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THE MATCHING EXTENDABILITY OF OPTIMAL 1-EMBEDDED GRAPHS ON THE PROJECTIVE PLANE

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Abstract

In this paper, we discuss matching extendability of optimal 1-projective plane graphs (abbreviated as O1PPG), which are drawn on the projective plane P^2 so that every edge crosses another edge at most once, and have n vertices and exactly $4n-4$ edges. We first show that every O1PPG of even order is 1-extendable. Next, we characterize 2-extendable O1PPG's in terms of a separating cycle consisting of only non-crossing edges. Moreover, we characterize O1PPG's having connectivity exactly 5. Using the characterization, we further identify three independent edges in those graphs that are not extendable.

Keywords: 1-embeddable graph, projective plane, perfect matching.

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1. Introduction

Our graphs dealt in this paper are all finite, simple and connected. We denote the vertex set and the edge set of a graph G by $V(G)$ and $E(G)$, respectively. The *order* of G means the number of vetices of G. A cycle of length k is a *k-cycle*. For a cycle C, an edge $e \in E(G)$ such that $V(e) \subset V(C)$ and $e \notin E(C)$ is called a *chord* of C. A cycle C in G is *separating* if $G - V(C)$ is a disconnected graph. We denote the induced subgraph of a graph G by $S \subset V(G)$ by $G[S]$. A set M of edges of a graph G is a *matching* if no two edges of M share a vertex. Let M be a matching of a graph G. Each vertex incident with an edge of M is *covered* by M. The set of vertices covered by M is denoted by $V(M)$. In particular, M

is *perfect* if M covers all vertices of G; that is, $V(G) = V(M)$. A matching M of G is *extendable* if G has a perfect matching containing M. Moreover, a graph G with at least $2k+2$ vertices is k-*extendable* if any matching M in G with $|M| = k$ is extendable. Matching extendability has been widely studied in literature (e.g., see [13]). In particular, matching extendability of graphs on closed surfaces was investigated in [1, 2, 4, 11]; for example, it was proven as a basic result that no planar graph is 3-extendable.

A graph G is 1-*embeddable* on a closed surface F^2 if it can be drawn on F^2 so that every edge of G crosses another edge at most once. The drawn image of G on F^2 is a 1-*embedded* graph on F^2. (We implicitly consider *good drawings*, that is, (i) vertices are on different points on the surface, (ii) no adjacent edges cross, (iii) no three edges cross at the same point, and (iv) any non-adjacent edges do not touch tangently.) The study of 1-*planar* graphs, which are 1-embeddable graphs on the plane or the sphere, was first introduced by Ringel [14], and recently developed in various points of view (see e.g., [6, 16]); the drawn image is called a 1-*plane graph*. It is known that if G is a 1-embedded graph on F^2 with at least three vertices, then $|E(G)| \leq 4|V(G)| - 4\chi(F^2)$ holds, where $\chi(F^2)$ stands for the Euler characteristic of F^2 (see [7] for example). In particular, a 1-embedded graph G on F^2 that satisfies the equality, that is $|E(G)| = 4|V(G)| - 4\chi(F^2)$, is *optimal*. An edge in a 1-embedded graph G is *crossing* if it crosses another edge, and *non-crossing* otherwise. Let G be an optimal 1-embedded graph on F^2, and let W be a closed walk consisting of only non-crossing edges that bounds a 2-cell D; where a 2-cell is homeomorphic to an open disc. If D contains an odd number of vertices, then we call D an *odd weighted region*. In particular, if W is a cycle, then W is a *barrier cycle*. A barrier cycle of length k is called a *barrier k-cycle*.

The matching extendability of 1-embedded graphs on F^2 was first addressed in [3], and the authors proved that every optimal 1-plane graph (abbreviated as O1PG) of even order is 1-extendable. Further in the same paper, they discussed 2-extendability of O1PG's, and established the following theorem.

Theorem 1 (Fujisawa *et al.* [3]). *An* O1PG *G of even order is* 2-*extendable unless G contains a barrier* 4-*cycle.*

Furthermore, they discussed extendable three edges in O1PG's and obtained the following result.

Theorem 2 (Fujisawa *et al.* [3]). *Let G be a* 5-*connected* O1PG *of even order, and M be a matching of G with $|M| = 3$. Then M is extendable unless G contains a barrier* 6-*cycle C such that $V(M) = V(C)$.*

We extend the topic to graphs on non-spherical closed surfaces. In this paper, we especially discuss matching extendability of optimal 1-embedded graphs on the projective plane; we denote the projective plane by P^2 briefly. An optimal

1-embedded graph on P^2 is also called an *optimal 1-projective plane graph*, and is abbreviated as O1PPG. Note that every O1PPG has exactly $4|V(G)|-4$ edges by the equality above with $\chi(P^2) = 1$. First, we discuss 1-extendability of O1PPG's, and show the following theorem, using Hamiltonian paths contained in those graphs.

Theorem 3. *Every* O1PPG *of even order is* 1*-extendable.*

Next, we discuss 2-extendability, and prove the following theorem. The statement looks similar to the spherical case, but we need to establish some lemmas specific to the case of the projective plane.

Theorem 4. *An* O1PPG G *of even order is* 2*-extendable if and only if* G *contains a barrier* 4*-cycle.*

The following corollary easily follows from Theorem 4.

