$. Since $|\mathcal{J}(T)| = 3$ and $\mathcal{J}(T) \subseteq \{v_1, v_2, v_3, v_4\}$, we have $\{v_1, v_3\} \subseteq \mathcal{J}(T)$ or $\{v_2, v_4\} \subseteq \mathcal{J}(T)$.

(1) Assume $\{v_1, v_3\} \subseteq \mathcal{J}(T)$. Recall that $v_1v_3 \notin E(T)$. By Observation 6(1,2), we have $\ell(F_i) \geq 8$, $i = 1, 2, 3$. By Observation 3(1), we have $w(T) \leq 0$.

(2) Assume $\{v_2, v_4\} \subseteq \mathcal{J}(T)$. Recall that $v_2v_4 \in E(T)$. By Observations 6(1) and 8, we have $\ell(F_i) \geq 8$ for $i = 1, 2$, and $\ell(F_3) = 4$ or $\ell(F_3) \geq 9$.

When $\ell(F_3) = 4$, by Observation 9(2), we have $|E(C_{F_3})| = \ell(F_3) = 4$, so $|E(C_{F_3}) \setminus E_T(C_{F_3})| = 2$. Let $E(C_{F_3}) \setminus E_T(C_{F_3}) = \{e_1, e_2\}$ and H_i be a triangular block containing e_i, $i = 1, 2$. By Observation 9(2), we have $H_i = G[e_i]$, $i = 1, 2$. Let $V(H_1) \cap V(H_2) = \{v_6\}$. Clearly, $t_G(v_6) \geq 2$. We next calculate $w(T)$ and $w(H_i)$ for $i = 1, 2$. Firstly, we calculate $w(T)$. By Observation 1, we have $n_G(T) \leq \frac{7}{2}$. Combining $|\mathcal{F}_3(T)| = 4$, $|E(T)| = 8$ and Observation 2, we have $w(T) \leq 36(4 + \frac{1}{8} + \frac{1}{8} + \frac{2}{4}) + 9n_G(T) - 25|E(T)| \leq \frac{5}{2}$. According to analysis similar to Subcase 3.1 (1.2), we have $w(H_i) \leq -\frac{5}{2}$, $i = 1, 2$. Let $\mathcal{T}_4' = \{T, H_1, H_2\}$. Then $\sum_{H \in \mathcal{T}_4'} w_G(H) \leq 0$.

When $\ell(F_3) \geq 9$, by Observation 3(1), we have $w(T) \leq 0$.

Subcase 3.3. $|\mathcal{J}(T)| = 4$. Notice that $\mathcal{F}(T) \setminus \mathcal{F}_3(T) = \{F_1, F_2, F_3, F_4\}$ and $|E_T(C_{F_i})| = 1$, by Observation 6(1), we have $\ell(F_i) \geq 8$ for $i = 1, 2, 3, 4$. By Observation 3(1), we have $w(T) \leq 0$.

Case 4. $T \cong T_5^4$. Clearly, $|E(T)| = 9$, $|\mathcal{F}_3(T)| = 5$ and $|\mathcal{J}(T)| \in \{2, 3\}$.

Subcase 4.1. $|\mathcal{J}(T)| = 2$. Let $\mathcal{J}(T) = \{u, v\}$. Notice that $\mathcal{F}(T) \setminus \mathcal{F}_3(T) = \{F_1, F_2\}$, without loss of generality, let $|E_T(C_{F_i})| = i$, $i = 1, 2$. If $\ell(F_i) \in \{4, 5\}$ for $i \in \{1, 2\}$, then $\ell(F_i) - 1 \in \mathcal{L}_{G[E(G) \setminus E(T)]}(u, v)$, so G contains $C_{\ell(F_i)+2}$ since $3 \in \mathcal{L}_T(u, v)$, a contradiction. Thus $\ell(F_i) \notin \{4, 5\}$, $i = 1, 2$. By Observation 5, we have $\ell(F_i) \geq 8$, $i = 1, 2$. Clearly, $E_T(C_{F_1}) \subseteq E_2(C_{F_1})$ and $E_T(C_{F_2}) \subseteq E_1(C_{F_2})$. By equation (2), we have $f_{F_i}(e) = \frac{1}{\ell(F_i)} \leq \frac{1}{8i}$ when $e \in E_T(C_{F_i})$, $i = 1, 2$. By Observation 1 and equation (4), we have $n_G(T) \leq 4$ and $f_G(T) = |\mathcal{F}_3(T)| + \sum_{i=1}^{2} \sum_{e \in E_T(C_{F_i})} f_{F_i}(e) \leq 5 + \sum_{i=1}^{2} i \cdot \frac{1}{8i} = \frac{21}{4}$. Thus $w(T) \leq 0$.

Subcase 4.2. $|\mathcal{J}(T)| = 3$. Recall that $\mathcal{F}(T) \setminus \mathcal{F}_3(T) = \{F_1, F_2, F_3\}$ and $|E_T(C_{F_i})| = 1$ for $i = 1, 2, 3$. If $\ell(F_i) \in \{4, 5\}$ for $i \in \{1, 2\}$, then $\ell(F_i) - 1 \in \mathcal{L}_{G[E(G) \setminus E(T)]}(u, v)$ for $\{u, v\} \subseteq \mathcal{J}(T)$, so G contains $C_{\ell(F_i)+2}$ since $3 \in \mathcal{L}_T(u, v)$, a contradiction. Thus $\ell(F_i) \notin \{4, 5\}$, $i = 1, 2, 3$. By Observation 5, we have $\ell(F_i) \geq 8$, $i = 1, 2, 3$. Clearly, $E_T(C_{F_i}) \subseteq E_2(C_{F_i})$, by equation (2), we have $f_{F_i}(e) = \frac{1}{\ell(F_i)} \leq \frac{1}{8}$ when $e \in E_T(C_{F_i})$, $i = 1, 2, 3$. By Observation 1 and equation

(4), we have $n_G(T) \leq \frac{7}{2}$ and $f_G(T) = |\mathcal{F}_3(T)| + \sum_{i=1}^{3} \sum_{e \in E_T(C_{F_i})} f_{F_i}(e) \leq 5 + \sum_{i=1}^{3} 1 \cdot \frac{1}{8} = \frac{43}{8}$. Thus $w(T) \leq 0$.

