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THE PLANAR TURÁN NUMBER OF $\{C_6, C_7\}$

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Abstract

Let \mathcal{H} be a set of graphs. A graph is \mathcal{H} -free if it does not contain any copy of H as a subgraph where $H \in \mathcal{H}$. The planar Turán number of \mathcal{H} , denoted by $ex_p(n, \mathcal{H})$, is the maximum number of edges in an \mathcal{H} -free planar graph on n vertices. The upper bounds of $ex_p(n, \{C_k, C_{k+1}\})$ are known when $3 \leq k \leq 5$, and these bounds are tight. In this paper, we give the upper bound of $ex_p(n, \{C_6, C_7\})$ for all integers $n \geq 76$, and this bound is sharp.

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1. INTRODUCTION

All graphs considered in this paper are finite, undirected and simple. Let G be a graph with vertex set V(G) and edge set E(G). For any $v \in V(G)$, we use $N_G(v)$ to denote the set of neighbours of v in G. We define $N_G[v] := N_G(v) \cup \{v\}$. Let $d_G(v)$ be the degree of vertex v in G. We use $\delta(G)$ to denote the minimum degree

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of G. Let $V_i(G)$ denote the set of vertices of degree *i* in G. We say a cycle is a Hamilton cycle of G if it contains all vertices of G. Let g(G) denote the length of the shortest cycle of G. For $V' \subseteq V(G)$ $(E' \subseteq E(G))$, G[V'] (G[E']) denotes a subgraph which is induced by V'(E') of G. We use C_t and K_t to denote the cycle and complete graph on *t* vertices, respectively. We use K_t^- to denote the graph obtained from K_t with one edge removed. A graph is *k*-degenerate if it can be reduced to K_1 by repeatedly deleting vertices of degree at most *k*. A graph is 2-connected if it is a connected graph on at least 3 vertices and without any cut vertex. For any integer *k*, let $[k] := \{1, 2, \ldots, k\}$.

A graph G is planar if it can be drawn in the plane so that its edges intersect only at their ends, and such a planar embedding of G is called a plane graph. The unbounded face of a plane graph is called the outer face. For any plane graph G, a vertex of G is called an internal vertex if it is not on the boundary of the outer face of G. We use F(G) to denote the set of all faces of G. We denote the face of degree k and more than k by k-face and k^+ -face, respectively. We denote the face by (v_1, v_2, \ldots, v_k) -face, if its boundary is a cycle with vertices v_1, v_2, \ldots, v_k in order. If there exists a sequence of 3-faces F_1, F_2, \ldots, F_k so that F_i and F_{i+1} exactly share one edge where $i = 1, 2, \ldots, k-1$ and $k \geq 2$, the 3-faces F_1 and F_k are equivalent. The terminology and notation used in this paper can refer to [1].

Let \mathcal{H} be a set of graphs. A graph is \mathcal{H} -free if it does not contain any copy of H as a subgraph where $H \in \mathcal{H}$. The Turán number of \mathcal{H} , denoted by $ex(n, \mathcal{H})$, is the maximum number of edges in an \mathcal{H} -free graph on n vertices. When $\mathcal{H} = \{H\}$, we write $ex(n, \mathcal{H})$ as ex(n, H). In 1941, Turán [18] proved a classical result in the field of extremal graph theory. He determined the exact value of $ex(n, K_k)$ for any integer n and all $k \geq 3$, and he also proved that the balanced complete (k-1)-partite graph on n vertices is the unique extremal graph. This has led to a considerable amount of research work including the classical Turán-type problem and the Turán-type problems when the host graphs are hypergraph, hypercube and random graph [10, 14, 15].

In recent years, the Turán-type problem when the host graph is planar have received much attention. In 2016, Dowden [3] considered the Turán-type problems when the host graph is a planar graph, i.e., how many edges can an \mathcal{H} -free planar graph on n vertices have? And the maximum number of edges is called the planar Turán number of \mathcal{H} , denoted by $ex_p(n, \mathcal{H})$. For $|\mathcal{H}| = 1$, let $\mathcal{H} = \{H\}$, we write $ex_p(n, \mathcal{H})$ as $ex_p(n, \mathcal{H})$. Early in 2007, Wang *et al.* [19] gave the upper bounds of $ex_p(n, C_k)$ when $3 \leq k \leq 7$, but they did not prove whether these bounds were tight. Dowden [3] determined the exact value of $ex_p(n, C_3)$ and its unique extremal graph, he also gave the tight upper bounds of $ex_p(n, C_k)$ when $k \in \{4, 5\}$. Lan *et al.* [12] obtained the upper bound of $ex_p(n, C_6)$. Ghosh *et al.* [6] improved the upper bound of $ex_p(n, C_6)$ given in [12] and obtained the tight upper bound of $ex_p(n, C_6)$. In 2023, Shi *et al.* [16] and Győri *et al.* [7] independently obtained the tight upper bound of $ex_p(n, C_7)$.

Theorem 1. Let n be an integer.

- (1) [3] For all $n \ge 3$, $ex_p(n, C_3) = 2n 4$.
- (2) [3] For all $n \ge 4$, $ex_p(n, C_4) \le \frac{15}{7}(n-2)$, the equality holds when $n \equiv 30 \pmod{70}$.
- (3) [3] For all $n \ge 11$, $ex_p(n, C_5) \le \frac{12n-33}{5}$, the equality holds for infinitely many n.
- (4) [6] For all $n \ge 18$, $ex_p(n, C_6) \le \frac{5}{2}n 7$, the equality holds when $n \equiv 10 \pmod{18}$.
- (5) [7, 16] For all n > 38, $ex_p(n, C_7) \le \frac{18}{7}n \frac{48}{7}$, the equality holds for infinitely many n.

Regarding the lower bound of $ex_p(n, C_k)$, Cranston *et al.* [2] obtained that $ex_p(n, C_k) \geq 3n - 6 - \frac{3n+6}{k}$, when *n* is a function of *k* and sufficiently large for all $k \geq 11$. Lan *et al.* [11] obtained that $ex_p(n, C_k) \geq 3n - \frac{3-\frac{2}{k-1}}{k-6+\lfloor\frac{k-1}{2}\rfloor}n + \frac{12+3r-\frac{8+2r}{k-2}}{k-6+\lfloor\frac{k-1}{2}\rfloor} + \frac{4}{k-1} - \min\{r+10, 11\}$, for all *n* and *k* with $n \geq k \geq 11$ and for *r* being the remainder of n-4 when divided by $k-6 + \lfloor\frac{k-1}{2}\rfloor$. Győri *et al.* [8] obtained that $ex_p(n, C_k) \geq 3n - 6 - \frac{6\cdot3^{\log_2 3}n}{k^{\log_2 3}}$ for *n* sufficiently large and for all *k*. Regarding the upper bound of $ex_p(n, C_k)$, Cranston *et al.* [2] conjectured that $ex_p(n, C_k) \leq 3n - 6 - \frac{Dn}{k^{\log_2 3}}$ for *n* sufficiently large and for all *k* where *D* is a constant. Shi *et al.* [17] verified the above conjecture with $D = \frac{1}{4}$.

Theorem 2 [17]. For all integers $n, k \ge 4$, $ex_p(n, C_k) \le 3n - 6 - \frac{n}{4k^{\log_2 3}}$.

For $|\mathcal{H}| \geq 2$, Du *et al.* [4] gave the tight upper bounds of $ex_p(n, \{C_k, C_{k+1}\})$ when $k \in \{3, 4\}$. Du *et al.* [5] obtained the tight upper bound of $ex_p(n, \{C_5, C_6\})$. By Theorem 1(1), we see that $ex_p(n, C_3) = 2n-4$ for all $n \geq 3$, so $ex_p(n, \{C_3, C_k\})$ $\leq 2n - 4$ for all $k \geq 5$. Since $K_{2,n-2}$ must be C_k -free for all $k \neq 4$, we have $ex_p(n, \{C_3, C_k\}) \geq 2n - 4$. Thus $ex_p(n, \{C_3, C_k\}) = 2n - 4$ when $k \geq 5$. The minimum degree of $K_{2,n-2}$ is 2, so Győri *et al.* [9] studied the maximum number of edges in a $\{C_3, C_{2k}\}$ -free planar graph on n vertices with the minimum degree more than 2, and obtained the upper bounds when $k \in \{3, 4\}$.

Theorem 3. Let n be an integer.

- (1) [4] For all $n \ge 4$, $ex_p(n, \{C_3, C_4\}) \le \frac{5}{3}(n-2)$, the equality holds when $n \equiv 5 \pmod{15}$.
- (2) [4] For all $n \ge 8$, $ex_p(n, \{C_4, C_5\}) \le 2n 6$, the equality holds when $n \equiv 3 \pmod{9}$.

(3) [5] For all $n \ge 14$, $ex_p(n, \{C_5, C_6\}) \le \frac{30n-84}{13}$, the equality holds when $n \equiv 7 \pmod{10}$.

In this paper, we obtain the sharp upper bound of the planar Turán number of $\{C_6, C_7\}$; the results are as follows.

Theorem 4. For all integers $n \ge 76$, we have $ex_p(n, \{C_6, C_7\}) \le \frac{27}{11}n - \frac{72}{11}$. **Theorem 5.** If $n \equiv 10 \pmod{22}$, then $ex_p(n, \{C_6, C_7\}) = \frac{27}{11}n - \frac{72}{11}$.

2. Proof of Theorem 4

To prove Theorem 4, we give some definitions and lemmas. Given a plane graph G, a triangular block of G is a subgraph induced by the edge set consisting of all edges on the boundaries of a 3-face and all 3-faces equivalent to it. If an edge of G is not on the boundary of any 3-face, then the subgraph induced by the edge is called a trivial triangular block of G. Let $\mathcal{T}(G)$ denote the set of all triangular blocks of G.

A triangular block on at most 5 vertices must be $\{C_6, C_7\}$ -free. According to the definition of the triangular block, the triangular blocks on 5 vertices are T_5^1, T_5^2, T_5^3 and T_5^4 , the triangular blocks on 4 vertices are T_4^1 and T_4^2 , the triangular block on *i* vertices is T_i for i = 2, 3, as shown in Figure 1.



Figure 1. The triangular blocks on at most 5 vertices.

Lemma 6. Any triangular block on k vertices is C_6 -free if and only if $k \leq 5$.

Proof. The sufficiency clearly holds for $k \leq 5$. Now we prove the necessity. A triangular block on at most 5 vertices must be C_6 -free. So we only prove that any triangular block on k vertices must contain C_6 when $k \geq 6$. According to the definition of the triangular block, a triangular block on at least 6 vertices must contain a triangular block on 6 vertices as a subgraph. Therefore, we only need to prove that any triangular block on 6 vertices must contain a triangular block on 6 vertices. Notice that T must contain a triangular block H_1 on 5 vertices as a subgraph. Let $V(T) \setminus V(H_1) = \{x_0\}$. According to the definition of the triangular block, we see that x_0 must be adjacent to the ends x_1 and x_2 of an edge on the boundary of the outer face of H_1 . All triangular blocks

on 5 vertices are depicted in Figure 1. Clearly, T_5^i has a Hamilton cycle containing e for any edge e on the boundary of the outer face of T_5^i when i = 1, 2, 3, 4. Thus H_1 has a Hamilton cycle containing edge x_1x_2 , we denote this cycle by $x_1Px_2x_1$. So T contains $C_6 = x_0x_1Px_2x_0$. Thus any triangular block on 6 vertices must contain a C_6 . This completes the proof of the necessity.

Corollary 1. Any $\{C_6, C_7\}$ -free triangular block has at most 5 vertices.

Proof. Any $\{C_6, C_7\}$ -free triangular block must be C_6 -free. By Lemma 1, we see that any $\{C_6, C_7\}$ -free triangular block has at most 5 vertices.

Let G be a plane graph and T be a triangular block of G. An edge of T is called an outer edge of T if it is on the boundary of a 3⁺-face of G. The ends of an outer edge of T are called the outer vertices of T. We use $t_G(v)$ to denote the number of the triangular blocks sharing v in G where $v \in V(G)$. For any $v \in V(G)$, when $t_G(v) \ge 2$, v is called a junction vertex; when $t_G(v) = 1$, v is called a non-junction vertex. We denote the set of all junction vertices of T by $\mathcal{J}(T)$. For any triangular block T of G, we define $n_G(T) := \sum_{v \in V(T)} \frac{1}{t_G(v)}$. For any $v \in V(G)$, since the number of the triangular blocks sharing v in G is $t_G(v)$, we have

(1)
$$\sum_{T \in \mathcal{T}(G)} n_G(T) = \sum_{T \in \mathcal{T}(G)} \sum_{v \in V(T)} \frac{1}{t_G(v)} = \sum_{v \in V(G)} t_G(v) \cdot \frac{1}{t_G(v)} = |V(G)|.$$

Observation 1. Let T be a triangular block of a plane graph G. Then $n_G(T) \leq |V(T)| - \frac{1}{2}|\mathcal{J}(T)|$.

Proof. According to the definition of the junction vertex, for any $v \in V(G)$, we have $t_G(v) \ge 2$ when v is a junction vertex and $t_G(v) = 1$ when v is a non-junction vertex. So by the definition of $n_G(T)$, then $n_G(T) \le \frac{1}{2} \cdot |\mathcal{J}(T)| + 1 \cdot (|V(T)| - |\mathcal{J}(T)|) = |V(T)| - \frac{1}{2}|\mathcal{J}(T)|$.