Corollary 5. *Any* 5*-connected* O1PPG *of even order is* 2*-extendable.*

Every O1PPG G has a vertex with degree 6, since the average degree of G is less than 8. (And since the minimum degree of G is at least 6, and every optimal 1-embedded graph on a closed surface is Eulerian. We mention these facts in Section 2.) Therefore, no O1PPG is 3-extendable; take three edges on the 6-cycle induced by neighbors of a vertex of degree 6. The following theorem characterizes three mutually independent edges that are not extendable in those graphs.

Theorem 6. *Let* G *be a* 5*-connected* O1PPG *of even order, and* M *be a matching of* G *with* $|M| = 3$*. Then* M *is not extendable if and only if* G *has either* (i) *an odd weighted region bounded by a closed walk* W *of length* 6 *such that* $V(W) \setminus V(M) = \emptyset$*, or* (ii) *a subgraph of* $Q(G)$ *shown as* $(a), \ldots, (f)$ *or* (g) *in Figure* 1*, each of whose face is an odd weighted region, where big gray vertices are covered by* M*.*

Note that each of $(a), (b), \ldots, (f)$ and (g) represents a graph on P^2. To obtain the projective plane P^2, identify each antipodal pair of points of the hexagon or the octagon in the figure. (Similarly, carry out the same identification for regular polygons or dashed-circles in other figures in the paper to obtain P^2.) For example (a) is a graph on P^2 that has seven vertices and nine edges.

This paper is organized as follows. In the next section, we first define terminology used in the paper, and discuss the fundamental results hold for optimal 1-embedded graphs on general closed surfaces. Next, we discuss connectivity of O1PPG's and separating short cycles in underlying quadrangulation consisting of non-crossing edges in Section 3. Furthermore, we characterize O1PPG's having connectivity exactly 5 in the section; note that there is no O1PG (on the sphere) having connectivity exactly 5. In Section 4, we discuss extendability of O1PPG's, and prove our main theorems.

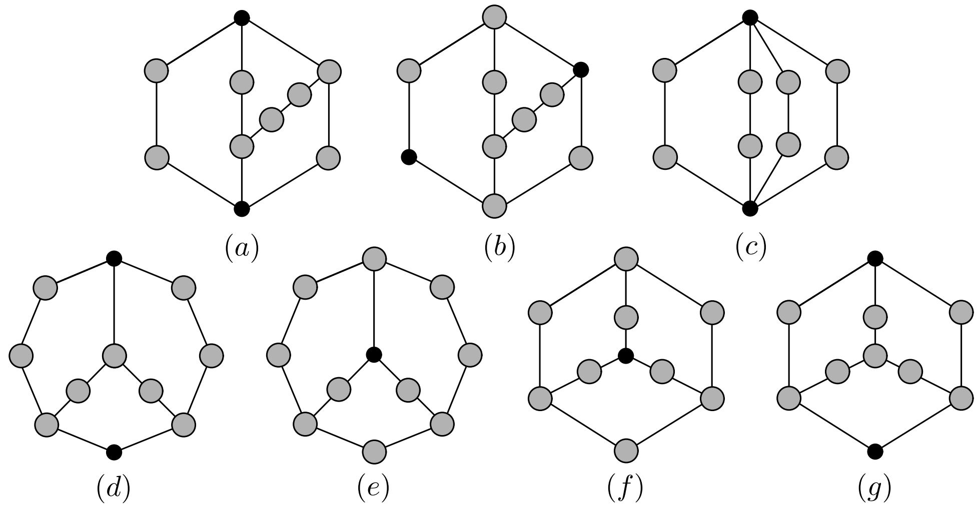

Figure 1. Specified subgraphs in $Q(G)$ in Theorem 6.

2. Preliminaries and Basic Results

A vertex set S of a connected graph G is a *cut* if $G-S$ has at least two connected components. A cut S of G is *minimal* if any proper subset of S is not a cut of G. For a cut S of G, if $|S| = k$, then we call S a *k-cut* of G. We denote the number of connected components of $G-S$ for $S \subset V(G)$ by $C(G-S)$. In particular, the number of *odd components* (respectively, even components), i.e., connected components having odd (respectively, even) number of vertices, is denoted by $C_o(G-S)$ (respectively, $C_e(G-S)$). That is, we have $C(G-S) = C_o(G-S) + C_e(G-S)$.

Let G be a graph embedded on a closed surface F^2. Then a connected component of $F^2 - G$, which is as a topological space, is a *face* of G, and we denote the face set of G by $F(G)$; that is, "a face" in this paper is not necessarily homeomorphic to an (open) 2-cell. A *boundary closed walk* W of a face f is a closed walk bounding f in G. (Actually, under our definition, the boundary of a face might be a union of closed walks.) A *k-gonal face* or simply a *k-face* means a face with boundary closed walk of length exactly k. If every face of G is homeomorphic to a 2-cell, then G is a 2*-cell embedding* or 2*-cell embedded graph* on F^2. Furthermore, a *region* bounded by a closed walk might contains some vertices and edges in its interior in our latter argument; that is, a face is always a region, but the converse does not hold in general.

