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ON A 3-COLORING OF PLANE GRAPHS WITHOUT MONOCHROMATIC FACIAL 3-PATHS

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Abstract

A facial path in a plane graph G is a subpath of the boundary walk of a face of G. The Four Color Theorem states that every plane graph contains a proper vertex 4-coloring in which each monochromatic path consists of exactly one vertex. Czap, Fabrici, and Jendroľ in 2021 conjectured that every plane graph G admits an improper vertex 3-coloring in which every monochromatic facial path in G has at most two vertices. In this paper we prove this conjecture. Our result is optimal.

Keywords: plane graph, facial path, vertex-coloring.

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1. Introduction and Notations

All graphs considered in this paper are connected plane graphs. They can contain loops and multiple edges. We use the standard graph theory terminology according to [3]. However, the most frequent notions of the paper are defined through it. Let G be a connected plane graph with a set of vertices $V(G)$, a set of edges $E(G)$, and a set of faces $F(G)$. The *boundary* of a face f is the collection of all the edges and the vertices contained in the closure of f that can be organized into a closed walk in G traversing a simple closed curve lying just inside the face f. This closed walk is called *boundary walk* of the face f. The *degree* of a face f is the length of its boundary walk. A *k-face* (k^+-face, respectively, k^--face) is a face of degree k (at least k, respectively, at most k).

A *facial path* on a face f is any path that is a consecutive part of the boundary walk of a face the face f.

Two edges of a plane graph G are *facially adjacent* if they are adjacent and consecutive on the boundary walk of a face.

Two faces of a plane graph G are *adjacent* if they share an edge and are *touched* if they have a common vertex but are not adjacent.

2. 3-Colorings

A *vertex k-coloring* (or simply a k-coloring) of a graph G is a mapping $\varphi : V(G) \to \{1, \ldots, k\}$. A graph is *$k$-colorable* if it has a k-coloring. Unless otherwise stated, the colorings in this paper are not necessarily proper.

We concentrate on the question of what can be achieved with three colors for plane graphs. We begin our short survey with a proper 3-coloring of plane graphs.

Since 1973, by Stockmeyer [18], we know that the planar 3-colorability is NP-complete.

Grötzsch's theorem [13] from 1959 states that all planar graphs without 3-cycles are properly 3-colorable. Motivated by this result, Steinberg in 1976 conjectured that any planar graph with no cycles of length 4 or 5 is 3-colorable; see [17]. For many years, this conjecture had been considered to be one of the major open problems in the coloring of planar graphs. The conjecture has been attracting substantial attention among graph theoretists; see [4]. Erdős in 1991 [17] suggested to determine the smallest k such that every planar graph with no cycle of lengths $4, \ldots, k$ is 3-colorable. Borodin *et al.* [6] in 2005, improving on [1] and [16], have shown that Erdős' $k \leq 7$.

In 2017 Cohen-Addad *et al.* [9] showed that Steinberg's conjecture is false.

Many other relaxations of the conjecture have been established. We refer the reader for further results on conjecture and other open problems related to the coloring of planar graphs to [4, 5], and [11].

Interesting results have been proved for improper 3-colorings of plane graphs.

Poh [15] in 1990 and, independently, Goddard [12] in 1991 proved that every planar graph can be colored with at most three colors so that each of its monochromatic components is a path.

Can these monochromatic paths be short? The answer is no. Chartrand, Geller, and Hedetniemi [8] proved in 1968 that for every positive integer t there exists a 4-chromatic plane triangulation G_t such that any its 3-coloring involves a monochromatic path of length t.

Broersma *et al.* [7] proved in 2006 that it is NP-hard to decide whether a planar graph has a 3-coloring without any monochromatic path P_n, $n \geq 3$, a path on n vertices.

Every planar graph has a proper 4-coloring [2] which equivalently means that every such graph has a 4-coloring in which any monochromatic facial path consists of exactly one vertex.

A connected plane graph G with $\delta(G) \geq 3$ and $\delta^*(G) \geq 3$ is called the *normal plane map*. Here $\delta(G)$, respectively $\delta^*(G)$, denotes the minimum vertex respectively face, degree of G.