Let G be a 2-connected plane graph and F be a face of G. Since G is 2connected, we know that the boundary of each face of G is a cycle, we denote the boundary of F by C_F . If two adjacent edges on the boundary of F are the outer edges of a T_5^4 , then we call that this T_5^4 is closely related to F, as shown in Figure 2 (the gray area is F). For any $F \in F(G)$, let k(F) be the number of T_5^4 s which are closely related to F, we define $E_1(C_F) := \{e \in E(C_F) | e \in E(T_5^4), T_5^4$ is closely related to F} and $E_2(C_F) := E(C_F) \setminus E_1(C_F)$. For any $F \in F(G)$ and $e \in E(C_F)$, let $\ell(F) = |E(C_F)| - k(F)$, and we define

(2)
$$f_F(e) := \begin{cases} \frac{1}{2\ell(F)}, & \text{if } e \in E_1(C_F); \\ \frac{1}{\ell(F)}, & \text{if } e \in E_2(C_F). \end{cases}$$

Since $|E(C_F)| = |E_1(C_F)| + |E_2(C_F)| = 2k(F) + |E_2(C_F)|$, we have $k(F) = \ell(F) - |E_2(C_F)|$.



Figure 2. T_5^4 is closely related to F.

If T_5^4 is closely related to F, then we have $|E(T_5^4) \cap E(C_F)| = 2$. Since the triangular blocks are edge-disjoint, we have $|E_1(C_F)| = 2k(F)$ and $|E_2(C_F)| = |E(C_F)| - 2k(F)$. So for all $F \in F(G)$, we have $\sum_{e \in E(C_F)} f_F(e) = \sum_{e \in E_1(C_F)} f_F(e) = \sum_{e \in E_1(C_F)} \frac{1}{2\ell(F)} + \sum_{e \in E_2(C_F)} \frac{1}{f_e(F)} f_F(e) = \sum_{e \in E_1(C_F)} \frac{1}{2\ell(F)} + \sum_{e \in E_2(C_F)} \frac{1}{\ell(F)} = 2k(F) \cdot \frac{1}{2\ell(F)} + (|E(C_F)| - 2k(F)) \cdot \frac{1}{\ell(F)} = \ell(F) \cdot \frac{1}{\ell(F)} = 1.$

For any triangular block T of G, we define $f_G(T) := \sum_{e \in E(T)} (f_{F_e^1}(e) + f_{F_e^2}(e))$, where two faces F_e^1 and F_e^2 are incident with e. The triangular blocks are edge-disjoint, so

(3)
$$\sum_{T \in \mathcal{T}(G)} f_G(T) = \sum_{T \in \mathcal{T}(G)} \sum_{e \in E(T)} (f_{F_e^1}(e) + f_{F_e^2}(e)) = \sum_{e \in E(G)} (f_{F_e^1}(e) + f_{F_e^2}(e))$$
$$= \sum_{F \in F(G)} \sum_{e \in E(C_F)} f_F(e) = \sum_{F \in F(G)} 1 = |F(G)|.$$

We denote the set of all faces of G incident with edges of T by $\mathcal{F}(T)$. According to the definition of $f_G(T)$, we have $f_G(T) = \sum_{F \in \mathcal{F}(T)} \sum_{e \in E_T(C_F)} f_F(e)$ where $E_T(C_F) = E(C_F) \cap E(T)$. We denote the set of all 3-faces of G incident with edges of T by $\mathcal{F}_3(T)$. If $F \in \mathcal{F}_3(T)$, then $E_1(C_F) = \emptyset$, k(F) = 0 and $\ell(F) = |E(C_F)|$. Thus $\sum_{e \in E(C_F)} f_F(e) = \sum_{e \in E(C_F)} \frac{1}{|E(C_F)|} = 1$. For any $F \in \mathcal{F}_3(T)$, we have $E_T(C_F) = E(C_F)$, so $\sum_{F \in \mathcal{F}_3(T)} \sum_{e \in E_T(C_F)} f_F(e) = \sum_{F \in \mathcal{F}_3(T)} \sum_{e \in E(C_F)} f_F(e) = \sum_{F \in \mathcal{F}_3(T)} 1 = |\mathcal{F}_3(T)|$. Thus

(4)
$$f_G(T) = \sum_{F \in \mathcal{F}_3(T)e \in E_T(C_F)} f_F(e) + \sum_{F \in \mathcal{F}(T) \setminus \mathcal{F}_3(T)e \in E_T(C_F)} \sum_{f_F(e)} f_F(e)$$
$$= |\mathcal{F}_3(T)| + \sum_{F \in \mathcal{F}(T) \setminus \mathcal{F}_3(T)e \in E_T(C_F)} \sum_{f_F(e)} f_F(e).$$

Let G be a 2-connected planar graph. So for any $T \in \mathcal{T}(G)$, we must have $|\mathcal{J}(T)| \geq 2$ and $|\mathcal{F}(T) \setminus \mathcal{F}_3(T)| = |\mathcal{J}(T)|$. Thus for any trivial triangular block $H \in \mathcal{T}(G)$, there are only two faces in $\mathcal{F}(H)$, both faces are 3⁺-faces, since two vertices of H must be junction vertices. Let $w(T) = 36f_G(T) + 9n_G(T) - 25|E(T)|$ for any $T \in \mathcal{T}(G)$.

Observation 2. Let G be a 2-connected planar graph, T be a triangular block of G on at most 5 vertices with $\mathcal{F}(T) \setminus \mathcal{F}_3(T) = \{F_1, F_2, \dots, F_{|\mathcal{J}(T)|}\}$ and $T \neq T_5^4$. Then $w(T) = 36 \left(|\mathcal{F}_3(T)| + \sum_{i=1}^{|\mathcal{J}(T)|} \frac{|E_T(C_{F_i})|}{\ell(F_i)} \right) + 9n_G(T) - 25|E(T)|.$

Proof. Since $T \neq T_5^4$, we have $E_T(C_{F_i}) \subseteq E_2(C_{F_i})$ for $i = 1, 2, ..., |\mathcal{J}(T)|$. By equation (2), we have $f_{F_i}(e) = \frac{1}{\ell(F_i)}$ for $e \in E_T(C_{F_i})$ and $i \in [|\mathcal{J}(T)|]$. By equation (4), we have $w(T) = 36(|\mathcal{F}_3(T)| + \sum_{i=1}^{|\mathcal{J}(T)|} \sum_{e \in E_T(C_{F_i})} f_{F_i}(e)) + 9n_G(T) - 25|E(T)| = 36(|\mathcal{F}_3(T)| + \sum_{i=1}^{|\mathcal{J}(T)|} \frac{|E_T(C_{F_i})|}{\ell(F_i)}) + 9n_G(T) - 25|E(T)|.$

Observation 3. Let G be a 2-connected planar graph and T be a triangular block of G with $\mathcal{F}(T) \setminus \mathcal{F}_3(T) = \{F_1, F_2, \dots, F_{|\mathcal{J}(T)|}\}$. Then $w(T) \leq 0$ if one of the following holds. (1) $T \cong T_5^j$ and $\ell(F_k) \geq 8$ for $j \in [3]$ and $k = 1, 2, \dots, |\mathcal{J}(T)|$; (2) $T \cong T_4^j$ and $\ell(F_k) \geq 8$ for $j \in [2]$ and $k = 1, 2, \dots, |\mathcal{J}(T)|$; (3) $T \cong T_5^1$, $|\mathcal{J}(T)| = 3, \ \ell(F_1) \geq 8$ and $\ell(F_k) \geq 5$ for k = 2, 3.

Proof. By Observation 1, we have $n_G(T) \leq |V(T)| - \frac{1}{2}|\mathcal{J}(T)| \leq |V(T)| - 1$ since $|\mathcal{J}(T)| \geq 2$. If (1) holds, then $n_G(T) \leq 4$. When $T \cong T_5^1$, since $\sum_{i=1}^{|\mathcal{J}(T)|} |E_T(C_{F_i})| = 5$, we have $\sum_{i=1}^{|\mathcal{J}(T)|} \frac{|E_T(C_{F_i})|}{\ell(F_i)} \leq \frac{5}{8}$, combining $|\mathcal{F}_3(T)| = 3$, |E(T)| = 7 and Observation 2, we have $w(T) \leq 0$. When $T \cong T_5^j$, $j \in \{2,3\}$, since $\sum_{i=1}^{|\mathcal{J}(T)|} |E_T(C_{F_i})| = 4$, we have $\sum_{i=1}^{|\mathcal{J}(T)|} \frac{|E_T(C_{F_i})|}{\ell(F_i)} \leq \frac{4}{8}$, combining $|\mathcal{F}_3(T)| = 4$, |E(T)| = 8 and Observation 2, we have $w(T) \leq 0$. If (2) holds, then $n_G(T) \leq 3$. When $T \cong T_4^j$, $j \in [2]$, since $\sum_{i=1}^{|\mathcal{J}(T)|} \frac{|E_T(C_{F_i})|}{\ell(F_i)} = 5 - j$, we have $\sum_{i=1}^{|\mathcal{J}(T)|} \frac{|E_T(C_{F_i})|}{\ell(F_i)} \leq \frac{5-j}{8}$, combining $|\mathcal{F}_3(T)| = j + 1$, |E(T)| = j + 4 and Observation 2, we have $w(T) \leq 0$. If (3) holds, then by $|\mathcal{J}(T)| = 3$, we have $n_G(T) \leq 5 - \frac{1}{2}|\mathcal{J}(T)| \leq \frac{7}{2}$. Notice that $\sum_{i=1}^{|\mathcal{J}(T)|} \frac{|E_T(C_{F_i})|}{\ell(F_i)} = \frac{|E_T(C_{F_i})|}{\ell(F_i)} + \sum_{i=2}^{|\mathcal{J}(T)|} \frac{|E_T(C_{F_i})|}{\ell(F_i)} \leq \frac{|E_T(C_{F_i})|}{8} + \frac{5-|E_T(C_{F_1})|}{4} = \frac{40-3|E_T(C_{F_1})|}{40} \leq \frac{37}{40}$. Combining $|\mathcal{F}_3(T)| = 3$, |E(T)| = 7 and Observation 2, we have $w(T) \leq 0$.

Observation 4. Let G be a 2-connected planar graph and T be a triangular block of G. For all $F \in \mathcal{F}(T) \setminus \mathcal{F}_3(T)$, if $k(F) \ge 1$, then G contains two cycles $C_{|E(C_F)|+1}$ and $C_{|E(C_F)|+2}$.

Proof. Let $u_1, u_2, \ldots, u_{|E(C_F)|}$ denote vertices of C_F in order. Since $k(F) \geq 1$, there must exist a T_5^4 closely related to C_F , i.e., there must exist two adjacent edges of C_F that are two outer edges of a T_5^4 , without loss of generality, let these two edges be u_1u_2 and u_2u_3 . Notice that there exists a path P_i of length i + 2 between u_1 and u_3 in T_5^4 , i = 1, 2, so G contains $C_{|E(C_F)|+i} = u_1P_iu_3u_4\cdots u_{|E(C_F)|}u_1$.

Observation 5. Let G be a 2-connected $\{C_6, C_7\}$ -free planar graph and T be a triangular block of G. For all $F \in \mathcal{F}(T) \setminus \mathcal{F}_3(T)$, we have $\ell(F) \geq 4$ and $\ell(F) \notin \{6,7\}$.

Proof. By the definition of $\ell(F)$, we obtain that $\ell(F) = |E(C_F)| - k(F) \ge 3$ and G contains cycles of length $\ell(F)$ and $|E(C_F)|$. Since G is $\{C_6, C_7\}$ -free, we have $\ell(F) \notin \{6,7\}$. Now we prove that $\ell(F) \ge 4$. Assume $\ell(F) = 3$. Recall that $k(F) = \ell(F) - |E_2(C_F)| \le \ell(F)$ and $|E(C_F)| = k(F) + 3$. If k(F) = 0, then $|E(C_F)| = 3$, this contradicts the fact that $F \in \mathcal{F}(T) \setminus \mathcal{F}_3(T)$. If k(F) = i, $i \in \{1,2\}$, then $|E(C_F)| = i+3$, by Observation 4, we know that G contains C_{i+5} , a contradiction. If k(F) = 3, then $|E(C_F)| = 6$, a contradiction. So $\ell(F) \ge 4$.

Let G be a 2-connected $\{C_6, C_7\}$ -free planar graph and T be a triangular block of G with $T \neq T_5^4$. For all $F \in \mathcal{F}(T) \setminus \mathcal{F}_3(T)$, let $V(C_F) \cap \mathcal{J}(T) = \{u_F^T, v_F^T\}$. Let $C_{\ell(F)}$ be a cycle of length $\ell(F)$ containing all edges of $E_2(C_F)$. Let P_F^T be a path in $C_{\ell(F)}$ with ends u_F^T and v_F^T and containing no edges of $E_T(C_F)$. Since $T \neq T_5^4$, we have $E_T(C_F) \subseteq E_2(C_F)$, so $|E(P_F^T)| = |E(C_{\ell(F)})| - |E_T(C_F)| =$ $\ell(F) - |E_T(C_F)|$. Let H be a subgraph of G. For $\{u, v\} \subseteq V(H)$, let $\mathcal{L}_H(u, v)$ be the set of lengths of all paths between u and v in H. When H is a triangular block of G with $|V(H)| \in \{4, 5\}$, it is easy to verify that for $\{u, v\} \subseteq V(H)$, we have $\{2, 3, \ldots, |V(H)| - 2\} \subseteq \mathcal{L}_H(u, v)$; moreover, we have $|V(H)| - 1 \in \mathcal{L}_H(u, v)$ if uv is an outer edge of H or $uv \notin E(H)$.

