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INDEPENDENT [k]-ROMAN DOMINATION ON GRAPHS

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Abstract

Given a function $f\colon V(G) \to \mathbb{Z}_{\geq 0}$ on a graph G, $AN(v)$ denotes the set of neighbors of $v \in V(G)$ that have positive labels under f. In 2021, Abdollahzadeh Ahangar *et al.* introduced the notion of [k]-Roman dominating function ([k]-RDF) of a graph G, which is a function $f\colon V(G) \to \{0, 1, \ldots, k+1\}$ such that $\sum_{u \in N[v]} f(u) \geq k + |AN(v)|$ for all $v \in V(G)$ with $f(v) < k$. The weight of f is $\sum_{v \in V(G)} f(v)$. The [k]-Roman domination number, denoted by $\gamma_{[kR]}(G)$, is the minimum weight of a [k]-RDF of G. The notion of [k]-RDF for $k = 1$ has been extensively investigated in the scientific literature since 2004, when introduced by Cockayne *et al.* as Roman domination. An independent [k]-Roman dominating function ([k]-IRDF) $f\colon V(G) \to \{0, 1, \ldots, k+1\}$ of a graph G is a [k]-RDF of G such that the set of vertices with positive labels is an independent set. The independent [k]-Roman domination number of G is the minimum weight of a [k]-IRDF of G and is denoted by $i_{[kR]}(G)$. In this paper, we propose the study of independent [k]-Roman domination on graphs for arbitrary $k \geq 1$. We prove that, for all $k \geq 3$, the decision problems associated with $i_{[kR]}(G)$ and $\gamma_{[kR]}(G)$ are $\mathcal{NP}$-complete for planar bipartite graphs with maximum degree 3. We also present lower and upper bounds for $i_{[kR]}(G)$. Moreover, we present lower and upper bounds for the parameter $i_{[kR]}(G)$ for a family of 3-regular graphs called generalized Blanuša snarks.

Keywords: Roman domination, dominating set, independent set.

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1. Introduction

Let $G = (V(G), E(G))$ be a graph with vertex set $V(G)$ and edge set $E(G)$. Two vertices $u, v \in V(G)$ are *adjacent* or *neighbors* if $uv \in E(G)$. We say that G is trivial if $|V(G)| = 1$. For every vertex $v \in V(G)$, the *open neighborhood* of v is the set $N(v) = \{u \in V(G) : uv \in E(G)\}$, and the *closed neighborhood* of v is the set $N[v] = \{v\} \cup N(v)$. The *degree* of a vertex $v \in V(G)$ is the number of neighbors of v and is denoted $d_G(v)$. A *leaf vertex* of G is a vertex $v \in V(G)$ with $d_G(v) = 1$. Graph G is r-regular if $d_G(v) = r$ for all $v \in V(G)$. The maximum degree of G is denoted by $\Delta(G)$. For any subset $S \subseteq V(G)$, the *induced subgraph* $G[S]$ is the graph whose vertex set is S and whose edge set consists of all edges in $E(G)$ that have both endpoints in S. As usual, P_n denotes a path on $n \geq 1$ vertices and C_n denotes a cycle on $n \geq 3$ vertices.

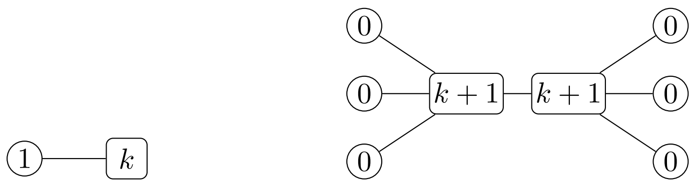
A set $S \subseteq V(G)$ is called a *dominating set* of G if every vertex $v \in V(G)\backslash S$ is adjacent to a vertex in S. The *domination number* of G, denoted $\gamma(G)$, is the minimum cardinality of a dominating set of G. A set of vertices $S \subseteq V(G)$ is called *independent* if no two vertices in S are adjacent. An *independent dominating set* of G is an independent set $S \subseteq V(G)$ that is also dominating. The *independent domination number* of G, denoted $i(G)$, is the minimum cardinality of an independent dominating set of G. The domination on graphs has been extensively studied in the scientific literature, giving rise to many variations [12], a well-known of them being the Roman domination [11].

Let G be a graph. For any function $f\colon V(G) \to \mathbb{Z}_{\geq 0}$ and $S \subseteq V(G)$, define $f(S) = \sum_{v \in S} f(v)$. A *Roman dominating function* (RDF) on G is a function $f\colon V(G) \to \{0, 1, 2\}$ such that every vertex $u \in V(G)$ with *label* $f(u) = 0$ is adjacent to at least one vertex $v \in V(G)$ with label $f(v) = 2$. The *weight* of an RDF f is defined as $\omega(f) = f(V(G)) = \sum_{v \in V(G)} f(v)$. The *Roman domination number* of G is the minimum weight over all RDFs on G and is denoted by $\gamma_R(G)$.

The conception of Roman domination on graphs was motivated by defense strategies devised by the Roman Empire during the reign of Emperor Constantine, 272–337 AD [21, 19]. The idea behind Roman domination is that labels 1 or 2 represent either one or two Roman legions stationed at a given Roman province (vertex v). A neighboring province (an adjacent vertex u) is considered to be *unsecured* if no legions are stationed there (i.e., $f(u) = 0$). An unsecured province u can be secured by sending a legion to u from an adjacent province v, by respecting the condition that a legion cannot be sent from a province v if doing so leaves that province without a legion. Thus, two legions must be stationed at a province ($f(v) = 2$) before one of the legions can be sent to an adjacent province.

Results on Roman domination and its variants have been collected in [6, 7, 8, 9, 10], summing up to more than two hundred papers. Many of these variants aim to increase the effectiveness of the defensive strategy modeled by Roman domination. In 2021, Abdollahzadeh Ahangar *et al.* [1] introduced the notion of $[k]$-Roman domination, a generalization of Roman domination, which groups many of these Roman domination's variants under the same definition. The idea behind $[k]$-Roman domination is that any unsecured province could be defended by at least k legions without leaving any secure neighboring province without military forces.

Let G be a graph and $f\colon V(G) \to \mathbb{Z}_{\geq 0}$ be a function. We say that a vertex v of G is *active* under f if $f(v) \geq 1$. For any vertex $v \in V(G)$, the *active neighborhood* of v under f, denoted by $AN(v)$, is the set of vertices $w \in N(v)$ such that $f(w) \geq 1$. For any integer $k \geq 1$, a $[k]$-*Roman dominating function* on G, also called $[k]$-RDF, is a function $f\colon V(G) \to \{0, 1, \ldots, k+1\}$ such that $f(N[v]) \geq k + |AN(v)|$ for every vertex $v \in V(G)$ with $f(v) < k$. The *weight* of a $[k]$-RDF f on G is defined as $\omega(f) = f(V(G)) = \sum_{v \in V(G)} f(v)$. The $[k]$-*Roman domination number* of G is the minimum weight that a $[k]$-RDF of G can have, and is denoted by $\gamma_{[kR]}(G)$. A $[k]$-RDF of G with weight $\gamma_{[kR]}(G)$ is called a $\gamma_{[kR]}$-*function* of G or $\gamma_{[kR]}(G)$-*function.* Given a $[k]$-RDF $f\colon V(G) \to \{0, 1, \ldots, k+1\}$ of a graph G, define $V_i = \{u \in V(G) : f(u) = i\}$ for $0 \leq i \leq k+1$. We call $(V_0, V_1, \ldots, V_{k+1})$ the *ordered partition* of $V(G)$ under f. Since there exists a 1-1 correspondence between the functions $f\colon V(G) \to \{0, 1, \ldots, k+1\}$ and the ordered partitions $(V_0, V_1, \ldots, V_{k+1})$ of $V(G)$, it is common to use the notation $f = (V_0, V_1, \ldots, V_{k+1})$ to refer to a $[k]$-RDF of G. By the definition of ordered partition, we can alternatively define the weight of a $[k]$-RDF f as $\omega(f) = \sum_{p=0}^{k+1} p|V_p|$. Figure 1 shows some graphs endowed with $[k]$-RDFs.



(a) Path with a $[k]$-RDF with weight $k+1$.

(b) Tree with a $[k]$-RDF with weight $2k+2$.

Figure 1. Two graphs endowed with $[k]$-Roman dominating functions.

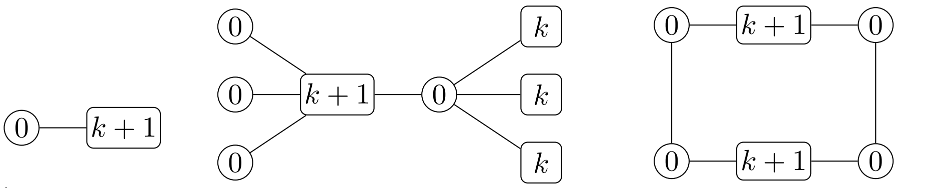
For every $k \geq 1$, the $[k]$-Roman Domination Problem is to determine $\gamma_{[kR]}(G)$ for an arbitrary graph G. Khalili *et al.* [13] proved that the decision version of the $[k]$-Roman Domination Problem is $\mathcal{NP}$-complete even when restricted to bipartite and chordal graphs. Moreover, Valenzuela-Tripodoro *et al.* [23] proved that the decision version of $[k]$-RDP is $\mathcal{NP}$-complete even when restricted to star convex and comb convex bipartite graphs.

Note that [1]-Roman domination is equivalent to the original Roman domination definition. In addition, [2]-Roman domination has been previously studied [4] under the name of Double Roman domination, as well as [3]-Roman domination has been investigated [1] under the name of Triple Roman domination, and [4]-Roman domination has been recently studied under the name of Quadruple Roman domination [3]. Recently, Khalili *et al.* [13] and Valenzuela-Tripodoro *et al.* [23] presented sharp upper and lower bounds for the $[k]$-Roman domination number for all $k \geq 1$.