Czap, Fabrici, and Jendroľ in 2021 published the following.

Theorem 1 [10]**.** *Every normal plane map admits a* 3*-coloring without monochromatic facial* 4*-paths.*

Conjecture 2 [10]**.** *Every normal plane map admits a* 3*-coloring without monochromatic facial* 3*-paths.*

In this paper, we prove Conjecture 2. We have the following.

Theorem 3. *Every plane graph admits a* 3*-coloring in which any monochromatic facial path has at most two vertices. Moreover, the bound* 2 *is optimal for* 4*-chromatic plane graphs.*

Observe that our result is in contrast with the result referred to above of Chartrand, Geller, and Hedetniemi (1968), see [8].

3. BASIC PROPERTIES OF A COUNTEREXAMPLE TO THEOREM 3

Let G be a counterexample to Theorem 3 with the minimum number of vertices and then with the minimum number of edges.

To obtain a contradiction, we are looking for a 3-coloring of G, called a *required* 3*-coloring*, which does not admit a monochromatic facial 3-path a facial path on three vertices. First, we describe the basic properties of G.

For a cycle C in a plane graph G we denote the subgraph of G induced by the vertices and edges of G lying inside C and outside C by $\mathrm{int}_G(C)$ and $\mathrm{ext}_G(C)$, respectively. We say that C is a *separating* cycle if both $\mathrm{int}_G(C)$ and $\mathrm{ext}_G(C)$ are not empty.

It is easy to prove the following.

Claim 4. *G does not contain* 1*-vertices,* 2*-vertices,* 1*-faces,* 2*-faces, and separating* 1*-cycles.*

Claim 5. *G does not contain any pair of adjacent* 3*-faces.*

Claim 6. *G contains no separating* 2*-cycle C with* $|V(C \cup \mathrm{int}_G(C))| \leq \frac{|V(G)|+7}{6}$.

Proof. Assume that G contains such a 2-cycle $C = [u, v]$. Let $B = C \cup \mathrm{int}_G(C)$ and let B have the smallest cardinality among all the separating 2-cycles C of G.

Consider a complete plane graph on four vertices K_4. Replace each edge xy of K_4 with the subgraph B identifying x and y with u and v, respectively. The resulting plane graph H is a normal plane map with $|V(H)| = 6|V(B)| - 8$ vertices. If $|V(H)| = 6|V(B)| - 8 \leq |V(G)| - 1$, that is, if $|V(B)| \leq \frac{|V(G)|+7}{6}$, the graph H has a required 3-coloring $\lambda(H)$ in which there is a colored copy of B with $\lambda(u) = \lambda(v)$ and another colored copy of B with $\lambda(u) \neq \lambda(v)$.

Next, we remove $\mathrm{int}_G(C)$ from G. The resulting graph $G^* = G \setminus \mathrm{int}_G(C)$ is not a counterexample, so it has a required 3-coloring $\varphi(G^*)$. If $\varphi(u)$ and $\varphi(v)$ are colors of u and v in $\varphi(G^*)$, we color the vertices in B by $\lambda(B)$, so that we get $\lambda(u) = \varphi(u)$ and $\lambda(v) = \varphi(v)$. If we insert such a colored B into $\lambda(G^*)$, we get the required 3-coloring of G. A contradiction. ■

Claim 7. *G does not contain any pair of adjacent 3-vertices.*

Proof. Suppose, to contrary, that G contains a pair of adjacent 3-vertices u and v. Let x and y be the other two neighbors of u. Let w and z be the other two neighbors of v chosen so that the vertices x, u, v, and w are incident to the same face. We distinguish two cases.

Case 1. Let uv be incident to a 3-face $f = [x, u, v]$ (i.e., the vertices x and w coincide). By Claim 5, each face adjacent to f is a 4^+-face. Contract the edge uy to the vertex y and the edge vz to the vertex z. We obtain a new edge yz if there is no other edge yz that forms a new 2-face. In the latter case, we remove one. The received graph G^* is smaller than G and therefore has a required 3-coloring $\lambda(G^*)$. This coloring provides a full required 3-coloring $\varphi(G)$ as follows $\varphi(t) = \lambda(t)$ for any $t \in V(G) \setminus \{u, v\}$, $\varphi(u) \in \{1,2,3\} \setminus \{\lambda(y), \lambda(x)\}$ and $\varphi(v) \in \{1,2,3\} \setminus \{\lambda(z), \lambda(w)\}$.

