Discussiones Mathematicae
Graph Theory xx (xxxx) 1–10
https://doi.org/10.7151/dmgt.2583

THE PROBABILISTIC UPPER BOUNDS ON THE ISOLATION NUMBER OF A GRAPH

Minhui Li [1], Shumin Zhang [1,2,*]

and

Chengfu Ye [1,2]

[1] *School of Mathematics and Statistics*
Qinghai Normal University, Xining, Qinghai 810008, China
[2] *Academy of Plateau Science and Sustainability,*
People's Government of Qinghai Province and Beijing Normal University, China

e-mail: liminhuimath@163.com
zhangshumin@qhnu.edu.cn
yechf@qhnu.edu.cn

Abstract

Let $G = (V, E)$ be a graph. A subset $S \subseteq V$ is *an isolating set* of G if the graph induced by the set $V \backslash N[S]$ contains no edge. The size of a smallest isolating set of G is called *the isolation number*, denoted by $\iota(G)$. In this paper, we obtain the upper bounds on $\iota(G)$ via probabilistic method, and improve the previous bound on $\iota(G)$ given by Caro and Hansberg.

Keywords: domination, isolating set, isolation number.

2020 Mathematics Subject Classification: 05C05, 05C12, 05C76.

1. Introduction

In this article, we consider a simple undirected graph G with the *vertex set* $V(G)$ and *edge set* $E(G)$. For any vertex $v \in V(G)$, the *open neighborhood* of v, denoted by $N_G(v)$ (or simply $N(v)$), is defined to be $\{u \in V(G) : uv \in E(G)\}$, and the *closed neighborhood* of v, denoted by $N_G[v]$ (or simply $N[v]$), is the set $N_G(v) \cup \{v\}$. The *degree* of a vertex v is $|N(v)|$, denoted by $d_G(v)$. The *maximum*

[*] Corresponding author.

degree and *minimum degree* of G are denote by $\Delta(G)$ and $\delta(G)$, respectively. A vertex v is called *an isolated vertex* of G if $d_G(v) = 0$.

For any subset $S \subseteq V(G)$, the *open neighborhood* of S, denoted by $N_G(S)$ (or simply $N(S)$), is defined to be $\bigcup_{v \in S} N(v)$, and the *closed neighborhood* of S, denoted by $N_G[S]$ (or simply $N[S]$), is the set $N(S) \cup S$. For any subset $S \subseteq V(G)$, let $G[S]$ denote the subgraph of G induced by S, and $V(G) \backslash S$ denote the subset removing all elements of S from $V(G)$. A subset $S \subseteq V(G)$ is a *dominating set* of G if every vertex in G is either in S or adjacent to a vertex in S. The minimum cardinality of a dominating set of G is called the *domination number* of G, denoted by $\gamma(G)$.

The study of domination in graphs has experienced rapid development since its introduction, owing to its wide range of applications in various research areas. Much of the research relating to domination can be found in [9, 10, 11]. Since the decision problems related to the domination number is NP-complete, it is meaningful to determine good upper bounds on domination number, where some basic bounds on the domination number are given in terms of the order and minimum degree of a graph. However, the tight bound on the domination number of a graph is not yet known for $\delta \geq 4$, but when δ is sufficiently large, probability method can be used to find the asymptotically optimal bound of the domination number. The bound of the following result gets increasingly tight as δ increases.

Theorem 1 [1]. *If G is a graph of order n with minimum degree $\delta \geq 1$, then*

$$\gamma(G) \leq \frac{\ln(\delta+1)+1}{\delta+1} n.$$

Rad [15] provided the following new upper bound on the domination number and improved on previous bounds by applying the Alon-Spencer procedure while removing unnecessary vertices from a dominating set.

Theorem 2 [15]. *If G is a graph on n vertices with minimum degree $\delta > 1$ and maximum degree Δ, then for any integer $k \geq 1$,*

$$\gamma(G) \leq \frac{n}{\delta+1}\left[\ln(\delta+1)+1-(\delta-\ln(1+\delta))\sum_{i=1}^{k}\left(\frac{\ln(1+\delta)}{1+\delta}\right)^{i(1+\Delta)}\right].$$

In recent years, many variants of domination have attracted a great deal of attention and research. Among them, we introduce a specific variant called isolation proposed by Caro and Hansberg [7] in 2017.