A simple closed curve γ on a closed surface F^2 is *trivial* if γ bounds a 2-cell on F^2, and *essential* otherwise. We apply these definition to cycles of graphs embedded on F^2, regarding them as simple closed curves. A simple closed curve

γ on a closed surface F^2 is *surface separating* if $F^2 - \gamma$ is disconnected as a topological space. We also apply the definition to cycles of graphs on F^2. Note that every trivial closed curve on a closed surface is surface separating. In addition, it is well-known that every surface separating simple closed curve on the sphere or the projective plane is trivial. The following proposition is known in topological graph theory, and is commonly used.

Proposition 7 (Nakamoto [8])**.** *Let G be a graph embedded on a closed surface F^2 so that each face is bounded by a closed walk of even length. Then the length of two cycles in G have the same parity if they are homotopic to each other on F^2. Furthermore, there is no surface separating odd cycle in G.*

The *representativity* $r(G)$ of a graph G embedded on a non-spherical closed surface F^2 is the minimum number of crossing points of G and γ, where γ ranges over all essential simple closed curves on F^2. A graph G embedded on F^2 is *k-representative* if $r(G) \geq k$. A graph G embedded on a non-spherical closed surface F^2 is *polyhedral* if G is 3-connected and 3-representative. In particular, a graph G embedded on the sphere is polyhedral if G is just 3-connected.

A *quadrangulation* (respectively, *triangulation*) is a simple 2-cell embedded graph on a closed surface such that every face is a 4-face (respectively, 3-face). It was shown in [7] that every simple optimal 1-embedded graph G on F^2 is obtained from a polyhedral quadrangulation H by adding a pair of crossing edges in each face of H. We call the quadrangulation H, which consists of all the non-crossing edges of G, the *quadrangular subgraph* of G, and denote it by $Q(G)(= H)$. By the property above, $\deg_G(v) = 2\deg_H(v)$ for any $v \in V(G)$, that is, G is Eulerian. For a vertex v of an optimal 1-embedded graph G on F^2, the union of all the faces (with boundaries) of $Q(G)$ incident to v forms a disc D containing the unique vertex v. We call the boundary cycle of D the *link* of v and denote it by $L_G(v)$; observe that the boundary corresponds to a cycle since $Q(G)$ is polyhedral.

Let F^2 be a closed surface. An *arc* in F^2 is the image of a continuous map $\alpha : [0, 1] \to F^2$; we denote the image $\alpha([0, 1])$ by α for brevity, if there is no misunderstanding. The arc α *joins* its endpoints $\alpha(0)$ and $\alpha(1)$. Let G be an optimal 1-embedded graph on F^2 and let H_1 and H_2 be connected subgraphs of G. Then a subgraph K of $Q(G)$ *separates* H_1 and H_2 on F^2 if $V(K) \cap V(H_i) = \emptyset$ for each $i \in \{1, 2\}$, and any arc α on F^2 that joins $x_1 \in V(H_1)$ and $x_2 \in V(H_2)$ has an intersection with K; i.e., $\alpha \cap K \neq \emptyset$. Note that $F^2 \setminus K$ is disconnected as a topological space.

Let G be an optimal 1-embedded graph on F^2 and let $\mathcal{C}(G)$ be the set of all the crossing points of G. We obtain the *associated graph* $G^\times$, which is embedded on F^2, from G by regarding every crossing point as a vertex of degree 4. (That is, $G^\times$ has $V(G^\times) = V(G) \cup \mathcal{C}(G)$ and $E(G^\times) = E(Q(G)) \cup \{xz, yz \mid xy \in E(G) \setminus E(Q(G)), z \in \mathcal{C}(G) \cap xy\}$.) We call new vertices, which correspond to crossing points of G, *false vertices* of $G^\times$. Note that $G^\times$ is a triangulation of F^2.

In the argument below, we often consider the induced subgraph of $Q(G)$ by a cut S of G, which is $Q(G)[S]$ under our definition. However, when the underlying graph G is clear, we use $Q[S]$ in place of $Q(G)[S]$, to simplify notation. In the following four lemmas, we assume that G is an optimal 1-embedded graph on F^2, and $S \subset V(G)$ is a cut of G.

Lemma 8. *Let $D_1, \dots, D_m$ $(m \geq 2)$ denote connected components of $G - S$. Then any two connected components D_i and D_j $(i \neq j)$ are separated by $Q[S]$. That is, each face of $Q[S]$ contains at most one connected component of $G - S$.*

Proof. Suppose to the contrary that $Q[S]$ does not separate connected components D_i and D_j $(i \neq j)$ of $G - S$. Then there exists an arc α on F^2 such that $\alpha \cap Q[S] = \emptyset$ and α joins $x \in V(D_i)$ and $y \in V(D_j)$. Since the associated graph $G^\times$ of G is a triangulation of F^2 and $\alpha \cap Q[S] = \emptyset$, we can fix α so that $\alpha \cap G^\times \subset V(G^\times) \setminus S$. We may assume that $\alpha \cap D_k = \emptyset$ for any $k \neq i, j$; otherwise, retake the closest D_i and D_j. Since $G^\times$ is a triangulation of F^2, there exists a path $P^\times$ in $G^\times$ between x and y along α. Now assume that $P^\times$ passes through a false vertex z corresponding to a crossing point created by a pair of crossing edges v_0v_2 and v_1v_3. If a 2-path v_izv_{i+2} is contained in $P^\times$, then we replace it by v_iv_{i+2}, which is a crossing edge of G, where the indices are taken modulo 4. On the other hand, if a 2-path v_izv_{i+1} is contained in $P^\times$, then we replace it by a non-crossing edge v_iv_{i+1} of G. We do the replacement above for all false vertices contained in $P^\times$, and obtain a path P in G between x and y not containing any vertex in S, a contradiction. ∎

Now, we show the following lemma that mentions the minimum degree of $Q[S]$ for a minimal cut S of optimal 1-embedded graphs on closed surfaces.

