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ANTI-RAMSEY NUMBER OF UNION OF 5-PATH AND MATCHING

Qing Jie and Zemin Jin[1]

School of Mathematical Sciences
Zhejiang Normal University
Jinhua 321004, China

e-mail: 2782193314@qq.com
zeminjin@zjnu.cn

Abstract

The anti-Ramsey number $AR(K_n, G)$ is defined as the maximum integer k such that there is an edge coloring of K_n using k colors, in which there is no rainbow copy of G, namely, a copy of G whose edges have distinct colors. In 2016, Gilboa and Roditty provided the upper and lower bounds of the anti-Ramsey number for $P_k \cup tP_2$ with $k \geq 5$. The problem on linear forests was considered in recent years. In this paper, we consider the case $k = 5$ and we determine the exact value of the anti-Ramsey number $AR(K_n, P_5 \cup tP_2)$ for $n \geq 2t + 6$.

Keywords: anti-Ramsey number, rainbow matching, disjoint union of graphs.

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1. Introduction

In this paper, we consider only finite simple undirected graphs. A subgraph of an edge-colored graph is called rainbow if all of its edges receive different colors. For a graph G and a positive n, the anti-Ramsey number $AR(K_n, G)$ is the maximum number of colors in an edge-coloring of G with no rainbow copy of G.

The concept of anti-Ramsey number was introduced by Erdős *et al.* [4]. They showed that these are closely related to Turán number. Since then, a significant number of results were established for a wide variety of graphs in complete graphs,

[1]Corresponding author.

especially for matchings, see [3, 6, 7, 10, 12, 21, 33]. In recent years, numerous scholars have focused their attention on the study of anti-Ramsey number for disjoint union of graphs. An interesting problem concerning anti-Ramsey number is the determination of anti-Ramsey number of matchings. Later, the host graphs are generalized to other graphs. Besides of complete graphs as host graphs, the anti-Ramsey number of matchings has also been studied extensively in planar graphs [2, 13, 14, 16, 22, 24, 30–32], hypergraphs [9, 18, 29, 35] and bipartite graphs [15, 17, 23, 25–28].

Bialostocki *et al.* [1] determined the anti-Ramsey number for all graphs with no more than four edges in their research. However, for more complex structures, Gilboa and Roditty [8] determined that the anti-Ramsey numbers of $L \cup tP_2$ and $L \cup kP_3$ when t and k are large enough and L is a graph satisfying some conditions, including $C_3 \cup tP_2, P_3 \cup tP_2, P_4 \cup tP_2$ and $P_2 \cup tP_3$, in K_n for large enough n. Furthermore, they obtain upper and lower bounds of the anti-Ramsey number for $P_k \cup tP_2$,where $k \geq 5$, $t \geq 1$ and n is large enough. The analogue results were present in [5, 34], where the authors obtained the anti-Ramsey number of several kinds of forests in K_n for large enough n. Jin and Gu [19] improved the bound on n and obtained the value of $AR(K_n, C_3 \cup tP_2)$ for all $n \geq 2t+3$ and they also determined the value of $AR(K_n, K_4 \cup tP_2)$ for all $n \geq 2t+4$. Additionally, He and Jin [11] improved the bound on n and obtained the value of $AR(K_n, P_3 \cup tP_2)$ for all $n \geq 2t+3$. Later, Jin *et al.* [20] improved the bound on n and obtained the value of $AR(K_n, P_4 \cup tP_2)$ for all $n \geq 2t+4$.

It is important to note that Gilboa and Roditty [8] only provided the upper and lower bounds of the anti-Ramsey number for $P_k \cup tP_2$ with $k \geq 5$. In this paper, we consider the case where $k = 5$. We obtain the precise anti-Ramsey number of $P_5 \cup tP_2$ for all $n \geq 2t+6$ in complete graphs.

The paper is organized as follows. In Section 2, we present some definitions, notations and preliminary results, which will be used in the subsequent sections. Our main results show that the anti-Ramsey number $AR(K_n, P_5 \cup tP_2)$ is not a single formula for different $t \geq 1$. Therefore, the proofs of our main results are divided into three sections. In Section 3, we obtain the value for $P_5 \cup P_2$ for $n \geq 7$. In Section 4, we get the anti-Ramsey number of $P_5 \cup tP_2$ for $n \geq 2t+6$ and $2 \leq t \leq 4$. In Section 5, the anti-Ramsey number of $P_5 \cup tP_2$ for $n \geq 2t+6$ and $t \geq 5$ is determined.