A subset $S \subseteq V(G)$ is an *isolating set* of G if the graph induced by the set $V(G) \backslash N[S]$ contains no edge. The minimum cardinality of an isolating set of G is called the *isolation number* of G, denoted by $\iota(G)$. We refer an isolating set of the cardinality $\iota(G)$ as an $\iota(G)$-set. For two subsets $A, B \subseteq V(G)$, we say that A *isolates* B if A is an isolating set of $G[B]$. Currently, the isolation related problems of graphs have garnered a lot research attention and yielded numerous results. In particular, Borg and his collaborators have conducted in-depth studies on the properties of certain isolation parameters (see [2, 3, 4, 5, 6]), more related results could be seen [8, 12, 16, 17, 18, 19].

Despite the some research on the isolation number, more refined upper bounds related to the minimum degree of a graph remain unknown, except for the following result. Theorem 3 established the upper bound of isolation number by the probabilistic method outlined in Theorem 1.

Theorem 3 [7]**.** *For any graph G of order n with minimum degree $\delta \geq 1$,*

$$\iota(G) \leq \frac{\ln(\delta+1)+\frac{1}{2}}{\delta+1}\,n.$$

In this paper, we obtain two upper bounds on $\iota(G)$ based on the fundamental probabilistic methods in [15] and [7], along with some additional ideas.

2. Main Results

In this section, we present two theorems concerning the upper bounds of the isolation number. First, we obtain a new upper bound of the isolation number in Theorem 6 by following the method in [7]. The following results are required to prove Theorem 6.

Lemma 4 [13]**.** *If G is an isolate-free graph of order n, then $\gamma(G) \leq \frac{1}{2}n$.*

Obviously, there is the following result from Lemma 4.

Lemma 5. *If G is a graph of order n, then there exists an isolating set D of G with $|D| \leq \frac{1}{2}n$ such that D dominates all isolated vertices of G.*

Proof. Let I be the isolated vertex set of G and $G_1, \ldots, G_t$ be the connected components of G with $|V(G_i)| \geq 2$, where $i = 1, \ldots, t$. By Lemma 4, there is a dominating set D_i of G_i such that $|D_i| \leq \frac{1}{2}|V(G_i)|$. Assume $D = D_1 \cup \cdots \cup D_t$. Then D is a dominating set of $G[V(G)\backslash I]$, that is, D dominates all isolated vertices of G. Clearly, D is an isolating set of G. Moreover, $|D| = \sum_{i=1}^{t} |D_i| \leq \sum_{i=1}^{t} \frac{1}{2}|V(G_i)| \leq \frac{1}{2}n$. (Note that t may not exist, that is, $V(G) = I$ and $D = \emptyset$.) ■

Theorem 6. *Let G be an n-vertex graph with minimum degree $\delta \geq 1$. Then*

$$\iota(G) \leq \frac{\ln(1+\delta)-\ln 2+1}{1+\delta}\,n.$$

Proof. Let A be a set of vertices in which each vertex $v \in A$ is independently and uniformly selected from $V(G)$ with probability p, where $p \in [0,1]$. Then the expected value of $|A|$ is $\mathbb{E}(|A|) = np$. Let I be the set of isolated vertices in $G[V(G)\backslash A]$ and $B = V(G)\backslash(N[A] \cup I)$. Consequently,

$$Pr(v \in B) = Pr(v \in V(G)\backslash N[A]) = (1-p)^{1+d_G(v)} \leq (1-p)^{1+\delta},$$

the first equation holds because $I \subseteq N(A)$. By Lemma 5, there is an isolating set D of $G[B]$ such that $|D| \leq \frac{1}{2}|B|$. It is not difficult to see that $A \cup D$ is an isolating set of G. By linearity of expectation,

$$\mathbb{E}(|A \cup D|) = \mathbb{E}(|A|) + \mathbb{E}(|D|) \leq \mathbb{E}(|A|) + \frac{1}{2}\mathbb{E}(|B|) \leq np + \frac{1}{2}(1-p)^{1+\delta}n.$$

