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BOUNDS ON COLORING TREES WITHOUT RAINBOW PATHS

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Abstract

For a graph with colored vertices, a rainbow subgraph is one where all vertices have different colors. For graph G, let $c_k(G)$ denote the maximum number of different colors in a coloring without a rainbow path on k vertices, and $cp_k(G)$ the maximum number of colors if the coloring is required to be proper. The parameter c_3 has been studied by multiple authors. We investigate these parameters for trees and $k \geq 4$. We first calculate them when G is a path, and determine when the optimal coloring is unique. Then for trees T of order n, we show that the minimum value of $c_4(T)$ and $cp_4(T)$ is $(n+2)/2$, and the trees with the minimum value of $cp_4T)$ are the coronas. Further, the minimum value of $c_5(T)$ and $cp_5(T)$ is $(n+3)/2$, and the trees with the minimum value of either parameter are octopuses.

Keywords: no-rainbow, path, tree, corona.

2020 Mathematics Subject Classification: 05C15.

1. Introduction

We consider undirected graphs where the vertices receive colors. We define a subgraph as *rainbow* if all its vertices receive different colors, and we study colorings where for some fixed graph H there is no rainbow subgraph isomorphic to H. This question was first studied for the path on three vertices by Bujtás *et al.* [4] and for stars in general by Bujtás *et al.* [3], and then studied in [6, 7, 8] inter alia. There has also been work on the case where H must be induced [1]. The problem is also a special case of more general questions introduced and studied earlier by Voloshin [10]. (The edge-coloring version is much more studied, where it is called anti-Ramsey theory.)

Our focus here is on the case that H is a path. And specifically on the maximum number of colors one can use on a graph and there not be a rainbow path. For graph G, let $c_k(G)$ denote the maximum number of different colors one can use without there being a rainbow P_k (meaning a path with k vertices), where the coloring is not required to be proper. Let $cp_k(G)$ denote the maximum number of colors with the additional constraint that adjacent vertices receive different colors; that is, it is a proper coloring. Note that $cp_k(G)$ might not exist; for example, it does not exist for the complete graph K_n where $n \geq k$.

As noted above, the function $c_3(G)$ has already been studied. See for example [4, 6, 7] and the references therein. The equivalent $cp_3(G)$ is uninteresting: the only way a P_3 can be properly colored without being rainbow is that the first and third vertex have the same color; so such a coloring of G exists only when G is bipartite. The parameter $c_4(G)$ is also briefly studied in [8].

We proceed as follows. In Section 2 we consider the colorings of paths and determine when the extremal colorings are unique. In Section 3 we show that the minimum value of $c_4(T)$ and $cp_4(T)$ for trees of order n is $(n+2)/2$, and the trees with the minimum value of $cp_4(T)$ are precisely the coronas. In Section 4 we show that the minimum value of $c_5(T)$ and $cp_5(T)$ for trees of order n is $(n+3)/2$, and the trees with the minimum value of either parameter are octopuses. In Section 5 we conclude with brief thoughts for future research.

2. Colorings Paths Without Rainbow Paths

We begin with the calculation of the parameters for paths and determining when the optimal coloring is unique. There has been previous work. Observation 19 of [9] considered the problem of coloring the path such that there is no specified rainbow subpath and no three consecutive vertices receive the same color; since the optimal coloring without a rainbow path clearly does not have three consecutive vertices of the same color, the formula given there applies to $c_k(P_n)$. The same formula can also be read out of the results on mixed interval hypergraphs in [5]. For proper colorings, the special case of $cp_4(P_n)$ was resolved in [8]. Nevertheless, we include proofs of all formulas, since this proof enables us to determine when the extremal coloring is unique.

For a graph G and vertex w, we define the operation of *attaching a* P_m as adding a copy of P_m and joining one end of the P_m to w. See Figure 1 for an example. We will need the following lemma, which is possibly interesting in its own right.

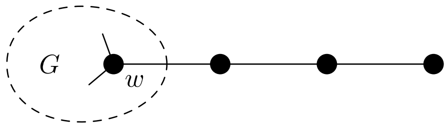

Figure 1. Attaching a P_3.