Case 2. Let both faces incident to uv be 4^+-faces. Now we contract the edge uv into a new vertex u^*. The received graph G^* is smaller than G and therefore has a required 3-coloring $\lambda(G^*)$. This coloring provides a full required 3-coloring $\varphi(G)$ as follows $\varphi(t) = \lambda(t)$ for any $t \in V(G) \setminus \{u, v\}$, $\varphi(u) \in \{1,2,3\} \setminus \{\lambda(y), \lambda(x)\}$, and $\varphi(v) \in \{1,2,3\} \setminus \{\lambda(w), \lambda(z)\}$. ■

A *k-vertex* is any vertex of degree k. A k-vertex v, $k \geq 3$, is said to be a $(a_1, a_2, \ldots, a_k)$-vertex if the faces $f_1, f_2, \ldots, f_k$, incident to v have degrees $a_1, a_2, \ldots, a_k$, respectively.

Claim 8. *G does not contain any $(3,4,4)$-vertex,*

Proof. Let u be a $(3,4,4)$-vertex. Let $G^* = G - u$. As G^* is smaller than G, it has a required 3-coloring $\lambda(G^*)$. Using this coloring, we can obtain the complete required 3-coloring $\varphi(G)$ of G as follows. We put $\varphi(t) = \lambda(t)$ for any vertex $t \in V(G) \setminus u$. For $\varphi(v)$ we choose a color that appears (in the coloring $\lambda(G^*)$) at most once on the vertices of the 5-face that arose when the vertex u was removed. A contradiction. ■

Claim 9. *G does not contain any $(4,4,4)$-vertex.*

Proof. Let u be a $(4,4,4)$-vertex. Now the 3-vertex u is incident to three 4-faces $f_1 = [v,u,x,s]$, $f_2 = [v,u,z,w]$, and $f_3 = [z,u,x,y]$. Observe that at least one pair of the vertices w and x or s and z does not belong to the same face of G. Let, without loss of generality, w and x not belong to the same face. Let G^* be the graph obtained from G by removing the vertex u followed by the identification of the vertices x and w in the vertex x. The received graph G^* is smaller than G and has a required 3-coloring $\lambda(G^*)$. It induces a partial required 3-coloring $\varphi(G)$ of G. Using it, we can get a required 3-coloring $\varphi(G)$ as follows $\varphi(t) = \lambda(t)$ for any $t \in V(G) \setminus \{u,w\}$. $\varphi(w) = \varphi(x)$, and for $\varphi(u)$ we choose a color that appears at most once in the color set $\{\varphi(s), \varphi(v), \varphi(y), \varphi(z)\}$. A contradiction ■

Claim 10. *G does not contain any 3-vertex.*

Proof. By Claims 8 and 9, assume that G contains a $(3^+,4^+,5^+)$-vertex u. Let the faces incident to u be a 3-face $[u,w,\dots,z]$, a 4^+-face $[u,w,r,\dots,x,v]$, and a 5^+-face $[u,z,p,q,\dots,v]$. Without loss of generality, the vertices x and z are not connected by any facial 3-path.

Case 1. If z is not joined to v by a facial 3-path (i.e., the path of length 2), we contract the edge vu to the vertex u followed by contracting the edge uz to the vertex z. The new smaller graph G^* so obtained has a required 3-coloring $\lambda(G^*)$. Using this coloring, we obtain the complete required 3-coloring $\varphi(G)$ of G as follows. We put $\varphi(t) = \lambda(t)$ for any vertex $t \in V(G) \setminus \{u,v\}$ and $\varphi(v) = \lambda(z)$. For u we choose a color that does not appear (in the coloring $\lambda(G^*)$) at the vertices w and z. A contradiction.