Given a $[k]$-Roman dominating function $f = (V_0, V_1, \ldots, V_{k+1})$ on a graph G, we observe that the set of vertices $S = V_1 \cup V_2 \cup \cdots \cup V_{k+1}$ is a dominating set of G since $V(G) \backslash S = V_0$ and every vertex in V_0 is adjacent to a vertex in S. This connection between dominating sets and the set of active vertices of a graph G under a $[k]$-Roman dominating function makes it possible to relate the parameters $\gamma(G)$ and $\gamma_{[kR]}(G)$ as well as to extend some restrictions traditionally imposed on dominating sets to $[k]$-Roman dominating functions. An example is the concept of independent dominating set: one may require the dominating set of active vertices of G to be also independent. Indeed, in their seminal paper, Cockayne *et al.* [11] introduced the notion of Roman dominating functions $f = (V_0, V_1, V_2)$ whose set of active vertices $V_1 \cup V_2$ is an independent set, which are called *independent Roman dominating functions*. In 2019, Maimani *et al.* [15] introduced the notion of *independent double Roman dominating function*, which is a [2]-Roman dominating function $f = (V_0, V_1, V_2, V_3)$ of a graph G such that the set of active vertices $V_1 \cup V_2 \cup V_3$ is an independent set. When studying independent Roman domination and independent double Roman domination, one can observe some differences but many similarities. Thus, based on the previous observations, we propose a generalization of independent Roman domination and independent double Roman domination, defined as follows.

A $[k]$-Roman dominating function $f = (V_0, V_1, \ldots, V_{k+1})$ on a graph G is called an *independent $[k]$-Roman dominating function*, or $[k]$-IRDF for short, if the set of active vertices $V_1 \cup V_2 \cup \cdots \cup V_{k+1}$ is an independent set. The *independent $[k]$-Roman domination number* $i_{[kR]}(G)$ is the minimum weight of a $[k]$-IRDF on G, and a $[k]$-IRDF of G with weight $i_{[kR]}(G)$ is called an *$i_{[kR]}$-function* of G or *$i_{[kR]}(G)$-function*. The *Independent $[k]$-Roman Domination Problem* consists in determining $i_{[kR]}(G)$ for an arbitrary graph G. Since every $[k]$-IRDF is a $[k]$-RDF, we trivially obtain that $\gamma_{[kR]}(G) \leq i_{[kR]}(G)$ for every graph G. As an example, Figure 2 shows some graphs with $i_{[kR]}$-functions.

From the definition of independent $[k]$-Roman domination, we know that the active vertices $v \in V(G)$ with $f(v) < k$ must have at least one active neighbor since the condition $f(N[v]) \geq k + |AN(v)|$ must be satisfied. In addition to the previous condition, an independent $[k]$-Roman domination function also imposes that the set of active vertices must be independent. However, these two condi-



(a) Path with a [k]-IRDF. (b) Tree with a [k]-IRDF. (c) Cycle with a [k]-IRDF.

Figure 2. Graphs endowed with independent [k]-Roman dominating functions.

tions considered simultaneously imply that an independent [k]-Roman dominating function does not assign labels from the set $\{1, 2, \ldots, k-1\}$ to the vertices of a graph G. These initial observations concerning [k]-IRDFs are explicitly stated in the following propositions.

Proposition 1. *If G is a graph, then $\gamma_{[kR]}(G) \leq i_{[kR]}(G)$.*

Proposition 2. *If $f = (V_0, V_1, \ldots, V_{k+1})$ is a [k]-IRDF of a graph G, then $V_i = \emptyset$ for all $i \in \{1, 2, \ldots, k-1\}$.*

By Proposition 2, we can represent a [k]-IRDF $f = (V_0, V_1, \ldots, V_{k+1})$ simply as $f = (V_0, V_k, V_{k+1})$. Moreover, note that the weight of a [k]-IRDF $f = (V_0, V_k, V_{k+1})$ is also given by $\omega(f) = k|V_k| + (k+1)|V_{k+1}|$.

In this paper, we propose the study of independent [k]-Roman domination on graphs for arbitrary $k \geq 1$. The next sections of this paper are organized as follows. In Section 2, we prove that, for all $k \geq 3$, the decision versions of the Independent [k]-Roman Domination Problem and [k]-Roman Domination Problem are $\mathcal{NP}$-complete, even when restricted to planar bipartite graphs with maximum degree 3. In Section 3, we present some sharp lower and upper bounds for the independent [k]-Roman domination number of arbitrary graphs. In Section 4, we present lower and upper bounds for the independent [k]-Roman domination number for an infinite family of 3-regular graphs called Generalized Blanuša Snarks. Section 5 presents our concluding remarks.

2. Complexity Results

In this section, we show that, for every integer $k \geq 3$, the decision versions of the [k]-Roman Domination Problem ([k]-ROM-DOM) and the Independent [k]-Roman Domination Problem ([k]-IROM-DOM) are $\mathcal{NP}$-complete when restricted to graphs with maximum degree 3. We remark that the $\mathcal{NP}$-completeness of [1]-ROM-DOM and [1]-IROM-DOM, when restricted to the same class of graphs, has already been established [14]. In the remaining of this section, we deal with $k \geq 3$. Consider the following decision problems.

$[k]$-ROM-DOM
Instance: A graph G and a positive integer ℓ.
Question: Does G have a $[k]$-RDF with weight at most ℓ?

$[k]$-IROM-DOM
Instance: A graph G and a positive integer ℓ.
Question: Does G have a $[k]$-IRDF with weight at most ℓ?

For $k \geq 3$, we show that $[k]$-ROM-DOM and $[k]$-IROM-DOM are $\mathcal{NP}$-complete when restricted to graphs with maximum degree 3 through a reduction from the vertex cover problem. A *vertex cover* of a graph G is a set of vertices $S \subseteq V(G)$ such that each edge of G is incident to some vertex in S. The *vertex covering number* of G, denoted $\tau(G)$, is the cardinality of a smallest vertex cover of G. Given a graph G and a positive integer ℓ, the *Vertex Cover Problem* (VCP) consists in deciding whether G has a vertex cover S with cardinality at most ℓ. The Vertex Cover Problem is $\mathcal{NP}$-complete even when restricted to 2-connected planar 3-regular graphs [16] and we use this result to construct a polynomial time reduction from the Vertex Cover Problem to $[k]$-ROM-DOM ($[k]$-IROM-DOM) as follows.

Construction: given a 2-connected planar 3-regular graph G, construct a new graph F from G by replacing each edge $e = uv \in E(G)$ by a gadget G_e illustrated in Figure 3. Note that F is a planar bipartite graph with maximum degree 3.

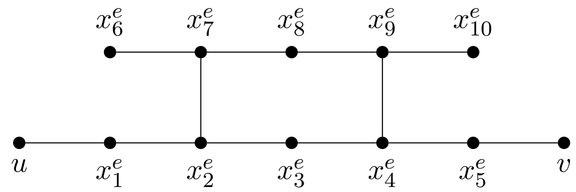


Figure 3. Gadget G_e used in the reduction.

In order to prove the $\mathcal{NP}$-completeness result, given in Theorem 9, we need the following auxiliary results.

Lemma 3 (Khalili *et al.* [13])**.** *If $k \geq 2$, then in a $\gamma_{[kR]}(G)$-function of a graph G, no vertex needs to be assigned the label* 1.

Proposition 4. *Let $k \geq 1$ be an integer. If G is a connected graph with at least 3 vertices, then in a $[k]$-Roman dominating function of G with weight $\gamma_{[kR]}(G)$, no leaf vertex of G needs to be assigned the label $k+1$.*

Proof. Let G be a connected graph on at least 3 vertices and f a $\gamma_{[kR]}$-function of G. Let $v \in V(G)$ be a leaf vertex and let $w \in V(G)$ be the neighbor of v. For the purpose of contradiction, suppose that f needed to assign $k+1$ to v, that is, $f(v) = k+1$. Since f is a $\gamma_{[kR]}$-function, $f(w) \leq k$ (otherwise, by assigning 0 to v we obtain a [k]-RDF with weight smaller then $\omega(f)$). We modify the labeling f by exchange the labels of the vertices v and w and maintaining the labels of all the other vertices the same. Note that f continues to be a [k]-RDF with the same weight as before and the vertex v does not have weight $k+1$ anymore. ■

Proposition 5. *Let $k \geq 1$ be an integer and G be a connected graph with at least 3 vertices. In any $\gamma_{[kR]}$-function f of G, no leaf vertex needs to be assigned a label different from 0 or k.*

Proof. Let G and f be as in the hypothesis and let $v \in V(G)$ be a leaf vertex with neighbor $w \in V(G)$. By Lemma 3 and Proposition 4, $f(v) \notin \{1, k+1\}$. If $f(v) \in \{0, k\}$, then f is the desired function. Thus, suppose that f needs to assign a label in the set $\{2, 3, \ldots, k-1\}$ to vertex v, that is, $f(v) \in \{2, 3, \ldots, k-1\}$. In this case, the neighbor w of v has $f(w) \neq 0$ and is, thus, an active neighbor of v. By the definition of [k]-RDF, $f(N[v]) = f(v)+f(w) \geq k+|AN(v)| = k+1$. Thus, $f(v) + f(w) \geq k+1$. We modify the labeling f by assigning label $f(v) + f(w)$ to vertex w, by assigning label 0 to vertex v, and maintaining the labels of all the remaining vertices of G the same. Note that f continues to be a [k]-RDF with the same weight as before and the new label of v does not belong to the set $\{2, 3, \ldots, k-1\}$, which is a contradiction. ■

Lemma 6. *Let $k \geq 2$ be an integer. Given a 2-connected planar 3-regular graph G, let F be the graph constructed from G by replacing each edge $e = uv$ in G by a gadget G_e shown in Figure 3. Then, any $\gamma_{[kR]}$-function f of F satisfies $(f(x_6^e), f(x_7^e)) \in \{(0, k+1), (k, 0)\}$ and $(f(x_9^e), f(x_{10}^e)) \in \{(k+1, 0), (0, k)\}$.*

Proof. Let f be a $\gamma_{[kR]}$-function of F. We only analyze the values $f(x_6^e)$ and $f(x_7^e)$ since the analysis for $f(x_9^e)$ and $f(x_{10}^e)$ is analogous and follows from the symmetry of G_e along the vertical axis.