Lemma 1. (a) *Assume $k \geq 2$. For any graph G and vertex w, if graph G_1 is obtained from G by attaching P_{k-1} to w, then $c_k(G_1) = c_k(G) + k - 2$.*

(b) *Assume $k \geq 3$. For any graph G and* end-vertex *w (that is, a vertex of degree 1), if graph G_2 is obtained from G by attaching P_{k-2} to w, then $cp_k(G_2) = cp_k(G) + k - 3$.*

Proof. (a) Let X denote the attached P_{k-1}. It cannot happen that every vertex in X gets its own unique color, since that creates a rainbow P_k with w. On the other hand, any coloring of G is extendable to G_1 by giving the first vertex of X the same color as w and giving the remaining $k-2$ vertices of X each their own unique color. This proves the formula.

(b) Let Y denote the attached P_{k-2}. It cannot happen that every vertex in Y gets its own unique color, since that creates a rainbow P_k with w and w's neighbor in G (which necessarily have different colors). On the other hand, any coloring of G is extendable to G_2 by giving the first vertex of Y the same color as w's neighbor in G, and giving the remaining $k-3$ vertices of Y each their own unique color. This proves the formula. ∎

Note that part (b) of Lemma 1 does not generalize to all w: for example, if $G = K_3$ then $cp_4(G) = 3$ but $cp_4(G_2) = 3$ too. (It is of course true that $cp_k(G_2) \leq cp_k(G) + k - 3$ for all w.) Lemma 1 enables us to prove the following.

Theorem 2. *Let $k \geq 2$.*

(a) *For $n \geq 1$ it holds that $c_k(P_n) = \lfloor (k-2)n/(k-1) \rfloor + 1$.*

(b) *For $n \geq k-1$ the optimal coloring is unique exactly when n is a multiple of $k-1$.*

Proof. (a) For fixed k we prove the bound by induction on n. For the base case, note that the formula gives n for $n \leq k-1$, which is correct. For $n \geq k$, the above lemma implies that $c_k(P_n) = c_k(P_{n-k+1}) + k - 2$, and so the formula follows from the inductive hypothesis. For example, for P_{11} without rainbow P_5, an optimal coloring uses nine colors such as

`12344567789`.

(b) Uniqueness also follows by induction. The base case is the range $k-1 \leq n \leq 2k-3$. For $n = k-1$ the optimal coloring is rainbow. Otherwise the optimal coloring has all vertices but two having their own color. Such a coloring is valid if and only if the two vertices with the same color are contained in positions $n-k+1$ through k, and thus the coloring is not unique. So assume $n \geq 2k-2$.

To have equality in the bound, by the recurrence it follows that the coloring of P_{n-k+1} must be optimal. On the other hand, any coloring of P_{n-k+1} is extendable to P_n by duplicating the color of the last vertex of P_{n-k+1} and then giving the remaining $k-2$ vertices of X each their own color. Hence for the optimal coloring of P_n to be unique, so must the optimal coloring of P_{n-k+1} and thus by the induction hypothesis the divisibility condition is necessary. To show that the condition is also sufficient, note that when the coloring of P_{n-k+1} is unique, its last $k-1$ vertices have different colors. So the only way to get $k-2$ colors on X is as described before: the first vertex of X must have the same color as the last vertex of the P_{n-k+1}. It follows that the optimal coloring of P_n is unique. ■

Theorem 3. *Let $k \geq 3$.*

(a) *For $n \geq 2$ it holds that $cp_k(P_n) = \lfloor((k-3)n+1)/(k-2)\rfloor + 1$.*

(b) *For $n \geq k-1$ the optimal coloring is unique exactly when n is 1 more than a multiple of $k-2$.*

Proof. (a) For fixed k we prove the bound by induction on n. For the base case, note that the formula gives n for $n \leq k-1$, which is correct. For $n \geq k$, the above lemma implies that $cp_k(P_n) = cp_k(P_{n-k+2}) + k - 3$, and so the formula follows from the induction hypothesis. For example, for P_{11} without rainbow P_5, the optimal proper coloring uses eight colors such as

`12343565787.`

(b) Uniqueness also follows by induction. The base case is the range $k-1 \leq n \leq 2k-4$. For $n = k-1$ the optimal coloring is rainbow. Otherwise the optimal coloring has all vertices but two having their own color. Such a coloring is valid if and only if the two vertices with the same color are contained in positions $n-k+1$ through k, and thus the coloring is not unique. So assume $n \geq 2k-3$.