Lemma 9. *If S is minimal, then the minimum degree of $Q[S]$ is at least 2.*

Proof. Let $v \in S$, and let D_1 and D_2 be connected components of $G - S$. Since S is minimal, G has edges vx_1 and vx_2 where $x_i \in V(D_i)$ for each $i \in \{1, 2\}$. Note that both x_1 and x_2 are on the link $L_G(v)$. Since $L_G(v)$ is a cycle, there must be two vertices $s_1, s_2 \in S \cap V(L_G(v))$ which separate x_1 and x_2 in $L_G(v)$. Since $\{s_1, s_2\}$ separates x_1 and x_2 in G as well, we have $\{s_1, s_2\} \subseteq N_{Q[S]}(v)$, and we got our desired conclusion. ∎

Lemma 10. *If $Q[S]$ has p faces such that the sum of the lengths of its boundary walks is at least $2q \geq 6$, then the following inequalities hold.*

(i) $|E(Q[S])| \geq 2|F(Q[S])| + (q - 2)p$,

(ii) $|S| - \chi(F^2) + (2 - q)p \geq |F(Q[S])|$.

Proof. Note that a face of $Q[S]$ is not necessarily a 2-cell, and does not necessarily have the unique boundary component. By Proposition 7, which actually holds for general boundaries of faces (that is, not restricted to cycles), the sum of lengths of boundary walks of $Q[S]$ is even. Thus, we have $2|E(Q[S])| \geq 4(|F(Q[S])| - p) + 2pq$, and hence (i) in the theorem holds. Furthermore, by combining Euler's formula $|S| - |E(Q[S])| + |F(Q[S])| \geq \chi(F^2)$, we can easily obtain (ii) in the statement. ∎

Lemma 11. *If $|S| \leq C_o(G - S) + 2k$ holds for some integer k, then we have the following*

$$2|F(Q[S])| + 2k - \chi(F^2) \geq |E(Q[S])|.$$

Proof. By Lemma 8, $|F(Q[S])| \geq C_o(G - S) \geq |S| - 2k$ holds. Then by $|S| - |E(Q[S])| + |F(Q[S])| \geq \chi(F^2)$, we obtain the inequality in the statement. ∎

3. MINIMAL CUTS AND SUBGRAPHS IN $Q(G)$

In this section, we describe properties of induced subgraphs by minimal cuts in O1PPG's. First of all, we show 4-connectedness of O1PPG's as follows.

Theorem 12. *Every* O1PPG *G is* 4*-connected. Furthermore, if G has a* 4*-cut S, then $Q[S]$ contains a separating trivial* 4*-cycle of G.*

Proof. In [10], it was proven that every quadrangulation G on a closed surface with $|V(G)| \geq 6$ can be extended to a 4-connected triangulation by adding a diagonal edge in every face of G. Furthermore, it was shown in [15] that every O1PPG has at least nine vertices. Combining the results above, we obtain the former half of the statement of the theorem.

Next, we discuss the latter half of the statement. Let S be a 4-cut of G, and first assume that $Q[S]$ is not a 2-cell embedding; note that $Q[S]$ is bipartite. Since $Q[S]$ has at least two faces by Lemma 8, $Q[S]$ contains a cycle C. Then $|C| = 4$ and C does not have any chord; otherwise $Q[S]$ would have a trivial cycle of length 3, a contradiction to that $Q[S]$ is bipartite. In this case, $Q[S]$ is just a 4-cycle, and has exactly two faces, one of which is a 2-cell and the other of which contains a cross cap. Each face contains exactly one connected component by Lemma 8, and we have our desired trivial 4-cycle in $Q[S]$.

Secondly, we assume that $Q[S]$ is a 2-cell embedding. By Euler's formula $4 - |E(Q(S))| + |F(Q[S])| = 1$, $|E(Q[S])|$ is either 5 or 6 since $|F(Q[S])| \geq 2$ by Lemma 8. In the case when $|E(Q[S])| = 6$, $Q[S] \cong K_4$, and it is known that K_4 is uniquely embedded on P^2 such that each face is bounded by a closed walk of even length, which is actually a cycle of length 4. On the other hand, if $|E(Q[S])| = 5$, then $Q[S]$ has exactly two faces, one of which is bounded by a

4-cycle and the other of which is bounded by a closed walk of length 6; note that the sum of the lengths of boundary walks must be $2|E(Q[S])| = 10$. Observe that in the embedding of $Q[S]$ above, any 3-cycle must be essential. (Therefore, the embedding of $Q[S]$ is uniquely determined.) In either case, we have our desired trivial 4-cycle in $Q[S]$ by Lemma 8. ∎

Next, we present some facts holding for 5-connected O1PPG's.