2. Preliminaries

Firstly, we present some definitions and notations necessary in the paper. Notice that throughout the paper, we consider the anti-Ramsey problem with complete graph as host graphs. Given an edge-colored graph G, denote by $c(G)$

the set of colors of all edges of G and by $c(e)$ the color of edge e. Furthermore, for a subset $E^{'}$ of $E(G)$, denote by $c(E^{'})$ the set of colors of all edges in $E^{'}$. Let S and T be two disjoint vertex subsets of $V(G)$, denote by $[S,T]_G$ the set of all edges between S and T in G. When $G=K_n$, we write $[S,T]$ for short. Moreover, if $S=\{v\}$, we write $[v,T]_G$ for short.

In addition to this, given a subset $S\subseteq V(K_n)$ denote by $l(S)$ the set of the colors only appearing at the edges incident with vertices in S, namely, $l(S)=c(K_n)\backslash c(K_n-S)$. When $S=\{v\}$, we write $l(v)$ for short and the number of colors in $l(v)$ is called the saturated degree of v in K_n. If $c(vx)\in l(v)$, then we call that x saturates v. In the proof of our main results, we always partition a complete graph K_n (or its vertices) into graphs (or sets) in the order H_1, H_2 and D. Also, let $H=H_1\cup H_2$. For each vertex $u\in V(K_n)$, denote $l_1(H_1,u)=l(u)\cap c([H_1,u])$, $l_2(H_2,u)=l(u)\cap c([H_2,u])\backslash l_1(H_1,u)$, $l(H,u)=l_1(H_1,u)\cup l_2(H_2,u)$, and $l_3(D,u)=l(u)\backslash l(H,u)$.

Secondly, we introduce some preliminary results which are useful. Bialostocki, Gilboa and Roditty [1] determined the anti-Ramsey number of graphs with no more than four edges. In particular, they proved the following theorem for P_5.

Theorem 1 [1]. *For $n\geq 5$, $AR(K_n,P_5)=n$.*

Furthermore, Gilboa and Roditty [8] proved the following theorem which enables us to get the upper bound of $AR(K_n, L\cup tP_2)$ from the upper bound on the disjoint union of L and a smaller matching for large enough n.

Theorem 2 [8]. *Given a graph L. Let $n_0\geq 2t_1+|V(L)|$, $t_1\geq 0$, and s and r be real numbers. Suppose that $AR(K_n, L\cup t_1P_2)\leq (t_1+s)\big(n-\frac{t_1+s+1}{2}\big)+r$ for any integers $n\geq n_0$. Then there exists a constant γ, which depends only on integers L, t_1, s, r and n_0, such that for any positive integers $t\geq t_1$ and $n>\frac{5t}{2}+\gamma$,*

$$AR(K_n, L\cup tP_2)\leq (t+s)\left(n-\frac{t+s+1}{2}\right)+r.$$

Based on Theorem 2, the authors got the results for some special graphs L, including P_3, P_4, C_3, etc., in K_n for large enough n. Moreover, the authors gave the upper and lower bound of $AR(K_n, P_k\cup tP_2)$ with $k\geq 5$ and they proved the following theorem as a corollary for $P_k\cup tP_2$ with $k\geq 5$.

Theorem 3 [8]. *For any integers $k\geq 5, t\geq 0$ and $n\geq 2t+k$,*

$$AR(n, P_k\cup tP_2)\geq \left(t+\left\lceil\frac{k-1}{2}\right\rceil-2\right)\left(n-\frac{t+\lceil\frac{k-1}{2}\rceil-1}{2}\right)+1.$$

and for any integer $k \geq 5$ there is a constant $\gamma_2(P_k)$ such that for any integers $t \geq 0$ and $n > \frac{5t}{2} + \gamma_2(P_k)$,

$$AR(n, P_k \cup tP_2) \leq \left(t + \left\lfloor \frac{k-1}{2} \right\rfloor - 1\right)\left(n - \frac{t + \left\lfloor \frac{k-1}{2} \right\rfloor}{2}\right) + 1 + (k - 1 \bmod 2).$$

In particular, He and Jin [11] improved the bound on n and obtained the exact value of $AR(K_n, P_3 \cup tP_2)$ for all $n \geq 2t + 3$ based on Theorem 2 when $L = P_3$. They got the following theorem.

Theorem 4 [11]**.** *Let $n(t) = \frac{5t+2}{2} + \frac{1}{t-1}$. For $n \geq 2t+3$ and $t \geq 2$, $AR(K_n, P_3 \cup tP_2) = f(n, t)$, where*

$$f(n,t) = \begin{cases} t(2t-1) + 1, & \text{if } 2t + 3 \leq n \leq \lfloor n(t) \rfloor, \\ (t-1)(n - \frac{t}{2}) + 1, & \text{if } n \geq \lceil n(t) \rceil. \end{cases}$$

Later, Fang *et al.* [5] considered the anti-Ramsey number of more general graphs which consist of small components, i.e., forests, including star-forest, linear forest and double stars. Especially for linear forest, they gave an approximate value of anti-Ramsey number of a linear forests with at least one even path component.