Note that x_6^e is a leaf vertex and $N(x_6^e) = \{x_7^e\}$. By Proposition 5, either $f(x_6^e) = 0$ or $f(x_6^e) = k$. If $f(x_6^e) = 0$, then $f(x_7^e) = k+1$ by the definition of [k]-RDF, and the result follows. Thus, suppose that $f(x_6^e) = k$. By Lemma 3, $f(x_7^e) \neq 1$. If $f(x_7^e) \geq 2$, then $f(x_6^e) + f(x_7^e) \geq k+2$. Hence, it would be possible to obtain a [k]-RDF with smaller weight by assigning label $f(x_6^e) + f(x_7^e) - 1$ to x_7^e and 0 to v, thus contradicting the choice of f. Therefore, we obtain that $f(x_7^e) = 0$, and the result follows. ■

Lemma 7. *Let $k \geq 3$ be an integer. Given a 2-connected planar 3-regular graph G, let F be a graph constructed from G by replacing each edge $e = uv$ in G by* *a gadget G_e illustrated in Figure 3. Let $U_e = \{x_2^e, x_3^e, x_4^e, x_6^e, x_7^e, x_8^e, x_9^e, x_{10}^e\} \subset V(G_e)$. Then, in any $\gamma_{[kR]}$-function f of the graph F, we have that the function f restricted to U_e is a $[k]$-RDF of $F[U_e]$ with weight $f(U_e) = 3k+2$. Moreover, $(f(x_2^e), f(x_4^e)) \in \{(0,0), (k+1,0), (0,k+1)\}$.*

Proof. Let $k \geq 3$ be an integer. Let G and F be as in the hypothesis. Let $f\colon V(F) \to \{0,1,\ldots,k+1\}$ be a $\gamma_{[kR]}$-function of F. For each gadget $G_e \subset F$, define $U_e = \left\{x_2^e, x_3^e, x_4^e, x_6^e, x_7^e, x_8^e, x_9^e, x_{10}^e\right\} \subset V(G_e)$. By Lemma 6, $(f(x_6^e), f(x_7^e)) \in \{(0,k+1), (k,0)\}$ and $(f(x_9^e), f(x_{10}^e)) \in \{(k+1,0), (0,k)\}$. Thus, there are four cases to analyze, depending on the values of the labels $f(x_6^e)$, $f(x_7^e)$, $f(x_9^e)$ and $f(x_{10}^e)$.

Case 1. $(f(x_6^e), f(x_7^e)) = (k,0)$ and $(f(x_9^e), f(x_{10}^e)) = (0,k)$. We claim that this case cannot occur. For the purpose of contradiction, suppose it occurs. Since f is a $\gamma_{[kR]}$-function and x_8^e has no active neighbor, we have that $f(x_8^e) = k$. Note that $\sum_{i=6}^{10} f(x_i^e) = 3k$. Thus, we can redefine the labels of some vertices of F so as to obtain another $[k]$-RDF f' of F with smaller weight than f such that $\sum_{i=6}^{10} f'(x_i^e) = 2k+2 < 3k$, as follows: let $(f'(x_6^e), f'(x_7^e), f'(x_8^e), f'(x_9^e), f'(x_{10}^e))$ $= (0, k+1, 0, k+1, 0)$ and make $f'(x) = f(x)$ for every remaining vertex x of F. This contradicts the choice of f as a $\gamma_{[kR]}$-function.

Case 2. $(f(x_6^e), f(x_7^e)) = (0,k+1)$ and $(f(x_9^e), f(x_{10}^e)) = (k+1,0)$. Since f is a $\gamma_{[kR]}$-function, we have that $f(x_8^e) = 0$. By the definition of $[k]$-RDF, we have that $f(N[x_3^e]) = f(x_2^e) + f(x_3^e) + f(x_4^e) \geq k + |AN(x_3^e)| \geq k$. All these facts imply that $f(U_e) = \sum_{w \in U_e} f(w) \geq 3k+2$. Moreover, a $[k]$-RDF of $F[U_e]$ with weight $3k+2$ is obtained by assigning labels $f(x_2^e) = 0$, $f(x_3^e) = k$ and $f(x_4^e) = 0$. Therefore, $f(U_e) = 3k+2$, $(f(x_2^e), f(x_4^e)) = (0,0)$, and the result follows.

Case 3. $(f(x_6^e), f(x_7^e)) = (0,k+1)$ and $(f(x_9^e), f(x_{10}^e)) = (0,k)$. Since f is a $\gamma_{[kR]}$-function, we have that $f(x_8^e) = 0$. Moreover, since $f(x_9^e) = 0$ and $f(x_8^e) + f(x_{10}^e) < k+1$ we obtain that $f(x_4^e) \neq 0$. By the definition of $[k]$-RDF and since $f(x_4^e) \neq 0$, we have that $f(N[x_3^e]) = f(x_2^e) + f(x_3^e) + f(x_4^e) \geq k + |AN(x_3^e)| \geq k+1$. All these facts imply that $f(U_e) = \sum_{w \in U_e} f(w) \geq 3k+2$. From the previous facts, we have that $f(U_e) = 3k+2$ only if $f(N[x_3^e]) = k + |AN(x_3^e)| = k+1$, which implies that $f(x_2^e) = 0$. Thus, $f(x_2^e) + f(x_3^e) + f(x_4^e) = 0 + f(x_3^e) + f(x_4^e) = k+1$, i.e., $f(x_3^e) + f(x_4^e) = k+1$. Since f is a $\gamma_{[kR]}$-function, we obtain that $f(x_3^e) = 0$ and $f(x_4^e) = k+1$. Therefore, $(f(x_2^e), f(x_4^e)) = (0, k+1)$ and the result follows.

Case 4. $(f(x_6^e), f(x_7^e)) = (k,0)$ and $(f(x_9^e), f(x_{10}^e)) = (k+1,0)$. The proof for this case is analogous to the proof of the previous case and follows from the symmetry of G_e along the vertical axis.

Therefore, in any $\gamma_{[kR]}$-function f of the graph F, we have that the function f restricted to U_e is a $[k]$-RDF of $F[U_e]$ with weight $f(U_e) = 3k+2$. Moreover, $(f(x_2^e), f(x_4^e)) \in \{(0,0), (k+1,0), (0,k+1)\}$. ■

Theorem 8. *Let $k \geq 3$ be an integer. Given a 2-connected planar 3-regular graph G, let F be a planar bipartite graph with $\Delta(F) = 3$ constructed from G by replacing each edge $e = uv$ in G by a gadget G_e illustrated in Figure 3. Then,*

$$\gamma_{[kR]}(F) = i_{[kR]}(F) = \tau(G) + k|V(G)| + (3k + 2)|E(G)|.$$

Proof. Let G and F be as in the statement of the theorem. Let C be a vertex cover of G with $|C| = \tau(G)$.

We initially prove that $i_{[kR]}(F) \leq \tau(G) + k|V(G)| + (3k+2)|E(G)|$. In order to do this, we construct an appropriate [k]-IRDF $f = (V_0, V_k, V_{k+1})$ of F as follows. First, define two empty sets D_k and D_{k+1}. For each gadget $G_e \subset F$, associated with an edge $e = uv \in E(G)$, do the following. If $v \in C$, then, add the vertex x^e_6 to D_k and add the vertices x^e_2 and x^e_9 to D_{k+1}; otherwise, add the vertex x^e_{10} to D_k and add the vertices x^e_4 and x^e_7 to D_{k+1}. Note that $|D_k| = |E(G)|$ and $|D_{k+1}| = 2|E(G)|$. Define the function $f = (V_0, V_k, V_{k+1})$ such that $V_0 = V(F)\backslash(V(G) \cup D_k \cup D_{k+1})$, $V_k = D_k \cup V(G)\backslash C$ and $V_{k+1} = D_{k+1} \cup C$. From the definition of f, we have that f is a [k]-IRDF of F with weight $\omega(f) = k|D_k \cup V(G)\backslash C| + (k + 1)|D_{k+1} \cup C| = k(|E(G)| + |V(G)| - \tau(G)) + (k+1)(2|E(G)| + \tau(G)) = \tau(G) + k|V(G)| + (3k+2)|E(G)|$. Therefore, $i_{[kR]}(F) \leq \omega(f) = \tau(G) + k|V(G)| + (3k+2)|E(G)|$.