Case 2. If z is connected to v by a facial 3-path, then neither the vertices w and p nor the vertices r and q share a common face. In this case, the vertex u is removed from G followed by identifying the edges wr and pq through the new face obtained to the new edge w^*r^*. The resulting graph G^* is smaller, so it has a required 3-coloring $\lambda(G^*)$. This coloring induces a partial required 3-coloring $\varphi(G)$ in which $\varphi(t) = \lambda(t)$ for all $t \in V(G) \setminus \{w,r,p,q\}$, $\varphi(w) = \varphi(p) = \lambda(w^*)$, and $\varphi(r) = \varphi(q) = \lambda(r^*)$. Now we can see that there is a color for the vertex u to complete the coloring $\varphi(G)$. A contradiction. ■

Claim 11. *G does not contain any $(3,4,3,4^+)$-vertex.*

Proof. We distinguish two basic cases.

Case 1. Let u be such a vertex in G. Let it be incident to two 3-faces $[v,u,w]$ and $[x,u,y]$ and to two 4-faces $[v,u,y,z]$ and $[w,u,x,s]$.

Observe that the pair of vertices v and y or the pair of vertices w and x is not contained in a separating 4-cycle via u. Without loss of generality, let the pair v and x not be contained in such a cycle. Remove the vertex u from G and through the new face obtained identify the vertices v and x in a new vertex u^*. The graph G^* obtained is smaller than G and has a required 3-coloring $\lambda(G^*)$. It induces a required 3-coloring $\varphi(G)$ as follows $\varphi(t) = \lambda(t)$ for any $t \in V(G) \setminus \{u, v, x\}$, $\varphi(v) = \varphi(x) = \lambda(u^*)$. It is easy to see that there is a suitable color for $\varphi(u)$. A contradiction.

Case 2. Assume that a vertex u in G is incident to two 3-faces $[u, v, w]$ and $[u, x, y]$, to a 4-face $[u, v, z, y]$, and to a 5^+-face $[u, w, r, \ldots, s, x]$. Similarly to the proof of Case 1 we can assume, without loss of generality, that there is no facial 3-path between vertices v and x. We distinguish two cases.

Case 2.1. Let there be no facial 3-path between the vertices w and s omitting the vertex r. Remove the vertex u from G and, through the face newly obtained, identify the vertices v and x to a new vertex v^* and the vertices w and s to a new vertex w^* to obtain a new edge v^*w^*. As the graph G^* newly obtained is smaller than G, it has a required 3-coloring $\lambda(G^*)$. It induces a partial required 3-coloring $\varphi(G)$ in which all the vertices of G are colored up to u; namely $\lambda(t) = \varphi(t)$ for any $t \in V(G) \setminus \{u, v, w, x, s\}$, $\varphi(v) = \varphi(x) = \lambda(v^*)$ and $\varphi(w) = \varphi(s) = \lambda(w^*)$. It is easy to see that there is a color for $\varphi(u)$ to get the full required 3-coloring $\varphi(G)$. A contradiction.

Case 2.2. Let there be a facial 3-path between the vertices w and s omitting the vertex r. Observe that now there is no facial 3-path between v and r and between r and x which does not pass s. Remove u from G and then identify the vertices v, x, and r in a new vertex u^*. As the graph G^* newly obtained is smaller than G, it has a required 3-coloring $\lambda(G^*)$. It induces a required partial 3-coloring $\varphi(G)$ in which all the vertices of G are colored up to u; namely $\varphi(t) = \lambda(t)$ for any $t \in V(G) \setminus \{u, v, r, x\}$, $\varphi(v) = \varphi(x) = \varphi(r) = \lambda(u^*)$. It is easy to see that there is a color for $\varphi(u)$ to get the full required 3-coloring $\varphi(G)$. A contradiction. ■

Claim 12. *G does not contain a $(3, 4, 4, 4)$-vertex.*

Proof. Assume that G contains such a 4-vertex u. Let u be incident to a 3-face $[u, v, y]$ and to three 4-faces $[u, v, r, w]$, $[u, w, s, x]$, and $[u, x, z, y]$. Then, without loss of generality, we can suppose that the vertices w and z are not joined by a facial 3-path. We distinguish two cases.