To have equality in the bound, by the recurrence it follows that the coloring of P_{n-k+2} must be optimal. On the other hand, any coloring of P_{n-k+2} is extendable to P_n by duplicating the color of the penultimate vertex and then giving the remaining $k-3$ vertices of Y each their own color. Hence for the coloring of P_n to be unique, so must the coloring of P_{n-k+2}, and so by the induction hypothesis the divisibility condition is necessary. To show that the condition is also sufficient, note that when the coloring of P_{n-k+2} is unique, its last $k-1$ vertices have different colors. So the only way to get $k-3$ colors on Y is as described before: the first vertex of Y must have the same color as the penultimate vertex of the P_{n-k+2}. It follows that the optimal coloring of P_n is unique. ■

One can also derive the formula for $c_k(P_n)$ using the fact that, in trees, the parameter is intimately related to the minimum number of edges one must remove to destroy all copies of P_k. For H a fixed graph, define a set F of edges in a graph G as *H-thwarting* if removing all of F from the graph G destroys all copies of H. The H-thwarting number, $\theta_H(G)$, is the minimum number of edges whose removal destroys all copies of H. In a coloring, we call an edge *monochromatic* if its two ends have the same color. Note that if the monochromatic edges form a H-thwarting set, then every H contains a monochromatic edge and hence the coloring is valid, that is, has no rainbow H. Hence in general graphs G there is the inequality $c_H(G) \geq n - \theta_H(G)$, where $c_H(G)$ denotes the maximum number of colors in a coloring of G without a rainbow copy of H. Theorem 16 in the paper [7] showed that in trees there is equality.

Theorem 4 [7]**.** *In any tree T of order n, it holds that $c_H(T) = n - \theta_H(T)$.*

3. Coloring Trees without a Rainbow P_4

A general lower bound for bipartite graphs in the P_4 case was obtained in [8]. Namely, it was observed that in a bipartite graph with bipartition (X, Y), if one gives each vertex in X its own unique color and gives all the vertices in Y the same color, the result is a valid proper coloring: every copy of P_4 has two vertices from Y. Since the bigger partite set has at least half the vertices, it follows that.

Theorem 5 [8]**.** *For any connected bipartite graph G on $n \geq 2$ vertices it holds that*

$$c_4(G) \geq cp_4(G) \geq \lceil n/2 \rceil + 1.$$

It was also noted in [8] that, if a graph G has a perfect matching, then $cp_4(G) \leq n/2 + 1$. The value for paths given in Theorem 3 is thus recovered.

3.1. Trees with extremal c_4

Perhaps surprisingly, the paths do not have the minimum value of $c_4(T)$ for a given order. Indeed we show that the trees T with the minimum value of $c_4(T)$ are precisely the coronas. The *corona* of a graph is defined by taking the graph, and for each vertex w adding one new vertex, called a *foot*, adjacent only to w. (This doubles the number of vertices.) The original graph is called the *core*. Figure 2 gives an example where the core is a tree (and the notched edges form a minimum P_4-thwarting set).

We will need the following lemma.

Lemma 6. *If T is a tree with end-vertex y', then there exists a minimum P_4-thwarting set that does not contain the edge incident with y'.*

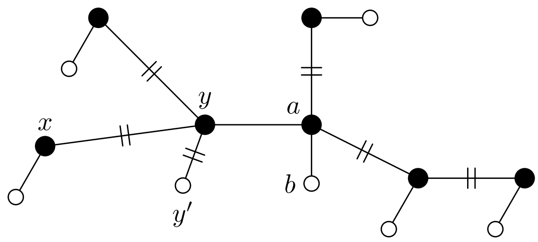

Figure 2. A corona and a minimum P_4-thwarting set.