Case 5. $T \cong T_4^1$. Let $V(T) = \{v_1, v_2, v_3, v_4\}$. The triangular block T is depicted in Figure 3(c). Clearly, $|E(T)| = 5$, $|\mathcal{F}_3(T)| = 2$ and $|\mathcal{J}(T)| \in \{2, 3, 4\}$.

Subcase 5.1. $|\mathcal{J}(T)| = 2$.

(1) Assume two junction vertices are the ends of an edge e of T. If e is not an outer edge of T, then $v_i \notin \mathcal{J}(T)$, so $d_G(v_i) = 2$, $i = 1, 3$. Since G is 2-connected, we obtain that v_1 or v_3 must be an internal vertex of G, a contradiction. So e is an outer edge of T. The 3^+-faces in $\mathcal{F}(T)$ are either only incident with e or with other three outer edges of T. Notice that $\mathcal{F}(T) \setminus \mathcal{F}_3(T) = \{F_1, F_2\}$, without loss of generality, let $|E_T(C_{F_1})| = 1$ and $|E_T(C_{F_2})| = 3$. Since $e \in E(T)$, we have $\ell(F_2) \neq 4$. By Observations 5 and 6(1), we have $\ell(F_1) \geq 8$, and $\ell(F_2) = 5$ or $\ell(F_2) \geq 8$.

When $\ell(F_2) = 5$, by Observation 9(4), we have $|E(C_{F_2})| = \ell(F_2) = 5$, so $|E(C_{F_2}) \setminus E_T(C_{F_2})| = 2$. Let $E(C_{F_2}) \setminus E_T(C_{F_2}) = \{e_1, e_2\}$ and H_i be a triangular block containing e_i, $i = 1, 2$. By Observation 9(4), we have $H_i = G[e_i]$, $i = 1, 2$. Let $V(H_1) \cap V(H_2) = \{v_5\}$. We calculate $w(T)$. Since $H_i = G[e_i]$ for $i = 1, 2$, we have $|V(T) \cup V(H_1) \cup V(H_2)| = |V(T) \cup \{v_5\}| = 5$. Since $n \geq 7$, G is 2-connected and $H_i = G[e_i]$, $i = 1, 2$, there exists $v \in \mathcal{J}(T)$ satisfying $t_G(v) \geq 3$. For all $u \in \mathcal{J}(T) \setminus \{v\}$ and $w \notin \mathcal{J}(T)$, we have $t_G(u) \geq 2$ and $t_G(w) = 1$. Thus $n_G(T) = \sum_{v \in V(T)} \frac{1}{t_G(v)} \leq \frac{1}{3} + \frac{1}{2} + 2 \cdot 1 = \frac{17}{6}$. Combining $|\mathcal{F}_3(T)| = 2$, $|E(T)| = 5$ and Observation 2, we have $w(T) \leq 36(2 + \frac{1}{8} + \frac{3}{5}) + 9n_G(T) - 25|E(T)| \leq 0$.

When $\ell(F_2) \geq 8$, by Observation 3(2), we have $w(T) \leq 0$.

(2) Assume two junction vertices are not the ends of any edge of T. Clearly, $\mathcal{J}(T) = \{v_1, v_3\}$ and $|E_T(C_{F_1})| = |E_T(C_{F_2})| = 2$. By Observation 7(1), we have $\ell(F_i) = 4$ or $\ell(F_i) \geq 8$, $i = 1, 2$. Without loss of generality, we assume that $\ell(F_1) \leq \ell(F_2)$.

When $\ell(F_i) = 4$, $i = 1, 2$, by Observation 9(3), we have $|E(C_{F_i})| = \ell(F_i) = 4$, so $|E(C_{F_i}) \setminus E_T(C_{F_i})| = |E(C_{F_i})| - |E_T(C_{F_i})| = 2$. Let $E(C_{F_i}) \setminus E_T(C_{F_i}) = \{e_{i,1}, e_{i,2}\}$ and $H_{i,j}$ be a triangular block containing $e_{i,j}$, $i = 1, 2$ and $j = 1, 2$. By Observation 9(3), we have $H_{i,j} = G[e_{i,j}]$, $i = 1, 2$ and $j = 1, 2$. Let $V(H_{i,1}) \cap V(H_{i,2}) = \{v_{i+4}\}$, $i = 1, 2$. Clearly, $t_G(v_j) \geq 2$, $j = 5, 6$. Notice that $\{H_{1,1}, H_{1,2}\} \cap \{H_{2,1}, H_{2,2}\} = \emptyset$, otherwise $\{H_{1,1}, H_{1,2}\} = \{H_{2,1}, H_{2,2}\}$, then $v_5 = v_6$ and $|V(G)| = |V(T) \cup V(H_{1,1}) \cup V(H_{1,2})| = |V(T) \cup \{v_5\}| = |V(T)| + 1 < 7$, a contradiction. We next calculate $w(T)$ and $w(H_{i,j})$ for $i = 1, 2$ and $j = 1, 2$. Since $H_{i,j}$ is a triangular block, $i = 1, 2$ and $j = 1, 2$, we have $t_G(v_k) \geq 3$ for $k = 1, 3$. Firstly, we calculate $w(T)$. For all $u \notin \mathcal{J}(T)$, we have $t_G(u) = 1$. Thus $n_G(T) = \sum_{v \in V(T)} \frac{1}{t_G(v)} \leq 2 \cdot \frac{1}{3} + 2 \cdot 1 = \frac{8}{3}$. Combining $|\mathcal{F}_3(T)| = 2$, $|E(T)| = 5$ and Observation 2, we have $w(T) \leq 36(2 + \frac{2}{4} + \frac{2}{4}) + 9n_G(T) - 25|E(T)| \leq 7$. Secondly, we calculate $w(H_{i,j})$ for $i = 1, 2$ and $j = 1, 2$. Without loss of generality, let $H_{1,1} = G[v_1v_5]$. Combining $F_1 \in \mathcal{F}(H_{1,1})$, $\ell(F_1) = 4$ and Observation 10(2), we have $w(H_{1,1}) \leq -4$. Similarly, $w(H_{i,j}) \leq -4$ for $i = 1, 2$ and $j = 1, 2$. Let $\mathcal{T}_5' = \{T, H_{1,1}, H_{1,2}, H_{2,1}, H_{2,2}\}$. Then $\sum_{H\in\mathcal{T}_5'} w_G(H) \leq 0$.