Theorem 5 [5]**.** *Let $F = \bigcup_{i=1}^{k} P_{k_i}$ be a linear forest with at least one even path component, where $k_i \geq 2$ for all $1 \leq i \leq k$. Then*

$$AR(K_n, F) = \left(\sum_{i=1}^{k} \left\lfloor \frac{k_i}{2} \right\rfloor - 2\right) n + O(1).$$

Soon after, on the base of Theorem 5, Xie *et al.* [34] determined the precise value of anti-Ramsey number of a linear forest when it contains even paths and obtained the following theorem.

Theorem 6 [34]**.** *Let $F = \bigcup_{i=1}^{k} P_{k_i}$ be a linear forest with at least one even path component, where $k \geq 2$ and $k_i \geq 2$ for all $1 \leq i \leq k$. Then for large enough n,*

$$AR(K_n, F) = \binom{\sum_{i=1}^{k} \left\lfloor \frac{k_i}{2} \right\rfloor - 2}{2} + \left(\sum_{i=1}^{k} \left\lfloor \frac{k_i}{2} \right\rfloor - 2\right)\left(n - \sum_{i=1}^{k} \left\lfloor \frac{k_i}{2} \right\rfloor + 2\right) + 1 + \varepsilon,$$

where $\varepsilon = 0$ if at least two k_i are even, or $\varepsilon = 1$ otherwise.

Theorem 3 only gave the lower and upper bounds of $AR(K_n, P_5 \cup tP_2)$. From Theorem 6, we can know the value of $AR(K_n, P_5 \cup tP_2)$ for large enough n. It is worth to pay attention to seek the exact value of $AR(K_n, P_5 \cup tP_2)$.

3. Result for $P_5 \cup P_2$

Lemma 7. $AR(K_7, P_5 \cup P_2) = 8$.

This lemma plays a role as the inductive basis in the proof of the following theorem. Since the proof of this lemma is cases analysis, we present its proof details in APPENDIX.

Theorem 8. *For $n \geq 7$, $AR(K_n, P_5 \cup P_2) = n + 1$.*

Proof. To show that $AR(K_n, P_5 \cup P_2) \geq n+1$, first we construct an edge-coloring of K_n without rainbow $P_5 \cup P_2$. Take a vertex $v \in V(K_n)$, all edges incident with the vertex v are colored by distinct colors, and all other edges are colored by two additional colors. We can see that the number of colors is $n + 1$ and there is no rainbow $P_5 \cup P_2$. This implies that $AR(n, P_5 \cup P_2) \geq n + 1$.

Next we will prove the upper bound, that is, the inequality $AR(K_n, P_5 \cup P_2) \leq n+1$. According to Lemma 7, we can know that $AR(K_7, P_5 \cup P_2) = 8$. It follows that we need to show $AR(K_n, P_5 \cup P_2) \leq n+1$ for any integer $n \geq 8$. Let c be an $(n+2)$-edge-coloring of K_n. We need to show that K_n contains a rainbow $P_5 \cup P_2$. Instead, we assume that K_n does not contain any rainbow $P_5 \cup P_2$. Evidently, any subgraph K_{n-1} has no rainbow $P_5 \cup P_2$. Then by the induction hypothesis on n, we have $|c(K_{n-1})| \leq n$ for any subgraph K_{n-1} in K_n with edge-coloring c. Therefore, the saturated degree of each vertex in K_n is at least 2. According to Theorem 1, we know that K_n contains a rainbow P_5. Let $H = v_1v_2v_3v_4v_5$ be a rainbow path P_5 and $D = V(K_n) \backslash V(H)$. As K_n does not contain any rainbow $P_5 \cup P_2$, we can easily get $c(K_n[D]) \subseteq c(H)$. Take three different vertices $u, v, w \in D$. Denote by Q the graph on vertex set $V(H) \cup \{u, v, w\}$ with the edges $E(H)$ and all the edges saturating u, v, w. Clearly, Q is rainbow and Q contains a rainbow $P_5 \cup P_2$ except for the graph in Figure 1, where $|l(u)| = |l(v)| = |l(w)| = 2$ and both v_2 and v_4 saturate all u, v, w. For convenience, the colors of edges of Q are illustrated in Figure 1.