Observation 6. Let G be a 2-connected $\{C_6, C_7\}$ -free planar graph and T be a triangular block of G with $T \neq T_5^4$. For all $F \in \mathcal{F}(T) \setminus \mathcal{F}_3(T)$, we have $\ell(F) \geq 8$, if one of the following holds. (1) $|V(T)| \in \{4, 5\}$ and $|E_T(C_F)| = 1$; (2) |V(T)| = 5, $|E_T(C_F)| = 2$ and $u_F^T v_F^T \notin E(T)$; (3) $T \in \{T_5^2, T_5^3\}$ and $|E_T(C_F)| = 3$.

Proof. Assume (1) holds. If $\ell(F) = i$, then $|E(P_F^T)| = \ell(F) - |E_T(C_F)| = i - 1$, $i \in \{4, 5\}$; we obtain that G contains C_{i+2} since $3 \in \mathcal{L}_T(u_F^T, v_F^T)$, a contradiction. So $\ell(F) \notin \{4, 5\}$. Assume (2) holds. If $\ell(F) = i$, then $|E(P_F^T)| = i - 2$, $i \in \{4, 5\}$; we obtain that G contains C_{i+2} since $4 \in \mathcal{L}_T(u_F^T, v_F^T)$, a contradiction. So $\ell(F) \notin \{4, 5\}$. Assume (3) holds. Since $|E_T(C_F)| = 3$, we have $\ell(F) \neq 4$. If $\ell(F) = 5$, then $|E(P_F^T)| = 2$; we obtain that G contains C_6 since $4 \in \mathcal{L}_T(u_F^T, v_F^T)$, a contradiction. So $\ell(F) \notin \{4, 5\}$. By Observation 5, if (1), (2) or (3) holds, then $\ell(F) \geq 8$.

Observation 7. Let G be a 2-connected $\{C_6, C_7\}$ -free planar graph and T be a triangular block of G. For all $F \in \mathcal{F}(T) \setminus \mathcal{F}_3(T)$, we have $\ell(F) = 4$ or $\ell(F) \ge 8$, if one of the following holds. (1) $T \cong T_i^1$, $|E_T(C_F)| = i - 2$ for $i \in \{4, 5\}$ and $u_F^T v_F^T \notin E(T)$; (2) $T \cong T_4^2$ and $|E_T(C_F)| = 2$.

Proof. Assume (1) or (2) holds. If $\ell(F) = 5$, then $|E(P_F^T)| = \ell(F) - |E_T(C_F)| = 5 - (|V(T)| - 2) = 7 - |V(T)|$, with $|V(T)| - 1 \in \mathcal{L}_T(u_F^T, v_F^T)$; we obtain that G

contains C_6 , a contradiction. So $\ell(F) \neq 5$. By Observation 5, we have $\ell(F) = 4$ or $\ell(F) \geq 8$.

Observation 8. Let G be a 2-connected $\{C_6, C_7\}$ -free planar graph and T be a triangular block of G. For all $F \in \mathcal{F}(T) \setminus \mathcal{F}_3(T)$, if $T \cong T_5^3$, $|E_T(C_F)| = 2$ and $u_F^T v_F^T \in E(T)$, then $\ell(F) = 4$ or $\ell(F) \ge 9$.

Proof. If $\ell(F) = i$, then $|E(P_F^T)| = \ell(F) - |E_T(C_F)| = i - 2, i \in \{5, 8\}$, with $\{1, 3\} \subseteq \mathcal{L}_T(u_F^T, v_F^T)$, and we obtain that G contains C_{i-1} and C_{i+1} , a contradiction. So $\ell(F) \notin \{5, 8\}$. By Observation 5, we have $\ell(F) = 4$ or $\ell(F) \ge 9$.

Observation 9. Let G be a 2-connected $\{C_6, C_7\}$ -free planar graph and T be a triangular block of G. For all $F \in \mathcal{F}(T) \setminus \mathcal{F}_3(T)$, we have $|E(C_F)| = \ell(F)$ and each edge of $E(C_F) \setminus E_T(C_F)$ is in a trivial triangular block, if one of the following holds. (1) $T = T_5^1$, $|E_T(C_F)| = 3$, $\ell(F) = 4$ and $u_F^T v_F^T \notin E(T)$; (2) $T \in \{T_5^3, T_4^2\}$, $|E_T(C_F)| = 2$, $\ell(F) = 4$ and $u_F^T v_F^T \in E(T)$; (3) $T = T_4^1$, $|E_T(C_F)| = 2$, $\ell(F) = 4$ and $u_F^T v_F^T \notin E(T)$; (4) $T = T_4^1$, $|E_T(C_F)| = 3$, $\ell(F) = 5$ and $u_F^T v_F^T \in E(T)$.

Proof. Firstly, we prove that $|E(C_F)| = \ell(F)$. Suppose $|E(C_F)| \neq \ell(F)$. Since $k(F) = |E(C_F)| - \ell(F)$ and $|E(C_F)| \geq \ell(F)$, we have $k(F) \geq 1$. Notice that $T \neq T_5^4$, since $k(F) = \ell(F) - |E_2(C_F)|$ and $|E_2(C_F)| \geq |E_T(C_F)|$, we have $k(F) \leq \ell(F) - |E_T(C_F)| \leq 2$. According to $|E(C_F)| = \ell(F) + k(F)$, we have $|E(C_F)| \in \{5, 6, 7\}$. By Observation 4, we obtain that G contains C_6 or C_7 , a contradiction. So $|E(C_F)| = \ell(F)$.

Secondly, we prove that each edge of $E(C_F) \setminus E_T(C_F)$ is in a trivial triangular block. If (1) holds, then $|E(C_F) \setminus E_T(C_F)| = |E(C_F)| - |E_T(C_F)| = \ell(F) - |E_T(C_F)| = 1$, so $E(C_F) \setminus E_T(C_F) = \{u_F^T v_F^T\}$. Let H be a triangular block containing $u_F^T v_F^T$. If $H \neq G[u_F^T v_F^T]$, then $u_F^T v_F^T$ must be incident with a 3-face of H, then $2 \in \mathcal{L}_H(u_F^T, v_F^T)$, we know that G contains C_6 since $4 \in \mathcal{L}_T(u_F^T, v_F^T)$, a contradiction. So $H = G[u_F^T v_F^T]$. If (2), (3) or (4) holds, then $|E(C_F) \setminus E_T(C_F)| =$ 2. Let $E(C_F) \setminus E_T(C_F) = \{u_F^T w, v_F^T w\}$. Now we prove that $u_F^T w$ and $v_F^T w$ are in different triangular blocks. Suppose $u_F^T w$ and $v_F^T w$ belong to a triangular block T'. Clearly, $T' \neq T_2$. Since $|E(C_F)| = \ell(F)$, we have k(F) = 0, i.e., $T' \neq T_5^4$. If $T' \in \{T_3, T_4^1\}$ and $u_F^T v_F^T \in E(T')$, then $d_G(w) = 2$, a contradiction. If $T' \in \{T_5^1, T_5^2, T_5^3, T_4^2\}$, or $T' = T_4^1$ and $u_F^T v_F^T \notin E(T')$, then $3 \in \mathcal{L}_{T'}(u_F^T, v_F^T)$, so G contains C_6 since $3 \in \mathcal{L}_T(u_F^T, v_F^T)$, a contradiction. So $u_F^T w$ and $v_F^T w$ are in different triangular blocks. Let H_u and H_v be triangular blocks containing $u_F^T w$ and $v_F^T w$, respectively. If $H_u \neq G[u_F^T w]$, then $u_F^T w$ must be incident with a 3-face of H_u , i.e., $3 \in \mathcal{L}_{H_u \cup H_v}(u_F^T, v_F^T)$, we know that G contains C_6 since $3 \in \mathcal{L}_T(u_F^T, v_F^T)$, a contradiction. So $H_u = G[u_F^T w]$. Similarly, $H_v = G[v_F^T w]$.

Observation 10. Let G be a 2-connected $\{C_6, C_7\}$ -free planar graph, H be a triangular block of G with $E(H) = \{v_1v_2\}, \mathcal{F}(H) = \{F_1, F_2\}$ and $\ell(F_1) = 4$.

Then (1) $w(H) \leq \frac{18}{r} - \frac{23}{2}$ if $t_G(v_i) \geq r$, i = 1, 2, where $r \in \{2, 3\}$; (2) $w(H) \leq -4$ if $t_G(v_1) \geq 2$ and $t_G(v_2) \geq 3$.

Proof. Since $\ell(F_1) = 4$, we have $|E(P_{F_1}^H)| = \ell(F_1) - |E_H(C_{F_1})| = 3$. If $\ell(F_2) = i$, $i \in \{4, 5\}$, then $|E(P_{F_2}^H)| = \ell(F_2) - |E_H(C_{F_2})| = i - 1$, so G contains $C_{i+2} = v_1 P_{F_1}^H v_2 P_{F_2}^H v_1$, a contradiction. By Observation 5, we have $\ell(F_2) \ge 8$. Since |E(H)| = 1, we have $|\mathcal{F}_3(H)| = 0$ and $|E_H(C_{F_i})| = 1$ for i = 1, 2. By Observation 2, we have $w(H) \le 36(0 + \frac{1}{4} + \frac{1}{8}) + 9n_G(H) - 25|E(H)| = 9n_G(H) - \frac{23}{2}$. Recall that $n_G(H) = \sum_{v \in V(H)} \frac{1}{t_G(v)} = \frac{1}{t_G(v_1)} + \frac{1}{t_G(v_2)}$. If $t_G(v_i) \ge r$, i = 1, 2, then $n_G(H) \le \frac{2}{r}$, so $w(H) \le \frac{18}{r} - \frac{23}{2}$ where $r \in \{2, 3\}$; if $t_G(v_1) \ge 2$ and $t_G(v_2) \ge 3$, then $n_G(H) \le \frac{5}{6}$, so $w(H) \le -4$. ■

Lemma 7 [13]. If G is a 2-degenerate graph on n vertices with $n \ge 2$, then $|E(G)| \le 2n-3$.

Lemma 8 [20]. If a connected graph on n vertices is composed of blocks G_1, G_2, \ldots, G_s , then $n = \sum_{i=1}^s |V(G_i)| - s + 1$.

Lemma 9. Let G be a 2-connected and $\{C_6, C_7\}$ -free planar graph on n vertices with $n \ge 7$. If the degree of any internal vertex of one plane graph of G is at least 3, then $|E(G)| \le \frac{27}{11}n - \frac{72}{11}$.

We will give the proof of Lemma 9 in Section 3. Now we prove Theorem 4.

Proof of Theorem 4. Let G be any $\{C_6, C_7\}$ -free planar graph. We need to prove that $|E(G)| \leq \frac{27}{11}|V(G)| - \frac{72}{11}$, i.e., $27|V(G)| - 11|E(G)| \geq 72$ when $|V(G)| \geq 76$. If G is 2-degenerate, by Lemma 7, we have $|E(G)| \leq 2|V(G)| - 3 \leq \frac{27}{11}|V(G)| - \frac{72}{11}$ when $|V(G)| \geq 76$. If G is not 2-degenerate, repeatedly delete vertices of degree at most 2 from G until the degree of any vertex of the remaining graph is at least 3, the remaining graph is denoted by G^- . Clearly, we have $\delta(G^-) \geq 3$ and $|E(G)| \leq |E(G^-)| + 2(|V(G)| - |V(G^-)|)$. So $27|V(G)| - 11|E(G)| \geq 27|V(G)| - 11(|E(G^-)| + 2(|V(G)| - |V(G^-)|)) = 5(|V(G)| - |V(G^-)|) + 27|V(G^-)| - 11|E(G^-)|$, i.e.,

(5)
$$27|V(G)| - 11|E(G)| \ge 5(|V(G)| - |V(G^-)|) + 27|V(G^-)| - 11|E(G^-)|.$$

Let T_1, T_2, \ldots, T_t be components of G^- and $H_{c_{i-1}+1}, H_{c_{i-1}+2}, \ldots, H_{c_i}$ be the blocks which belong to T_i of G^- where $c_0 = 0$. Clearly, the number of the blocks in G^- is c_t . Since the blocks are edge-disjoint, we have $|E(G^-)| = \sum_{i=1}^{c_t} |E(H_i)|$. By Lemma 8, we have $|V(T_i)| = \sum_{j=c_{i-1}+1}^{c_i} |V(H_j)| - (c_i - c_{i-1}) + 1$. So

(6)
$$|V(G^{-})| = \sum_{i=1}^{t} |V(T_i)| = \sum_{i=1}^{c_t} |V(H_i)| - c_t + t.$$

Let b_j and $b_{\geq 7}$ be the number of the blocks on j vertices and at least 7 vertices in G^- , respectively, when j = 2, 3, ..., 6. So $c_t = \sum_{i=2}^6 b_i + b_{\geq 7}$ and $\sum_{i=1}^{c_t} |V(H_i)| = \sum_{i\geq 2} ib_i$. For any block H_i of G^- , by Lemma 9, we have $27|V(H_i)| - 11|E(H_i)| - 27 \geq 72 - 27 = 45$ when $|V(H_i)| \geq 7$. Clearly, $|E(H_i)| = 1$ when $|V(H_i)| = 2$ and $|E(H_i)| \leq 3|V(H_i)| - 6$ when $3 \leq |V(H_i)| \leq 6$, then $27|V(H_i)| - 11|E(H_i)| - 27 \geq 3$ when $2 \leq |V(H_i)| \leq 6$. By inequality (5), we have $27|V(G)| - 11|E(G)| \geq 5(|V(G)| - |V(G^-)|) + 27|V(G^-)| - 11|E(G^-)| = 5(|V(G)| - |V(G^-)|) + 27(\sum_{i=1}^{c_t} |V(H_i)| - c_t + t) - 11\sum_{i=1}^{c_t} |E(H_i)| = 5(|V(G)| - |V(G^-)|) + \sum_{i=1}^{c_t} (27|V(H_i)| - 11|E(H_i)| - 27) + 27t \geq \frac{3}{5}(|V(G)| - |V(G^-)|) + 45b_{\geq 7} + 3\sum_{i=2}^{6} b_i + 27t$, i.e.,

$$27|V(G)| - 11|E(G)| \ge \frac{3}{5}(|V(G)| - |V(G^{-})|) + 45b_{\ge 7} + 3\sum_{i=2}^{6}b_i + 27t.$$

If $b_{\geq 7} \geq 1$, then by $t \geq 1$, we have $27 |V(G)| - 11 |E(G)| \geq 72$. If $b_{\geq 7} = 0$, then $c_t = \sum_{i=2}^{6} b_i$ and $\sum_{i=1}^{c_t} |V(H_i)| = \sum_{i=2}^{6} ib_i$. By equation (6), we have $|V(G^-)| = \sum_{i=2}^{6} (i-1)b_i + t \leq 5 \sum_{i=2}^{6} b_i + t$, so $27|V(G)| - 11|E(G)| \geq \frac{3}{5}(|V(G)| - (5 \sum_{i=2}^{6} b_i + t)) + 3 \sum_{i=2}^{6} b_i + 27t = \frac{3}{5}|V(G)| + \frac{132}{5}t$. Thus $27 |V(G)| - 11 |E(G)| \geq 72$ when $|V(G)| \geq 76$.