Proof. Let F be a minimum thwarting set. Say y' has neighbor y. Assume the edge yy' is in F. The minimality of F means that if we add yy' to $T - F$, there will be a P_4, say $y'yab$, where only edge $y'y$ is in F. It follows that every other edge incident with y must be in F, else we have a P_4 starting bay. So one can change F by replacing $y'y$ by ya and still have a minimum thwarting set of T. ■

Theorem 7. *For a corona H derived from core tree B, it holds that $c_4(H) = n/2 + 1$, where n is the order of H.*

Proof. It suffices to show that $\theta_{P_4}(H) = n/2-1$. Such a thwarting set is achieved by taking all the edges in the core. It remains to show that there is no smaller thwarting set.

We prove that $\theta_{P_4} \geq n/2 - 1$ by induction on the number of vertices of the core graph. The base case of the induction is that B has one vertex. That corresponds to H being K_2; this has $\theta_{P_4} = 0$. So assume B has at least two vertices.

Consider a vertex x that is an end-vertex in B. Say its neighbor in B is y. By Lemma 6, there is a minimum thwarting set F of H that does not contain the edge joining y to its foot, say y'. It follows that some edge incident with x is in F. Let H' be obtained from H by removing x and its foot neighbor. Then the portion of F restricted to H' is a thwarting set of H', and hence by the induction hypothesis, has size at least $(n - 2)/2 - 1$. But F also contains an edge incident with x, and so has size at least $n/2 - 1$, as required. ■

We show next that coronas are the only examples of trees where $c_4 = n/2+1$. We will need the following lemma.

Lemma 8. *If T is a corona, then there is a minimum P_4-thwarting set that contains any one designated leaf edge.*

Proof. Assume the designated edge joins end-vertex y' with neighbor y. Then a thwarting set can be constructed by taking yy' together with all edges of the core, except for one incident with y. (See Figure 2 earlier.) ■

Theorem 9. *If T is a tree on $n \geq 2$ vertices that is not a corona, then $c_4(T) > n/2 + 1$.*

Proof. By Theorem 5, we already know this for trees of odd order. So assume that n is even. It suffices to show that $\theta_{P_4}(T) < n/2 - 1$. The proof is by induction on the diameter of T. If the diameter is 1 then $n = 2$ but the tree is a corona. If the diameter is 2, then the tree is a star and $\theta_4 = 0$; so the result is true.

So assume the diameter of the non-corona tree T is at least 3. Then since P_4 is a corona, we know T is not P_4 and hence $n \geq 6$. Consider a longest (diametrical) path Q in the tree. Say the path starts with vertices $abcd$.

Case 1. *Vertex b has degree* 2. Then $T' = T - \{a, b\}$ is a tree. If T' is not a corona, then $\theta_{P_4}(T') < (n-2)/2 - 1$ by the induction hypothesis, and we can extend to a thwarting set of T by adding the edge bc.

So assume T' is a corona. Then since T is not a corona, it must be that c is an end-vertex of T'. At the same time, by Lemma 8, there is a minimum thwarting set of T' (that is, of size $(n-2)/2 - 1$) that uses edge cd. This is also a thwarting set of T.

Case 2. *Vertex b has degree $r > 2$.* Then by the maximality of the path Q, every neighbor of b except c is an end-vertex. Let T' be the graph obtained from T by deleting b and all its end-vertex neighbors. Note that T' is a tree. By Theorem 5, it has a thwarting set of size at most $(n-r)/2 - 1$. One can extend that set to a thwarting set of T of size $n/2 - r/2 < n/2 - 1$ by adding edge bc. ■

At the other extreme, the question of the maximum value of c_4 is trivial, since the value is n if and only of the graph does not contain a copy of P_4. One can also readily determine the trees where the value is $n - 1$. We define a *multi-corona* as a graph that results from adding one or more feet to every vertex of a base graph.

Theorem 10. *For a tree T of order n containing P_4, it holds that $c_4(T) = n - 1$ if and only if T is a subgraph of a multi-corona of P_4.*

Proof. Let U be a multi-corona of P_4. It is immediate that the central edge of U forms a thwarting set of U and of any subgraph of U that contains a P_4.