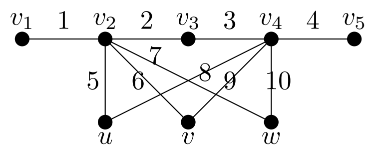

Figure 1. The graph Q with no rainbow $P_5 \cup P_2$.

Notice that K_n does not contain any rainbow $P_5 \cup P_2$. Consider a copy of $P_5 \cup P_2$, where $P_5 = v_1v_2uv_4v_3$ and $P_2 = wv_5$, we have $c(wv_5) \in \{1, 5, 8, 3\}$. Similarly, consider another copy of $P_5 \cup P_2$, where $P_5 = uv_2vv_4v_3$ and $P_2 =$

wv_5, we have $c(wv_5) \in \{5, 6, 9, 3\}$. Consider the third copy of $P_5 \cup P_2$, where $P_5 = v_1v_2vv_4u$ and $P_2 = wv_5$, we have $c(wv_5) \in \{1, 6, 9, 8\}$. Since $\{1, 5, 8, 3\} \cap \{5, 6, 9, 3\} \cap \{1, 6, 9, 8\} = \emptyset$, a contradiction. This completes the proof of the theorem. ■

4. Result for $P_5 \cup tP_2$ with $2 \leq t \leq 4$

Lemma 9. *For any integers $2 \leq t \leq 4$ and $n = 2t + 6$,*

$$AR(K_n, P_5 \cup tP_2) = \frac{3t^2}{2} + \frac{11t}{2} + 1.$$

This lemma is necessary and also plays a role as the inductive basis in the proof of the following theorem. The proof idea of this lemma is similar to the proof of result for $t \geq 5$ in next section. In order to improve the accessibility of the paper for readers, we present its proof details in APPENDIX.

Theorem 10. *For any integers $2 \leq t \leq 4$ and $n \geq 2t + 6$,*

$$AR(K_n, P_5 \cup tP_2) = t(n - t) + \binom{t}{2} + 1.$$

Proof. To show that $AR(K_n, P_5 \cup tP_2) \geq t(n - t) + \binom{t}{2} + 1$, first we construct an edge-coloring of K_n without rainbow $P_5 \cup tP_2$. Take a complete subgraph $G = K_{n-t}$ of K_n. Color all the edges of G by the same color and then color the other edges by distinct new colors. Then we obtain a $(t(n - t) + \binom{t}{2} + 1)$-edge-colored graph K_n with no rainbow $P_5 \cup tP_2$. This implies that $AR(K_n, P_5 \cup tP_2) \geq t(n - t) + \binom{t}{2} + 1$.

Next we will prove the upper bound, that is, the inequality $AR(K_n, P_5 \cup tP_2) \leq t(n - t) + \binom{t}{2} + 1$. According to Lemma 9, we only need to show that $AR(K_n, P_5 \cup tP_2) \leq t(n - t) + \binom{t}{2} + 1$ for any integer $n \geq 2t + 7$. Let c be a $(t(n - t) + \binom{t}{2} + 2)$-edge-coloring of K_n. We need to show that K_n contains a rainbow $P_5 \cup tP_2$. On the contrary, assume that K_n does not contain any rainbow $P_5 \cup tP_2$. Evidently, any subgraph K_{n-1} has no rainbow $P_5 \cup tP_2$. Then by the induction hypothesis on n, we have $|c(K_{n-1})| \leq t(n - 1 - t) + \binom{t}{2} + 1$. Thus the saturated degree of each vertex v of K_n satisfies

$$|l(v)| \geq t(n - t) + \binom{t}{2} + 2 - t(n - 1 - t) - \binom{t}{2} - 1 = t + 1 \geq 3.$$

Therefore, the saturated degree of each vertex in K_n is at least 3. By Theorem 4, we have $|c(K_n)| > AR(K_n, P_3 \cup (t + 1)P_2)$ for $2 \leq t \leq 4$. So K_n must contain a rainbow subgraph $P_3 \cup (t+1)P_2$, say H, where $H = H_1 \cup H_2$. Let $H_1 = (t+1)P_2$

with $E(H_1) = \{e_i | e_i = x_i y_i, 1 \leq i \leq t+1\}$ and $H_2 = v_1v_2v_3$. Let $D = K_n - V(H)$. Since $n \geq 2t+7$, we can get that $|V(D)| \geq 2$. Take a vertex $u \in V(D)$. Now we consider the value of $|l_3(D,u)|$.