In summary, for all $n \ge 76$, we have $ex_p(n, \{C_6, C_7\}) \le \frac{27}{11}n - \frac{72}{11}$.

3. Proof of Lemma 9

Let G be a plane graph satisfying the condition of Lemma 9. To prove that $|E(G)| \leq \frac{27}{11}|V(G)| - \frac{72}{11}$, by Euler formula, we prove that $11|E(G)| \leq 27|V(G)| - 36(|V(G)| - |E(G)| + |F(G)|)$, i.e., $36|F(G)| + 9|V(G)| - 25|E(G)| \leq 0$. Since the triangular blocks are edge-disjoint, we have $|E(G)| = \sum_{T \in \mathcal{T}(G)} |E(T)|$. By equations (1) and (3), we have $36|F(G)| + 9|V(G)| - 25|E(G)| = \sum_{T \in \mathcal{T}(G)} (36f_G(T) + 9n_G(T) - 25|E(T)|)$. Thus we only need to prove that $\sum_{T \in \mathcal{T}(G)} w(T) \leq 0$. By Corollary 1, we see that T is isomorphic to a triangular block in Figure 1 for all $T \in \mathcal{T}(G)$. We consider eight cases based on the structure of the triangular blocks. For any $T \in \mathcal{T}(G)$, since G is 2-connected, we have $|\mathcal{J}(T)| \geq 2$ and $|\mathcal{F}(T) \setminus \mathcal{F}_3(T)| = |\mathcal{J}(T)|$, let $\mathcal{F}(T) \setminus \mathcal{F}_3(T) = \{F_1, F_2, \dots, F_{|\mathcal{J}(T)|}\}$.

Case 1. $T \cong T_5^1$. Let $V(T) = \{v_1, v_2, v_3, v_4, v_5\}$. The triangular block T is depicted in Figure 3(a). Clearly, |E(T)| = 7, $|\mathcal{F}_3(T)| = 3$ and $|\mathcal{J}(T)| \in \{2, 3, 4, 5\}$.

Subcase 1.1. $|\mathcal{J}(T)| = 2.$

(1) Assume two junction vertices are the ends of an edge e of T. If e is not an outer edge of T, then $v_i \notin \mathcal{J}(T)$, so $d_G(v_i) = 2$, i = 1, 4. Since G is 2-connected, we obtain that v_1 or v_4 must be an internal vertex of G, a contradiction. So e



Figure 3. Three cases of T.

is an outer edge of T. The 3⁺-faces in $\mathcal{F}(T)$ are either only incident with e or with other four outer edges of T. Notice that $\mathcal{F}(T) \setminus \mathcal{F}_3(T) = \{F_1, F_2\}$, without loss of generality, let $|E_T(C_{F_1})| = 1$ and $|E_T(C_{F_2})| = 4$. Since $e \in E(T)$, we have $\ell(F_2) \notin \{4, 5\}$. By Observations 5 and 6(1), we have $\ell(F_i) \geq 8$, i = 1, 2. By Observation 3(1), we have $w(T) \leq 0$.

(2) Assume two junction vertices are not the ends of any edge of T. Let $\mathcal{J}(T) = \{u, w\}$. Notice that $\mathcal{F}(T) \setminus \mathcal{F}_3(T) = \{F_1, F_2\}$, without loss of generality, let $|E_T(C_{F_1})| = 2$ and $|E_T(C_{F_2})| = 3$. By Observations 6(2) and 7(1), we have $\ell(F_1) \geq 8$, and $\ell(F_2) = 4$ or $\ell(F_2) \geq 8$.

When $\ell(F_2) = 4$, by Observation 9(1), we have $|E(C_{F_2})| = \ell(F_2) = 4$. Since $|E(C_{F_2}) \setminus E_T(C_{F_2})| = 1$, we have $uw \in E(C_{F_2})$. Let H_1 be a triangular block containing uw. By Observation 9(1), we have $H_1 = G[uw]$. We next calculate w(T) and $w(H_1)$. Since $V(H_1) \subseteq V(T)$, we have $|V(T) \cup V(H_1)| = 5$. Since $n \ge 7$ and G is 2-connected, we have $t_G(u) \ge 3$ and $t_G(w) \ge 3$. Firstly, we calculate w(T). For all $v \notin \mathcal{J}(T)$, we have $t_G(v) = 1$. Thus $n_G(T) = \sum_{v \in V(T)} \frac{1}{t_G(v)} \le 2 \cdot \frac{1}{3} + 3 \cdot 1 = \frac{11}{3}$. Combining $|\mathcal{F}_3(T)| = 3$, |E(T)| = 7 and Observation 2, we have $w(T) \le 36(3 + \frac{2}{8} + \frac{3}{4}) + 9n_G(T) - 25|E(T)| \le 2$. Secondly, we calculate $w(H_1)$. Notice that $H_1 = G[uw]$, $F_2 \in \mathcal{F}(H_1)$ and $\ell(F_2) = 4$, by Observation 10(1), we have $w(H_1) \le -\frac{11}{2}$. Let $\mathcal{T}'_1 = \{T, H_1\}$. Then $\sum_{H \in \mathcal{T}'_1} w(H) \le 0$.

When $\ell(F_2) \ge 8$, by Observation 3(1), we have $w(T) \le 0$.

Subcase 1.2. $|\mathcal{J}(T)| = 3$. Notice that $\{v_1, v_4\} \cap \mathcal{J}(T) \neq \emptyset$, otherwise $v_i \notin \mathcal{J}(T)$ for i = 1, 4, we obtain that $d_G(v_i) = 2$ and v_1 or v_4 must be an internal vertex of G since G is 2-connected, a contradiction.

(1) Assume $|\{v_1, v_4\} \cap \mathcal{J}(T)| = 1$. Without loss of generality, let $v_1 \in \mathcal{J}(T)$ and $v_4 \notin \mathcal{J}(T)$.

(1.1) If $v_5 \notin \mathcal{J}(T)$, then $\mathcal{J}(T) = \{v_1, v_2, v_3\}$. Let $E_T(C_{F_1}) = \{v_1v_2\}$, $E_T(C_{F_2}) = \{v_2v_3\}$ and $E_T(C_{F_3}) = \{v_3v_4, v_4v_5, v_1v_5\}$. By Observations 6(1) and 7(1), we have $\ell(F_i) \geq 8$ for i = 1, 2 and $\ell(F_3) = 4$ or $\ell(F_3) \geq 8$.

When $\ell(F_3) = 4$, by Observation 9(1), we have $|E(C_{F_3})| = \ell(F_3) = 4$. Since $|E(C_{F_3}) \setminus E_T(C_{F_3})| = 1$, we have $v_1v_3 \in E(C_{F_3})$. Let H_1 be a triangular block containing v_1v_3 . By Observation 9(1), we have $H_1 = G[v_1v_3]$. By $V(H_1) \subseteq V(T)$, we have $|V(T) \cup V(H_1)| = 5$. Since G is 2-connected, $n \geq 7$ and $H_1 = G[v_1v_3]$,

there exists $v \in \mathcal{J}(T)$ satisfying $t_G(v) \geq 3$. For all $u \in \mathcal{J}(T) \setminus \{v\}$ and $w \notin \mathcal{J}(T)$, we have $t_G(u) \geq 2$ and $t_G(w) = 1$. So $n_G(T) = \sum_{v \in V(T)} \frac{1}{t_G(v)} \leq 1 \cdot \frac{1}{3} + 2 \cdot \frac{1}{2} + 2 \cdot 1 = \frac{10}{3}$. Combining $|\mathcal{F}_3(T)| = 3$, |E(T)| = 7 and Observation 2, we have $w(T) \leq 36(3 + \frac{1}{8} + \frac{1}{8} + \frac{3}{4}) + 9n_G(T) - 25|E(T)| \leq 0$.

When $\ell(F_3) \ge 8$, by Observation 3(1), we have $w(T) \le 0$.

(1.2) If $v_5 \in \mathcal{J}(T)$, then $\mathcal{J}(T) = \{v_1, v_2, v_5\}$ or $\mathcal{J}(T) = \{v_1, v_3, v_5\}$. When $\mathcal{J}(T) = \{v_1, v_2, v_5\}$, without loss of generality, let $|E_T(C_{F_1})| = |E_T(C_{F_2})| = 1$ and $|E_T(C_{F_3})| = 3$. Since $v_2v_5 \in E(T)$, we have $\ell(F_3) \neq 4$. By Observations 5 and 6(1), we have $\ell(F_i) \geq 8$ for i = 1, 2 and $\ell(F_3) \geq 5$. By Observation 3(3), we have $w(T) \leq 0$.

When $\mathcal{J}(T) = \{v_1, v_3, v_5\}$, let $E_T(C_{F_1}) = \{v_1v_5\}$, $E_T(C_{F_2}) = \{v_1v_2, v_2v_3\}$ and $E_T(C_{F_3}) = \{v_3v_4, v_4v_5\}$. By Observations 5 and 6(1,2), we have $\ell(F_i) \ge 8$ for i = 1, 2 and $\ell(F_3) \ge 4$. By Observation 1, we have $n_G(T) \le \frac{7}{2}$. Combining $|\mathcal{F}_3(T)| = 3, |E(T)| = 7$ and Observation 2, we have $w(T) \le 36(3 + \frac{1}{8} + \frac{2}{8} + \frac{2}{4}) + 9n_G(T) - 25|E(T)| \le 0$.

(2) Assume $|\{v_1, v_4\} \cap \mathcal{J}(T)| = 2$, i.e., $\{v_1, v_4\} \subseteq \mathcal{J}(T)$. According to the symmetry of T, there are two cases as follows.

When $\mathcal{J}(T) = \{v_1, v_2, v_4\}$, let $E_T(C_{F_1}) = \{v_1v_2\}$, $E_T(C_{F_2}) = \{v_1v_5, v_4v_5\}$ and $E_T(C_{F_3}) = \{v_2v_3, v_3v_4\}$. By Observation 6(1,2), we have $\ell(F_i) \geq 8$ for i = 1, 2, 3. By Observation 3(1), we have $w(T) \leq 0$.

When $\mathcal{J}(T) = \{v_1, v_4, v_5\}$, let $E_T(C_{F_1}) = \{v_1v_5\}$, $E_T(C_{F_2}) = \{v_4v_5\}$ and $E_T(C_{F_3}) = \{v_1v_2, v_2v_3, v_3v_4\}$. According to analysis similar to Subcase 1.2 (1.1), we have $w(T) \leq 0$.

Subcase 1.3. $|\mathcal{J}(T)| = k$ for $k \in \{4,5\}$. Notice that $\mathcal{F}(T) \setminus \mathcal{F}_3(T) = \{F_1, F_2, \ldots, F_k\}$, then the number of 3⁺-faces with exactly one edge of T is k-1 when k = 4 and k when k = 5. Without loss of generality, let $|E_T(C_{F_i})| = 1$ for $i = 1, 2, \ldots, k-1$ and $|E_T(C_{F_k})| = 6-k$. By Observations 5 and 6(1), we have $\ell(F_j) \geq 8$ for $j = 1, 2, \ldots, k-1$ and $\ell(F_k) \geq 4$. By Observation 1, we have $n_G(T) \leq 5 - \frac{k}{2}$. Combining $|\mathcal{F}_3(T)| = 3$, |E(T)| = 7 and Observation 2, we have $w(T) \leq 36(3 + \frac{k-1}{8} + \frac{6-k}{4}) + 9n_G(T) - 25|E(T)| \leq 0$.

Case 2. $T \cong T_5^2$. Recall that $\mathcal{F}(T) \setminus \mathcal{F}_3(T) = \{F_1, F_2, \dots, F_{|\mathcal{J}(T)|}\}$ and $|\mathcal{J}(T)| \in \{2, 3, 4\}$. According to the structure of T_5^2 , we see that $u_{F_i}^T v_{F_i}^T \notin E(T)$ when $|E_T(F_i)| = 2, i \in \{1, 2, \dots, |\mathcal{J}(T)|\}$. By Observation 6, we have $\ell(F_i) \ge 8$ for $i = 1, 2, \dots, |\mathcal{J}(T)|$. By Observation 3(1), we have $w(T) \le 0$.