On the other hand, consider a tree T with a thwarting set consisting of only the edge ab. Then all copies of P_4 in T contain the edge ab. Thus vertex a cannot have two neighbors of degree more than 1, and if it does have a neighbor a' of degree more than 1, that vertex can only have end-vertex neighbors. Similarly b has at most one neighbor b' of degree more than 1. Thus T is a subgraph of a multi-corona of $a'abb'$. ■

3.2. Trees with minimum cp_4

In contrast to the case for c_4, it seems that there is no simple description of the trees where $cp_4 = n/2+1$. We have seen that this is true of paths. It is also true for the *double star* D_b, defined by taking two stars each with b end-vertices and adding one edge joining the two centers c_1 and c_2. See Figure 3 for an example.

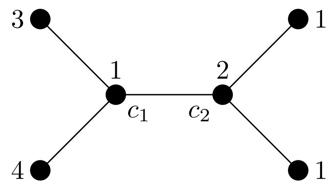

Figure 3. The double star D_2 with $cp_4 = 4$.

Lemma 11. *For a double star D_b it holds that $cp_4(D_b) = b+2$.*

Proof. The lower bound follows from Theorem 5. Consider a valid coloring of D_b. Then the central vertices, say c_1 and c_2, receive different colors. Consider some vertex with a third color; say vertex v adjacent to c_1. Then by the vc_1c_2w path, every vertex w adjacent to c_2 must have either the color of v or the color of c_1; that is, not a new color. So the number of colors other than that of c_1 and c_2 is at most the number of leaf-neighbors of c_1, which is b. Hence the double-star has $cp_4 = b+2$. ∎

We next show that, if T is a tree with $cp_4(T) = |T|/2+1$, then so is one with K_2 attached. We observed earlier that part (b) of Lemma 1 does not extend to general attachers w. However, it turns out that it does extend if the underlying graph is a tree, at least in the case $k=4$.

We will need the following idea. For a coloring, define a vertex x as *boring* if either (i) all neighbors of x have the same color, or (ii) all vertices at distance 2 from x have the same color as x, or both. For example, in the coloring of D_2 in Figure 3 every vertex is boring. We claim that in a tree T, one can choose an optimal cp_4-coloring such that every vertex is boring.

Lemma 12. *If T is a tree, then there exists an optimal cp_4-coloring such that every vertex is boring.*

Proof. Consider the optimal coloring of T with the most boring vertices, and suppose there is a vertex x_3 that is not boring. Then there is a vertex x_1 at distance 2 from x_3 with a different color. Let x_2 be their common neighbor. Say x_1 has color 1, x_2 has color 2, and x_3 has color 3. Further, there must be a neighbor of x_3, say x_4, that has a color different from x_2. Since $x_1x_2x_3x_4$ is not rainbow, vertex x_4 must have color 1. Indeed, all neighbors of x_3 must have color 1 or 2. Furthermore, all vertices at distance two must have color 1, 2, or 3. See Figure 4 for an example.

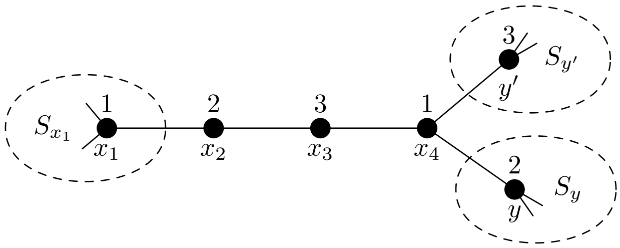

Figure 4. A possible coloring.

Now, we re-color the tree T. For each vertex y at distance 2 from x_3, let S_y be the subtree consisting of y and all vertices whose path to x_3 goes via y. If y has color 1, then in S_y change every vertex with color 1 to color 3, and vice versa. If y has color 2, then in S_y change every vertex with color 2 to color 3, and vice versa. One does not lose any color in the process, as $x_2x_3x_4$ still contains all three colors.

We claim that the new coloring is still a valid coloring. Every copy of P_4 intersects at least one of the S_y subtrees. Within S_y only names of colors have been changed and so there cannot now be a rainbow P_4 contained within S_y. Also, if w is the common neighbor of y and x_3, then vertices with the same color as w inside S_y did not change color, and so there cannot now be a rainbow P_4 contained within $S_y \cup \{w\}$. Further, since x_3 and all vertices at distance two from it now have color 3, any P_4 containing both y and x_3, or containing both y and another vertex at distance two from x_3, has two vertices of color 3, and so is not now rainbow.