By using the inequality $1 - p \leq e^{-p}$ for $p \in [0,1]$, we have

$$\mathbb{E}(|A \cup D|) \leq \left(p + \frac{1}{2}e^{-p(1+\delta)}\right)n.$$

Furthermore, since the function $f(x) = x + \frac{1}{2}e^{-x(1+\delta)}$ attains its minimum when $x = \frac{\ln(1+\delta)-\ln 2}{1+\delta}$, we take $p = \frac{\ln(1+\delta)-\ln 2}{1+\delta}$. Thus, $\mathbb{E}(|A \cup D|) \leq \frac{\ln(1+\delta)-\ln 2+1}{1+\delta}n$. Hence $\iota(G) \leq |A \cup D| \leq \frac{\ln(1+\delta)-\ln 2+1}{1+\delta}n$. ∎

Now, we give another upper bound on the isolation number of a graph, which constitutes the main result of this section.

Theorem 7. *Let G be an n-vertex graph with minimum degree $\delta > 1$ and maximum degree Δ. Then for every integer $k \geq 1$,*

$$\iota(G) \leq \frac{n}{1+\delta}\Bigg[\ln(1+\delta) - \ln 2 + 1 - (\delta - \ln(1+\delta) + \ln 2)\sum_{i=1}^{k}\left(\frac{\ln(1+\delta)-\ln 2}{1+\delta}\right)^{i(1+\Delta)}\Bigg].$$

The bound of Theorem 7 will be derived by applying the following lemmas.

Lemma 8. *Let G be an n-vertex graph with minimum degree $\delta > 1$ and maximum degree Δ. Let A be a subset of $V(G)$ in which each vertex $v \in A$ is independently chosen from $V(G)$ with probability p, where $p \in (0,1)$. And let $A' = \{v \in V(G) : N[v] \subseteq A\}$ and $A'' = \{v \in V(G) : N[v] \subseteq A'\}$. For every integer $x \geq 1$, there is a subset $S \subseteq A'$ such that S isolates A'' and $|S| \leq f(x-1)|A'|$, where*

$$f(0) = p + \frac{1}{2}(1-p)^{1+\delta} \text{ and } f(j) = f(0) - (1 - f(0))\sum_{i=1}^{j} p^{i(1+\Delta)}$$

for any integer $j \geq 1$.

Proof. We proceed by induction on $x \geq 1$. For $x = 1$, we show that there is a subset $S \subseteq A'$ such that S isolates A'' and $|S| \leq f(0)|A'| = (p + \frac{1}{2}(1-p)^{1+\delta})|A'|$. Let $A_1 \subseteq A'$ and $v \in A_1$ be independently chosen from A' with probability p, and let $B_1 \subseteq (A'' \backslash N[A_1])$ be the set of all vertices in A'' that are not isolated by A_1. Then by Lemma 5, one can find a set $I_1 \subseteq B_1$ to isolate B_1 with $|I_1| \leq \frac{1}{2}|B_1|$. Assume $S_1 = A_1 \cup I_1$. Then it is easy to see that S_1 isolates A''. Clearly, the expected value of $|A_1|$ is $\mathbb{E}(|A_1|) = |A'|p$. Furthermore, we have

$$Pr(v \in B_1) = Pr(v \in A'' \backslash N[A_1]) = (1-p)^{1+d_{G[A']}(v)} = (1-p)^{1+d_G(v)} \leq (1-p)^{1+\delta},$$

where the second equation holds because $d_{G[A']}(v) = d_G(v)$ for any vertex $v \in A''$. Thus $\mathbb{E}(|B_1|) \leq |A'|(1-p)^{1+\delta}$.

So by linearity of expectation,

$$\begin{aligned} \mathbb{E}(|S_1|) = \mathbb{E}(|A_1 \cup I_1|) &= \mathbb{E}(|A_1|) + \frac{1}{2}\mathbb{E}(|B_1|) \leq |A'|p + \frac{1}{2}|A'|(1-p)^{1+\delta} \\ &= (p + \frac{1}{2}(1-p)^{1+\delta})|A'| = f(0)|A'|. \end{aligned} \tag{1}$$

Hence, when $x = 1$, there is a subset $S \subseteq A'$ such that S isolates A'' and $|S| \leq f(0)|A'|$.