Case 1. Let the vertices v and z not be joined by a facial 3-path.

Case 1.1. If the vertices r and x are also not joined by a facial 3-path, then the vertex u is removed from G followed by identification the edges vr and zx through newly obtained face to the new edge v^*r^*. The resulting graph G^* is smaller, so it has a required 3-coloring λG^*. This coloring induces a partial required 3-coloring $\varphi(G)$ in which $\varphi(t) = \lambda(t)$ for all $t \in V(G) \setminus \{r, v, x, z\}$, $\varphi(v) = \varphi(z) = \lambda(v^*)$, and $\varphi(r) = \varphi(x) = \lambda(r^*)$. Now we can see that we have a color for the vertex u to complete the coloring $\varphi(G)$. A contradiction.

Case 1.2. If the vertices r and x are joined by a facial 3-path, the vertex u is removed from G followed by identification the edges yz and ws through the face newly obtained to the new edge y^*z^*. The resulting graph G^* is smaller, so it has a required 3-coloring λG^*. This coloring induces a partial required 3-coloring $\varphi(G)$ in which $\varphi(t) = \lambda(t)$ for all $t \in V(G) \setminus \{y, z, w, s\}$, $\varphi(y) = \varphi(w) = \lambda(y^*)$, and $\varphi(z) = \varphi(s) = \lambda(z^*)$. Now we can see that we have a color for the vertex u to complete the coloring $\varphi(G)$. A contradiction.

Case 2. Let the vertices v and z be joined by a facial 3-path. As the vertices w and z are not joined by a facial 3-path and the vertices y and r are not in a common face we remove the vertex u from G followed by identifying the edges yz and rw through the face newly obtained to the new edge y^*z^*. The resulting graph G^* is smaller, so it has a required 3-coloring $\lambda(G^*)$. This coloring induces a partial required 3-coloring $\varphi(G)$ in which $\varphi(t) = \lambda(t)$ for all $t \in V(G) \setminus \{y, z, w, r\}$, $\varphi(y) = \varphi(r) = \lambda(y^*)$, and $\varphi(z) = \varphi(w) = \lambda(z^*)$. Now we can see that we have a color for the vertex u to complete the coloring $\varphi(G)$. A contradiction. ■

Claim 13. *G does not contain any $(3, 5^+, 3, 5^+)$-vertex.*

Proof. In contrast, let there be a $(3, 5^+, 3, 5^+)$-vertex u in G. Let it be incident to two 3-faces $[u, v, w]$, $[u, x, y]$, to a 5^+-face $[u, v, p, \ldots, q, y]$, and to a 5^+-face $[u, x, s, \ldots, r, w]$. We distinguish two cases.

Case 1. Let there be neither a facial 3-path between the vertices v and y nor between the vertices w and x omitting the vertex u. Remove the vertex u from G and, through the face newly obtained, identify the edges vw and yx to a new edge v^*w^*. As the graph G^* newly obtained is smaller than G, it has a required 3-coloring $\lambda(G^*)$. It induces a partial required 3-coloring $\varphi(G)$ in which all the vertices of G are colored up to u; namely $\lambda(t) = \varphi(t)$ for any $t \in V(G) \setminus \{u, v, w, x, y\}$, $\varphi(v) = \varphi(y) = \lambda(v^*)$ and $\varphi(w) = \varphi(x) = \lambda(w^*)$. It is easy to see that there is a color for $\varphi(u)$ to get the full required 3-coloring $\varphi(G)$. A contradiction.

Case 2. Without loss of generality, let there be no facial 3-path between the vertices v and y and there be a facial 3-path between w and x. Observe that now there is no facial 3-path between r and v and between r and y. Remove u from G first and then identify the vertices v, y, and r to a new vertex v^*. As the graph G^* newly obtained is smaller than G, it has a required 3-coloring $\lambda(G^*)$. It induces a required partial 3-coloring $\varphi(G)$ in which all the vertices of

G are colored up to u; namely $\varphi(t) = \lambda(t)$ for any $t \in V(G) \setminus \{u, v, r, y\}$ and $\varphi(v) = \varphi(y) = \varphi(r) = \lambda(v^*)$. It is easy to see that there is a color for $\varphi(u)$ to get the full required 3-coloring $\varphi(G)$. A contradiction.