Suppose that $|l_3(D,u)| = 0$ for each vertex $u \in V(D)$. Notice that $|l_1(H_1,v_1)| = 0$ and $|l_2(H_2,v_1)| \leq 2$. So $|l_3(D,v_1)| \geq 1$. Take a vertex $w \in V(D)$ with $w \neq u$. Since K_n has no rainbow $P_5 \cup tP_2$, we can get that $|l(H,w)| \leq 2$. So $3 \leq |l(w)| = |l_3(D,w)| + |l(H,w)| \leq 2$, a contradiction. Suppose that $|l_3(D,u)| \geq 1$ for each vertex $u \in V(D)$. Since K_n does not contain rainbow $P_5 \cup tP_2$, we can obtain that $|l_1(H_1,v_1)| = 0$ and $|l_3(D,v_1)| = 0$. So $3 \leq |l(v_1)| = |l_1(H_1,v_1)| + |l_2(H_2,v_1)| + |l_3(D,v_1)| \leq 2$, a contradiction. Hence we can assume that there are two distinct vertices $v, w \in V(D)$ such that $|l_3(D,v)| = 0$ and $|l_3(D,w)| \geq 1$. If there is a vertex $v \in V(K_n)$ such that $|l(v)| \geq 4$, it is easy to obtain a rainbow $P_5 \cup tP_2$. Hence $|l(v)| = 3$ for each vertex $v \in V(K_n)$. Since K_n has no rainbow $P_5 \cup tP_2$, we can deduce that each component of K_n is K_4. Now, take two distinct K_4, say $K_4[v_i, v_j, v_k, v_l]$ and $K_4[v_i', v_j', v_k', v_l']$. Then we can consider the color of edge $v_iv_{i'}$. Regardless of the color of edge $v_iv_{i'}$, we can get a rainbow $P_5 \cup tP_2$, a contradiction.

This completes the proof of the theorem. ■

5. Result for $P_5 \cup tP_2$ with $t \geq 5$

Theorem 11. *Let $n_0(t) = \frac{5t+7}{2} + \frac{1}{t}$. For $n \geq 2t+6$ and $t \geq 5$, $AR(K_n, P_5 \cup tP_2) = f(n,t)$, where*

$$f(n,t) = \begin{cases} (t+1)(2t+1)+1, & \text{if } 2t+6 \leq n \leq \lfloor n_0(t) \rfloor, \\ t(n-t) + \binom{t}{2} + 1, & \text{if } n \geq \lceil n_0(t) \rceil. \end{cases}$$

It is worth noting that $(t+1)(2t+1)+1 = t(n-t) + \binom{t}{2} + 1$ if and only if $n = n_0(t)$. Furthermore, $(t+1)(2t+1)+1 \geq t(n-t) + \binom{t}{2} + 1$ if and only if $n \leq n_0(t)$. We prove Theorem 11 in the following subsections.

5.1. Lower bound of $AR(K_n, P_5 \cup tP_2)$

Proof. In order to prove the lower bound of $AR(K_n, P_5 \cup tP_2)$, we give two edge-colorings of K_n without rainbow $P_5 \cup tP_2$, which contains exactly $(t+1)(2t+1)+1$ and $t(n-t) + \binom{t}{2} + 1$ colors, respectively.

For the first coloring, take a subgraph K_{2t+2} of K_n. Color the subgraph K_{2t+2} into rainbow and the remaining edges are colored with one new color. Then we can get exactly a $((t+1)(2t+1)+1)$-edge-coloring of K_n without rainbow $P_5 \cup tP_2$.

In the second coloring, take a subgraph K_{n-t} of K_n. Color all the edges of K_{n-t} by the same color, which is never used again. The remaining edges are each colored with a unique color. Then we can get exactly a $(t(n-t)+\binom{t}{2}+1)$-edge-coloring of K_n without any rainbow $P_5 \cup tP_2$. ■

5.2. Upper bound of $AR(K_n, P_5 \cup tP_2)$

Proof. We prove this by induction on n. Let c be an edge-coloring of K_n and $|c(K_n)| = f(n,t)+1$. By Theorem 4, we have $|c(K_n)| > AR(K_n, P_3 \cup (t+1)P_2)$. So K_n must contain a rainbow subgraph $P_3 \cup (t+1)P_2$, say H, where $H = H_1 \cup H_2$. Let $H_1 = (t+1)P_2$ with $E(H_1) = \{e_i | e_i = x_i y_i, 1 \le i \le t+1\}$ and $H_2 = v_1v_2v_3$. Let $D = K_n - V(H)$. Then we can get $1 \le |V(D)| \le n-(2t+5)$.

We need to prove that there is a rainbow $P_5 \cup tP_2$ in K_n. On the contrary, we suppose that there is no rainbow $P_5 \cup tP_2$. Obviously, any subgraph K_{n-1} in K_n has no rainbow $P_5 \cup tP_2$. Below in the following subsection, we will complete the proof by considering the value of n.