Case 3. $T \cong T_5^3$. Let $V(T) = \{v_1, v_2, v_3, v_4, v_5\}$. The triangular block T is depicted in Figure 3(b). Clearly, |E(T)| = 8, $|\mathcal{F}_3(T)| = 4$ and $|\mathcal{J}(T)| \in \{2, 3, 4\}$.

Subcase 3.1. $|\mathcal{J}(T)| = 2$.

(1) Assume two junction vertices are the ends of an edge e of T. According to whether e is an outer edge, there are two cases as follows.

(1.1) If e is an outer edge of T, then the 3⁺-faces in $\mathcal{F}(T)$ are either only incident with e or with other three outer edges of T. Notice that $\mathcal{F}(T) \setminus \mathcal{F}_3(T) =$ $\{F_1, F_2\}$, without loss of generality, let $|E_T(C_{F_1})| = 1$ and $|E_T(C_{F_2})| = 3$. By Observation 6(1,3), we have $\ell(F_i) \geq 8$, i = 1, 2. By Observation 3(1), we have $w(T) \leq 0$.

(1.2) If e is not an outer edge of T, then $\mathcal{J}(T) = \{v_2, v_4\}$ and $|E_T(C_{F_1})| = |E_T(C_{F_2})| = 2$. By Observation 8, we have $\ell(F_i) = 4$ or $\ell(F_i) \ge 9$ for i = 1, 2. Without loss of generality, we assume that $\ell(F_1) \le \ell(F_2)$.

When $\ell(F_i) = 4$, i = 1, 2, by Observation 9(2), we have $|E(C_{F_i})| = \ell(F_i) = 4$, so $|E(C_{F_i}) \setminus E_T(C_{F_i})| = |E(C_{F_i})| - |E_T(C_{F_i})| = 2$. Let $E(C_{F_i}) \setminus E_T(C_{F_i}) = \{e_{i,1}, e_{i,2}\}$ and $H_{i,j}$ be a triangular block containing $e_{i,j}$, i = 1, 2 and j = 1, 2. Let $V(H_{i,1}) \cap V(H_{i,2}) = \{v_{i+5}\}$, i = 1, 2. Clearly, $t_G(v_j) \ge 2$, j = 6, 7. Notice that $\{H_{1,1}, H_{1,2}\} \cap \{H_{2,1}, H_{2,2}\} = \emptyset$, otherwise $\{H_{1,1}, H_{1,2}\} = \{H_{2,1}, H_{2,2}\}$, then $v_6 = v_7$ and $|V(G)| = |V(T) \cup V(H_{1,1}) \cup V(H_{1,2})| = |V(T) \cup \{v_6\}| = |V(T)| + 1 < 7$, a contradiction. We next calculate w(T) and $w(H_{i,j})$ for i = 1, 2 and j = 1, 2. Since $H_{i,j}$ is a triangular block, i = 1, 2 and j = 1, 2, we have $t_G(v_k) \ge 3$ for k = 2, 4. Firstly, we calculate w(T). For all $v \notin \mathcal{J}(T)$, we have $t_G(v) = 1$. Thus $n_G(T) = \sum_{v \in V(T)} \frac{1}{t_G(v)} \le 2 \cdot \frac{1}{3} + 3 \cdot 1 = \frac{11}{3}$. Combining $|\mathcal{F}_3(T)| = 4$, |E(T)| = 8 and Observation 2, we have $w(T) \le 36(4 + \frac{2}{4} + \frac{2}{4}) + 9n_G(T) - 25|E(T)| \le 13$. Secondly, we calculate $w(H_{i,j})$ for i = 1, 2 and j = 1, 2. Without loss of generality, let $H_{1,1} = G[v_2v_6]$. Combining $F_1 \in \mathcal{F}(H_{1,1}), \ell(F_1) = 4$ and Observation 10(2), we have $w(H_{1,1}) \le -4$. Similarly, $w(H_{i,j}) \le -4$ for i = 1, 2 and j = 1, 2. Let $T_2' = \{T, H_{1,1}, H_{1,2}, H_{2,2}\}$. Then $\sum_{H \in T_2'} w_G(H) \le 0$.

When $\ell(F_1) = 4$ and $\ell(F_2) \ge 9$, by Observation 9(2), we have $|E(C_{F_1})| = \ell(F_1) = 4$, so $|E(C_{F_1}) \setminus E_T(C_{F_1})| = 2$. Let $E(C_{F_1}) \setminus E_T(C_{F_1}) = \{e_1, e_2\}$ and H_i be a triangular block containing e_i , i = 1, 2. By Observation 9(2), we have $H_i = G[e_i]$, i = 1, 2. Let $V(H_1) \cap V(H_2) = \{v_6\}$. Clearly, $t_G(v_6) \ge 2$. We next calculate w(T) and $w(H_i)$ for i = 1, 2. Firstly, we calculate w(T). Since $H_i = G[e_i]$ for i = 1, 2, we have $|V(T) \cup V(H_1) \cup V(H_2)| = |V(T) \cup \{v_6\}| = 6$. Since $n \ge 7$, G is 2-connected and $H_i = G[e_i]$, i = 1, 2, there exists $v \in \mathcal{J}(T)$ satisfying $t_G(v) \ge 3$. For all $u \in \mathcal{J}(T) \setminus \{v\}$ and $w \notin \mathcal{J}(T)$, we have $t_G(u) \ge 2$ and $t_G(w) = 1$. Thus $n_G(T) = \sum_{v \in V(T)} \frac{1}{t_G(v)} \le \frac{1}{3} + \frac{1}{2} + 3 \times 1 = \frac{23}{6}$. Combining $|\mathcal{F}_3(T)| = 4$, |E(T)| = 8 and Observation 2, we have $w(T) \le 36(4 + \frac{2}{4} + \frac{2}{9}) + 9n_G(T) - 25|E(T)| \le \frac{9}{2}$. Secondly, we calculate $w(H_i)$ for i = 1, 2. Without loss of generality, let $H_1 = G[v_2v_6]$. Since $v_2 \in \mathcal{J}(T)$, we have $t_G(v_2) \ge 2$. Combining $F_1 \in \mathcal{F}(H_1), \ell(F_1) = 4$ and Observation 10(1), we have $w(H_1) \le -\frac{5}{2}$. Similarly, $w(H_2) \le -\frac{5}{2}$. Let $\mathcal{T}'_3 = \{T, H_1, H_2\}$. Then $\sum_{H \in \mathcal{T}'_3} w_G(H) \le 0$.

When $\ell(F_i) \ge 9$, i = 1, 2, by Observation 3(1), we have $w(T) \le 0$.

(2) Assume two junction vertices are not the ends of any edge of T. Clearly,

 $\mathcal{J}(T) = \{v_1, v_3\}$ and $|E_T(C_{F_1})| = |E_T(C_{F_2})| = 2$. By Observation 6(2), we have $\ell(F_i) \ge 8, i = 1, 2$. By Observation 3(1), we have $w(T) \le 0$.

Subcase 3.2. $|\mathcal{J}(T)| = 3$. Notice that $\mathcal{F}(T) \setminus \mathcal{F}_3(T) = \{F_1, F_2, F_3\}$, without loss of generality, let $|E_T(C_{F_i})| = 1$ for i = 1, 2 and $|E_T(C_{F_3})| = 2$. Since $|\mathcal{J}(T)| = 3$ and $\mathcal{J}(T) \subseteq \{v_1, v_2, v_3, v_4\}$, we have $\{v_1, v_3\} \subseteq \mathcal{J}(T)$ or $\{v_2, v_4\} \subseteq \mathcal{J}(T)$.

(1) Assume $\{v_1, v_3\} \subseteq \mathcal{J}(T)$. Recall that $v_1v_3 \notin E(T)$. By Observation 6(1,2), we have $\ell(F_i) \geq 8$, i = 1, 2, 3. By Observation 3(1), we have $w(T) \leq 0$.

(2) Assume $\{v_2, v_4\} \subseteq \mathcal{J}(T)$. Recall that $v_2v_4 \in E(T)$. By Observations 6(1) and 8, we have $\ell(F_i) \geq 8$ for i = 1, 2, and $\ell(F_3) = 4$ or $\ell(F_3) \geq 9$.

When $\ell(F_3) = 4$, by Observation 9(2), we have $|E(C_{F_3})| = \ell(F_3) = 4$, so $|E(C_{F_3}) \setminus E_T(C_{F_3})| = 2$. Let $E(C_{F_3}) \setminus E_T(C_{F_3}) = \{e_1, e_2\}$ and H_i be a triangular block containing e_i , i = 1, 2. By Observation 9(2), we have $H_i = G[e_i]$, i = 1, 2. Let $V(H_1) \cap V(H_2) = \{v_6\}$. Clearly, $t_G(v_6) \ge 2$. We next calculate w(T) and $w(H_i)$ for i = 1, 2. Firstly, we calculate w(T). By Observation 1, we have $n_G(T) \le \frac{7}{2}$. Combining $|\mathcal{F}_3(T)| = 4$, |E(T)| = 8 and Observation 2, we have $w(T) \le 36(4 + \frac{1}{8} + \frac{1}{8} + \frac{2}{4}) + 9n_G(T) - 25|E(T)| \le \frac{5}{2}$. According to analysis similar to Subcase 3.1 (1.2), we have $w(H_i) \le -\frac{5}{2}$, i = 1, 2. Let $\mathcal{T}'_4 = \{T, H_1, H_2\}$. Then $\sum_{H \in \mathcal{T}'_4} w_G(H) \le 0$.

When $\ell(F_3) \ge 9$, by Observation 3(1), we have $w(T) \le 0$.

Subcase 3.3. $|\mathcal{J}(T)| = 4$. Notice that $\mathcal{F}(T) \setminus \mathcal{F}_3(T) = \{F_1, F_2, F_3, F_4\}$ and $|E_T(C_{F_i})| = 1$, by Observation 6(1), we have $\ell(F_i) \geq 8$ for i = 1, 2, 3, 4. By Observation 3(1), we have $w(T) \leq 0$.

Case 4. $T \cong T_5^4$. Clearly, |E(T)| = 9, $|\mathcal{F}_3(T)| = 5$ and $|\mathcal{J}(T)| \in \{2,3\}$.

Subcase 4.1. $|\mathcal{J}(T)| = 2$. Let $\mathcal{J}(T) = \{u, v\}$. Notice that $\mathcal{F}(T) \setminus \mathcal{F}_3(T) = \{F_1, F_2\}$, without loss of generality, let $|E_T(C_{F_i})| = i, i = 1, 2$. If $\ell(F_i) \in \{4, 5\}$ for $i \in \{1, 2\}$, then $\ell(F_i) - 1 \in \mathcal{L}_{G[E(G) \setminus E(T)]}(u, v)$, so G contains $C_{\ell(F_i)+2}$ since $3 \in \mathcal{L}_T(u, v)$, a contradiction. Thus $\ell(F_i) \notin \{4, 5\}, i = 1, 2$. By Observation 5, we have $\ell(F_i) \geq 8, i = 1, 2$. Clearly, $E_T(C_{F_1}) \subseteq E_2(C_{F_1})$ and $E_T(C_{F_2}) \subseteq E_1(C_{F_2})$. By equation (2), we have $f_{F_i}(e) = \frac{1}{\ell(F_i)} \leq \frac{1}{8i}$ when $e \in E_T(C_{F_i}), i = 1, 2$. By Observation 1 and equation (4), we have $n_G(T) \leq 4$ and $f_G(T) = |\mathcal{F}_3(T)| + \sum_{i=1}^2 \sum_{e \in E_T(C_{F_i})} f_{F_i}(e) \leq 5 + \sum_{i=1}^2 i \cdot \frac{1}{8i} = \frac{21}{4}$. Thus $w(T) \leq 0$.

Subcase 4.2. $|\mathcal{J}(T)| = 3$. Recall that $\mathcal{F}(T) \setminus \mathcal{F}_3(T) = \{F_1, F_2, F_3\}$ and $|E_T(C_{F_i})| = 1$ for i = 1, 2, 3. If $\ell(F_i) \in \{4, 5\}$ for $i \in \{1, 2\}$, then $\ell(F_i) - 1 \in \mathcal{L}_{G[E(G) \setminus E(T)]}(u, v)$ for $\{u, v\} \subseteq \mathcal{J}(T)$, so G contains $C_{\ell(F_i)+2}$ since $3 \in \mathcal{L}_T(u, v)$, a contradiction. Thus $\ell(F_i) \notin \{4, 5\}$, i = 1, 2, 3. By Observation 5, we have $\ell(F_i) \geq 8$, i = 1, 2, 3. Clearly, $E_T(C_{F_i}) \subseteq E_2(C_{F_i})$, by equation (2), we have $f_{F_i}(e) = \frac{1}{\ell(F_i)} \leq \frac{1}{8}$ when $e \in E_T(C_{F_i})$, i = 1, 2, 3. By Observation 1 and equation

(4), we have $n_G(T) \leq \frac{7}{2}$ and $f_G(T) = |\mathcal{F}_3(T)| + \sum_{i=1}^3 \sum_{e \in E_T(C_{F_i})} f_{F_i}(e) \leq 5 + \sum_{i=1}^3 1 \cdot \frac{1}{8} = \frac{43}{8}$. Thus $w(T) \leq 0$.