Finally, we note that x_3 is now boring (since all vertices at distance 2 have the same color), as are all its neighbors. Further, every other vertex that was boring remains so. This re-coloring increases the number of boring vertices, and so contradicts the choice of coloring. That is, the supposition that the coloring had a vertex that is not boring is false. ■

Using Lemma 12 we can prove a result about attachments.

Lemma 13. *For any nontrivial tree T and vertex w, let tree T_2 be obtained from T by attaching P_2. Then $cp_4(T_2) = cp_4(T) + 1$.*

Proof. We noted earlier that $cp_4(T_2) \leq cp_4(T) + 1$, since the two new vertices cannot both get a unique color. So it remains to find a suitable coloring.

By Lemma 12 there exists an optimal coloring of T where w is boring. We color the P_2 as follows. If all neighbors of w have the same color, say red, then color the first vertex of P_2 red and give the other vertex a new color. If all vertices at distance 2 from w have the same color as w, say blue, then give the first vertex of P_2 a new color and color the other vertex blue. In either case the result is a valid coloring. ∎

But there are many other trees T with $cp_4(T) = |T|/2 + 1$. Figure 5 shows an example.

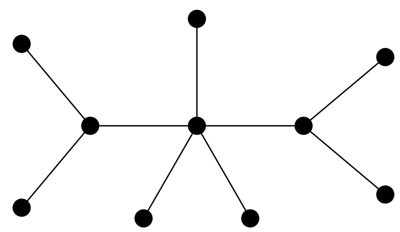

Figure 5. A tree with $cp_4(T) = 6$.

At the other extreme, the question of the maximum value of cp_4 can also be considered. Using Theorem 10 one can show.

Theorem 14. *For a tree T of order n containing P_4, it holds that $cp_4(T) = n-1$ if and only if T is a subgraph of a multi-corona of P_3 whose middle vertex has degree 2.*

4. Coloring Trees Without a Rainbow P_5

In this section we determine the trees with the minimum value of the parameters for colorings without a rainbow P_5.

Theorem 15. *For a tree T of order $n \geq 3$, it holds that $c_5(T) \geq cp_5(T) \geq (n+3)/2$, and this is best possible.*

Proof. Since a coloring without a rainbow P_4 also does not have a rainbow P_5, we know from Theorem 5 that $cp_5(T) \geq (n+2)/2$. Thus it suffices to show that achieving $(n+2)/2$ is not possible.

As noted in the lead-in to Theorem 5, one obtains a proper no-rainbow-P_4 coloring in T by choosing one partite set X and giving every vertex in X its own unique color while giving all vertices in the other partite set Y the same color. So $cp_5(T) \geq cp_4(T) > n/2 + 1$ unless both partite sets have size $n/2$. So assume that is the case.

Choose some end-vertex y; say it is in Y. Then use the same coloring as above, except that y also gets its own unique color. Then the total number of colors is $n/2+2$. Every P_5 that contains y contains two other vertices of Y, and hence remains not rainbow. That is, $cp_5(T) > (n+2)/2$. ■

The value in Theorem 15 is achieved by the octopus O_b produced by taking the star with b edges and subdividing every edge. Figure 6 shows O_5.

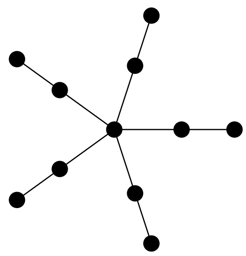

Figure 6. The octopus O_5.

Lemma 16. *For the octopus O_b with $b \geq 2$, it holds that $c_5(O_b) = cp_5(O_b) = b+2$.*

Proof. It is immediate that a minimum P_5-thwarting set is obtained by taking one edge from $b-1$ of the arms; thus $c_5(O_b) = (2b+1)-(b-1) = b+2$. The value of $cp_5(O_b)$ then follows by Theorem 15. ■

We conclude this section by showing that the octopus is the unique extremal graph for both parameters.