5.2.1. $2t+6 \le n \le \lfloor n_0(t) \rfloor$.

By the definition of $n_0(t)$, we have

$$\lfloor n_0(t) \rfloor = \begin{cases} \frac{5t+6}{2}, & \text{if } t \text{ is even and } t \ge 6, \\ \frac{5t+7}{2}, & \text{if } t \text{ is odd and } t \ge 5. \end{cases}$$

Then we have

$$|V(D)| \le \begin{cases} \frac{t-4}{2}, & \text{if } t \text{ is even and } t \ge 6, \\ \frac{t-3}{2}, & \text{if } t \text{ is odd and } t \ge 5. \end{cases}$$

Let G ba a rainbow spanning subgraph of K_n with $|c(K_n)|$ edges and $H \subseteq G$. Since K_n has no rainbow $P_5 \cup tP_2$, we can deduce that $c([\{v_1,v_3\}, V(H_1)]) \subseteq c(H)$. We can see that there is no P_5 in $G[V(D)]$. Otherwise, we can obtain a rainbow $P_5 \cup tP_2$. So we can get that each component of the graph $G[V(D)]$ consists of stars, cycles, K_4 or isolated vertices. So we can deduce that $|E(G[V(D)])| \le \frac{3|V(D)|}{2}$. Moreover, for each vertex $v \in V(D)$, it mush hold that v is adjacent to at most one component of H in G and this implies that $|[v, V(H)]_G| \le 3$. According to $c([\{v_1,v_3\}, V(H_1)]) \subseteq c(H)$, we can get that $|[V(H_2), V(H_1)]_G| \le 2t+2$, i.e. $|[v_2, V(H_1)]_G| \le 2t+2$. Below, we will discuss the value of $|[v_2, V(H_1)]_G|$.

Case 1. $|[v_2, V(H_1)]_G| \ge 1$. Without loss of generality, let $v_2x_1 \in E(G)$. Then we can get $c(v_1v_3) \in c(H_1) \cup c(v_2x_1)$. And we have the following claims.

Claim 1. $[\{v_1, v_3, y_1\}, V(D)]_G = \emptyset$.

Proof. Suppose that $[\{v_1, v_3, y_1\}, V(D)]_G \ne \emptyset$. Then we can see that there is a rainbow P_5 on the vertex set $V(H_2) \cup \{x_1, y_1, v\}$, the union of this P_5 and $H_1 - x_1y_1$

forms a rainbow $P_5 \cup tP_2$, a contradiction. Thus $[\{v_1, v_3, y_1\}, V(D)]_G = \emptyset$ and so the claim holds. $\square$

From Claim 1, we can deduce that $|[v, V(H)]_G| \le 2$ for each vertex $v \in V(D)$.

Claim 2. $|E(D) \cap E(G)| = 0$.

Proof. Suppose that $E(D) \cap E(G) \neq \emptyset$. Let $vw \in E(G)$ with $v, w \in V(D)$. Then we can get that $[V(e_1), V(H_1 - e_1)]_G = \emptyset$. So we have $|E(G[V(H_1)])| \le \binom{2t}{2} + 1$. Then we have

$$\begin{aligned}
|E(G)| &= |E(G[V(H_1)])| + |E(G[V(H_2)])| + |[V(H_2), V(H_1)]_G| \\
&\quad + |[V(D), V(H)]_G| + |E(G[V(D)])| \\
&\le \binom{2t}{2} + 1 + 2 + 2(t+1) + 2|V(D)| + \frac{3|V(D)|}{2} \\
&= 2t^2 + t + 5 + \frac{7|V(D)|}{2} \\
&\le \begin{cases} 2t^2 + \frac{11t}{4} - 2, & \text{if } t \text{ is even and } t \ge 6, \\ 2t^2 + \frac{11t}{4} - \frac{1}{4}, & \text{if } t \text{ is odd and } t \ge 5. \end{cases}
\end{aligned}$$

By $|c(K_n)| = f(n,t) + 1 = 2t^2 + 3t + 3$, we can see that $|E(G)| < |c(K_n)|$, a contradiction. Thus $|E(D) \cap E(G)| = 0$ and so the claim holds. $\square$

Claim 3. $|E(G[V(H_1)])| \le 2t^2 + 1$.

Proof. By $[\{v_1, v_3, y_1\}, V(D)]_G = \emptyset$, we have $c([\{v_1, v_3\}, V(D)]) \subseteq c(H) \cup c(v_2x_1)$. Now we will consider the color of edge v_1v.