Case 3. Let there be a facial 3-path between the vertices v and y and also a facial 3-path between the vertices w and x, both avoiding the vertex u. Observe that now neither the vertices w and p nor the vertices s and y are joined by a facial 3-path. Remove the vertex u from G and identify above the pair w and p, respectively s and y, to new vertices w^*, respectively y^*, through the newly obtained face. The resulting smaller graph G^* has a required 3-coloring $\lambda(G^*)$ which yields to a partial required 3-coloring $\varphi(G)$ in which all the vertices of G are colored up to u; namely $\varphi(t) = \lambda(t)$ for any $t \in V(G) \setminus \{u, w, p, s, y\}$, $\varphi(w) = \varphi(p) = \lambda(w^*)$ and $\varphi(s) = \varphi(y) = \lambda(y^*)$. It is easy to see that there is a color for $\varphi(u)$ to get the full required 3-coloring $\varphi(G)$. A contradiction. ■

4. Proof of the Main Result

By Claim 4, we see that G is normal plane map. The basic information on normal plane maps is provided by the classical Lebesgue theorem [14]. From it and Claims 5, 10, 11, 12, and 13 we have for G the following.

Claim 14. *G is a normal plane map of $\delta^*(G) = 4$ without adjacent 3-faces and contains some $(3, 4, a, b)$-vertices for $(a, b) \in \{(4, 5), (5, 4)\}$.*

This claim says that in G there are lot of necessary 4-vertices of that type as no other types of necessary k-vertices, $3 \leq k \leq 5$, according to Lebesgue's theorem, are present in G. Next we investigate structures of them in G.

Claim 15. *Every $(3, 4, 5, 4)$-vertex and every $(3, 4, 4, 5)$-vertex in G is contained in a separating 4-cycle.*

Proof. We distinguish two cases.

Case 1. Let G contain a $(3, 4, 5, 4)$-vertex u which is not contained in any separating 4-cycle. Let it be incident to a 3-face $[u, v, y]$, two 4-faces $[u, v, r, w]$ and $[u, y, z, x]$, and a 5-face $[u, w, p, q, x]$. Remove the vertex u from G and through the face newly obtained contract the edge vy in v^* and identify the vertices w and x in a new vertex w^*. The obtained smaller graph G^* has a required 3-coloring $\lambda(G^*)$ which yields a partial required 3-coloring $\varphi(G)$ in which all the vertices of G are colored up to u; namely $\varphi(t) = \lambda(t)$ for any $t \in V(G) \setminus \{u, v, w, x, y\}$, $\varphi(w) = \varphi(x) = \lambda(w^*)$ and $\varphi(v) = \varphi(y) = \lambda(v^*)$. It is easy to see that there is a color for $\varphi(u)$ to get the full required 3-coloring $\varphi(G)$. A contradiction.

Case 2. Let G contain a $(3, 4, 4, 5)$-vertex u which is not contained in any separating 4-cycle. Let it be incident to a 3-face $[u, v, y]$, two 4-faces $[u, v, r, w]$ and

$[u, w, p, x]$, and a 5-face $[u, x, s, z, y]$. Remove the vertex u from G and through the face newly obtained identify the vertices x, y and r in new vertex x^*. The obtained smaller graph G^* has a required 3-coloring $\lambda(G^*)$ which yields to a partial required 3-coloring $\varphi(G)$ in which all the vertices of G are colored up to u; namely $\varphi(t) = \lambda(t)$ for any $t \in V(G) \setminus \{u, r, x, y\}$, $\varphi(x) = \varphi(y) = \varphi(r) = \lambda(x^*)$. It is easy to see that there is a color for $\varphi(u)$ to get the full required 3-coloring $\varphi(G)$. A contradiction. ■