When $\ell(F_1) = 4$ and $\ell(F_2) \geq 8$, by Observation 9(3), we have $|E(C_{F_1})| = \ell(F_1) = 4$, so $|E(C_{F_1}) \setminus E_T(C_{F_1})| = 2$. Let $E(C_{F_1}) \setminus E_T(C_{F_1}) = \{e_1, e_2\}$ and H_i be a triangular block containing e_i, $i = 1, 2$. By Observation 9(3), we have $H_i = G[e_i]$, $i = 1, 2$. Let $V(H_1) \cap V(H_2) = \{v_5\}$. We calculate $w(T)$. Since $H_i = G[e_i]$ for $i = 1, 2$, we have $|V(T) \cup V(H_1) \cup V(H_2)| = |V(T) \cup \{v_5\}| = 5$. Since $n \geq 7$, G is 2-connected and $H_i = G[e_i]$, $i = 1, 2$, there exists $v \in \mathcal{J}(T)$ satisfying $t_G(v) \geq 3$. For all $u \in \mathcal{J}(T) \setminus \{v\}$ and $w \notin \mathcal{J}(T)$, we have $t_G(u) \geq 2$ and $t_G(w) = 1$. Thus $n_G(T) = \sum_{v\in V(T)} \frac{1}{t_G(v)} \leq \frac{1}{3} + \frac{1}{2} + 2 \cdot 1 = \frac{17}{6}$. Combining $|\mathcal{F}_3(T)| = 2$, $|E(T)| = 5$ and Observation 2, we have $w(T) \leq 36(2 + \frac{2}{4} + \frac{2}{8}) + 9n_G(T) - 25|E(T)| \leq 0$.

When $\ell(F_i) \geq 8$, $i = 1, 2$, by Observation 3(2), we have $w(T) \leq 0$.

Subcase 5.2. $|\mathcal{J}(T)| = k$, $k \in \{3, 4\}$. Notice that $\mathcal{F}(T) \setminus \mathcal{F}_3(T) = \{F_1, F_2, \ldots, F_k\}$, then the number of 3^+-faces with exactly one edge of T is $k - 1$ when $k = 3$ and k when $k = 4$. Without loss of generality, let $|E_T(C_{F_i})| = 1$ for $i = 1, 2, \ldots, k - 1$ and $|E_T(C_{F_k})| = 5 - k$. By Observations 5 and 6(1), we have $\ell(F_i) \geq 8$ for $i = 1, 2, \ldots, k - 1$ and $\ell(F_k) \geq 4$. By Observation 1, we have $n_G(T) \leq 4 - \frac{k}{2}$. Combining $|\mathcal{F}_3(T)| = 2$, $|E(T)| = 5$ and Observation 2, we have $w(T) \leq 36(2 + \frac{k-1}{8} + \frac{5-k}{4}) + 9n_G(T) - 25|E(T)| \leq 0$.

Case 6. $T \cong T_4^2$. Clearly, $|E(T)| = 6$, $|\mathcal{F}_3(T)| = 3$ and $|\mathcal{J}(T)| \in \{2, 3\}$.

Subcase 6.1. $|\mathcal{J}(T)| = 2$. Notice that $\mathcal{F}(T) \setminus \mathcal{F}_3(T) = \{F_1, F_2\}$, without loss of generality, let $|E_T(C_{F_1})| = 1$ and $|E_T(C_{F_2})| = 2$. Let $\mathcal{J}(T) = \{v_1, v_2\}$. By Observations 6(1) and 7(2), we have $\ell(F_1) \geq 8$, and $\ell(F_2) = 4$ or $\ell(F_2) \geq 8$.

When $\ell(F_2) = 4$, by Observation 9(2), we have $|E(C_{F_2})| = \ell(F_2) = 4$, so $|E(C_{F_2}) \setminus E_T(C_{F_2})| = 2$. Let $E(C_{F_2}) \setminus E_T(C_{F_2}) = \{e_1, e_2\}$ and H_i be a triangular block containing e_i, $i = 1, 2$. By Observation 9(2), we have $H_i = G[e_i]$, $i = 1, 2$. Let $V(H_1) \cup V(H_2) = \{v_5\}$. Clearly, $t_G(v_5) \geq 2$. We next calculate $w(T)$ and $w(H_i)$ for $i = 1, 2$. Since $H_i = G[e_i]$ for $i = 1, 2$, we have $|V(T) \cup V(H_1) \cup V(H_2)| = |V(T) \cup \{v_5\}| = 5$. Since $n \geq 7$, G is 2-connected and $H_i = G[e_i]$, $i = 1, 2$, there exists $v \in \mathcal{J}(T)$ satisfying $t_G(v) \geq 3$. For $u \in \mathcal{J}(T) \setminus \{v\}$, we have $t_G(u) \geq 2$. Firstly, we calculate $w(T)$. For $w \notin \mathcal{J}(T)$, we have $t_G(w) = 1$. Thus $n_G(T) = \sum_{v\in V(T)} \frac{1}{t_G(v)} \leq \frac{1}{3} + \frac{1}{2} + 2 \cdot 1 = \frac{17}{6}$. Combining $|\mathcal{F}_3(T)| = 3$, $|E(T)| = 6$ and Observation 2, we have $w(T) \leq 36(3 + \frac{1}{8} + \frac{2}{4}) + 9n_G(T) - 25|E(T)| \leq 6$. Secondly, we calculate $w(H_i)$ for $i = 1, 2$. Without loss of generality, let $H_1 = G[v_1v_5]$ and $H_2 = G[v_2v_5]$. Combining $F_2 \in \mathcal{F}(H_i)$ for $i = 1, 2$, $\ell(F_2) = 4$ and Observation 10(1,2), we have $w(H_1) \leq -\frac{5}{2}$ and $w(H_2) \leq -4$. Let $\mathcal{T}_6' = \{T, H_1, H_2\}$. Then $\sum_{P\in\mathcal{T}_6'} w_G(P) \leq 0$.

When $\ell(F_2) \geq 8$, by Observation 3(2), we have $w(T) \leq 0$.

Subcase 6.2. $|\mathcal{J}(T)| = 3$. Notice that $\mathcal{F}(T) \setminus \mathcal{F}_3(T) = \{F_1, F_2, F_3\}$ and $|E(C_{F_i})| = 1$ for $i = 1, 2, 3$, by Observation 6(1), we have $\ell(F_i) \geq 8$ for $i = 1, 2, 3$. By Observation 3(2), we have $w(T) \leq 0$.