Lemma 13. *Let G be a 5-connected* O1PPG, *and let S be a minimal cut with $|S| \in \{5, 6\}$. Then the followings hold.*

(i) $|E(Q[S])| \geq 2|F(Q[S])| + 2$.

(ii) *If $Q[S]$ is not a 2-cell embedding, then $|S| = 6$. Furthermore, $Q[S]$ is a trivial 6-cycle.*

(iii) *If $|S| = 6$, then $Q[S]$ has a 2-cell face bounded by a 6-cycle.*

Proof. The inequality (i) easily follows from (i) of Lemma 10 with $p \geq 2$ and $q \geq 3$ since $C(G - S) \geq 2$, and since G is 5-connected. Next, assume that $Q[S]$ is not a 2-cell embedding. Then $Q[S]$ is a bipartite graph. If $|S| = 5$, then $Q[S] \cong K_{2,3}$ by Lemma 9. This $Q[S]$ is embedded on P^2 having three 4-faces; observe that exactly one of them is not a 2-cell face. By Lemma 8, at least two faces above contain vertices of G, and G would have a 4-cut, a contradiction. Thus assume that $|S| = 6$. In this case, either $Q[S] \cong K_{2,4}$ or $Q[S]$ contains a 6-cycle. In the former case, there exists a 4-face containing a vertex of G, a contradiction; similar to the case when $|S| = 5$. Hence $Q[S]$ contains a 6-cycle C. Under the condition, C has at most two chords since $Q[S]$ is a planar graph. However, in any case, $Q[S]$ has at most one k-face with $k \geq 6$, contradicting Lemma 8. Thus (ii) in the statement holds.

Finally, we discuss (iii), and assume that $|S| = 6$. We may assume that $Q[S]$ is a 2-cell embedding by the result (ii) above. Suppose that G has no k-face with $k = 6$. Then $Q[S]$ has at least two faces bounded by closed walks of length at least 8 by Lemma 8. By (ii) of Lemma 10 with $p \geq 2$ and $q \geq 4$, we have $1 \geq |F(Q[S])|$, contradicting $|F(Q[S])| \geq 2$. Thus $Q[S]$ has a 6-face f bounded by a closed walk W. Since $Q[S]$ has no chord inside f, f contains a vertex of G. If W is not a cycle, then it is contrary to S being a minimal cut. Thus (iii) holds. ∎

Next, see Figure 2. We call the graph embedded on P^2 as shown in the figure a *projective-bowtie*. Note that the projective-bowtie has five vertices, six edges and two faces bounded by closed walks of length 6. Actually, in [3], it was proven that every 5-connected O1PG G is 6-connected. That is, there is no O1PG having connectivity exactly 5. The next lemma (and the theorem) illustrates a distinct property for O1PPG's, in contrast to the aforementioned fact for O1PG's.

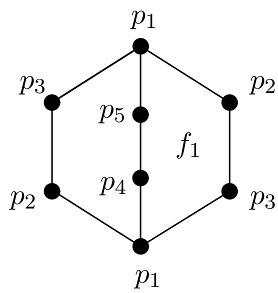

Figure 2. Projective-bowtie.

Lemma 14. *Let G be a 5-connected* O1PPG, *and let S be a 5-cut of G. Then $Q[S]$ is a projective-bowtie.*

Proof. Let $S = \{p_1, \ldots, p_5\}$ be a 5-cut of G; clearly, S is minimal. By (ii) of Lemma 13, we may assume that $Q[S]$ is a 2-cell embedding. It easily follows from (i) of Lemma 13 that $|E(Q[S])| \geq 6$ by $|F(Q[S])| \geq 2$. By substituting the term of the number of faces in Euler's formula $5 - |E(Q[S])| + |F(Q[S])| = 1$ for (i) in Lemma 13, we have $|E(Q[S])| \leq 6$. Consequently, $|E(Q[S])| = 6$, and further $|F(Q[S])| = 2$ holds. This implies that $Q[S]$ has exactly two 6-gonal faces denoted by f_1 and f_2 by Lemma 8. Note that f_i is not bounded by a cycle for each $i \in \{1, 2\}$. That is, there exists a vertex of S, say p_1 without loss of generality, which appears on the boundary closed walk of f_1, denoted by W_1, at least twice. If the distance between two p_1's on W_1 is at most 2, then either $Q[S]$ is not simple or $Q[S]$ has a vertex of degree 1, contradicting Lemma 9.