Observe that there can be several facial 3-paths between v and y (and, of course, between w and x) omitting u. These facial 3-paths together with the facial 3-path vuy (wux) form separating 4-cycles. We are interested in such a separating 4-cycle $C^* = uvmy$ (or $C^* = uwnx$) for which the subgraph $B^* = C^* \cup \mathrm{int}_G(C^*)$ does not contain as a proper subgraph any other subgraph $B' = C' \cup \mathrm{int}_G(C')$ for a separating 4-cycle $C' = uvoy$ (or $C' = uw\bar{o}x$) through the edges uv and uy (or uw and ux). This separating 4-cycle C^* is called the *light-separating* 4-*cycle* for the triple $(u; \{v, y\})$ (for the triple $(u; \{w, x\})$).

Note that the subpath vuy appears in one or two light-separating 4-cycles. In the later case, these two separating cycles have disjoint interiors. Observe, that there can be other light-separating 4-cycles through u that use another pairs of edges incident to u.

From now on, let C^* denote the *lightest*-separating 4-cycle, that is, the light-separating 4-cycle with the minimum number of vertices and then with the minimum number of edges in the subgraph $B^* = C^* \cup \mathrm{int}_G(C^*)$ among all light-separating 4-cycles on all 4-vertices of G. Without loss of generality, let $C^* = uvmy$.

Now we are interested in what is inside $B = B^*$ of C^*. To find out this, consider it. Observe that $\deg_B(v) \geq 3$, $\deg_B(y) \geq 3$, $\deg_B(m) \geq 2$, and $2 \leq \deg_G(u) \leq 3$. We distinguish two cases.

Case 1. Let $\deg_B(u) = 2$. Suppress the vertex u. Denote by D the resulting graph. Construct an auxiliary graph H as follows. The construction starts with a plane graph W of a 6-sided double-wheel W, all faces of which are 3-faces $[v, y, m]$ with $\deg_W(m) = 6$, $\deg_W(v) = \deg_W(y) = 4$. We insert the graph D into each 3-face of W so that the boundary cycle vmy of each face of W is identified to the outer cycle vmy of D. Observe that the normal plane map H obtained has the following properties : $\deg_H(m) \geq 6$, $\deg_H(v) \geq 8$, $\deg_H(y) \geq 8$, and any vertex that could appear in H by Lebesgue's theorem [14] has to be present inside a copy of B^* in H, and therefore in G. Applying Claims 5, 10, and Claim 14 on H it follows that $\delta^*(H) = 4$. Observe that H does not contain $(3, 4, 3, 4^+)$-vertices (by Claim 11), $(3, 4, 4, 4)$-vertices (by Claim 12), and $(3, 5^+, 3, 5^+)$-vertices (by Claim 13). This means that H must contain $(3, 4, 5, 4)$-vertices and/or $(3, 4, 4, 5)$-vertices. All these vertices are present inside C^* of G by the statement of Claim

14. The graph B contains inside the $(3,4,5,4)$-vertices and/or the $(3,4,4,5)$-vertices. A light-separating 4-cycle $\bar{C}$ through such a 4-vertex inside B in G bounds (according to Claim 15) less vertices than the lightest-separating 4-cycle C^* in the interior of C^*. A contradiction.

Case 2. Let $\deg_B(u) \geq 3$. To learn a structure of B we construct an auxiliary graph H. Consider the subgraph B. Observe that $\deg_B(v) \geq 3$, $\deg_B(y) \geq 3$, $3 \deg_B(u) \leq 4$, and $\deg_B(m) \geq 2$.

The construction of H starts with a normal plane map $A = A_6$, the dual plane graph to the 6-sided anti-prism all faces of which are 4-faces $[u, v, m, y]$ with $\deg_A(u) = \deg_A(v) = \deg_Q(y) = 3$ and $\deg_A(m) = 6$. We insert in each 4-face of A the subgraph B so that the boundary cycle $uvmy$ of it is identified to the boundary cycle $uvmy$ of B. Observe that the normal plane map H so obtained has the following properties: Each of the vertices u, v, m, and y has in H the degree at least 6.

To get a contradiction in this case, we proceed in the rest of this proof in the same way as in Case 1 above.

This finishes the proof of Theorem 3.

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