Case 7. $T \cong T_3$. Let $V(T) = \{v_1, v_2, v_3\}$. Clearly, $|E(T)| = 3$, $|\mathcal{F}_3(T)| = 1$ and $|\mathcal{J}(T)| \in \{2, 3\}$.

Subcase 7.1. $|\mathcal{J}(T)| = 2$. Let $\mathcal{J}(T) = \{v_1, v_2\}$, $E_T(C_{F_1}) = \{v_1v_2\}$ and $E_T(C_{F_2}) = \{v_2v_3, v_1v_3\}$. If $\ell(F_1) = 5$, then $|E(P^T_{F_1})| = \ell(F_1) - |E_T(C_{F_1})| = 4$, so G contains $C_6 = v_1P^T_{F_1}v_2v_3v_1$, a contradiction. By Observation 5, we have $\ell(F_1) = 4$ or $\ell(F_1) \geq 8$, and $\ell(F_2) \in \{4, 5\}$ or $\ell(F_2) \geq 8$. Now we prove that $|E(C_{F_i})| = \ell(F_i)$ when $\ell(F_i) = 4$, $i \in \{1, 2\}$. Suppose $|E(C_{F_i})| \neq \ell(F_i)$ for $i \in \{1, 2\}$. Similar to the proof of Observation 9, we have $1 \leq k(F_i) \leq \ell(F_i) - |E_T(C_{F_i})| \leq 3$ and $|E(C_{F_i})| \in \{5, 6, 7\}$, $i \in \{1, 2\}$. By Observation 4, we obtain that G contains C_6 or C_7, a contradiction. So $|E(C_{F_i})| = \ell(F_i)$ when $\ell(F_i) = 4$, $i \in \{1, 2\}$.

We consider the following two cases based on the value of $\ell(F_1)$.

(1) Assume $\ell(F_1) = 4$. Recall that $|E(C_{F_1})| = \ell(F_1) = 4$. Let v_1, v_4, v_5, v_2 denote vertices of C_{F_1} in order.

Now we prove that v_1v_4, v_4v_5 and v_2v_5 are in trivial triangular blocks, respectively. We claim that any two edges in $\{v_1v_4, v_4v_5, v_2v_5\}$ are not in the same triangular block. Suppose at least two edges in $\{v_1v_4, v_4v_5, v_2v_5\}$ are in the same triangular block. Assume all edges in $\{v_1v_4, v_4v_5, v_2v_5\}$ are in the same triangular block T'. Since $v_1v_2 \in E(T)$, we have $v_1v_2 \notin E(T')$, thus T' contains at least five outer edges. Clearly, $T' = T^1_5$. Thus $4 \in \mathcal{L}_{T'}(v_1, v_2)$. Since $2 \in \mathcal{L}_T(v_1, v_2)$, we see that G contains C_6, a contradiction. Assume exactly two edges in $\{v_1v_4, v_4v_5, v_2v_5\}$ are in the same triangular block T' and the other edge is in the triangular block T'', $T' \neq T''$. Notice that $T' \neq T_2$, we see that one edge in $E(T') \cap \{v_1v_4, v_4v_5, v_2v_5\}$ is incident with a 3-face of T'. Since the degree of any internal vertex of G is at least 3, we see that two edges in $E(T') \cap \{v_1v_4, v_4v_5, v_2v_5\}$ are not incident with the same 3-face of T'. So $4 \in \mathcal{L}_{T' \cup T''}(v_1, v_2)$. Since $2 \in \mathcal{L}_T(v_1, v_2)$, we see that G contains C_6, a contradiction. So any two edges in $\{v_1v_4, v_4v_5, v_2v_5\}$ are not in the same triangular block. Let H_1, H_2 and H_3 be triangular blocks containing v_1v_4, v_4v_5 and v_2v_5, respectively. Suppose H_1 is not in a trivial triangular block. There exists a 3-face incident with v_1v_4, then $4 \in \mathcal{L}_{H_1 \cup H_2 \cup H_3}(v_1, v_2)$, we see that G contains C_6 since $2 \in \mathcal{L}_T(v_1, v_2)$, a contradiction. So H_1 is in a trivial triangular block. Similarly, H_i is in a trivial triangular block for $i = 2, 3$. So v_1v_4, v_4v_5 and v_2v_5 are in trivial triangular blocks, respectively.

Clearly, $t_G(v_i) \geq 2$, $i = 4, 5$. Now we prove that $\ell(F_2) \neq 5$. Suppose $\ell(F_2) = 5$. Clearly, $|E(P^T_{F_2})| = \ell(F_2) - |E_T(C_{F_2})| = 3$. Notice that $E(P^T_{F_2}) \neq \{v_1v_4, v_4v_5, v_2v_5\}$, otherwise $|V(G)| = |V(T) \cup V(H_1) \cup V(H_2) \cup V(H_3)| = |\{v_1, v_2, v_3, v_4, v_5\}| < 7$, a contradiction. Thus there exists $C_6 = v_1v_4v_5v_2P^T_{F_2}v_1$, a contradiction. So $\ell(F_2) \neq 5$, then $\ell(F_2) = 4$ or $\ell(F_2) \geq 8$.