Thus we may assume that f_1 is bounded by a closed walk $W = p_1xyp_1zwp_1$ where $\{x, y, z, w\} \subseteq \{p_2, \ldots, p_5\}$ without loss of generality. Observe that the 3-cycle p_1xyp_1 is essential on P^2 by Proposition 7. Under the condition, if $x = z$, then there exists a 2-cell region bounded by a 4-cycle p_1wxyp_1. Since $|V(Q[S])| = 5$ and $|E(Q[S])| = 6$, one of the two regions contains the unique inner vertex and the unique inner edge of $Q[S]$, contradicting Lemma 9. Therefore, x, y, z and w are distinct vertices, and hence we may assume that $W = p_1p_2p_3p_1p_4p_5p_1$. Now, all the edges of $Q[S]$ appeared, and the outside region actually corresponds to the second face f_2 of $Q[S]$. That is, $Q[S]$ is the projective-bowtie. ■

Theorem 15. *Let G be a 5-connected* O1PPG. *Then G is 6-connected if and only if $Q(G)$ does not contain a projective-bowtie as a subgraph.*

Proof. First, we prove the necessity. Assume that $Q(G)$ contains a projective-bowtie H with $V(H) = \{p_1, p_2, p_3, p_4, p_5\}$ as shown in Figure 2, where one face of H, say f_1, is bounded by a closed walk $W = p_1p_2p_3p_1p_4p_5p_1$. Suppose that the region R_1, which corresponds to f_1, does not contain any vertex of G. Then there exists a non-crossing edge p_2p_4 or p_3p_5 of G in R_1; now say p_2p_4, up to symmetry. Then there exists a crossing edge p_1p_2, which crosses p_3p_4 in R_1, contrary to G being simple. Hence each face of H contains a vertex of G. Then $V(H)$ is a 5-cut of G, that is, G is not 6-connected. The sufficiency immediately follows from Lemma 14. ■

The following lemma describes the shape of $Q[S]$ for a minimal 6-cut S of a 5-connected O1PPG. Note that in Figure 3, antipodal points of each dashed disk are identified to obtain P^2, as we mentioned in the introduction.

Lemma 16. *Let G be a 5-connected* O1PPG *and S be a minimal 6-cut of G. Then $Q[S]$ is one of* (I), (II), (III) *and* (IV) *as shown in Figure* 3.

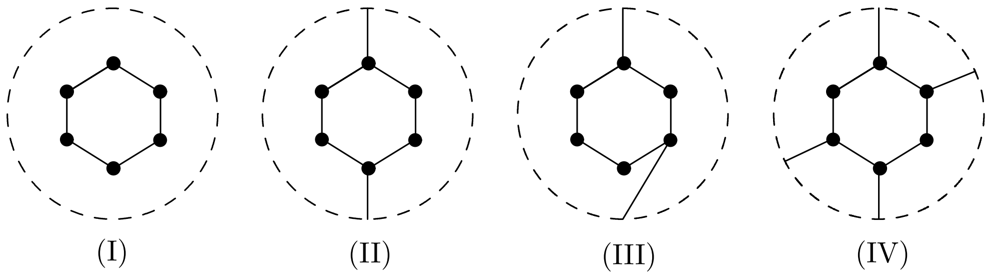

Figure 3. $Q[S]$ obtained by a minimal 6-cut S.

Proof. By (iii) in Lemma 13, $Q[S]$ has a 2-cell face f bounded by a cycle C of length 6. By categorizing based on the number of cords of C outside f, we obtain (I), (II), (III) and (IV) in Figure 3. Note that by Lemma 8, $Q[S]$ has at least two faces bounded by closed walks with length at least 6. Further, consider Proposition 7. ■

4. Proof of the Main Theorems

The following famous result plays an important role in the proof of Theorem 3.

Theorem 17 (Kawarabayashi and Ozeki [5])**.** *Every 4-connected graph embedded on P^2 is Hamilton-connected.*

Now, we prove our first main result in the paper.

***Proof of Theorem* 3.** Let G be an O1PPG, and let $e = uv$ be an edge of G. By the result in [10] as mentioned in the proof of Theorem 12, G has a

4-connected triangulation T as a spanning subgraph. Then T has a Hamiltonian path $P = x_1x_2 \cdots x_{|V(G)|}$, where $u = x_1$ and $v = x_{|V(G)|}$ by Theorem 17. In G, $\{x_2x_3, x_4x_5, \ldots, x_{|V(G)|-2}x_{|V(G)|-1}, x_{|V(G)|}x_1\}$ is a perfect matching that contains e. ∎

Actually, the following lemma is a generalization of Lemma 2.3 in [12] that is often used in matching theory; in particular, if $k = 1$, then S in the lemma is called an $\{e_1, e_2\}$-*blocker*.

Lemma 18 (Fujisawa *et al.* [3])**.** *Let G be a k-extendable graph and $\{e_1, \ldots, e_{k+1}\}$ be a matching of G which is not extendable. Then there exists $S \subseteq V(G)$ such that*

(i) $S \supset \bigcup_{i=1}^{k+1} V(e_i)$ *and*

(ii) $|S| = C_o(G - S) + 2k$.

In the remaining part of the section, we prove the following two main results using tools proven in the previous section.

***Proof of Theorem* 4.** The necessity is trivial and hence we prove the sufficiency of the statement. Let G be an O1PPG that is not 2-extendable, and assume that e_1 and e_2 are independent edges of G that are not extendable. By Theorem 3, G is 1-extendable, and hence there exists $S \subset V(G)$ that satisfies (i) and (ii) of Lemma 18 for $k = 1$; for e_1 and e_2. Now we consider $Q[S]$ on P^2. By (i) of Lemma 10 with $p = 0$, we have $|E(Q[S])| \geq 2|F(Q[S])|$. On the other hand, by Lemma 11 with $k = 1$, we have $2|F(Q[S])| + 1 \geq |E(Q[S])|$. Thus, either $|E(Q[S])| = 2|F(Q[S])|$ or $|E(Q[S])| = 2|F(Q[S])| + 1$ holds.