Assume that the result holds for all positive integers x' with $x' \leq x$ by the inductive hypothesis. It suffices to prove that the result holds for $x + 1$. Let $A_1 \subseteq A'$ and $v \in A_1$ be is independently chosen from A' with probability p, and let $B_1 \subseteq (A'' \backslash N[A_1])$ be the set of all vertices in A'' that are not isolated by A_1 and $I_1 \subseteq B_1$ isolate B_1 with $|I_1| \leq \frac{1}{2}|B_1|$. Let $A_1' = \{v \in A_1 : N_{G[A']}[v] \subseteq A_1\}$, and $A_1'' = \{v \in A_1 : N_{G[A']}[v] \subseteq A_1'\}$. Since $G[A']$ is a graph with minimum degree $\delta > 1$, we apply the inductive hypothesis to the graph $G[A']$. Thus, there is a subset $S_x \subseteq A_1'$ such that S_x isolates A_1'' and $|S_x| \leq f(x-1)|A_1'|$. Assume $S_{x+1} = (A_1 \backslash A_1') \cup S_s \cup I_1$. Then it is easy to see that S_{x+1} isolates A'', see Figure 1.

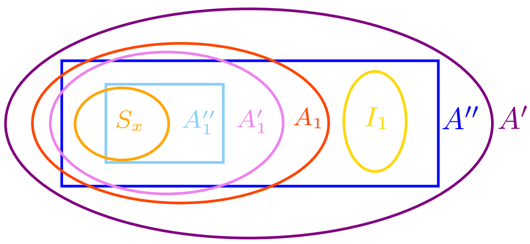

Figure 1. The illustration of Lemma 8.

Moreover, we have

$$\mathbb{E}(|A_1|) = |A'|p, \ \mathbb{E}(|B_1|) \leq |A'|(1-p)^{1+\delta} \text{ and } \mathbb{E}(|A_1'|) \geq |A'|p^{1+\Delta}.$$

Thus by linearity of expectation,

$$
\begin{aligned}
\mathbb{E}\left(|S_{x+1}|\right) &= \mathbb{E}\left(|(A_1\backslash A_1')\cup S_x\cup I_1|\right)\\
&= \mathbb{E}\left(|A_1|\right) - \mathbb{E}\left(|A_1'|\right) + \mathbb{E}\left(|S_x|\right) + \frac{1}{2}\,\mathbb{E}\left(|B_1|\right)\\
&\leq |A'|p - \mathbb{E}\left(|A_1'|\right) + f(x-1)\,\mathbb{E}\left(|A_1'|\right) + \frac{1}{2}|A'|(1-p)^{1+\delta}\\
&= |A'|p + \frac{1}{2}|A'|(1-p)^{1+\delta} - (1-f(x-1))\,\mathbb{E}\left(|A_1'|\right)\\
&\leq |A'|p + \frac{1}{2}|A'|(1-p)^{1+\delta} - (1-f(x-1))|A'|p^{1+\Delta}.
\end{aligned} \tag{2}
$$

In addition, according to the definition of f,

$$
\begin{aligned}
(1-f(x-1))p^{1+\Delta} = \Bigg[&1-p-\frac{1}{2}(1-p)^{1+\delta}\\
&+\left(1-p-\frac{1}{2}(1-p)^{1+\delta}\right)\sum_{i=1}^{x-1}p^{i(1+\Delta)}\Bigg]p^{1+\Delta}\\
=&\left(1-p-\frac{1}{2}(1-p)^{1+\delta}\right)\sum_{i=1}^{x}p^{i(1+\Delta)}.
\end{aligned} \tag{3}
$$