When $\ell(F_2) = 4$, then $|E(C_{F_2})| = \ell(F_2) = 4$. Let v_1, v_3, v_2, v_6 denote vertices of C_{F_2} in order. Now we prove that v_1v_6 and v_2v_6 are in trivial triangular blocks, respectively. We claim that v_1v_6 and v_2v_6 are in different triangular blocks. Suppose v_1v_6 and v_2v_6 belong to a triangular block T'. Clearly, $T' \neq T_2$. Since $|E(C_{F_2})| = \ell(F_2)$, we have $k(F_2) = 0$, i.e., $T' \neq T_5^4$. Since $v_1v_2 \in E(T)$, we have $v_1v_2 \notin E(T')$, then $T' \neq T_3$. So $T' \in \{T_5^1, T_5^2, T_5^3, T_4^1, T_4^2\}$. Notice that $3 \in \mathcal{L}_{T'}(v_1, v_2)$, so G contains C_6 since there exists a path $v_1v_4v_5v_2$ in C_{F_1}, a contradiction. So v_1v_6 and v_2v_6 are in different triangular blocks. Let H_4 and H_5 be triangular blocks containing v_1v_6 and v_2v_6, respectively. Suppose H_4 is not in a trivial triangular block. There exists a 3-face incident with v_1v_6, then $3 \in \mathcal{L}_{H_4 \cup H_5}(v_1, v_2)$, we see that G contains C_6 since there exists a path $v_1v_4v_5v_2$ in C_{F_1}, a contradiction. So H_4 is in a trivial triangular block. Similarly, H_5 is in a trivial triangular block. So v_1v_6 and v_2v_6 are in trivial triangular blocks, respectively. Clearly, $t_G(v_6) \geq 2$. We next calculate $w(T)$ and $w(H_i)$ for $i = 1, 2, \ldots, 5$. Since H_i is a triangular block for $i = 1, 3, 4, 5$, we have $t_G(v_i) \geq 3$, $i = 1, 2$. Firstly, we calculate $w(T)$. Since $v_3 \notin \mathcal{J}(T)$, we have $t_G(v_3) = 1$. Thus $n_G(T) = \sum_{v \in V(T)} \frac{1}{t_G(v)} \leq 2 \cdot \frac{1}{3} + 1 \cdot 1 = \frac{5}{3}$. Combining $|\mathcal{F}_3(T)| = 1$, $|E(T)| = 3$ and Observation 2, we have $w(T) \leq 36(1 + \frac{1}{4} + \frac{2}{4}) + 9n_G(T) - 25|E(T)| \leq 3$. Secondly, we calculate $w(H_i)$ for $i = 1, 2, \ldots, 5$. Recall that $H_1 = G[v_1v_4]$. Since $v_1 \in \mathcal{J}(T)$, we have $t_G(v_1) \geq 2$. Combining $F_1 \in \mathcal{F}(H_1)$, $\ell(F_1) = 4$ and Observation 10(1), we have $w(H_1) \leq -\frac{5}{2}$. Similarly, $w(H_i) \leq -\frac{5}{2}$ for $i = 2, 3, 4, 5$. Let $\mathcal{T}_7' = \{T, H_1, H_2, H_3, H_4, H_5\}$. Then $\sum_{H \in \mathcal{T}_7'} w_G(H) \leq 0$.

When $\ell(F_2) \geq 8$, by Observation 1, we have $n_G(T) \leq 2$. Combining $|\mathcal{F}_3(T)| = 1$, $|E(T)| = 3$ and Observation 2, we have $w(T) \leq 36(1 + \frac{1}{4} + \frac{2}{8}) + 9n_G(T) - 25|E(T)| \leq 0$.

(2) Assume $\ell(F_1) \geq 8$. When $\ell(F_2) = 4$, we have $|E(C_{F_2})| = \ell(F_2) = 4$. Let v_1, v_3, v_2, v_4 denote vertices of C_{F_2} in order. Let H_1 and H_2 be triangular blocks containing v_1v_4 and v_2v_4, respectively. Now we prove that $|V(T) \cup V(H_1) \cup V(H_2)| \leq 6$. Assume $H_1 = H_2$. By Corollary 1, we have $|V(H_1)| \leq 5$, we have $|V(T) \cup V(H_1) \cup V(H_2)| = |V(T) \cup V(H_1)| = |V(H_1)| + 1 \leq 6$. Assume $H_1 \neq H_2$. Clearly, $|V(H_1) \cap V(H_2)| = 1$ and $|V(H_i) \cap V(T)| = 1$ for $i = 1, 2$. Since v_iv_4 is an outer edge of H_i, $i = 1, 2$, we have $\{1, 2, \ldots, |V(H_i)| - 1\} \subseteq \mathcal{L}_{H_i}(v_i, v_4)$, then $\{2, 3, \ldots, |V(H_1)| + |V(H_2)| - 2\} \subseteq \mathcal{L}_{H_1 \cup H_2}(v_1, v_2)$. By $\{1, 2\} \subseteq \mathcal{L}_T(v_1, v_2)$, we see that G contains C_j, $j = 3, 4, \ldots, |V(H_1)| + |V(H_2)|$, so $|V(H_1)| + |V(H_2)| \leq 5$. So $|V(T) \cup V(H_1) \cup V(H_2)| = |V(H_1)| + |V(H_2)| \leq 5$. So $|V(T) \cup V(H_1) \cup V(H_2)| \leq 6$. Since $n \geq 7$ and G is 2-connected, there exists $v \in \mathcal{J}(T)$ satisfying $t_G(v) \geq 3$. For all $u \in \mathcal{J}(T) \setminus \{v\}$ and $w \notin \mathcal{J}(T)$, we have $t_G(u) \geq 3$ and $t_G(w) \geq 2$. Thus $n_G(T) = \sum_{v \in V(T)} \frac{1}{t_G(v)} \leq \frac{1}{3} + \frac{1}{2} + 1 = \frac{11}{6}$. Combining $|\mathcal{F}_3(T)| = 1$, $|E(T)| = 3$ and Observation 2, we have $w(T) \leq 36(1 + \frac{1}{8} + \frac{1}{2}) + 9n_G(T) - 25|E(T)| \leq 0$.

When $\ell(F_2) \geq 5$, by Observation 1, we have $n_G(T) \leq 2$. Combining $|\mathcal{F}_3(T)| =$

1, $|E(T)| = 3$ and Observation 2, we have $w(T) \leq 36(1 + \frac{1}{8} + \frac{2}{5}) + 9n_G(T) - 25|E(T)| \leq 0$.

Subcase 7.2. $|\mathcal{J}(T)| = 3$. Recall that $\mathcal{F}(T) \setminus \mathcal{F}_3(T) = \{F_1, F_2, F_3\}$. Clearly, $|E_T(C_{F_i})| = 1$ for $i = 1, 2, 3$. Without loss of generality, we assume that $\ell(F_1) \leq \ell(F_2) \leq \ell(F_3)$. By Observation 5, we have $\ell(F_i) \geq 4$ for $i = 1, 2, 3$. If $\ell(F_2) = 4$, then $\ell(F_1) = 4$, notice that $|E(P^T_{F_j})| = \ell(F_j) - |E_T(C_{F_j})| = 3$ for $j = 1, 2$, so G contains C_7 since $|E(T)\setminus(E(C_{F_1}) \cup E(C_{F_2}))| = 1$, a contradiction; if $\ell(F_2) = 5$, then $|E(P^T_{F_2})| = \ell(F_2) - |E_T(C_{F_2})| = 4$, so G contains C_6 since $2 \in \mathcal{L}_T(u^T_{F_2}, v^T_{F_2})$, a contradiction. By Observation 5, we have $\ell(F_1) \geq 4$ and $\ell(F_i) \geq 8$, $i = 2, 3$. By Observation 1, we have $n_G(T) \leq \frac{3}{2}$. Combining $|\mathcal{F}_3(T)| = 1$, $|E(T)| = 3$ and Observation 2, we have $w(T) \leq 36(1 + \frac{1}{4} + \frac{1}{8} + \frac{1}{8}) + 9n_G(T) - 25|E(T)| \leq 0$.