Case 5. $T \cong T_4^1$. Let $V(T) = \{v_1, v_2, v_3, v_4\}$. The triangular block T is depicted in Figure 3(c). Clearly, |E(T)| = 5, $|\mathcal{F}_3(T)| = 2$ and $|\mathcal{J}(T)| \in \{2, 3, 4\}$.

Subcase 5.1. $|\mathcal{J}(T)| = 2$.

(1) Assume two junction vertices are the ends of an edge e of T. If e is not an outer edge of T, then $v_i \notin \mathcal{J}(T)$, so $d_G(v_i) = 2$, i = 1, 3. Since G is 2-connected, we obtain that v_1 or v_3 must be an internal vertex of G, a contradiction. So e is an outer edge of T. The 3⁺-faces in $\mathcal{F}(T)$ are either only incident with e or with other three outer edges of T. Notice that $\mathcal{F}(T) \setminus \mathcal{F}_3(T) = \{F_1, F_2\}$, without loss of generality, let $|E_T(C_{F_1})| = 1$ and $|E_T(C_{F_2})| = 3$. Since $e \in E(T)$, we have $\ell(F_2) \neq 4$. By Observations 5 and 6(1), we have $\ell(F_1) \geq 8$, and $\ell(F_2) = 5$ or $\ell(F_2) \geq 8$.

When $\ell(F_2) = 5$, by Observation 9(4), we have $|E(C_{F_2})| = \ell(F_2) = 5$, so $|E(C_{F_2}) \setminus E_T(C_{F_2})| = 2$. Let $E(C_{F_2}) \setminus E_T(C_{F_2}) = \{e_1, e_2\}$ and H_i be a triangular block containing e_i , i = 1, 2. By Observation 9(4), we have $H_i = G[e_i]$, i = 1, 2. Let $V(H_1) \cap V(H_2) = \{v_5\}$. We calculate w(T). Since $H_i = G[e_i]$ for i = 1, 2, we have $|V(T) \cup V(H_1) \cup V(H_2)| = |V(T) \cup \{v_5\}| = 5$. Since $n \ge 7$, G is 2-connected and $H_i = G[e_i]$, i = 1, 2, there exists $v \in \mathcal{J}(T)$ satisfying $t_G(v) \ge 3$. For all $u \in \mathcal{J}(T) \setminus \{v\}$ and $w \notin \mathcal{J}(T)$, we have $t_G(u) \ge 2$ and $t_G(w) = 1$. Thus $n_G(T) = \sum_{v \in V(T)} \frac{1}{t_G(v)} \le \frac{1}{3} + \frac{1}{2} + 2 \cdot 1 = \frac{17}{6}$. Combining $|\mathcal{F}_3(T)| = 2$, |E(T)| = 5 and Observation 2, we have $w(T) \le 36(2 + \frac{1}{8} + \frac{3}{5}) + 9n_G(T) - 25|E(T)| \le 0$.

When $\ell(F_2) \ge 8$, by Observation 3(2), we have $w(T) \le 0$.

(2) Assume two junction vertices are not the ends of any edge of T. Clearly, $\mathcal{J}(T) = \{v_1, v_3\}$ and $|E_T(C_{F_1})| = |E_T(C_{F_2})| = 2$. By Observation 7(1), we have $\ell(F_i) = 4$ or $\ell(F_i) \ge 8$, i = 1, 2. Without loss of generality, we assume that $\ell(F_1) \le \ell(F_2)$.

When $\ell(F_i) = 4$, i = 1, 2, by Observation 9(3), we have $|E(C_{F_i})| = \ell(F_i) = 4$, so $|E(C_{F_i}) \setminus E_T(C_{F_i})| = |E(C_{F_i})| - |E_T(C_{F_i})| = 2$. Let $E(C_{F_i}) \setminus E_T(C_{F_i}) = \{e_{i,1}, e_{i,2}\}$ and $H_{i,j}$ be a triangular block containing $e_{i,j}$, i = 1, 2 and j = 1, 2. By Observation 9(3), we have $H_{i,j} = G[e_{i,j}]$, i = 1, 2 and j = 1, 2. Let $V(H_{i,1}) \cap V(H_{i,2}) = \{v_{i+4}\}$, i = 1, 2. Clearly, $t_G(v_j) \ge 2$, j = 5, 6. Notice that $\{H_{1,1}, H_{1,2}\} \cap \{H_{2,1}, H_{2,2}\} = \emptyset$, otherwise $\{H_{1,1}, H_{1,2}\} = \{H_{2,1}, H_{2,2}\}$, then $v_5 = v_6$ and $|V(G)| = |V(T) \cup V(H_{1,1}) \cup V(H_{1,2})| = |V(T) \cup \{v_5\}| = |V(T)| + 1 < 7$, a contradiction. We next calculate w(T) and $w(H_{i,j})$ for i = 1, 2 and j = 1, 2. Since $H_{i,j}$ is a triangular block, i = 1, 2 and j = 1, 2, we have $t_G(v_k) \ge 3$ for k = 1, 3. Firstly, we calculate w(T). For all $u \notin \mathcal{J}(T)$, we have $t_G(u) = 1$. Thus $n_G(T) = \sum_{v \in V(T)} \frac{1}{t_G(v)} \le 2 \cdot \frac{1}{3} + 2 \cdot 1 = \frac{8}{3}$. Combining $|\mathcal{F}_3(T)| = 2$, |E(T)| = 5 and Observation 2, we have $w(T) \le 36(2 + \frac{2}{4} + \frac{2}{4}) + 9n_G(T) - 25|E(T)| \le 7$. Secondly, we calculate $w(H_{i,j})$ for i = 1, 2 and j = 1, 2. Without loss of generality, let $H_{1,1} = G[v_1v_5]$. Combining $F_1 \in \mathcal{F}(H_{1,1}), \ \ell(F_1) = 4$ and Observation 10(2), we have $w(H_{1,1}) \leq -4$. Similarly, $w(H_{i,j}) \leq -4$ for i = 1, 2 and j = 1, 2. Let $\mathcal{T}'_5 = \{T, H_{1,1}, H_{1,2}, H_{2,1}, H_{2,2}\}$. Then $\sum_{H \in \mathcal{T}'_{\mathcal{T}}} w_G(H) \leq 0$.

When $\ell(F_1) = 4$ and $\ell(F_2) \ge 8$, by Observation 9(3), we have $|E(C_{F_1})| = \ell(F_1) = 4$, so $|E(C_{F_1}) \setminus E_T(C_{F_1})| = 2$. Let $E(C_{F_1}) \setminus E_T(C_{F_1}) = \{e_1, e_2\}$ and H_i be a triangular block containing e_i , i = 1, 2. By Observation 9(3), we have $H_i = G[e_i]$, i = 1, 2. Let $V(H_1) \cap V(H_2) = \{v_5\}$. We calculate w(T). Since $H_i = G[e_i]$ for i = 1, 2, we have $|V(T) \cup V(H_1) \cup V(H_2)| = |V(T) \cup \{v_5\}| = 5$. Since $n \ge 7$, G is 2-connected and $H_i = G[e_i]$, i = 1, 2, there exists $v \in \mathcal{J}(T)$ satisfying $t_G(v) \ge 3$. For all $u \in \mathcal{J}(T) \setminus \{v\}$ and $w \notin \mathcal{J}(T)$, we have $t_G(u) \ge 2$ and $t_G(w) = 1$. Thus $n_G(T) = \sum_{v \in V(T)} \frac{1}{t_G(v)} \le \frac{1}{3} + \frac{1}{2} + 2 \cdot 1 = \frac{17}{6}$. Combining $|\mathcal{F}_3(T)| = 2$, |E(T)| = 5 and Observation 2, we have $w(T) \le 36(2 + \frac{2}{4} + \frac{2}{8}) + 9n_G(T) - 25|E(T)| \le 0$.

When $\ell(F_i) \ge 8$, i = 1, 2, by Observation 3(2), we have $w(T) \le 0$.

Subcase 5.2. $|\mathcal{J}(T)| = k, k \in \{3, 4\}$. Notice that $\mathcal{F}(T) \setminus \mathcal{F}_3(T) = \{F_1, F_2, \ldots, F_k\}$, then the number of 3⁺-faces with exactly one edge of T is k - 1 when k = 3 and k when k = 4. Without loss of generality, let $|E_T(C_{F_i})| = 1$ for $i = 1, 2, \ldots, k - 1$ and $|E_T(C_{F_k})| = 5 - k$. By Observations 5 and 6(1), we have $\ell(F_i) \geq 8$ for $i = 1, 2, \ldots, k - 1$ and $\ell(F_k) \geq 4$. By Observation 1, we have $n_G(T) \leq 4 - \frac{k}{2}$. Combining $|\mathcal{F}_3(T)| = 2$, |E(T)| = 5 and Observation 2, we have $w(T) \leq 36(2 + \frac{k-1}{8} + \frac{5-k}{4}) + 9n_G(T) - 25|E(T)| \leq 0$.

Case 6. $T \cong T_4^2$. Clearly, |E(T)| = 6, $|\mathcal{F}_3(T)| = 3$ and $|\mathcal{J}(T)| \in \{2, 3\}$.

Subcase 6.1. $|\mathcal{J}(T)| = 2$. Notice that $\mathcal{F}(T) \setminus \mathcal{F}_3(T) = \{F_1, F_2\}$, without loss of generality, let $|E_T(C_{F_1})| = 1$ and $|E_T(C_{F_2})| = 2$. Let $\mathcal{J}(T) = \{v_1, v_2\}$. By Observations 6(1) and 7(2), we have $\ell(F_1) \ge 8$, and $\ell(F_2) = 4$ or $\ell(F_2) \ge 8$.

When $\ell(F_2) = 4$, by Observation 9(2), we have $|E(C_{F_2})| = \ell(F_2) = 4$, so $|E(C_{F_2}) \setminus E_T(C_{F_2})| = 2$. Let $E(C_{F_2}) \setminus E_T(C_{F_2}) = \{e_1, e_2\}$ and H_i be a triangular block containing e_i , i = 1, 2. By Observation 9(2), we have $H_i = G[e_i]$, i = 1, 2. Let $V(H_1) \cup V(H_2) = \{v_5\}$. Clearly, $t_G(v_5) \ge 2$. We next calculate w(T) and $w(H_i)$ for i = 1, 2. Since $H_i = G[e_i]$ for i = 1, 2, we have $|V(T) \cup V(H_1) \cup V(H_2)| = |V(T) \cup \{v_5\}| = 5$. Since $n \ge 7$, G is 2-connected and $H_i = G[e_i]$, i = 1, 2, there exists $v \in \mathcal{J}(T)$ satisfying $t_G(v) \ge 3$. For $u \in \mathcal{J}(T) \setminus \{v\}$, we have $t_G(u) \ge 2$. Firstly, we calculate w(T). For $w \notin \mathcal{J}(T)$, we have $t_G(w) = 1$. Thus $n_G(T) = \sum_{v \in V(T)} \frac{1}{t_G(v)} \le \frac{1}{3} + \frac{1}{2} + 2 \cdot 1 = \frac{17}{6}$. Combining $|\mathcal{F}_3(T)| = 3$, |E(T)| = 6 and Observation 2, we have $w(T) \le 36(3 + \frac{1}{8} + \frac{2}{4}) + 9n_G(T) - 25|E(T)| \le 6$. Secondly, we calculate $w(H_i)$ for i = 1, 2. Without loss of generality, let $H_1 = G[v_1v_5]$ and $H_2 = G[v_2v_5]$. Combining $F_2 \in \mathcal{F}(H_i)$ for i = 1, 2, $\ell(F_2) = 4$ and Observation 10(1,2), we have $w(H_1) \le -\frac{5}{2}$ and $w(H_2) \le -4$. Let $\mathcal{T}'_6 = \{T, H_1, H_2\}$. Then $\sum_{P \in \mathcal{T}'_6} w_G(P) \le 0$.

When $\ell(F_2) \ge 8$, by Observation 3(2), we have $w(T) \le 0$.

Subcase 6.2. $|\mathcal{J}(T)| = 3$. Notice that $\mathcal{F}(T) \setminus \mathcal{F}_3(T) = \{F_1, F_2, F_3\}$ and $|E(C_{F_i})| = 1$ for i = 1, 2, 3, by Observation 6(1), we have $\ell(F_i) \ge 8$ for i = 1, 2, 3. By Observation 3(2), we have $w(T) \le 0$.

Case 7. $T \cong T_3$. Let $V(T) = \{v_1, v_2, v_3\}$. Clearly, |E(T)| = 3, $|\mathcal{F}_3(T)| = 1$ and $|\mathcal{J}(T)| \in \{2,3\}$.

Subcase 7.1. $|\mathcal{J}(T)| = 2$. Let $\mathcal{J}(T) = \{v_1, v_2\}, E_T(C_{F_1}) = \{v_1v_2\}$ and $E_T(C_{F_2}) = \{v_2v_3, v_1v_3\}$. If $\ell(F_1) = 5$, then $|E(P_{F_1}^T)| = \ell(F_1) - |E_T(C_{F_1})| = 4$, so G contains $C_6 = v_1P_{F_1}^Tv_2v_3v_1$, a contradiction. By Observation 5, we have $\ell(F_1) = 4$ or $\ell(F_1) \ge 8$, and $\ell(F_2) \in \{4, 5\}$ or $\ell(F_2) \ge 8$. Now we prove that $|E(C_{F_i})| = \ell(F_i)$ when $\ell(F_i) = 4$, $i \in \{1, 2\}$. Suppose $|E(C_{F_i})| \ne \ell(F_i)$ for $i \in \{1, 2\}$. Similar to the proof of Observation 9, we have $1 \le k(F_i) \le \ell(F_i) - |E_T(C_{F_i})| \le 3$ and $|E(C_{F_i})| \in \{5, 6, 7\}, i \in \{1, 2\}$. By Observation 4, we obtain that G contains C_6 or C_7 , a contradiction. So $|E(C_{F_i})| = \ell(F_i)$ when $\ell(F_i) = 4$, $i \in \{1, 2\}$.