Next, we show that $\gamma_{[kR]}(F) \geq \tau(G) + k|V(G)| + (3k + 2)|E(G)|$. Let $f = (V_0, \emptyset, V_2, \ldots, V_{k+1})$ be a $\gamma_{[kR]}$-function of F. Let G_e be a gadget of F, for any edge $e = uv \in E(G)$. Define the set $U_e = \{x^e_2, x^e_3, x^e_4, x^e_6, x^e_7, x^e_8, x^e_9, x^e_{10}\} \subset V(G_e) \subset V(F)$. By Lemma 7, the function f restricted to U_e has weight $3k + 2$ and is a [k]-RDF of the subgraph induced by U_e. Moreover, $(f(x^e_2), f(x^e_4)) \in \{(0, k + 1), (k + 1, 0), (0, 0)\}$. Let $S = \{x \in U_e : f(x) \neq 0, e \in E(G)\}$. Let $V' \subset V(F)$ be the set of vertices that are not adjacent to some vertex in S and are not in S, that is, $V' = V(F)\backslash N[S]$. Let F' be the induced subgraph $F[V']$. For each $e \in E(G)$, all the vertices in U_e and at most one of the vertices x^e_1 and x^e_5 are not in V'. This implies that F' is a forest of trees with $|V(G)|$ components such that each component is a star whose central vertex is a vertex $z \in V(G)$. Let T be a component of F'. If T is a single vertex (i.e. $V(T) = \{z\}$), then $f(z) = k$. On the other hand, if T is not a single vertex, then z is the central vertex of the star T and $f(z) = k+1$. Let $D = V(G) \cap V_{k+1}$. From the above discussion, we conclude that D is a vertex cover of G. Since each subset U_e contributes with $3k+2$ to the weight of f and there are $|E(G)|$ of these subsets, then they contribute to a total of $(3k + 2)|E(G)|$ to the weight of f. From these facts we obtain that $\omega(f) = (k+1)|D| + k(|V(G)| - |D|) + (3k+2)|E(G)| = |D| + k|V(G)| + (3k+2)|E(G)|$. Thus, $\tau(G) \leq |D| = \omega(f) - k|V(G)| - (3k + 2)|E(G)| = \gamma_{[kR]}(G) - k|V(G)| - (3k + 2)|E(G)|$. Therefore, $\gamma_{[kR]}(G) \geq \tau(G) + k|V(G)| + (3k + 2)|E(G)|$.

Since $\gamma_{[kR]}(G) \leq i_{[kR]}(G)$ (see Proposition 1), we have that $\tau(G) + k|V(G)| + (3k + 2)|E(G)| \leq \gamma_{[kR]}(G) \leq i_{[kR]}(G) \leq \tau(G) + k|V(G)| + (3k + 2)|E(G)|$, and the result follows. ■

Theorem 9. *Let $k \geq 3$ be an integer. Then, $[k]$-ROM-DOM (respectively, $[k]$-IROM-DOM) is $\mathcal{NP}$-complete even when restricted to planar bipartite graphs G with $\Delta(G) = 3$.*

Proof. We first show that $[k]$-ROM-DOM (respectively, $[k]$-IROM-DOM) is a member of $\mathcal{NP}$. Given any instance (G, ℓ) of $[k]$-ROM-DOM (respectively, $[k]$-IROM-DOM) and a certificate function $f \colon V(G) \to \{0, 1, \ldots, k{+}1\}$, we can verify (in polynomial time) if $\sum_{v \in V(G)} f(v) \leq \ell$ and if $\sum_{u \in N[v]} f(u) \geq |AN(v)| + k$ for every $v \in V(G)$ (respectively, in the case of $[k]$-IROM-DOM, it is also necessary to check if $V_k \cup V_{k+1}$ is an independent set). Next, we show that $[k]$-ROM-DOM (respectively, $[k]$-IROM-DOM) is $\mathcal{NP}$-hard. Recall that we showed how to construct a planar bipartite graph F with $\Delta(F) = 3$ from a given 2-connected planar 3-regular graph G in polynomial time on $|E(G)|$. From Theorem 8, we deduce that there exists a polynomial time algorithm that calculates $\tau(G)$ if and only if there exists a polynomial time algorithm that calculates $\gamma_{[kR]}(F)$ (respectively, $i_{[kR]}(F)$). However, since VCP is $\mathcal{NP}$-complete even when restricted to 2-connected planar 3-regular graphs, we obtain, from this reduction, that $[k]$-ROM-DOM (respectively, $[k]$-IROM-DOM) is $\mathcal{NP}$-complete even when restricted to planar bipartite graphs with maximum degree 3. ■

3. Bounds for the Independent $[k]$-Roman Domination Number

In this section, we present some lower and upper bounds for the independent $[k]$-Roman domination number of arbitrary graphs. Since the set $V_k \cup V_{k+1}$ is an independent dominating set in every $[k]$-IRDF $f = (V_0, V_k, V_{k+1})$ of a graph G, it seems reasonable that $i_{[kR]}(G)$ and $i(G)$ are related, such as shown in the following proposition.

Proposition 10. *Let $k \geq 1$ be an integer. If G is a graph, then $k \cdot i(G) \leq i_{[kR]}(G) \leq (k+1) \cdot i(G)$.*

Proof. Given a minimum independent dominating set S of a graph G, we define a $[k]$-IRDF $f = (V_0, V_k, V_{k+1})$ of G with weight $(k+1)i(G)$ by making $V_{k+1} = S$, $V_k = \emptyset$ and $V_0 = V(G)\backslash S$. Hence, $i_{[kR]}(G) \leq \omega(f) = (k+1)i(G)$.

Now, let $f = (V_0, V_k, V_{k+1})$ be an $i_{[kR]}$-function of a graph G. Since $i(G) \leq |V_k| + |V_{k+1}|$, we have that $k \cdot i(G) \leq k(|V_k| + |V_{k+1}|) \leq k|V_k| + (k+1)|V_{k+1}| = i_{[kR]}(G)$, and the result follows. ■

The lower bound presented in Proposition 10 is tight since it is attained by empty graphs. Moreover, graphs whose independent $[k]$-Roman domination number equals the upper bound given in Proposition 10 receive a specific name. We say that a graph G is *independent* [k]*-Roman* when $i_{[kR]}(G) = (k+1)i(G)$. The next lemma is a generalization of a result of Shao *et al.* [20] and presents a characterization of independent [k]-Roman graphs.

Lemma 11. *Let G be a graph. Then G is independent* [k]*-Roman if and only if G has an $i_{[kR]}$-function $f = (V_0, V_k, V_{k+1})$ such that $V_k = \emptyset$.*

Proof. Let G be a graph. First, suppose that G has an $i_{[kR]}(G)$-function $f = (V_0, V_k, V_{k+1})$ such that $V_k = \emptyset$. This implies that $i(G) \leq |V_k| + |V_{k+1}| = |V_{k+1}|$ and that $i_{[kR]}(G) = (k+1)|V_{k+1}|$. Thus, $(k+1)i(G) \leq (k+1)|V_{k+1}| = i_{[kR]}(G)$. By Proposition 10, $i_{[kR]}(G) \leq (k+1)i(G)$. Therefore, $i_{[kR]}(G) = (k+1)i(G)$ and G is independent [k]-Roman.

Now, consider G independent [k]-Roman. For the purpose of contradiction, suppose that every $i_{[kR]}(G)$-function $f = (V_0, V_k, V_{k+1})$ has $V_k \neq \emptyset$. Let $f = (V_0, V_k, V_{k+1})$ be an $i_{[kR]}$-function of G with $|V_k|$ as minimum as possible. From the definition of [k]-IRDF, we know that $i(G) \leq |V_k \cup V_{k+1}| = |V_k| + |V_{k+1}|$.

In fact, we claim that $i(G) = |V_k| + |V_{k+1}|$. In order to prove this claim, suppose that there exists a minimum independent dominating set S of G such that $|S| < |V_k| + |V_{k+1}|$. Let $g = (V_0^g, V_k^g, V_{k+1}^g)$ be a function with $V_k^g = \emptyset$, $V_{k+1}^g = S$ and $V_0^g = V(G)\backslash S$. Thus, g is a [k]-IRDF of G with $|V_{k+1}^g| = i(G)$. Since G is independent [k]-Roman, we have that $(k+1)i(G) = i_{[kR]}(G)$. Moreover, by the definition of [k]-IRDF, we know that $i_{[kR]}(G) = k|V_k| + (k+1)|V_{k+1}|$. Then, we have that $\omega(g) = (k+1)|V_{k+1}^g| = (k+1)i(G) = i_{[kR]}(G) = k|V_k| + (k+1)|V_{k+1}| = \omega(f)$. Thus, we have $\omega(g) = \omega(f) = i_{[kR]}(G)$, but $V_k^g = \emptyset$. In other words, we found an $i_{[kR]}(G)$-function $g = (V_0^g, V_k^g, V_{k+1}^g)$ with $V_k^g = \emptyset$, which is a contradiction. Therefore, $i(G) = |V_k| + |V_{k+1}|$ as claimed.

Since G is independent [k]-Roman, we have that $(k+1)i(G) = i_{[kR]}(G)$ and, thus, $(k+1)(|V_k| + |V_{k+1}|) = (k+1)i(G) = i_{[kR]}(G) = k|V_k| + (k+1)|V_{k+1}|$, implying that $|V_k| = 0$, which is a contradiction.

Therefore, we conclude that G has an $i_{[kR]}(G)$-function $f = (V_0, V_k, V_{k+1})$ such that $V_k = \emptyset$. ■

Another useful kind of lower bound connects the parameter with the maximum degree and number of vertices of the graph. As an example, in what concerns the [k]-Roman domination number, Valenzuela-Tripodoro *et al.* [23] presented the following lower bound for the [k]-Roman domination number of nontrivial connected graphs.

Theorem 12 (Valenzuela-Tripodoro *et al.* [23])**.** *Let $k \geq 1$ be an integer. Let G be a nontrivial connected graph with maximum degree $\Delta(G) \geq k$. Then $\gamma_{[kR]}(G) \geq \frac{|V(G)|(k+1)}{\Delta(G)+1}$.*

Since $i_{[kR]}(G) \geq \gamma_{[kR]}(G)$, we immediatelly obtain the following corollary from Theorem 12.

Corollary 13. *Let $k \geq 1$ be an integer. Let G be a nontrivial connected graph with maximum degree $\Delta(G) \geq k$. Then $i_{[kR]}(G) \geq \frac{|V(G)|(k+1)}{\Delta(G)+1}$.*

We remark that Corollary 13 only applies for the cases where $k \leq \Delta(G)$. In the next theorem we present a new lower bound for the independent $[k]$-Roman domination number of connected graphs G for all $k \geq 4$.

Theorem 14. *Let $k \geq 4$ be an integer. If G is a nontrivial connected graph with $\Delta(G) \geq 1$, then*

$$i_{[kR]}(G) \geq \frac{|V(G)|(k+1)}{\Delta(G)+1}.$$

Moreover, if $i_{[kR]}(G) = \frac{|V(G)|(k+1)}{\Delta(G)+1}$, then G is independent $[k]$-Roman.