In the former case, every face of $Q[S]$ is bounded by a 4-cycle, and at most one of them contains a cross cap. That is, the other faces are all 2-cell, and by Lemma 8, we can find an odd weighted face since $G - S$ has at least two odd components. In the latter case, the equality of Euler's formula (in Lemma 11) holds, and hence $Q[S]$ is a 2-cell embedding. Furthermore, by the argument above, $Q[S]$ has the unique 6-gonal face and all the others are 4-gonal. Similarly, we find our desired barrier cycle. ∎

***Proof of Theorem* 6.** First, we show the necessity. If a 5-connected O1PPG G has (i) in the statement, then $G - V(M)$ has an odd component, where M is the set of specified three independent edges. On the other hand, if G has (ii) in the statement, then $G - V(M)$ has a cut vertex v such that $G - (V(M) \cup \{v\})$ has exactly three odd components. In either case, $G - V(M)$ does not have a perfect matching.

Next, we discuss the sufficiency. Assume that M is a matching of a 5-connected O1PPG G with $|M| = 3$ that is not extendable. By Corollary 5,

G is 2-extendable. Hence there exists $S \subset V(G)$ which satisfies (i) and (ii) of Lemma 18 for $k = 2$. Note that (i) of Lemma 18 implies $|S| \geq 6$.

First, we show that $|S| \leq 7$. Suppose to the contrary that $|S| \geq 8$. Then $C_o(G - S) \geq 4$ by (ii) of Lemma 18, and hence $Q[S]$ has at least four faces bounded by a closed walk with length at least 6 by Lemma 8. Hence we obtain $|E(Q[S])| \geq 2|F(Q[S])| + 4$ by (i) of Lemma 10 with $p \geq 4$ and $q \geq 3$. On the other hand, we have $2|F(Q[S])| + 3 \geq |E(Q[S])|$ by Lemma 11 and (ii) of Lemma 18, a contradiction. Therefore, we have $|S| \in \{6, 7\}$. We will now divide the proof into the following two cases.

Case (α). There exists a connected component D of $G - S$ having exactly five neighbors in S. Let $S' \subset S$ denote the set of five neighbors of D. Then $Q[S']$ is the projective-bowtie by Lemma 14. If $S' \subset V(M)$, then G contains a region that satisfies (i) in the statement. Thus we assume that $S' \setminus V(M) \neq \emptyset$. Under the situation, note that we have $|S| = 7$; we have $C_o(G - S) = 3$ by Lemma 18. Then, by (ii) of Lemma 10 with $q = 3$ and $p = 3$, we have $|F(Q[S])| \leq 3$, and hence $|F(Q[S])| = 3$. This means that the equality of (ii) of Lemma 10 holds, and hence $Q[S]$ has exactly three 6-gonal faces. Note that $|E(Q[S])| = 9$ by (i) of Lemma 10 and Lemma 11, and that $C_e(G - S) = 0$ by Lemma 8.

Now we put $S \setminus S' = \{x, y\}$. Observe that $Q[S']$, which is a projective-bowtie, has exactly six edges. By Lemma 9, and since $|E(Q[S]) \setminus E(Q[S'])| = 3$, we have $\deg_{Q[S]}(x) = \deg_{Q[S]}(y) = 2$, and this implies that $s_1 x y s_2$ is a path of length 3, where $s_1, s_2 \in S'$. Since $Q[S]$ has exactly three 6-gonal faces by the argument above, $Q[S]$ is a graph shown as (1) or (2) in Figure 4, up to homeomorphism. Finally, we consider one vertex $z \in S \setminus V(M)$. If $\deg_{Q[S]}(z) = 2$, then $Q[S]$ has a region that satisfies (i) in the statement. Thus $\deg_{Q[S]}(z) \geq 3$, and hence $Q[S]$ with big gray vertices of $V(M)$ is (a), (b) or (c) in Figure 1.

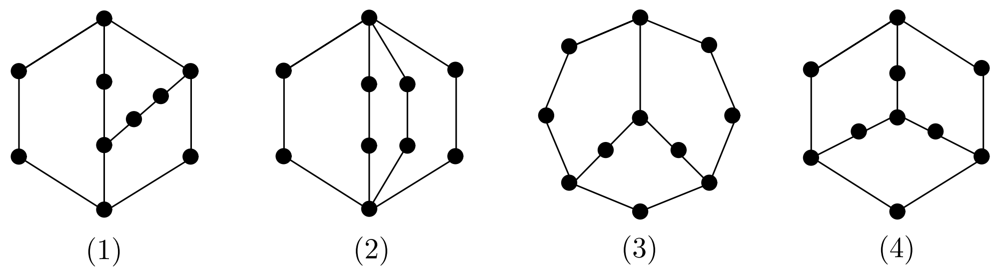

Figure 4. $Q[S]$ in the case of $|S| = 7$.

Case (β). There exists no connected component of $G - S$ having exactly five neighbors in S. First, assume that $|S| = 6$, that is, $S = V(M)$. Then by Lemma 18, $C_o(G - S) = 2$. By the hypothesis of Case (β), S is a minimal 6-cut of G. By Lemma 16, $Q[S]$ is one of (I), (II), (III) and (IV) shown in Figure 3.