We consider the following two cases based on the value of $\ell(F_1)$.

(1) Assume $\ell(F_1) = 4$. Recall that $|E(C_{F_1})| = \ell(F_1) = 4$. Let v_1, v_4, v_5, v_2 denote vertices of C_{F_1} in order.

Now we prove that v_1v_4 , v_4v_5 and v_2v_5 are in trivial triangular blocks, respectively. We claim that any two edges in $\{v_1v_4, v_4v_5, v_2v_5\}$ are not in the same triangular block. Suppose at least two edges in $\{v_1v_4, v_4v_5, v_2v_5\}$ are in the same triangular block. Assume all edges in $\{v_1v_4, v_4v_5, v_2v_5\}$ are in the same triangular block T'. Since $v_1v_2 \in E(T)$, we have $v_1v_2 \notin E(T')$, thus T' contains at least five outer edges. Clearly, $T' = T_5^1$. Thus $4 \in \mathcal{L}_{T'}(v_1, v_2)$. Since $2 \in \mathcal{L}_T(v_1, v_2)$, we see that G contains C_6 , a contradiction. Assume exactly two edges in $\{v_1v_4, v_4v_5, v_2v_5\}$ are in the same triangular block T' and the other edge is in the triangular block T'', $T' \neq T''$. Notice that $T' \neq T_2$, we see that one edge in $E(T') \cap \{v_1v_4, v_4v_5, v_2v_5\}$ is incident with a 3-face of T'. Since the degree of any internal vertex of G is at least 3, we see that two edges in $E(T') \cap \{v_1v_4, v_4v_5, v_2v_5\}$ are not incident with the same 3-face of T'. So $4 \in \mathcal{L}_{T'\cup T''}(v_1, v_2)$. Since $2 \in \mathcal{L}_T(v_1, v_2)$, we see that G contains C_6 , a contradiction. So any two edges in $\{v_1v_4, v_4v_5, v_2v_5\}$ are not in the same triangular block. Let H_1 , H_2 and H_3 be triangular blocks containing v_1v_4 , v_4v_5 and v_2v_5 , respectively. Suppose H_1 is not in a trivial triangular block. There exists a 3-face incident with v_1v_4 , then $4 \in \mathcal{L}_{H_1 \cup H_2 \cup H_3}(v_1, v_2)$, we see that G contains C_6 since $2 \in \mathcal{L}_T(v_1, v_2)$, a contradiction. So H_1 is in a trivial triangular block. Similarly, H_i is in a trivial triangular block for i = 2, 3. So v_1v_4, v_4v_5 and v_2v_5 are in trivial triangular blocks, respectively.

Clearly, $t_G(v_i) \geq 2$, i = 4, 5. Now we prove that $\ell(F_2) \neq 5$. Suppose $\ell(F_2) = 5$. Clearly, $|E(P_{F_2}^T)| = \ell(F_2) - |E_T(C_{F_2})| = 3$. Notice that $E(P_{F_2}^T) \neq \{v_1v_4, v_4v_5, v_2v_5\}$, otherwise $|V(G)| = |V(T) \cup V(H_1) \cup V(H_2) \cup V(H_3)| = |\{v_1, v_2, v_3, v_4, v_5\}| < 7$, a contradiction. Thus there exists $C_6 = v_1v_4v_5v_2P_{F_2}^Tv_1$, a contradiction.

diction. So $\ell(F_2) \neq 5$, then $\ell(F_2) = 4$ or $\ell(F_2) \geq 8$.

When $\ell(F_2) = 4$, then $|E(C_{F_2})| = \ell(F_2) = 4$. Let v_1, v_3, v_2, v_6 denote vertices of C_{F_2} in order. Now we prove that v_1v_6 and v_2v_6 are in trivial triangular blocks, respectively. We claim that v_1v_6 and v_2v_6 are in different triangular blocks. Suppose v_1v_6 and v_2v_6 belong to a triangular block T'. Clearly, $T' \neq T_2$. Since $|E(C_{F_2})| = \ell(F_2)$, we have $k(F_2) = 0$, i.e., $T' \neq T_5^4$. Since $v_1v_2 \in E(T)$, we have $v_1v_2 \notin E(T')$, then $T' \neq T_3$. So $T' \in \{T_5^1, T_5^2, T_5^3, T_4^1, T_4^2\}$. Notice that $3 \in \mathcal{L}_{T'}(v_1, v_2)$, so G contains C_6 since there exists a path $v_1 v_4 v_5 v_2$ in C_{F_1} , a contradiction. So v_1v_6 and v_2v_6 are in different triangular blocks. Let H_4 and H_5 be triangular blocks containing v_1v_6 and v_2v_6 , respectively. Suppose H_4 is not in a trivial triangular block. There exists a 3-face incident with v_1v_6 , then $3 \in \mathcal{L}_{H_4 \cup H_5}(v_1, v_2)$, we see that G contains C_6 since there exists a path $v_1v_4v_5v_2$ in C_{F_1} , a contradiction. So H_4 is in a trivial triangular block. Similarly, H_5 is in a trivial triangular block. So v_1v_6 and v_2v_6 are in trivial triangular blocks, respectively. Clearly, $t_G(v_6) \ge 2$. We next calculate w(T) and $w(H_i)$ for $i = 1, 2, \ldots, 5$. Since H_i is a triangular block for i = 1, 3, 4, 5, we have $t_G(v_i) \geq 3$, i = 1, 2. Firstly, we calculate w(T). Since $v_3 \notin \mathcal{J}(T)$, we have $t_G(v_3) = 1$. Thus $n_G(T) = \sum_{v \in V(T)} \frac{1}{t_G(v)} \le 2 \cdot \frac{1}{3} + 1 \cdot 1 = \frac{5}{3}$. Combining $|\mathcal{F}_3(T)| = 1$, |E(T)| = 3and Observation 2, we have $w(T) \leq 36(1 + \frac{1}{4} + \frac{2}{4}) + 9n_G(T) - 25|E(T)| \leq 3$. Secondly, we calculate $w(H_i)$ for i = 1, 2, ..., 5. Recall that $H_1 = G[v_1v_4]$. Since $v_1 \in \mathcal{J}(T)$, we have $t_G(v_1) \geq 2$. Combining $F_1 \in \mathcal{F}(H_1), \ \ell(F_1) = 4$ and Observation 10(1), we have $w(H_1) \leq -\frac{5}{2}$. Similarly, $w(H_i) \leq -\frac{5}{2}$ for i = 2, 3, 4, 5. Let $\mathcal{T}'_7 = \{T, H_1, H_2, H_3, H_4, H_5\}$. Then $\sum_{H \in \mathcal{T}'_7} w_G(H) \leq 0$.

When $\ell(F_2) \geq 8$, by Observation 1, we have $n_G(T) \leq 2$. Combining $|\mathcal{F}_3(T)| = 1$, |E(T)| = 3 and Observation 2, we have $w(T) \leq 36(1 + \frac{1}{4} + \frac{2}{8}) + 9n_G(T) - 25|E(T)| \leq 0$.

(2) Assume $\ell(F_1) \geq 8$. When $\ell(F_2) = 4$, we have $|E(C_{F_2})| = \ell(F_2) = 4$. Let v_1, v_3, v_2, v_4 denote vertices of C_{F_2} in order. Let H_1 and H_2 be triangular blocks containing v_1v_4 and v_2v_4 , respectively. Now we prove that $|V(T) \cup V(H_1) \cup V(H_2)| \leq 6$. Assume $H_1 = H_2$. By Corollary 1, we have $|V(H_1)| \leq 5$, we have $|V(T) \cup V(H_1) \cup V(H_2)| = |V(T) \cup V(H_1)| = |V(H_1)| + 1 \leq 6$. Assume $H_1 \neq H_2$. Clearly, $|V(H_1) \cap V(H_2)| = 1$ and $|V(H_i) \cap V(T)| = 1$ for i = 1, 2. Since v_iv_4 is an outer edge of H_i , i = 1, 2, we have $\{1, 2, \ldots, |V(H_i)| - 1\} \subseteq \mathcal{L}_{H_i}(v_i, v_4)$, then $\{2, 3, \ldots, |V(H_1)| + |V(H_2)| - 2\} \subseteq \mathcal{L}_{H_1 \cup H_2}(v_1, v_2)$. By $\{1, 2\} \subseteq \mathcal{L}_T(v_1, v_2)$, we see that G contains $C_j, j = 3, 4, \ldots, |V(H_1)| + |V(H_2)|$, so $|V(H_1)| + |V(H_2)| \leq 5$. So $|V(T) \cup V(H_1) \cup V(H_2)| = |V(H_1)| + |V(H_2)| \leq 5$. So $|V(T) \cup V(H_1) \cup V(H_2)| \leq 6$. Since $n \geq 7$ and G is 2-connected, there exists $v \in \mathcal{J}(T)$ satisfying $t_G(v) \geq 3$. For all $u \in \mathcal{J}(T) \setminus \{v\}$ and $w \notin \mathcal{J}(T)$, we have $t_G(u) \geq 3$ and $t_G(w) \geq 2$. Thus $n_G(T) = \sum_{v \in V(T)} \frac{1}{t_G(v)} \leq \frac{1}{3} + \frac{1}{2} + 1 = \frac{11}{6}$. Combining $|\mathcal{F}_3(T)| = 1$, |E(T)| = 3 and Observation 2, we have $w(T) \leq 36(1 + \frac{1}{8} + \frac{1}{2}) + 9n_G(T) - 25|E(T)| \leq 0$.

When $\ell(F_2) \ge 5$, by Observation 1, we have $n_G(T) \le 2$. Combining $|\mathcal{F}_3(T)| =$

1, |E(T)| = 3 and Observation 2, we have $w(T) \leq 36(1 + \frac{1}{8} + \frac{2}{5}) + 9n_G(T) - 25|E(T)| \leq 0.$

Subcase 7.2. $|\mathcal{J}(T)| = 3$. Recall that $\mathcal{F}(T) \setminus \mathcal{F}_3(T) = \{F_1, F_2, F_3\}$. Clearly, $|E_T(C_{F_i})| = 1$ for i = 1, 2, 3. Without loss of generality, we assume that $\ell(F_1) \leq \ell(F_2) \leq \ell(F_3)$. By Observation 5, we have $\ell(F_i) \geq 4$ for i = 1, 2, 3. If $\ell(F_2) = 4$, then $\ell(F_1) = 4$, notice that $|E(P_{F_j}^T)| = \ell(F_j) - |E_T(C_{F_j})| = 3$ for j = 1, 2, so G contains C_7 since $|E(T) \setminus (E(C_{F_1}) \cup E(C_{F_2}))| = 1$, a contradiction; if $\ell(F_2) = 5$, then $|E(P_{F_2}^T)| = \ell(F_2) - |E_T(C_{F_2})| = 4$, so G contains C_6 since $2 \in \mathcal{L}_T(u_{F_2}^T, v_{F_2}^T)$, a contradiction. By Observation 5, we have $\ell(F_1) \geq 4$ and $\ell(F_i) \geq 8$, i = 2, 3. By Observation 1, we have $n_G(T) \leq \frac{3}{2}$. Combining $|\mathcal{F}_3(T)| = 1$, |E(T)| = 3 and Observation 2, we have $w(T) \leq 36(1 + \frac{1}{4} + \frac{1}{8} + \frac{1}{8}) + 9n_G(T) - 25|E(T)| \leq 0$.

Case 8. $T \cong T_2$. Recall that $|\mathcal{J}(T)| = 2$ and $\mathcal{F}(T) \setminus \mathcal{F}_3(T) = \{F_1, F_2\}$. Clearly, $|E_T(C_{F_i})| = 1$ for i = 1, 2. Without loss of generality, we assume that $\ell(F_1) \leq \ell(F_2)$. By Observation 5, we have $\ell(F_i) \geq 4$ for i = 1, 2. If $\ell(F_2) = 4$, then $\ell(F_1) = 4$, notice that $|E(P_{F_i}^T)| = \ell(F_i) - |E_T(C_{F_i})| = 3$ for i = 1, 2, so G contains C_6 , a contradiction. By Observation 5, we have $\ell(F_2) = 5$ or $\ell(F_2) \geq 8$. If $\ell(F_2) = 5$, then $\ell(F_1) = 5$, otherwise $\ell(F_1) = 4$, then $|E(P_{F_i}^T)| = \ell(F_i) - |E_T(C_{F_i})| = i + 2$ for i = 1, 2, so G contains C_7 , a contradiction. By Observation 1, we have $n_G(T) \leq 1$. Combining $|\mathcal{F}_3(T)| = 0$, |E(T)| = 1 and Observation 2, we have $w(T) \leq 36(\frac{1}{\ell(F_1)} + \frac{1}{\ell(F_2)}) + 9n_G(T) - 25|E(T)| = 36(\frac{1}{\ell(F_1)} + \frac{1}{\ell(F_2)}) - 16$. So $w(T) \leq 36(\frac{1}{5} + \frac{1}{5}) - 16 \leq 0$ when $\ell(F_2) = 5$ and $w(T) \leq 36(\frac{1}{4} + \frac{1}{8}) - 16 \leq 0$ when $\ell(F_2) \geq 8$.