Assume that $c(v_1v) \in c(H_2)$. If $c(v_1v) = c(v_2v_3)$, there is a rainbow $P_5 = vv_1v_2x_1y_1$, and the union of this P_5 and $H_1 - x_1y_1$ forms a rainbow $P_5 \cup tP_2$, a contradiction. Hence we can assume that $c(v_1v) = c(v_1v_2)$. Then we have $[V(e_1), V(H_1 - e_1)]_G = \emptyset$. Otherwise we can obtain that there is a rainbow $P_5 \cup P_2$ on the vertex set $V(H_2) \cup \{x_1, y_1, x_i, y_i, v\}$ with $2 \le i \le t+1$, the union of this $P_5 \cup P_2$ and $H_1 - e_1 - e_i$ forms a rainbow $P_5 \cup tP_2$, a contradiction. So $|E(G[V(H_1)])| \le \binom{2(t+1)}{2} - 4t = 2t^2 - t + 1 < 2t^2 + 1$.

Assume that $c(v_1v) = c(v_2x_1)$. Then we have $[V(e_1), V(H_1 - e_1)]_G = \emptyset$. Suppose that $[V(e_1), V(H_1 - e_1)]_G \neq \emptyset$. Without loss of generality, let $x_1x_2 \in E(G)$. Since K_n has no rainbow $P_5 \cup tP_2$, we can get that $[V(e_2), V(e_i)]_G = \emptyset$ with $3 \le i \le t+1$. So $|E(G[V(H_1)])| \le \binom{2(t+1)}{2} - 4(t-1) = 2t^2 - t + 5$. Then we have

$$\begin{aligned}|E(G)| &= |E(G[V(H_1)])| + |E(G[V(H_2)])| + |[V(H_2), V(H_1)]_G| \\ &\quad + |[V(D), V(H)]_G| + |E(G[V(D)])| \\ &\le 2t^2 - t + 5 + 2 + 2(t+1) + 2|V(D)| - 1 = 2t^2 + t + 8 + 2|V(D)| \\ &\le \begin{cases} 2t^2 + 2t + 4, & \text{if } t \text{ is even and } t \ge 6, \\ 2t^2 + 2t + 5, & \text{if } t \text{ is odd and } t \ge 5. \end{cases}\end{aligned}$$

By $|c(K_n)| = f(n,t) + 1 = 2t^2 + 3t + 3$, we can see that $|E(G)| < |c(K_n)|$ for all $t \ge 5$, a contradiction. So $|E(G[V(H_1)])| \le \binom{2(t+1)}{2} - 4t = 2t^2 - t + 1 < 2t^2 + 1$.

Hence we can assume that $c(v_1v) \in c(H_1)$. Suppose that $c(v_1v) = c(e_1)$. We can obtain that $[V(e_1), V(H_1 - e_1)]_G = \emptyset$. Suppose that $[V(e_1), V(H_1 - e_1)]_G \ne \emptyset$. Clearly, $[x_1, V(H_1 - e_1)]_G = \emptyset$. Hence without loss of generality, let $y_1x_2 \in E(G)$. Then we have $|[V(e_2), V(e_i)]_G| \le 2$ with $3 \le i \le t+1$. So $|E(G[V(H_1)])| \le \binom{2(t+1)}{2} - 2t - 2(t-1) = 2t^2 - t + 3$. Then

$$\begin{aligned}|E(G)| &= |E(G[V(H_1)])| + |E(G[V(H_2)])| + |[V(H_2), V(H_1)]_G| \\ &\quad + |[V(D), V(H)]_G| + |E(G[V(D)])| \\ &\le 2t^2 - t + 3 + 2 + 2(t+1) + 2|V(D)| = 2t^2 + t + 7 + 2|V(D)| \\ &\le \begin{cases} 2t^2 + 2t + 3, & \text{if } t \text{ is even and } t \ge 6, \\ 2t^2 + 2t + 4, & \text{if } t \text{ is odd and } t \ge 5. \end{cases}\end{aligned}$$