Proof. Let $k \geq 4$ be an integer and G be a nontrivial connected graph with maximum degree $\Delta \geq 1$. Let $f\colon V(G) \to \{0, k, k+1\}$ be an $i_{[kR]}(G)$-function. Recall that $V_i = \{w \in V(G) : f(w) = i\}$ for $i \in \{0, k, k+1\}$. In this proof, we use a discharging procedure similar to the approach used by Shao *et al.* [20]. Our discharging procedure is described as follows. Firstly, each vertex $v \in V(G)$ is assigned the initial charge $s(v) = f(v)$. Next, we apply the discharging procedure defined by means of the following two rules.

Rule 1. Every vertex $v \in V(G)$ with $s(v) = k+1$ sends a charge of $\frac{k+1}{\Delta+1}$ to each vertex in $N(v) \cap V_0$.

Rule 2. Every vertex $v \in V(G)$ with $s(v) = k$ sends a charge of $\frac{(k-2)(k+1)}{k(\Delta+1)}$ to each vertex in $N(v) \cap V_0$.

Denote by $s'(v)$ the final charge of vertex v after applying the discharging procedure. Note the following.

I. For each vertex $v \in V(G)$ with $f(v) = k+1$, since it sends charge to at most $d_G(v)$ vertices, by Rule 1 we obtain that the final charge of v is $s'(v) \geq s(v) - d_G(v)\frac{k+1}{\Delta+1} \geq (k+1) - \frac{\Delta(k+1)}{\Delta+1} = \frac{k+1}{\Delta+1}$, that is, $s'(v) \geq \frac{k+1}{\Delta+1}$.

II. For each vertex $v \in V(G)$ with $f(v) = k$, since it sends charge to at most $d_G(v)$ vertices, by Rule 2 we obtain that the final charge of v is $s'(v) \geq s(v) - d_G(v)\frac{(k-2)(k+1)}{k(\Delta+1)} \geq k - \frac{\Delta(k-2)(k+1)}{k(\Delta+1)} = \frac{k^2+\Delta k+2\Delta}{k(\Delta+1)} > \frac{k^2+\Delta k}{k(\Delta+1)} = \frac{k+\Delta}{\Delta+1} \geq \frac{k+1}{\Delta+1}$, that is, $s'(v) > \frac{k+1}{\Delta+1}$.

From the previous analysis, we obtain that $s'(v) \geq \frac{k+1}{\Delta+1}$ for all $v \in V(G)$ with $f(v) > 0$. Now, let us analyze an arbitrary vertex $v \in V(G)$ with $f(v) = 0$. Since f is a $[k]$-IRDF, we have that $f(N[v]) = f(N(v)) \geq |AN(v)| + k$. So, either v has at least one neighbor $w \in V_{k+1}$ or v has at least two neighbors $u_1, u_2 \in V_k$. If v has at least one neighbor $w \in V_{k+1}$, then $s'(v) \geq \frac{k+1}{\Delta+1}$ since w sent a charge of $\frac{k+1}{\Delta+1}$ to v. On the other hand, if v has at least two neighbors $u_1, u_2 \in V_k$, each of these neighbors sent a charge of $\frac{(k-2)(k+1)}{k(\Delta+1)}$ to v and, thus, $s'(v) \geq 2 \cdot \frac{(k-2)(k+1)}{k(\Delta+1)} \geq \frac{k+1}{\Delta+1}$ for $k \geq 4$. Hence, we obtain that $s'(v) \geq \frac{k+1}{\Delta+1}$ for all $v \in V(G)$. Moreover, since the discharging procedure does not change the total value of charge in G, we obtain that $i_{[kR]}(G) = \omega(f) = \sum_{v \in V(G)} f(v) = \sum_{v \in V(G)} s(v) = \sum_{v \in V(G)} s'(v) \geq \sum_{v \in V(G)} \frac{k+1}{\Delta+1} = \frac{|V(G)|(k+1)}{\Delta+1}$. Therefore, $i_{[kR]}(G) = \omega(f) \geq \frac{|V(G)|(k+1)}{\Delta+1}$.

From now on, suppose that $\omega(f) = \frac{|V(G)|(k+1)}{\Delta+1}$. In this case, by the previous inequality chain, we have that $s'(v) = \frac{k+1}{\Delta+1}$ for all $v \in V(G)$. This implies that no vertex of G was assigned label k since $s'(w) > \frac{k+1}{\Delta+1}$ for every vertex $w \in V(G)$ with $f(w) = k$. Hence, by Lemma 11, G is independent [k]-Roman. ■

Since a nontrivial connected graph G with $\Delta(G) = 1$ has exactly two vertices, an $i_{[kR]}(G)$-function is obtained by assigning $k+1$ to a vertex of G and 0 to the other. The next two results present lower bounds for $i_{[kR]}(G)$ when G has $\Delta(G) \geq 2$.

Theorem 15. *Let $k \geq 1$ be an integer. If G is a nontrivial connected graph with $\Delta(G) = 2$, then $i_{[kR]}(G) \geq \frac{|V(G)|(k+1)}{\Delta(G)+1}$.*

Proof. The case when $1 \leq k \leq 2$ follows from Corollary 13 and the case when $k \geq 4$ follows from Theorem 14. The case when $k = 3$ follows from the [3]-Roman domination number of cycles and paths obtained by Abdollahzadeh Ahangar *et al.* [1]. Note that, since G is connected and $\Delta(G) = 2$, then G is a path or a cycle. Abdollahzadeh Ahangar *et al.* proved that every cycle or path G on n vertices has $\gamma_{[3R]}(G) \geq \frac{4n}{3}$. Therefore, $i_{[3R]}(G) \geq \gamma_{[3R]}(G) \geq \frac{4n}{3}$. ■

Theorem 16. *Let $k \geq 1$ be an integer. If G is a nontrivial connected graph with $\Delta(G) \geq 3$, then $i_{[kR]}(G) \geq \frac{|V(G)|(k+1)}{\Delta(G)+1}$.*

Proof. The case when $1 \leq k \leq 3$ follows from Corollary 13 and the case when $k \geq 4$ follows from Theorem 14. ■

The lower bound presented in Theorem 16 is tight, which can be seen by analyzing Cartesian products of some paths and cycles. Given arbitrary graphs G and H, the *Cartesian product* of G and H is the graph $G \square H$ with vertex set $V(G \square H) = \{(u, v) : u \in V(G), v \in V(H)\}$. Two vertices (u_1, v_1) and (u_2, v_2) of $G \square H$ are adjacent if and only if either $u_1 = u_2$ and $v_1 v_2 \in E(H)$; or $v_1 = v_2$ and $u_1 u_2 \in E(G)$. Let $P_2 = (w_1, w_2)$ be a path with two vertices and $C_{4p} = (v_1, v_2, \ldots, v_{4p})$ be a cycle with $4p$ vertices, $p \geq 1$. As an example, Figure 4 shows the graph $P_2 \square C_8$. By Theorem 16, $i_{[kR]}(P_2 \square C_{4p}) \geq 2p(k+1)$. In addition, a [$k$]-IRDF of $P_2 \square C_{4p}$ with weight $2p(k+1)$ is easily obtained by assigning label $k+1$

to the set of vertices $\{(w_1v_i)\colon i = 2t, t \equiv 1 \pmod 2\} \cup \{(w_2v_j)\colon j = 4t, t \geq 1\}$. Therefore, $i_{[kR]}(P_2\square C_{4p}) = 2p(k+1)$ and, by Theorem 14, $P_2\square C_{4p}$ is independent $[k]$-Roman for all $k \geq 4$.

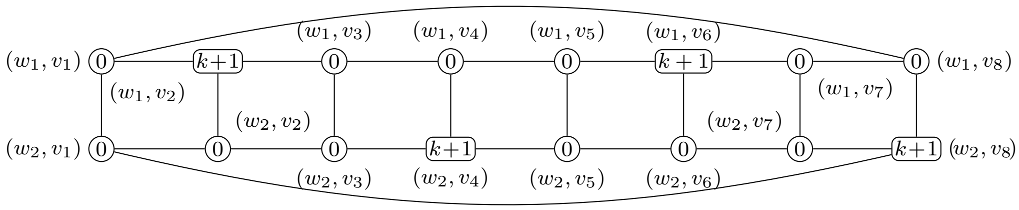


Figure 4. Cartesian product $P_2\square C_8$ with an $i_{[kR]}$-function.

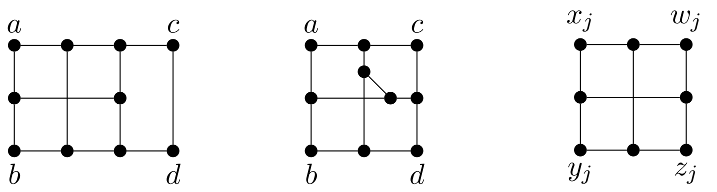
4. The Infinite Family of Generalized Blanuša Snarks

A *cut-edge* of a graph G is an edge whose deletion increases the number of connected components of G. A *snark* is a connected 3-regular graph G without cut-edges that does not admit a proper edge coloring with three colors. The origin of snarks is connected with the Four-Color Problem [22] and their study began in 1898 when the first snark was constructed by Petersen [18]. In 1946, Blanuša constructed two snarks, called Blanuša snarks [5]. From Blanuša snarks, Watkins constructed two infinite families of snarks, called Generalized Blanuša Snarks [24], which are considered in this section.

Luiz [14] determined the exact value of the parameter $i_{[1R]}(G)$ for every generalized Blanuša snark G. Therefore, in this section, we only analyze values of $i_{[kR]}$ for the generalized Blanuša snarks for values of $k \geq 2$.