Let $\mathcal{T}_1^i, \mathcal{T}_2^i, \ldots, \mathcal{T}_{p_i}^i$ be the sets of the triangular blocks, such that $\bigcup_{T \in \mathcal{T}_j^i} T$ isomorphic to $\bigcup_{T \in \mathcal{T}_i'} T$ is the subgraph of $G, j = 1, 2, \ldots, p_i$ and $i = 1, 2, \ldots, 7$. Since $\sum_{T \in \mathcal{T}_i'} w(T) \leq 0$, we have $\sum_{T \in \mathcal{T}_j^i} w(T) \leq 0$, $j = 1, 2, \ldots, p_i$ and $i = 1, 2, \ldots, 7$. Let $\mathcal{T}^* = \mathcal{T}(G) \setminus \bigcup_{i=1}^7 \bigcup_{j=1}^{p_i} \mathcal{T}_j^i$. According to the above proof, we have $w(T) \leq 0$ for any $T \in \mathcal{T}^*$. Thus $36 |F(G)| + 9 |V(G)| - 25 |E(G)| = \sum_{T \in \mathcal{T}(G)} (36 f_G(T) + 9 n_G(T) - 25 |E(T)|) = \sum_{T \in \mathcal{T}(G)} w(T) \leq 0$, as desired.

4. Proof of Theorem 5

By Theorem 4 and n = 22k + 10 > 21 for all $k \ge 1$, we have $ex_p(n, \{C_6, C_7\}) \le \frac{27}{11}n - \frac{72}{11}$. Thus we only need to prove that $ex_p(n, \{C_6, C_7\}) \ge \frac{27}{11}n - \frac{72}{11}$ for n = 22k + 10. So now we prove that there exists a $\{C_6, C_7\}$ -free planar graph with n vertices and $\frac{27}{11}n - \frac{72}{11}$ edges.

For the convenience of proof, we construct a graph M on 12k-5 vertices. Let P_1, P_2, P_3 and P_4 be four paths. Let $P_i = v_1^i v_2^i \dots v_{2k-1+\frac{i-1}{2}}^i$ when i = 1, 3 and $P_i = v_1^i v_2^i \dots v_{4k+1-i}^i$ when i = 2, 4. Clearly, $|P_1| = 2k-1, |P_2| = 4k-1, |P_3| = 2k$

and $|P_4| = 4k - 3$. Let M be the graph obtained from the union of P_1, P_2, P_3 and P_4 by adding edges $v_j^1 v_{2j-1}^2, v_j^3 v_{2j+1}^2$ and $v_{j+1}^3 v_{2j-1}^4$ for j = 1, 3, ..., 2k - 1. The graph M is depicted in the figure composed of thin edges in Figure 4. By the construction of M, we have $|V(M)| = \sum_{i=1}^4 |P_i| = (2k-1) + (4k-1) + 2k + (4k-3) = 12k - 5$. Let $A_1 = \{v_j^1 : j = 3, 5, ..., 2k - 3\}, A_2 = \{v_j^2 : j = 3, 5, ..., 4k - 3\}, A_3 = \{v_j^3 : j = 2, 3, ..., 2k - 1\}$ and $A_4 = \{v_j^4 : j = 5, 9, ..., 4k - 7\}$. Clearly, $|A_1| = |A_4| = k - 2$ and $|A_2| = |A_3| = 2k - 2$. Notice that $V_3(M) = \bigcup_{i=1}^4 A_i$, so $|V_3(M)| = \sum_{i=1}^4 |A_i| = 6k - 8$. To construct a $\{C_6, C_7\}$ -free planar graph on n vertices, we consider the following two cases based on the value of n.

Case 1. n = 44k - 12. In the first step, we construct a plane graph G_1 , which satisfies that each face is an 8-face and the degree of each vertex is 2 or 3. Let G_1 be the graph obtained from M by adding edges $v_{2k-1}^1 v_{4k-4}^4$ and $v_{\frac{j+5}{2}}^1 v_j^4$ for $j = 3, 7, \ldots, 4k - 9$; then adding a new vertex v and joining v and v_1^i for i = 1, 4. The graph G_1 is depicted in Figure 4.

By the construction of G_1 , we have $g(G_1) = 8$ and $|V(G_1)| = |V(M)| + 1 = 12k - 4$. By Handshaking Theorem and each face of G_1 is an 8-face, we have $8 |F(G_1)| = 2 |E(G_1)|$, i.e., $|F(G_1)| = \frac{1}{4} |E(G_1)|$. By Euler formula, we have $|V(G_1)| - 2 = |E(G_1)| - |F(G_1)| = \frac{3}{4} |E(G_1)|$ and therefore $|E(G_1)| = \frac{4}{3}(|V(G_1)| - 2) = 16k - 8$. Let $B_1 = V(P_1) \setminus (A_1 \cup \{v_2^1\})$ and $B_2 = \{v_j^4 : j = 3, 7, \ldots, 4k - 9\} \cup \{v_1^4, v_{4k-4}^4\}$. Clearly, $|B_1| = |B_2| = k$. By the construction of G_1 , we have $V_3(G_1) = V_3(M) \cup B_1 \cup B_2$, so $|V_3(G_1)| = |V_3(M)| + |B_1| + |B_2| = 8k - 8$. Since $V(G_1) = V_2(G_1) \cup V_3(G_1)$, we observe that $|V_2(G_1)| = |V(G_1)| - |V_3(G_1)| = (12k - 4) - (8k - 8) = 4k + 4$.



Figure 4. The graph G_1 in Case 1.

In the second step, we construct a plane graph G_2 based on G_1 . Let G_2 be the plane graph obtained from G_1 by adding a new vertex to each edge of G_1 , and joining any two new vertices added to two adjacent edges of G_1 . The

graph G_2 is depicted in Figure 5. By the construction of G_2 , we have $|V(G_2)| = |V(G_1)| + |E(G_1)| = 28k - 12$; the degree of each new vertex of G_2 is at least 4, so $V_i(G_2) = V_i(G_1)$ for i = 2, 3 and $|V_2(G_2) \cup V_3(G_2)| = |V(G_1)| = 12k - 4$; we have $G_2[N_{G_2}[y]] \cong K_{i+1}$ for any $y \in V_i(G_2), i = 2, 3$; for any two adjacent vertices u and v of G_1 , we see that the graphs $G_2[N_{G_2}[u]]$ and $G_2[N_{G_2}[v]]$ are edge-disjoint and their common vertices belong to $V(G_2) \setminus V(G_1)$.



Figure 5. The graph G_2 in Case 1.

In the third step, we construct a plane graph G_3 based on G_2 . For any $z \in V_2(G_2) \cup V_3(G_2)$, let $N_{G_2}(z) = \{z^{(1)}, z^{(2)}, \dots, z^{(d_{G_2}(z))}\}$. For any $x \in V_2(G_2)$ and $y \in V_3(G_2)$, let G_3 be the plane graph obtained from G_2 by doing the following. (1) for any $x \in V_2(G_2)$, adding two adjacent vertices $x^{(3)}$ and $x^{(4)}$ to the interior of $(x, x^{(1)}, x^{(2)})$ -face, adding edges $xx^{(3)}, x^{(i)}x^{(3)}$ and $x^{(i)}x^{(4)}$ for i = 1, 2; (2) for any $y \in V_3(G_2)$, adding a vertex $y^{(4)}$ to the interior of $(y, y^{(2)}, y^{(3)})$ -face and adding edges $yy^{(4)}$ and $y^{(i)}y^{(4)}$ for i = 2, 3. For some $x_0 \in V_2(G_2)$ and $y_0 \in V_3(G_2), G_3[x_0, x_0^{(1)}, x_0^{(2)}, x_0^{(3)}, x_0^{(4)}]$ and $G_3[y_0, y_0^{(1)}, y_0^{(2)}, y_0^{(3)}, y_0^{(4)}]$ are depicted in Figure 6(a) and Figure 6(b), respectively. By the construction of G_3 , for any $z \in V_2(G_2) \cup V_3(G_2)$, we have $G_3[z, z^{(1)}, z^{(2)}, z^{(3)}, z^{(4)}] \cong K_5^-$. So G_3 is composed of the union of K_5^- s, where these K_5^- s are edge-disjoint and their common vertices (if have) belong to $V(G_2) \setminus V(G_1)$. Since $|E(K_5^-)| = 9$, we have $|V(G_3)| = |V(G_2)| + 2|V_2(G_2)| + |V_3(G_2)| = (28k - 12) + 2(4k + 4) + (8k - 8) = 44k - 12 = n$ and $|E(G_3)| = 9|V_2(G_2) \cup V_3(G_2)| = 108k - 36$. Notice that n = 44k - 12, we have $|E(G_3)| = 108k - 36 = \frac{27}{11}n - \frac{72}{11}$.

Now we prove that G_3 is a $\{C_6, C_7\}$ -free plane graph. Notice that except the 3-faces in K_5^- , the degree of all other faces of G_3 is more than 7. For any $e \in E(G_3)$, the length of any cycle containing e is more than 7, unless all edges of the cycle belong to one K_5^- . Clearly, K_5^- is $\{C_6, C_7\}$ -free. So G_3 is a $\{C_6, C_7\}$ -free plane graph on n vertices and $|E(G_3)| = \frac{27}{11}n - \frac{72}{11}$ for n = 44k - 12.



Figure 6. The operations on G_2 in the third step.

Case 2. n = 44k + 10. In the first step, we construct a plane graph G_1 , which satisfies that each face is an 8-face and the degree of each vertex is 2 or 3. Let G_1 be the graph obtained from M by adding the path $v_1^0 v_2^0 \dots v_7^0$; then adding edges $v_1^0 v_1^1$, $v_3^0 v_1^3$, $v_6^0 v_2^1$, $v_7^0 v_1^4$, $v_{2k-1}^1 v_{4k-4}^4$ and $v_{\frac{j+5}{2}}^1 v_j^4$ for $j = 3, 7, \dots, 4k-9$. The graph G_1 is depicted in Figure 7.

By the construction of G_1 , we have $g(G_1) = 8$ and $|V(G_1)| = |V(M)| + 7 = 12k + 2$. By Handshaking Theorem and each face of G_1 is an 8-face, we have $8 |F(G_1)| = 2 |E(G_1)|$, i.e., $|F(G_1)| = \frac{1}{4} |E(G_1)|$. By Euler formula, we have $|V(G_1)| - 2 = |E(G_1)| - |F(G_1)| = \frac{3}{4} |E(G_1)|$ and therefore $|E(G_1)| = \frac{4}{3}(|V(G_1)| - 2) = 16k$. Let $C_1 = \{v_3^0, v_6^0, v_1^3\}$, $C_2 = V(P_1) \setminus A_1$ and $C_3 = \{v_j^4 : j = 3, 7, \dots, 4k - 9\} \cup \{v_1^4, v_{4k-4}^4\}$. Clearly, $|C_1| = 3, |C_2| = k + 1$ and $|C_3| = k$. By the construction of G_1 , we have $V_3(G_1) = \bigcup_{i=1}^3 C_i \cup V_3(M)$, so $|V_3(G_1)| = \sum_{i=1}^3 |C_i| + |V_3(M)| = 8k - 4$. Since $V(G_1) = V_2(G_1) \cup V_3(G_1)$, we observe that $|V_2(G_1)| = |V(G_1)| - |V_3(G_1)| = (12k + 2) - (8k - 4) = 4k + 6$.



Figure 7. The graph G_1 in Case 2.

The second and third steps are the same as in Case 1, the resulting plane

graph obtained through the second and third steps are denoted by G_2 and G_3 , respectively. By the construction of G_2 , we have $|V(G_2)| = |V(G_1)| + |E(G_1)| = 28k + 2$; the degree of each new vertex of G_2 is at least 4, so $V_i(G_2) = V_i(G_1)$ for i = 2, 3 and $|V_2(G_2) \cup V_3(G_2)| = |V(G_1)| = 12k + 2$; we have $G_2[N_{G_2}[y]] \cong K_{i+1}$ for any $y \in V_i(G_2)$, i = 2, 3; for any two adjacent vertices u and v of G_1 , we see that the graphs $G_2[N_{G_2}[u]]$ and $G_2[N_{G_2}[v]]$ are edge-disjoint and their common vertices belong to $V(G_2) \setminus V(G_1)$. By the construction of G_3 , for any $z \in V_2(G_2) \cup V_3(G_2)$, we have $G_3[z, z^{(1)}, z^{(2)}, z^{(3)}, z^{(4)}] \cong K_5^-$. So G_3 is composed of the union of K_5^- s, where these K_5^- s are edge-disjoint and their common vertices (if have) belong to $V(G_2) \setminus V(G_1)$. Since $|E(K_5^-)| = 9$, we have $|V(G_3)| = |V(G_2)| + 2|V_2(G_2)| + |V_3(G_2)| = (28k + 2) + 2(4k + 6) + (8k - 4) = 44k + 10 = n$ and $|E(G_3)| = 9|V_2(G_2) \cup V_3(G_2)| = 108k + 18$. Notice that n = 44k + 10, we have $|E(G_3)| = 108k + 18 = \frac{27}{11}n - \frac{72}{11}$. By the same analysis as in Case 1, we know that G_3 is a $\{C_6, C_7\}$ -free plane graph on n vertices and $|E(G_3)| = 108k + 18 = \frac{27}{11}n - \frac{72}{11}$ for n = 44k + 10.

In summary, for any integer k, we have $ex_p(n, \{C_6, C_7\}) \geq \frac{27}{11}n - \frac{72}{11}$ for n = 22k + 10. By Theorem 4, we have $ex_p(n, \{C_6, C_7\}) = \frac{27}{11}n - \frac{72}{11}$ for $n \equiv 10 \pmod{22}$, as desired.

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