So

$$
\begin{aligned}
\mathbb{E}\left(|S_{x+1}|\right) &\leq |A'|p + \frac{1}{2}|A'|(1-p)^{1+\delta} - (1-f(x-1))|A'|p^{1+\Delta}\\
&= |A'|p + \frac{1}{2}|A'|(1-p)^{1+\delta} - |A'|\left(1-p-\frac{1}{2}(1-p)^{1+\delta}\right)\sum_{i=1}^{x}p^{i(1+\Delta)}\\
&= |A'|\left[p+\frac{1}{2}(1-p)^{1+\delta} - \left(1-p-\frac{1}{2}(1-p)^{1+\delta}\right)\sum_{i=1}^{x}p^{i(1+\Delta)}\right]\\
&= f(x)|A'|.
\end{aligned} \tag{4}
$$

Hence, there is a subset $S \subseteq A'$ such that S isolates A'' and $|S| \leq f(x)|A'|$. ∎

By the inequality $1-p \leq e^{-p}$ for $0 \leq p \leq 1$, the following consequence follows.

Lemma 9. *Let G be an n-vertex graph with minimum degree $\delta > 1$ and maximum degree Δ. Let A be a subset of $V(G)$ in which each vertex $v \in A$ is independently chosen from $V(G)$ with probability p, where $p \in (0,1)$. And let $A' = \{v \in V(G) : N[v] \subseteq A\}$ and $A'' = \{v \in V(G) : N[v] \subseteq A'\}$. For every integer $x \geq 1$, there is a subset $S \subseteq A'$ such that S isolates A'' and $|S| \leq f(x-1)|A'|$, where*

$$f(0) = p + \frac{1}{2}e^{-p(1+\delta)}$$

and

$$f(j) = p + \frac{1}{2}e^{-p(1+\delta)} - \left(1 - p - \frac{1}{2}e^{-p(1+\delta)}\right)\sum_{i=1}^{j} p^{i(1+\Delta)},$$

for any integer $j \geq 1$.

Next we apply Lemma 9 to prove Theorem 7.

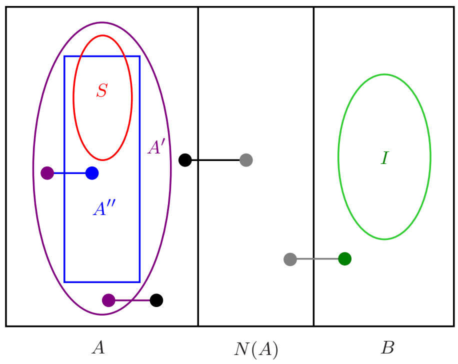

Figure 2. The illustration of Theorem 7.

Proof of Theorem 7. Let $B = V(G)\backslash N[A]$, $A' = \{v \in V(G) : N[v] \subseteq A\}$ and $A'' = \{v \in V(G) : N[v] \subseteq A'\}$. Note that $A'' \subseteq A' \subseteq A$, $d_{G[A'](v)} = d_G(v)$ for any vertex $v \in A''$. Furthermore, by the definition of A', A'', every vertex of $A'\backslash A''$ has at least one neighbor in $A\backslash A'$. Thus $A\backslash A'$ dominates $A'\backslash A''$. By Lemma 9, we know that there is a subset $S \subseteq A'$ such that S isolates A'' and $|S| \leq f(k-1)|A'|$, where

$$f(0) = p + \frac{1}{2}e^{-p(1+\delta)}$$

and

$$f(j) = p + \frac{1}{2}e^{-p(1+\delta)} - \left(1 - p - \frac{1}{2}e^{-p(1+\delta)}\right)\sum_{i=1}^{j} p^{i(1+\Delta)},$$

for any integer $j \geq 1$.

Now let I be a minimum isolating set of B, then $|I| \leq \frac{1}{2}|B|$ by Lemma 5. It is easy to see that $(A\backslash A') \cup I \cup S$ is an isolating set of G, see Figure 2.

$$\begin{aligned} |(A\backslash A') \cup I \cup S| &= |A\backslash A'| + |I| + |S| \leq |A| - |A'| + \frac{1}{2}|B| + |S| \\ &\leq |A| + \frac{1}{2}|B| - |A'| + f(k-1)|A'| \\ &= |A| + \frac{1}{2}|B| - (1 - f(k-1))|A'|. \end{aligned}$$