Case 8. $T \cong T_2$. Recall that $|\mathcal{J}(T)| = 2$ and $\mathcal{F}(T) \setminus \mathcal{F}_3(T) = \{F_1, F_2\}$. Clearly, $|E_T(C_{F_i})| = 1$ for $i = 1, 2$. Without loss of generality, we assume that $\ell(F_1) \leq \ell(F_2)$. By Observation 5, we have $\ell(F_i) \geq 4$ for $i = 1, 2$. If $\ell(F_2) = 4$, then $\ell(F_1) = 4$, notice that $|E(P^T_{F_i})| = \ell(F_i) - |E_T(C_{F_i})| = 3$ for $i = 1, 2$, so G contains C_6, a contradiction. By Observation 5, we have $\ell(F_2) = 5$ or $\ell(F_2) \geq 8$. If $\ell(F_2) = 5$, then $\ell(F_1) = 5$, otherwise $\ell(F_1) = 4$, then $|E(P^T_{F_i})| = \ell(F_i) - |E_T(C_{F_i})| = i + 2$ for $i = 1, 2$, so G contains C_7, a contradiction. By Observation 1, we have $n_G(T) \leq 1$. Combining $|\mathcal{F}_3(T)| = 0$, $|E(T)| = 1$ and Observation 2, we have $w(T) \leq 36(\frac{1}{\ell(F_1)} + \frac{1}{\ell(F_2)}) + 9n_G(T) - 25|E(T)| = 36(\frac{1}{\ell(F_1)} + \frac{1}{\ell(F_2)}) - 16$. So $w(T) \leq 36(\frac{1}{5} + \frac{1}{5}) - 16 \leq 0$ when $\ell(F_2) = 5$ and $w(T) \leq 36(\frac{1}{4} + \frac{1}{8}) - 16 \leq 0$ when $\ell(F_2) \geq 8$.

Let $\mathcal{T}^i_1, \mathcal{T}^i_2, \ldots, \mathcal{T}^i_{p_i}$ be the sets of the triangular blocks, such that $\bigcup_{T \in \mathcal{T}^i_j} T$ isomorphic to $\bigcup_{T \in \mathcal{T}'_i} T$ is the subgraph of G, $j = 1, 2, \ldots, p_i$ and $i = 1, 2, \ldots, 7$. Since $\sum_{T \in \mathcal{T}'_i} w(T) \leq 0$, we have $\sum_{T \in \mathcal{T}^i_j} w(T) \leq 0$, $j = 1, 2, \ldots, p_i$ and $i = 1, 2, \ldots, 7$. Let $\mathcal{T}^* = \mathcal{T}(G) \setminus \bigcup_{i=1}^{7} \bigcup_{j=1}^{p_i} \mathcal{T}^i_j$. According to the above proof, we have $w(T) \leq 0$ for any $T \in \mathcal{T}^*$. Thus $36\,|F(G)| + 9\,|V(G)| - 25\,|E(G)| = \sum_{T \in \mathcal{T}(G)} (36 f_G(T) + 9n_G(T) - 25|E(T)|) = \sum_{T \in \mathcal{T}(G)} w(T) \leq 0$, as desired.

4. Proof of Theorem 5

By Theorem 4 and $n = 22k + 10 > 21$ for all $k \geq 1$, we have $ex_p(n, \{C_6, C_7\}) \leq \frac{27}{11}n - \frac{72}{11}$. Thus we only need to prove that $ex_p(n, \{C_6, C_7\}) \geq \frac{27}{11}n - \frac{72}{11}$ for $n = 22k + 10$. So now we prove that there exists a $\{C_6, C_7\}$-free planar graph with n vertices and $\frac{27}{11}n - \frac{72}{11}$ edges.

For the convenience of proof, we construct a graph M on $12k - 5$ vertices. Let P_1, P_2, P_3 and P_4 be four paths. Let $P_i = v^i_1 v^i_2 \ldots v^i_{2k-1+\frac{i-1}{2}}$ when $i = 1, 3$ and $P_i = v^i_1 v^i_2 \ldots v^i_{4k+1-i}$ when $i = 2, 4$. Clearly, $|P_1| = 2k - 1, |P_2| = 4k - 1, |P_3| = 2k$

and $|P_4| = 4k-3$. Let M be the graph obtained from the union of P_1, P_2, P_3 and P_4 by adding edges $v_j^1 v_{2j-1}^2$, $v_j^3 v_{2j+1}^2$ and $v_{j+1}^3 v_{2j-1}^4$ for $j = 1, 3, \ldots, 2k-1$. The graph M is depicted in the figure composed of thin edges in Figure 4. By the construction of M, we have $|V(M)| = \sum_{i=1}^{4} |P_i| = (2k-1)+(4k-1)+2k+(4k-3) = 12k-5$. Let $A_1 = \{v_j^1 : j = 3, 5, \ldots, 2k-3\}$, $A_2 = \{v_j^2 : j = 3, 5, \ldots, 4k-3\}$, $A_3 = \{v_j^3 : j = 2, 3, \ldots, 2k-1\}$ and $A_4 = \{v_j^4 : j = 5, 9, \ldots, 4k-7\}$. Clearly, $|A_1| = |A_4| = k-2$ and $|A_2| = |A_3| = 2k-2$. Notice that $V_3(M) = \bigcup_{i=1}^{4} A_i$, so $|V_3(M)| = \sum_{i=1}^{4} |A_i| = 6k-8$. To construct a $\{C_6, C_7\}$-free planar graph on n vertices, we consider the following two cases based on the value of n.

Case 1. $n = 44k-12$. In the first step, we construct a plane graph G_1, which satisfies that each face is an 8-face and the degree of each vertex is 2 or 3. Let G_1 be the graph obtained from M by adding edges $v_{2k-1}^1 v_{4k-4}^4$ and $v_{\frac{j+5}{2}}^1 v_j^4$ for $j = 3, 7, \ldots, 4k-9$; then adding a new vertex v and joining v and v_1^i for $i = 1, 4$. The graph G_1 is depicted in Figure 4.