The members of the family of *generalized Blanuša snarks* are graphs formed from subgraphs called *construction blocks*, denoted B_0^1, B_0^2 and L (see Figure 5). A generalized Blanuša snark contains as subgraphs one of the graphs B_0^1, B_0^2 and i copies of the graph L, called $L_1, L_2, \ldots, L_i$. Vertices a, b, c and d, belonging to both B_0^1 and B_0^2, and the vertices x_j, y_j, w_j and z_j, belonging to L, are called *border vertices.*

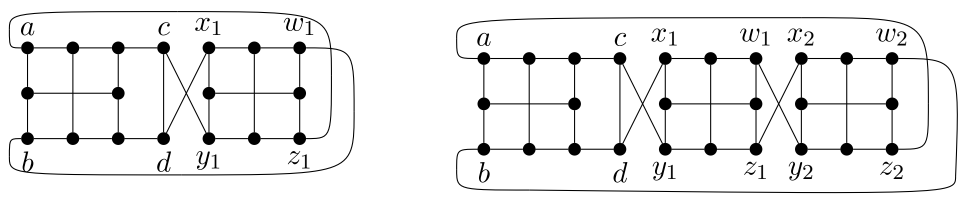
In the next paragraphs, we define these families of graphs based on a recursive construction. Let $\mathfrak{B}^1 = \{B_1^1, B_2^1, B_3^1, \ldots\}$ and $\mathfrak{B}^2 = \{B_1^2, B_2^2, B_3^2, \ldots\}$ be the first and the second families of generalized Blanuša snarks, respectively. The first member of $\mathfrak{B}^1$, the snark B_1^1, has vertex set $V(B_1^1) = V(B_0^1) \cup V(L_1)$ and edge set $E(B_1^1) = E(B_0^1) \cup E(L_1) \cup \{cy_1, dx_1, az_1, bw_1\}$ (see Figure 6(a)). The second snark in $\mathfrak{B}^1$, snark B_2^1, has vertex set $V(B_2^1) = V(B_0^1) \cup V(L_1) \cup V(L_2)$ and edge set $E(B_2^1) = E(B_0^1) \cup E(L_1) \cup E(L_2) \cup \{cy_1, dx_1, w_1y_2, z_1x_2, az_2, bw_2\}$ (see Figure 6(b)). The smallest snark of family $\mathfrak{B}^2$, graph B_1^2, has vertex set $V(B_1^2) =$



(a) Block B_0^1. (b) Block B_0^2. (c) Block L.

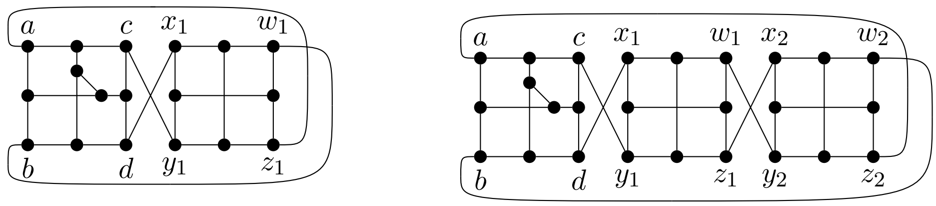
Figure 5. Construction blocks B_0^1, B_0^2 and L of the generalized Blanuša snarks.

$V(B_0^2) \cup V(L_1)$ and edge set $E(B_1^2) = E(B_0^2) \cup E(L_1) \cup \{bw_1, az_1, cy_1, dx_1\}$ (see Figure 7(a)). The second snark in $\mathfrak{B}^2$, B_2^2, has vertex set $V(B_2^2) = V(B_0^2) \cup V(L_1) \cup V(L_2)$ and edge set $E(B_2^2) = E(B_0^2) \cup E(L_1) \cup E(L_2) \cup \{bw_2, az_2, cy_1, dx_1, z_1x_2, w_1y_2\}$ (see Figure 7(b)).



(a) Generalized Blanuša snark B_1^1. (b) Generalized Blanuša snark B_2^1.

Figure 6. The first two smallest members of the family $\mathfrak{B}^1$.



(a) Generalized Blanuša snark B_1^2. (b) Generalized Blanuša snark B_2^2.

Figure 7. The first two smallest members of the family $\mathfrak{B}^2$.

In order to construct larger generalized Blanuša snarks, we use a subgraph LG_i, called *link graph*, with vertex set $V(LG_i) = V(L_{i-1}) \cup V(L_i)$ and edge set $E(LG_i) = E(L_{i-1}) \cup E(L_i) \cup \{w_{i-1}y_i, z_{i-1}x_i\}$ (see Figure 8). Let $t \in \{1, 2\}$. For each integer i, with $i \geq 3$, the snark B_i^t is obtained recursively from the snark B_{i-2}^t and the link graph LG_i according to the following rules:

(i) $V(B_i^t) = V(B_{i-2}^t) \cup V(LG_i)$;

(ii) $E(B_i^t) = (E(B_{i-2}^t) \backslash E_{i-2}^{out}) \cup E(LG_i) \cup E_i^{in}$, where $E_{i-2}^{out} = \{az_{i-2}, bw_{i-2}\}$ and

$E_i^{in} = \{w_{i-2}y_{i-1}, z_{i-2}x_{i-1}, az_i, bw_i\}$.

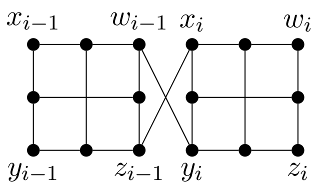


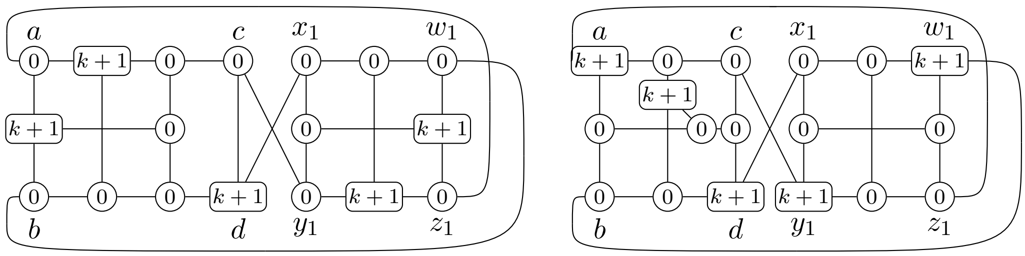
Figure 8. The link graph LG_i.

By using the previous recursive construction, Theorem 17 establishes an upper bound for $i_{[kR]}(B_i^t)$.

Theorem 17. *Let $k \geq 2$ be an integer. If B_i^t is a generalized Blanuša snark, with $t \in \{1, 2\}$ and $i \geq 1$, then*

$$i_{[kR]}(B_i^t) \leq \begin{cases} (k+1)(2i+2) + 2k & \text{if } t = 1 \text{ and } i \geq 3 \text{ with } i \text{ odd,} \\ (k+1)(2i+3) & \text{otherwise.} \end{cases}$$

Proof. Initially, we separately show that the snark B_i^t, with $i = 1$ and $t \in \{1, 2\}$ has a $[k]$-IRDF with weight equal to $5(k+1) = (k+1)(2i+3)$. This special case is shown in Figure 9, with B_1^1 and B_1^2 endowed with their respective $[k]$-IRDFs.

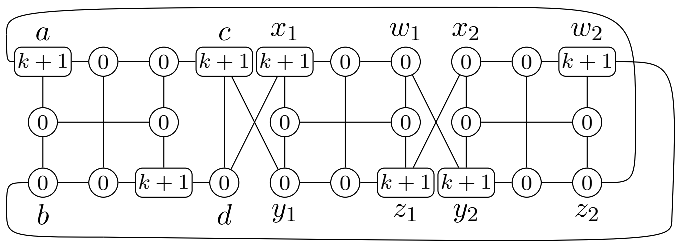


(a) Snark B_1^1 with a $[k]$-IRDF. (b) Snark B_1^2 with a $[k]$-IRDF.

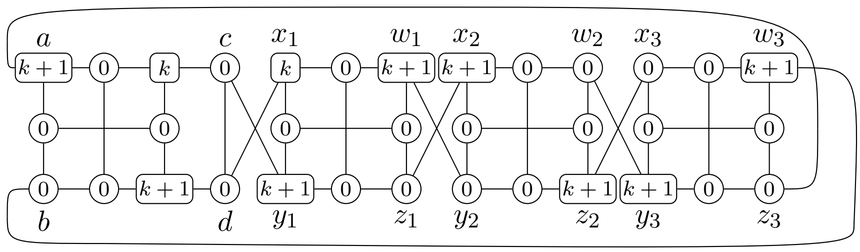
Figure 9. Independent $[k]$-Roman domination functions of snarks B_1^1 and B_1^2 with weight $5(k+1)$.

Next, we prove by strong induction on i that every snark B_i^t, with $t \in \{1, 2\}$ and $i \geq 2$, has a $[k]$-IRDF f_i with the following properties: (i) $f_i(a) = k+1$, $f_i(b) = 0$, $f_i(w_i) = k+1$ and $f_i(z_i) = 0$; (ii) $\omega(f_i) = (k+1)(2i+2) + 2k$ if $t = 1$, $i \geq 3$ and i odd; or $\omega(f_i) = (k+1)(2i+3)$ otherwise. We call *special* a $[k]$-IRDF f_i of B_i^t that satisfies the previous two properties. The induction is based on the recursive construction of the families $\mathfrak{B}^1$ and $\mathfrak{B}^2$.

For the base case, consider the snarks B_i^t with $i \in \{2, 3\}$ and $t \in \{1, 2\}$. For $i = 2$, Figures 10(a) and 11(a) exhibit the snarks B_2^1 and B_2^2, respectively, with their special [k]-IRDFs with weight $7(k+1) = (k+1)(2i+3)$. For $i = 3$, the snark B_3^1 is illustrated in Figure 10(b) with a special [k]-IRDF f_3 with weight $8(k+1)+2k$; and the snark B_3^2 is illustrated in Figure 11(b) with a special [k]-IRDF f_3 with weight $9(k+1)$.