By the linearity of expectation, we get

$$\mathbb{E}\left(|(A\backslash A')\cup D\cup S|\right)\leq \mathbb{E}\left(|A|\right)+\frac{1}{2}\mathbb{E}\left(|B|\right)-(1-f(k-1))\mathbb{E}\left(|A'|\right).$$

In addition,

$$\mathbb{E}\left(|A|\right)=np,\ \mathbb{E}\left(|B|\right)=n(1-p)^{1+\delta} \text{ and } \mathbb{E}\left(|A'|\right)\geq np^{1+\Delta}.$$

So,

$$\begin{aligned}\mathbb{E}\left(|(A\backslash A')\cup D\cup S|\right)&\leq \mathbb{E}\left(|A|\right)+\frac{1}{2}\mathbb{E}\left(|B|\right)-(1-f(k-1))\mathbb{E}\left(|A'|\right)\\&\leq np+\frac{1}{2}n(1-p)^{1+\delta}-n(1-f(k-1))p^{1+\Delta}\\&\leq np+\frac{1}{2}n(1-p)^{1+\delta}-n\left(1-p-\frac{1}{2}(1-p)^{1+\delta}\right)\sum_{i=1}^{k}p^{i(1+\Delta)}\\&\leq np+\frac{1}{2}ne^{-p(1+\delta)}-n\left(1-p-\frac{1}{2}e^{-p(1+\delta)}\right)\sum_{i=1}^{k}p^{i(1+\Delta)}.\end{aligned}$$

We take $p=\frac{\ln(1+\delta)-\ln 2}{1+\delta}$, then

$$\begin{aligned}&\mathbb{E}\left(|(A\backslash A')\cup D\cup S|\right)\\&\leq np+\frac{1}{2}ne^{-p(1+\delta)}-n\left(1-p-\frac{1}{2}e^{-p(1+\delta)}\right)\sum_{i=1}^{k}p^{i(1+\Delta)}\\&\leq \frac{n}{1+\delta}\left[\ln\left(\frac{1+\delta}{2}\right)+1-\left(\delta-\ln\left(\frac{1+\delta}{2}\right)\right)\sum_{i=1}^{k}\left(\frac{\ln(1+\delta)-\ln 2}{1+\delta}\right)^{i(1+\Delta)}\right].\end{aligned}$$

Hence,

$$\begin{aligned}\iota(G)\leq \frac{n}{1+\delta}\Bigg[&\ln(1+\delta)-\ln 2+1-(\delta-\ln(1+\delta)+\ln 2)\\&\sum_{i=1}^{k}\left(\frac{\ln(1+\delta)-\ln 2}{1+\delta}\right)^{i(1+\Delta)}\Bigg].\end{aligned}$$

■

It is worth noting that the isolation of a graph is closely related to the vertex-edge domination proposed by Peters in [14]. The vertex-edge domination problem is a variant of domination on "mixing" vertices and edges. We say that a vertex v *ve-dominates* each edge incident to a vertex in $N[v]$. A subset $S\subseteq V(G)$ is a *vertex-edge dominating set*, abbreviated *ve-dominating set*, if there exists a vertex $v\in S$ such that v *ve*-dominates e for each edge $e\in E(G)$.

The minimum cardinality of a ve-dominating set of G is called the *vertex-edge domination number*, abbreviated *ve-domination number*, denoted by $\gamma_{ve}(G)$. It is easy to know that a set S is an $\iota(G)$-set of G if and only if S is a $\gamma_{ve}(G)$-set. Thus, the relevant problems of ve-domination can be solved by studying the isolation problem.

3. CONCLUSION

We provide two new upper bounds on the isolation number of a graph. It is clear that the bounds in Theorem 6 and Theorem 7 are better than the bound given in Theorem 3, since $1 - \ln 2 < \frac{1}{2}$. In addition, the bound in Theorem 6 is better than the bound given in Theorem 7 for all $\delta \geq 2$, since $\delta - \ln(\frac{\delta+1}{2}) > 0$.

Acknowledgments

The authors are grateful to the editors and referees for their valuable comments and suggestions, which have helped to improve the presented paper. This work is supported by the National Science Foundation of China (Nos. 12261074, 12461065).