By the construction of G_1, we have $g(G_1) = 8$ and $|V(G_1)| = |V(M)| + 1 = 12k-4$. By Handshaking Theorem and each face of G_1 is an 8-face, we have $8\,|F(G_1)| = 2\,|E(G_1)|$, i.e., $|F(G_1)| = \frac{1}{4}\,|E(G_1)|$. By Euler formula, we have $|V(G_1)| - 2 = |E(G_1)| - |F(G_1)| = \frac{3}{4}\,|E(G_1)|$ and therefore $|E(G_1)| = \frac{4}{3}(|V(G_1)| - 2) = 16k-8$. Let $B_1 = V(P_1) \setminus (A_1 \cup \{v_2^1\})$ and $B_2 = \{v_j^4 : j = 3, 7, \ldots, 4k-9\} \cup \{v_1^4, v_{4k-4}^4\}$. Clearly, $|B_1| = |B_2| = k$. By the construction of G_1, we have $V_3(G_1) = V_3(M) \cup B_1 \cup B_2$, so $|V_3(G_1)| = |V_3(M)| + |B_1| + |B_2| = 8k-8$. Since $V(G_1) = V_2(G_1) \cup V_3(G_1)$, we observe that $|V_2(G_1)| = |V(G_1)| - |V_3(G_1)| = (12k-4) - (8k-8) = 4k+4$.

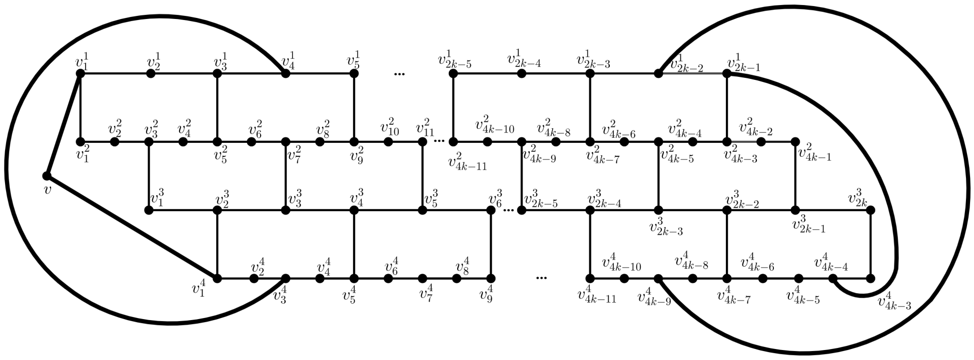


Figure 4. The graph G_1 in Case 1.

In the second step, we construct a plane graph G_2 based on G_1. Let G_2 be the plane graph obtained from G_1 by adding a new vertex to each edge of G_1, and joining any two new vertices added to two adjacent edges of G_1. The graph G_2 is depicted in Figure 5. By the construction of G_2, we have $|V(G_2)| = |V(G_1)| + |E(G_1)| = 28k - 12$; the degree of each new vertex of G_2 is at least 4, so $V_i(G_2) = V_i(G_1)$ for $i = 2, 3$ and $|V_2(G_2) \cup V_3(G_2)| = |V(G_1)| = 12k - 4$; we have $G_2[N_{G_2}[y]] \cong K_{i+1}$ for any $y \in V_i(G_2), i = 2, 3$; for any two adjacent vertices u and v of G_1, we see that the graphs $G_2[N_{G_2}[u]]$ and $G_2[N_{G_2}[v]]$ are edge-disjoint and their common vertices belong to $V(G_2) \setminus V(G_1)$.

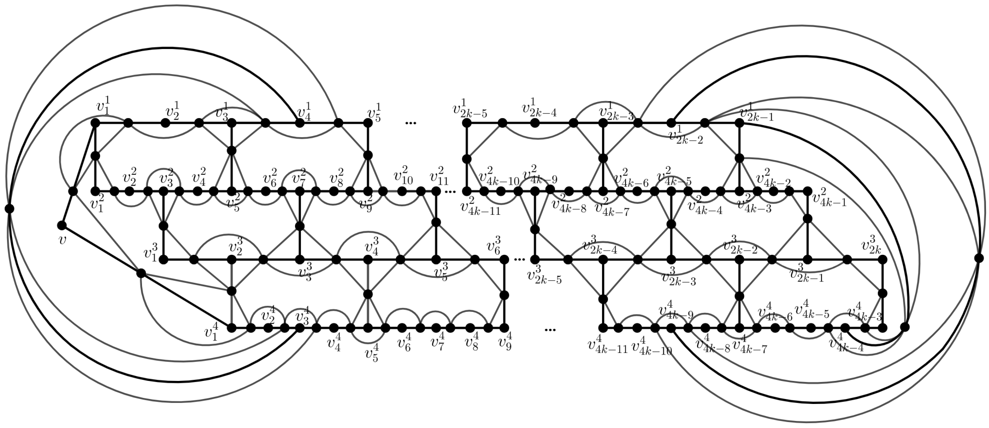
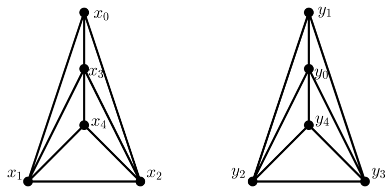


Figure 5. The graph G_2 in Case 1.