(a) Snark B_2^1 with a special [k]-IRDF with weight $7(k+1)$.



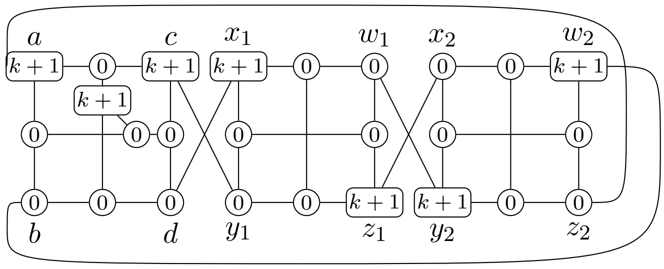
(b) Snark B_3^1 with a special [k]-IRDF with weight $8(k+1)+2k$.

Figure 10. Special [k]-IRDFs for the snarks B_2^1 and B_3^1.

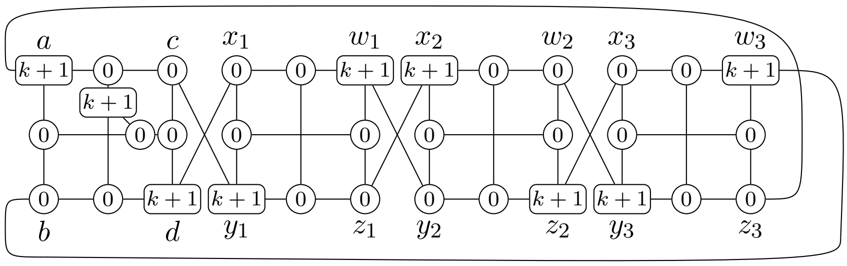
For the inductive step, consider a snark B_i^t with $i \geq 4$ and $t \in \{1,2\}$. By the recursive construction of generalized Blanuša snarks, we know that B_i^t can be constructed from the link graph LG_i and the snark B_{i-2}^t. Figure 12 shows the link graph LG_i with a vertex labeling $\varphi\colon V(LG_i) \to \{0, k+1\}$ with weight $4(k+1)$. Also, by induction hypothesis, the snark B_{i-2}^t has a special [k]-IRDF f_{i-2} with weight $\omega(f_{i-2}) = (k+1)(2(i-2)+2)+2k$ when $t = 1$ and $i \geq 3$, i odd; or with weight $\omega(f_{i-2}) = (k+1)(2(i-2)+3)$ otherwise. Since f_{i-2} is special, we also have that $f_{i-2}(a) = k+1$, $f_{i-2}(b) = 0$, $f_{i-2}(w_{i-2}) = k+1$, $f_{i-2}(z_{i-2}) = 0$, for $a, b, w_{i-2}, z_{i-2} \in V(B_{i-2}^t)$. Thus, we define a vertex labeling f_i for B_i^t as follows. For every vertex $v \in V(B_i^t)$,

$$f_i(v) = \begin{cases} f_{i-2}(v) & \text{if } v \in V(B_{i-2}^t) \cap V(B_i^t), \\ \varphi(v) & \text{if } v \in V(LG_i) \cap V(B_i^t). \end{cases}$$

Next, we prove that f_i is an [k]-IRDF of B_i^t. By induction hypothesis, the [k]-IRDF f_{i-2} of B_{i-2}^t is such that $f_{i-2}(a) = k+1$, $f_{i-2}(b) = 0$, $f_{i-2}(w_{i-2}) = k+1$, $f_{i-2}(z_{i-2}) = 0$. This implies that the labeling f_i restricted to subgraph $B_{i-2}^t - E_{i-2}^{out} \subset B_i^t$ is almost a [k]-IRDF of $B_{i-2}^t - E_{i-2}^{out}$ since z_{i-2} and b are the



(a) Snark B_2^2 with a special $[k]$-IRDF with weight $7(k+1)$.



(b) Snark B_3^2 with a special $[k]$-IRDF with weight $9(k+1)$.

Figure 11. Special $[k]$-IRDFs for the snarks B_2^2 and B_3^2.

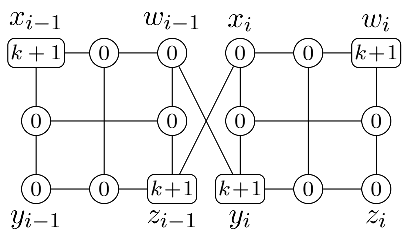


Figure 12. Link graph LG_i with a vertex labeling φ. Note that the vertices y_{i-1}, z_i and its neighbors have label 0.

only vertices with label 0 in $B_{i-2}^t - E_{i-2}^{out}$ such that $f(N[z_{i-2}]) < |AN(z_{i-2})| + k$ and $f(N[b]) < |AN(b)| + k$. Also, by construction, the labeling f_i restricted to subgraph $LG_i \subset B_i^t$ assigns label 0 to vertices y_{i-1} and z_i, and these are the only vertices with label 0 in LG_i that have $f(N[y_{i-1}]) < |AN(y_{i-1})| + k$ and $f(N[z_i]) < |AN(z_i)| + k$. Additionally, no two vertices with label $k+1$ in LG_i are adjacent. Thus, f_i restricted to LG_i is almost a $[k]$-IRDF of LG_i since y_{i-1} and z_i are the only vertices of LG_i that have label 0 and $f(N[y_{i-1}]) = f(N[z_i]) = 0$. Therefore, in order to prove that f_i is a $[k]$-IRDF of B_i^t, it suffices to show that the vertices y_{i-1}, z_{i-2}, z_i, b have a neighbor in B_i^t with label $k+1$. This comes down to analyzing the labels of the endpoints of the edges in the set $E_i^{in} = \{w_{i-2}y_{i-1}, z_{i-2}x_{i-1}, az_i, bw_i\}$ and verify if the vertices w_{i-2}, x_{i-1}, a, w_i have label

$k+1$. From the definition of f_i, we have that $f_i(w_{i-2}) = f_{i-2}(w_{i-2}) = k+1$, $f_i(x_{i-1}) = \varphi(x_{i-1}) = k+1$, $f_i(a) = f_{i-2}(a) = k+1$ and $f_i(w_i) = \varphi(w_i) = k+1$. Thus, the vertices y_{i-1}, z_{i-2}, z_i, b (that have label 0) are adjacent to vertices with label $k+1$ in B_i^t, that is, the function f_i is a $[k]$-IRDF of B_i^t.

Now, we prove that f_i is special. The weight of f_i is given by the sum of the weights of the functions f_{i-2} and φ. Thus, if $t = 1$, $i \geq 5$ and i odd, then $\omega(f_i) = \omega(f_{i-2}) + \omega(\varphi) = (k+1)(2(i-2)+2)+2k+4(k+1) = (k+1)(2i+2)+2k$; otherwise, we have that $\omega(f_i) = \omega(f_{i-2}) + \omega(\varphi) = (k+1)(2(i-2)+3)+4(k+1) = (k+1)(2i+3)$. Note that $f_i(a) = k+1$, $f_i(b) = 0$, $f_i(w_i) = k+1$ and $f_i(z_i) = 0$ since these are the labels of each of these vertices in the subgraphs B_{i-2}^t and LG_i. Therefore, f_i is a special $[k]$-IRDF of B_i^t, and the result follows. ■

By Theorem 14, $i_{[kR]}(B_i^t) \geq (k+1)(2i+2.5)$ for $k \geq 4$. However, for increasingly larger values of k, this lower bound moves away from the upper bounds given in Theorem 17. Therefore, better lower bounds are needed. Theorems 21 and 22 establish better lower bounds for the parameter $i_{[kR]}(B_i^t)$. In order to prove these results, we first present some additional definitions and auxiliary lemmas and theorems.

Given a graph G and two disjoint sets $S_1 \subset V(G)$ and $S_2 \subset V(G)$, we denote by $E(S_1, S_2)$ the set of edges $uv \in E(G)$ such that $u \in S_1$ and $v \in S_2$. Also, given $S \subseteq V(G)$, we denote by $N(S)$ the set of vertices $\{w \in V(G)\backslash S : uw \in E(G)$ and $u \in S\}$. We also define $N[S] = S \cup N(S)$.

Lemma 18. *Let $k \geq 2$ be an integer. If G is a 3-regular graph with n vertices and $f = (V_0, V_k, V_{k+1})$ is an $i_{[kR]}$-function of G, then $|V_k| \leq \frac{8i_{[kR]}(G)-2(k+1)n}{3k-5}$ and $|V_{k+1}| \geq \frac{2kn-5i_{[kR]}(G)}{3k-5}$.*

Proof. Let G be a 3-regular graph with n vertices and $f = (V_0, V_k, V_{k+1})$ be an $i_{[kR]}$-function of G. Thus, $i_{[kR]}(G) = \omega(f) = k|V_k| + (k+1)|V_{k+1}|$. This fact implies that

$$(1) \qquad |V_{k+1}| = \frac{i_{[kR]}(G) - k|V_k|}{k+1} \quad \text{and} \quad |V_k| = \frac{i_{[kR]}(G) - (k+1)|V_{k+1}|}{k}.$$

Since $k \geq 2$, each vertex $v \in V(G)$ with $f(v) = 0$ has at least one neighbor with label $k+1$ or at least two neighbors with label k. Let $S = V_0 \cap N(V_{k+1})$ and $T = V_0 \backslash S$. Since G is 3-regular, each vertex in V_{k+1} is adjacent to at most 3 vertices in S. Thus, $|S| \leq 3|V_{k+1}|$. Similarly, since each vertex in V_k is adjacent to at most 3 vertices in T and since each vertex in T has at least two neighbors in V_k, we obtain that $2|T| \leq |E(V_k, T)| \leq 3|V_k|$, which implies that $|T| \leq \frac{3|V_k|}{2}$. Therefore, $|V_0| = |S| + |T| \leq 3|V_{k+1}| + \frac{3|V_k|}{2}$.