Theorem 17. *For odd $n \geq 5$, the octopus is the unique tree T of order n with $cp_5(T) = (n+3)/2$ (and hence unique for c_5 too).*

Proof. It is immediate that the only tree of order 5 with $cp_5(T) < n$ is P_5 itself (which is the same as O_2). So assume the tree T has odd order $n \geq 7$ with bipartition (X, Y) where $|X| > |Y|$. It is immediate that T is not a star. As noted in the proof of Theorem 15, if there is an end-vertex in Y then $cp_5(T) \geq |X|+2$. It follows that all end-vertices of T must be in X. In particular, the diameter is at least four.

Consider a longest path in T, say starting $abcde$. Let A_1 denote the neighbors of c other than d, and let A_2 denote the vertices at distance two from c whose path to d goes via c. Since all end-vertices of T are in X, no vertex of A_1 is an end-vertex, and so $|A_2| \geq |A_1|$. Let $T' = T - (A_1 \cup A_2)$. Note that c is an end-vertex in tree T'. Any valid coloring of T' can be extended to one of T by giving each vertex of A_1 the color of vertex d, and giving each vertex of A_2 its own unique color. Hence $cp_5(T) \geq cp_5(T') + |A_2|$.

It follows that $cp_5(T) > (n+3)/2$, unless $cp_5(T') = (|T'|+3)/2$ and $|A_1| = |A_2|$. Suppose that T' has at least five vertices. Then, by the inductive hypothesis, the subtree T' is an octopus with center e while vertex c is an end-vertex thereof. But such a graph has $cp_5 \geq (n+5)/2$: give every vertex of X its own unique color except that c and e share colors, give d its own unique color, all other neighbors of c share one color, and all other neighbors of e share another color. The number of colors used is $(|X|-1)+3$, a contradiction. Hence in fact T' has order 3. Since c is an end-vertex of T', it follows that T is an octopus. ■

5. Conclusion

For future work, one open question is to determine all trees with the minimum value of cp_4. It would also be of interest to consider bounds for other graph families, such as planar graphs or regular graphs; for example, in [8] it is conjectured that $cp_4(G) \leq n/2 + 1$ for every connected cubic graph G of order n. And, of course, it would also be worthwhile to establish analogous bounds for c_k and cp_k for larger k.

References

[1] M. Axenovich and R. Martin, *Avoiding rainbow induced subgraphs in vertex-colorings*, Electron. J. Combin. **15(1)** (2008) #R12.

[2] Cs. Bujtás and Zs. Tuza, *Maximum number of colors: C-coloring and related problems*, J. Geom. **101** (2011) 83–97.
https://doi.org/10.1007/s00022-011-0082-2

[3] Cs. Bujtás, E. Sampathkumar, Zs. Tuza, C. Dominic and L. Pushpalatha, *Vertex coloring without large polychromatic stars*, Discrete Math. **312** (2012) 2102–2108.
https://doi.org/10.1016/j.disc.2011.04.013

[4] Cs. Bujtás, E. Sampathkumar, Zs. Tuza, M.S. Subramanya and C. Dominic, *3-consecutive C-colorings of graphs*, Discuss. Math. Graph Theory **30** (2010) 393–405.
https://doi.org/10.7151/dmgt.1502

[5] E. Bulgaru and V.I. Voloshin, *Mixed interval hypergraphs*, Discrete Appl. Math. **77** (1997) 29–41.
https://doi.org/10.1016/S0166-218X(97)89209-8

[6] J.D. Chandler, W.J. Desormeaux, T.W. Haynes and S.T. Hedetniemi, *Neighborhood-restricted* [$\leq$2]-*achromatic colorings*, Discrete Appl. Math. **207** (2016) 39–44.
https://doi.org/10.1016/j.dam.2016.02.023

[7] W. Goddard and H. Xu, *Vertex colorings without rainbow subgraphs*, Discuss. Math. Graph Theory **36** (2016) 989–1005.
https://doi.org/10.7151/dmgt.1896

[8] W. Goddard and H. Xu, *Vertex colorings without rainbow or monochromatic subgraphs*, J. Combin. Math. Combin. Comput. **102** (2017) 109–122.

[9] W. Goddard, K. Wash and H. Xu, *WORM colorings*, Discuss. Math. Graph Theory **35** (2015) 571–584.
https://doi.org/10.7151/dmgt.1814

[10] V. Voloshin, *On the upper chromatic number of a hypergraph*, Australas. J. Combin. **11** (1995) 25–45.

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