In the third step, we construct a plane graph G_3 based on G_2. For any $z \in V_2(G_2) \cup V_3(G_2)$, let $N_{G_2}(z) = \{z^{(1)}, z^{(2)}, \ldots, z^{(d_{G_2}(z))}\}$. For any $x \in V_2(G_2)$ and $y \in V_3(G_2)$, let G_3 be the plane graph obtained from G_2 by doing the following. (1) for any $x \in V_2(G_2)$, adding two adjacent vertices $x^{(3)}$ and $x^{(4)}$ to the interior of $(x, x^{(1)}, x^{(2)})$-face, adding edges $xx^{(3)}, x^{(i)}x^{(3)}$ and $x^{(i)}x^{(4)}$ for $i = 1, 2$; (2) for any $y \in V_3(G_2)$, adding a vertex $y^{(4)}$ to the interior of $(y, y^{(2)}, y^{(3)})$-face and adding edges $yy^{(4)}$ and $y^{(i)}y^{(4)}$ for $i = 2, 3$. For some $x_0 \in V_2(G_2)$ and $y_0 \in V_3(G_2)$, $G_3[x_0, x_0^{(1)}, x_0^{(2)}, x_0^{(3)}, x_0^{(4)}]$ and $G_3[y_0, y_0^{(1)}, y_0^{(2)}, y_0^{(3)}, y_0^{(4)}]$ are depicted in Figure 6(a) and Figure 6(b), respectively. By the construction of G_3, for any $z \in V_2(G_2) \cup V_3(G_2)$, we have $G_3[z, z^{(1)}, z^{(2)}, z^{(3)}, z^{(4)}] \cong K_5^-$. So G_3 is composed of the union of K_5^-s, where these K_5^-s are edge-disjoint and their common vertices (if have) belong to $V(G_2) \setminus V(G_1)$. Since $|E(K_5^-)| = 9$, we have $|V(G_3)| = |V(G_2)| + 2|V_2(G_2)| + |V_3(G_2)| = (28k - 12) + 2(4k + 4) + (8k - 8) = 44k - 12 = n$ and $|E(G_3)| = 9|V_2(G_2) \cup V_3(G_2)| = 108k - 36$. Notice that $n = 44k - 12$, we have $|E(G_3)| = 108k - 36 = \frac{27}{11}n - \frac{72}{11}$.

Now we prove that G_3 is a $\{C_6, C_7\}$-free plane graph. Notice that except the 3-faces in K_5^-, the degree of all other faces of G_3 is more than 7. For any $e \in E(G_3)$, the length of any cycle containing e is more than 7, unless all edges of the cycle belong to one K_5^-. Clearly, K_5^- is $\{C_6, C_7\}$-free. So G_3 is a $\{C_6, C_7\}$-free plane graph on n vertices and $|E(G_3)| = \frac{27}{11}n - \frac{72}{11}$ for $n = 44k - 12$.



$G_3[x_0, x_1, x_2, x_3, x_4]$ $G_3[y_0, y_1, y_2, y_3, y_4]$

(a) (b)

Figure 6. The operations on G_2 in the third step.

Case 2. $n = 44k+10$. In the first step, we construct a plane graph G_1, which satisfies that each face is an 8-face and the degree of each vertex is 2 or 3. Let G_1 be the graph obtained from M by adding the path $v_1^0 v_2^0 \dots v_7^0$; then adding edges $v_1^0 v_1^1$, $v_3^0 v_1^3$, $v_6^0 v_2^1$, $v_7^0 v_1^4$, $v_{2k-1}^1 v_{4k-4}^4$ and $v_{\frac{j+5}{2}}^1 v_j^4$ for $j = 3, 7, \dots, 4k-9$. The graph G_1 is depicted in Figure 7.

By the construction of G_1, we have $g(G_1) = 8$ and $|V(G_1)| = |V(M)| + 7 = 12k + 2$. By Handshaking Theorem and each face of G_1 is an 8-face, we have $8\,|F(G_1)| = 2\,|E(G_1)|$, i.e., $|F(G_1)| = \frac{1}{4}\,|E(G_1)|$. By Euler formula, we have $|V(G_1)| - 2 = |E(G_1)| - |F(G_1)| = \frac{3}{4}\,|E(G_1)|$ and therefore $|E(G_1)| = \frac{4}{3}(|V(G_1)| - 2) = 16k$. Let $C_1 = \{v_3^0, v_6^0, v_1^3\}$, $C_2 = V(P_1) \setminus A_1$ and $C_3 = \{v_j^4 : j = 3, 7, \dots, 4k-9\} \cup \{v_1^4, v_{4k-4}^4\}$. Clearly, $|C_1| = 3, |C_2| = k+1$ and $|C_3| = k$. By the construction of G_1, we have $V_3(G_1) = \bigcup_{i=1}^3 C_i \cup V_3(M)$, so $|V_3(G_1)| = \sum_{i=1}^3 |C_i| + |V_3(M)| = 8k - 4$. Since $V(G_1) = V_2(G_1) \cup V_3(G_1)$, we observe that $|V_2(G_1)| = |V(G_1)| - |V_3(G_1)| = (12k+2) - (8k-4) = 4k + 6$.

The second and third steps are the same as in Case 1, the resulting plane graph obtained through the second and third steps are denoted by G_2 and G_3, respectively. By the construction of G_2, we have $|V(G_2)| = |V(G_1)| + |E(G_1)| = 28k + 2$; the degree of each new vertex of G_2 is at least 4, so $V_i(G_2) = V_i(G_1)$ for $i = 2, 3$ and $|V_2(G_2) \cup V_3(G_2)| = |V(G_1)| = 12k + 2$; we have $G_2[N_{G_2}[y]] \cong K_{i+1}$ for any $y \in V_i(G_2), i = 2, 3$; for any two adjacent vertices u and v of G_1, we see that the graphs $G_2[N_{G_2}[u]]$ and $G_2[N_{G_2}[v]]$ are edge-disjoint and their common vertices belong to $V(G_2) \setminus V(G_1)$. By the construction of G_3, for any $z \in V_2(G_2) \cup V_3(G_2)$, we have $G_3[z, z^{(1)}, z^{(2)}, z^{(3)}, z^{(4)}] \cong K_5^-$. So G_3 is composed of the union of K_5^-s, where these K_5^-s are edge-disjoint and their common vertices (if have) belong to $V(G_2) \setminus V(G_1)$. Since $\left|E(K_5^-)\right| = 9$, we have $|V(G_3)| = |V(G_2)| + 2|V_2(G_2)| + |V_3(G_2)| = (28k+2) + 2(4k+6) + (8k-4) = 44k + 10 = n$ and $|E(G_3)| = 9\,|V_2(G_2) \cup V_3(G_2)| = 108k+18$. Notice that $n = 44k+10$, we have $|E(G_3)| = 108k+18 = \frac{27}{11}n - \frac{72}{11}$. By the same analysis as in Case 1, we know that G_3 is a $\{C_6, C_7\}$-free plane graph on n vertices and $|E(G_3)| = 108k+18 = \frac{27}{11}n - \frac{72}{11}$

for $n = 44k + 10$.

In summary, for any integer k, we have $ex_p(n, \{C_6, C_7\}) \geq \frac{27}{11}n - \frac{72}{11}$ for $n = 22k + 10$. By Theorem 4, we have $ex_p(n, \{C_6, C_7\}) = \frac{27}{11}n - \frac{72}{11}$ for $n \equiv 10 \pmod{22}$, as desired.

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