REFERENCES

[1] N. Alon, J.H. Spencer, The Probabilistic Method (John Wiley & Sons, Inc., Chichester, 2000).
https://doi.org/10.1002/0471722154

[2] P. Borg, *Isolation of connected graphs*, Discrete Appl. Math. **339** (2023) 154–165.
https://doi.org/10.1016/j.dam.2023.05.004

[3] P. Borg, *Isolation of cycles*, Graphs Combin. **36** (2020) 631–637.
https://doi.org/10.1007/s00373-020-02143-2

[4] P. Borg and P. Kaemawichanurat, *Partial domination of maximal outerplanar graphs*, Discrete Appl. Math. **283** (2020) 306–314.
https://doi.org/10.1016/j.dam.2020.01.005

[5] P. Borg, K. Fenech and P. Kaemawichanurat, *Isolation of k-cliques* II, Discrete Math. **345(7)** (2022) 112641.
https://doi.org/10.1016/j.disc.2021.112641

[6] P. Borg and P. Kaemawichanurat, *Extensions of the Art Gallery Theorem*, Ann. Comb. **27** (2023) 31–50.
https://doi.org/10.1007/s00026-022-00620-4

[7] Y. Caro and A. Hansberg, *Partial domination–the isolation number of a graph*, Filomat **31** (2017) 3925–3944.
https://doi.org/10.2298/FIL1712925C

[8] J. Chen and S-J. Xu, *P_5-isolation in graphs*, Discrete Appl. Math. **340** (2023) 331–349.
https://doi.org/10.1016/j.dam.2023.07.018

[9] T.W. Haynes, S.T. Hedetniemi and P.J. Slater, Fundamentals of Domination in Graphs (Marcel Dekker, New York, 1998).
https://doi.org/10.1201/9781482246582

[10] T.W. Haynes, S.T. Hedetniemi and M.A. Henning, Structures of Domination in Graphs, Dev. Math. **66** (Springer, Cham, 2021).
https://doi.org/10.1007/978-3-030-58892-2

[11] T.W. Haynes, S.T. Hedetniemi and M.A. Henning, Domination in Graphs: Core Concepts, Springer Monogr. Math. (Springer, Cham, 2023).
https://doi.org/10.1007/978-3-031-09496-5

[12] P. Kaemawichanurat and O. Favaron, *Partial domination and irredundance numbers in graphs*, Appl. Math. Comput. **457** (2023) 128153.
https://doi.org/10.1016/j.amc.2023.128153

[13] O. Ore, Theory of Graphs, in: Amer. Math. Soc. Colloq. Publ. **38** (Providence, RI, 1962) 206–212.
https://doi.org/10.1090/coll/038

[14] J.W. Peters, Theoretical and Algorithmic Results on Domination and Connectivity, PhD Thesis (Clemson University, Clemson, 1986).

[15] N.J. Rad, *New probabilistic upper bounds on the domination number of a graph*, Electron. J. Combin. **26(3)** (2019) #P3.28.
https://doi.org/10.37236/8345

[16] S. Tokunaga, T. Jiarasuksakun and P. Kaemawichanurat, *Isolation number of maximal outerplanar graphs*, Discrete Appl. Math. **267** (2019) 215-218.
https://doi.org/10.1016/j.dam.2019.06.011

[17] J. Yan, *Isolation of the Diamond Graph*, Bull. Malays. Math. Sci. Soc. **45** (2022) 1169–1181.
https://doi.org/10.1007/s40840-022-01248-6

[18] Y. Yin, X. An and B. Wu, *$K_{1,2}$-isolation number of claw-free cubic graphs*, Bull. Malays. Math. Sci. Soc. **47** (2024) 75.
https://doi.org/10.1007/s40840-024-01672-w

[19] G. Zhang and B. Wu, *A Note on the cycle isolation number of graphs*, Bull. Malays. Math. Sci. Soc. **47** (2024) 57.
https://doi.org/10.1007/s40840-024-01655-x

Received 18 November 2024
Revised 24 March 2025
Accepted 24 March 2025
Available online 3 April 2025

This article is distributed under the terms of the Creative Commons Attribution-NonCommercial-NoDerivatives 4.0 International License https://creativecommons.org/licenses/by-nc-nd/4.0/