Then $Q[S]$ contains a region that satisfies (i) in the statement.

Next, assume that $|S| = 7$. By Lemma 18, $C_o(G-S) = 3$. Then by the same argument as in Case (α), we have $|F(Q[S])| = 3$ and $|E(Q[S])| = 9$; observe that the equality in (ii) of Lemma 10 holds, and hence $Q[S]$ is a 2-cell embedding. Let f_1, f_2 and f_3 denote the three faces of $Q[S]$, each of which is a 6-gonal face. Then, f_i contains exactly one odd component of $G - S$ for each $i \in \{1, 2, 3\}$ by Lemma 8. Note that f_i is bounded by a 6-cycle for each $i \in \{1, 2, 3\}$; otherwise, there would be an odd component satisfying the condition of Case (α).

Let C denote the boundary 6-cycle of f_1, and put $S' = V(C)$. Then $Q[S']$ is (I), (II), (III) or (IV) in Figure 3 by Lemma 16, up to homeomorphism. Here, denote the unique vertex in $S \setminus S'$ by v. If $Q[S']$ is (I) of Figure 3, then v is adjacent to exactly three vertices of S' since $|E(Q[S]) \setminus E(Q[S'])| = 3$. Although $\deg_{Q[S]}(v) = 3$, only f_2 and f_3 can be incident to v. That is, v appears twice on the boundary closed walk of either f_2 or f_3, contradicting that every face of $Q[S]$ is bounded by a cycle. Moreover, (IV) of Figure 3 is not the case, since v is adjacent to exactly one vertex of $V(C)$, contradicting Lemma 9.

Therefore $Q[S']$ is either (II) or (III) of Figure 3. Observe that (II) is bipartite and (III) is non-bipartite. By Proposition 7, if $Q[S']$ is (II) (respectively, (III)), then f_2 and f_3 also have configurations (II) (respectively, (III)). See Figure 5. The center configuration illustrates (II) in an alternative form. There are exactly two ways, up to symmetry, to add v inside the unique 8-gonal face such that $\deg(v) = 2$ and the 8-gonal face is divided into two 6-gonal faces; see the left-hand side one and the right-hand side one in the figure. However, the configuration on the right-hand side has multiple edges incident to v, a contradiction. Therefore, $Q[S]$ corresponds to the configuration on the left-hand side, which is (3) in Figure 4. Similarly, (4) is derived from (III); the cases that do not lead to (4) contradict our assumptions. Finally, we specify a small black vertex, which is not in $V(M)$ as well as Case (α), and obtain (d), (e), (f) and (g) in Figure 1. ■

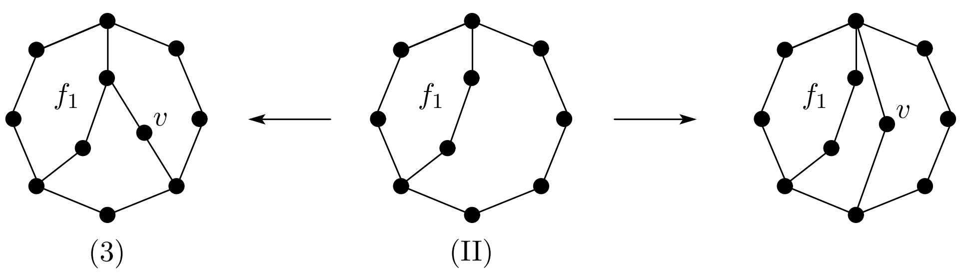

Figure 5. Adding v of degree 2 inside the 8-gonal region.

5. Remarks

In this paper, we have discussed matching extendability of O1PPG's. Then, how about for optimal 1-embedded graphs on the torus (or the Klein bottle)? We are aware that the situation changes significantly, particularly in contrast to the discussion on the sphere and the projective plane. At least, there exist optimal 1-embedded graphs on the torus that are not 1-extendable. (Consider an optimal 1-embedded graph G on the torus such that $Q(G)$ has a subgraph H that is also a quadrangulation of the torus, and that every face of H is an odd weighted region of G. Then every non-crossing edge of H is not extendable; observe that $|V(H)| = C_o(G - V(H))$.)

Moreover, every O1PG and every O1PPG is not 3-extendable since the graph has a vertex of degree exactly 6. However, the property does not hold for optimal 1-embedded graphs on the torus (or the Klein bottle); there exists infinitely many 8-regular graphs whose quadrangular subgraphs are 4-regular. At the end of the paper, we establish the following conjectures for those graphs:

Conjecture 19. *Let G be an optimal 1-embedded graph on the torus or the Klein bottle. If $Q(G)$ has no quadrangulation as a subgraph each of whose face corresponds to an odd weighted region of G, then G is 1-extendable.*

Conjecture 20. *Every 8-regular optimal 1-embedded graph on the torus or the Klein bottle is 3-extendable.*

Note that for 2-extendability, the statement for optimal 1-embedded graphs on the torus and that for optimal 1-embedded graphs on the Klein bottle might be substantially different; since the Klein bottle admits a separating simple closed curve that is not trivial, which is known as an *equator* of the surface.

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