From the definition of $[k]$-IRDF, it follows that $n = |V_0| + |V_k| + |V_{k+1}|$. Hence, $n = |V_0| + |V_k| + |V_{k+1}| \leq 3|V_{k+1}| + \frac{3|V_k|}{2} + |V_k| + |V_{k+1}| = 4|V_{k+1}| + \frac{5|V_k|}{2}$, that is,

$$n \le 4|V_{k+1}| + \frac{5|V_k|}{2}. \tag{2}$$

From Equation (1) and Inequality (2), we have that $n \le 4 \cdot \frac{i_{[kR]}(G)-k|V_k|}{k+1} + \frac{5|V_k|}{2} = \frac{8i_{[kR]}(G)-(3k-5)|V_k|}{2(k+1)}$. From the last inequality, we conclude that $|V_k| \le \frac{8i_{[kR]}(G)-2(k+1)n}{3k-5}$. Also, from Equation (1) and Inequality (2), we have that $n \le 4|V_{k+1}|+\frac{5i_{[kR]}(G)-5(k+1)|V_{k+1}|}{2k} = \frac{8k|V_{k+1}|+5i_{[kR]}(G)-5(k+1)|V_{k+1}|}{2k} = \frac{5i_{[kR]}(G)+(3k-5)|V_{k+1}|}{2k}$. From the last inequality, we conclude that $|V_{k+1}| \ge \frac{2kn-5i_{[kR]}(G)}{3k-5}$. ■

Lemma 19. *Let G be a graph and $k \ge 1$ be an integer. For any $i_{[kR]}$-function $f = (V_0, V_k, V_{k+1})$ of G, we have that $|V_{k+1}| \le i_{[kR]}(G) - k \cdot i(G)$ and $|V_k| \ge (k+1)i(G) - i_{[kR]}(G)$.*

Proof. Let G be a graph with an $i_{[kR]}$-function $f = (V_0, V_k, V_{k+1})$. Since $V_k \cup V_{k+1}$ is an independent dominating set of G, we have $i(G) \le |V_k| + |V_{k+1}|$. Hence, $k \cdot i(G) \le k|V_k| + k|V_{k+1}| = k|V_k| + (k+1)|V_{k+1}| - |V_{k+1}| = i_{[kR]}(G) - |V_{k+1}|$. This implies that $|V_{k+1}| \le i_{[kR]}(G) - k \cdot i(G)$. In addition, $(k+1)i(G) \le (k+1)|V_k| + (k+1)|V_{k+1}| = i_{[kR]}(G) + |V_k|$. This implies that $|V_k| \ge (k+1)i(G) - i_{[kR]}(G)$, and the result follows. ■

The next result is used in our proofs and determines the domination number and independent domination number for generalized Blanuša snarks.

Theorem 20 (Pereira [17])**.** *Let B_i^t be a generalized Blanuša snark with $t \in \{1,2\}$ and $i \ge 1$. Then*

$$i(B_i^t) = \gamma(B_i^t) = \begin{cases} 2i+4 & \text{if } t = 1 \text{ and } i \ge 3 \text{ with } i \text{ odd}, \\ 2i+3 & \text{otherwise.} \end{cases}$$

Theorem 21. *Let $k \ge 2$ be an integer. Let B_i^t be a generalized Blanuša snark such that $t = 1$ and $i \ge 3$ with i odd. Then $i_{[kR]}(B_i^t) \ge (k+1)(2i+2) + 2k - 2$.*

Proof. By the definition of B_i^t, we have that $|V(B_i^t)| = 8i+10$. Define $n = 8i+10$. Let $f = (V_0, V_k, V_{k+1})$ be an $i_{[kR]}$-function of B_i^t. For the purpose of contradiction, suppose that $i_{[kR]}(B_i^t) \le (k+1)(2i+2) + 2k - 3$. Next, we find a lower bound for $|V_k|$. By Theorem 20 and Theorem 19, $|V_k| \ge (k+1)i(G) - i_{[kR]}(G) \ge (k+1)(2i+4) - [(k+1)(2i+2)+2k-3] = 5$. Thus, $|V_k| \ge 5$. Next, we find an upper bound for $|V_k|$. By Lemma 18, $|V_k| \le \frac{8i_{[kR]}(B_i^t)-2(k+1)n}{3k-5} \le \frac{8((k+1)(2i+2)+2k-3)-2(k+1)(8i+10)}{3k-5} = \frac{12k-28}{3k-5} < 4$ for all $k \ge 2$. That is, $|V_k| < 4$. However, these facts imply that $5 \le |V_k| < 4$, which is a contradiction. ■

Theorem 22. *Let $k \geq 4$ be an integer. Let B_i^t be a generalized Blanuša snark such that $t = 2$, or $t = 1$ with $i = 1$, or $t = 1$ with i even. Then*

$$i_{[kR]}(B_i^t) = (k+1)(2i+3).$$

Proof. Let $k \geq 4$ be an integer. By Theorem 17, $i_{[kR]}(B_i^t) \leq (k+1)(2i+3)$. So, in order to conclude the proof, it suffices to prove that $i_{[kR]}(B_i^t) \geq (k+1)(2i+3)$. By the definition of B_i^t, we have that $|V(B_i^t)| = 8i+10$. Define $n = 8i+10$. Let $f = (V_0, V_k, V_{k+1})$ be an $i_{[kR]}$-function of B_i^t. For the purpose of contradiction, suppose that $i_{[kR]}(B_i^t) \leq (k+1)(2i+3) - 1$.

By Lemma 18, $|V_k| \leq \frac{8i_{[kR]}(B_i^k)-2(k+1)n}{3k-5} \leq \frac{8((k+1)(2i+3)-1)-2(k+1)(8i+10)}{3k-5} = \frac{(k+1)[8(2i+3)-2(8i+10)]-8}{3k-5} = \frac{4k-4}{3k-5}$. That is, $|V_k| \leq \frac{4k-4}{3k-5}$. For $k \geq 4$, we have that $\frac{4k-4}{3k-5} < 2$. This implies that $|V_k| < 2$ for all $k \geq 4$. On the other hand, by Lemma 19 and Theorem 20, $|V_k| \geq (k+1)i(B_i^t) - i_{[kR]}(B_i^t) \geq (k+1)(2i+3) - (k+1)(2i+3) + 1 = 1$. These facts imply that $|V_k| = 1$.

By the definition of $[k]$-RDF, $i_{[kR]}(B_i^t) = k|V_k| + (k+1)|V_{k+1}| = k + (k+1)|V_{k+1}|$. Moreover, since $i(B_i^t) = 2i+3$, we have that $2i+3 = i(B_i^t) \leq |V_k| + |V_{k+1}| = 1 + |V_{k+1}|$, which implies that $|V_{k+1}| \geq 2i+2$. Hence, $i_{[kR]}(B_i^t) = (k+1)|V_{k+1}| + k \geq (k+1)(2i+2)+k$. From these facts, we have that $(k+1)(2i+2)+k \leq i_{[kR]}(B_i^t) \leq (k+1)(2i+3)-1$. However, since $(k+1)(2i+2)+k = (k+1)(2i+3)-1$, we obtain that $i_{[kR]}(B_i^t) = (k+1)(2i+2)+k$. Since $i_{[kR]}(B_i^t) = (k+1)(2i+2)+k$ and $|V_k| = 1$, we obtain that $|V_{k+1}| = 2i+2$.

Since B_i^t is 3-regular, each vertex in V_{k+1} dominates at most 3 vertices in V_0. Thus, $|N(V_{k+1})| \leq 3|V_{k+1}|$. This implies that $|N[V_{k+1}]| = |V_{k+1}| + |N(V_{k+1})| \leq (2i+2) + 3(2i+2) = 8i+8$. In other words, there are at most $8i+8$ vertices that are either in V_{k+1} or are dominated by vertices in V_{k+1}. Since $|V(B_i^t)| = 8i+10$, there are at least 2 vertices in the set $V_0 \cup V_k$ that are not dominated by vertices with label $k+1$. One of these vertices belongs to the set V_k, since $|V_k| = 1$, and the other vertex, say w, belongs to the set V_0. Since $f(w) = 0$ and vertex w has no neighbor in the set V_{k+1}, we conclude that $f(N[w]) < k + |AN(w)|$, which is a contradiction. ∎

Corollary 23 follows from Theorems 20 and 22.

Corollary 23. *Let $k \geq 4$ be an integer. If B_i^t is a generalized Blanuša snark, with $t = 2$, or $t = 1$ with $i = 1$, or $t = 1$ with i even, then B_i^t is an independent $[k]$-Roman graph.*

5. Closing Remarks

In this work, we prove that, for all $k \geq 3$, the independent $[k]$-Roman domination problem and the $[k]$-Roman domination problem are $\mathcal{NP}$-complete even when restricted to planar bipartite graphs with maximum degree 3 and also present lower and upper bounds for the parameter $i_{[kR]}(G)$. Moreover, we investigate $i_{[kR]}(G)$ for a family of 3-regular graphs called generalized Blanuša snarks.

In Corollary 23, we present an infinite family of independent $[k]$-Roman graphs, which are graphs that have $i_{[kR]}(G) = (k+1)i(G)$. An interesting open problem is finding other classes of independent $[k]$-Roman graphs.

Adabi *et al.* [2] proved that any graph G with $\Delta(G) \leq 3$ has $\gamma_{[kR]}(G) = i_{[kR]}(G)$ for $k = 1$. We remark that the family of planar bipartite graphs with maximum degree 3 constructed in the reduction shown in Section 2 is an example of infinite family of graphs with $\Delta(G) = 3$ for which $\gamma_{[kR]}(G) = i_{[kR]}(G)$ for all $k \geq 1$. Thus, another interesting line of research is finding other classes of graphs with $\Delta(G) \leq 3$ for which $\gamma_{[kR]}(G) = i_{[kR]}(G)$ for $k \geq 2$. In fact, we conjecture that this property holds for all generalized Blanuša snarks.

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