By $|c(K_n)| = f(n,t) + 1 = 2t^2 + 3t + 3$, we can see that $|E(G)| < |c(K_n)|$ for all $t \ge 5$, a contradiction. So $|E(G[V(H_1)])| \le \binom{2(t+1)}{2} - 4t = 2t^2 - t + 1 < 2t^2 + 1$. Hence without loss of generality, we can assume that $c(v_1v) = c(e_2)$. Since K_n has no rainbow $P_5 \cup tP_2$, we can obtain that $|[V(e_2), V(e_i)]_G| \le 1$ with $1 \le i \le t+1$ and $i \ne 2$. Suppose $|[V(e_2), V(e_i)]_G| \ge 3$, there are two independent edges in $[V(e_2), V(e_i)]_G$, say x_2x_i and y_2y_i. Then $vv_1v_2x_ix_2$, y_2y_i and $H_1 - e_1 - e_2$ form a rainbow $P_5 \cup tP_2$ with $i = 1$ or $vv_1v_2x_1y_1$, x_2x_i, y_2y_i and $H_1 - e_1 - e_2 - e_i$ form a rainbow $P_5 \cup tP_2$ with $i \ge 2$, a contradiction. Hence we can assume that there is an edge e_i such that $|[V(e_2), V(e_i)]_G| = 2$ with $1 \le i \le t+1$ and $i \ne 2$. Without loss of generality, let $x_2x_1, y_2x_1 \in E(G)$, then we have $[V(e_2), V(e_j)]_G = \emptyset$ with $3 \le j \le t+1$. So $|E(G[V(H_1)])| \le \binom{2(t+1)}{2} - 4(t-1) = 2t^2 - t + 5$. Then we have

$$\begin{aligned}|E(G)| &= |E(G[V(H_1)])| + |E(G[V(H_2)])| + |[V(H_2), V(H_1)]_G| \\ &\quad + |[V(D), V(H)]_G| + |E(G[V(D)])| \\ &\le 2t^2 - t + 5 + 2 + 2(t+1) + 2|V(D)| = 2t^2 + t + 9 + 2|V(D)| \\ &\le \begin{cases} 2t^2 + 2t + 5, & \text{if } t \text{ is even and } t \ge 6, \\ 2t^2 + 2t + 4, & \text{if } t \text{ is odd and } t \ge 5. \end{cases}\end{aligned}$$

By $|c(K_n)| = f(n,t) + 1 = 2t^2 + 3t + 3$, we can see that $|E(G)| < |c(K_n)|$ for all $t \geq 5$, a contradiction. Thus $|[V(e_2), V(e_i)]_G| \leq 1$ with $1 \leq i \leq t+1$ and $i \neq 2$. So $|E(G[V(H_1)])| \leq \binom{2(t+1)}{2} - 3t = 2t^2 + 1$. This implies that $|E(G[V(H_1)])| \leq 2t^2 + 1$ and so the claim holds. □

Combined with the above claims, we can obtain that

$$\begin{aligned}
|E(G)| &= |E(G[V(H_1)])| + |E(G[V(H_2)])| + |[V(H_2), V(H_1)]_G| \\
&\quad + |[V(D), V(H)]_G| + |E(G[V(D)])| \\
&\leq 2t^2 + 1 + 2 + 2(t+1) + 2|V(D)| = 2t^2 + 3t + 5 + 2|V(D)| \\
&\leq \begin{cases} 2t^2 + 3t + 1, & \text{if } t \text{ is even and } t \geq 6, \\ 2t^2 + 3t + 2, & \text{if } t \text{ is odd and } t \geq 5. \end{cases}
\end{aligned}$$

By $|c(K_n)| = f(n,t) + 1 = 2t^2 + 3t + 3$, we can see that $|E(G)| < |c(K_n)|$ for all $t \geq 5$, a contradiction.

Case 2. $|[v_2, V(H_1)]_G| = 0$. By $[\{v_1, v_3\}, V(H_1)]_G = \emptyset$ and $[v_2, V(H_1)]_G = \emptyset$, we can obtain that $[V(H_2), V(H_1)]_G = \emptyset$. And we have the following claims.

Claim 4. For each vertex $v \in V(D)$, $|[v, V(H_2)]_G| \leq 2$.

Proof. Suppose that there is a vertex $v \in V(D)$ such that $|[v, V(H_2)]_G| = 3$. Without loss of generality, let $\{vv_1, vv_2, vv_3\} \subseteq E(G)$. Then we can obtain that $[w, V(H_2)]_G = \emptyset$ for each vertex $w \in V(D)$ with $w \neq v$. Thus we have $|[V(D), V(H)]_G| \leq 3 + 2(|V(D)| - 1) = 2|V(D)| + 1$.

Now consider the color of edge y_1v_3. Suppose that $c(y_1v_3) \in c(H_2)$, there is a rainbow P_5 on the vertex set $V(H_2) \cup \{v\}$, the union of this P_5 and $H_1 - x_1y_1$ forms a rainbow $P_5 \cup tP_2$, a contradiction. Hence $c(y_1v_3) \in c(H_1)$. Assume that $c(y_1v_3) = c(e_1)$, we can get a rainbow $P_5 = v_2v_1vv_3y_1$, the union of this P_5 and $H_1 - x_1y_1$ forms a rainbow $P_5 \cup tP_2$, a contradiction. Thus, without loss of generality, assume that $c(y_1v_3) = c(e_2)$, then we have $E(G[V(D)]) = \emptyset$ and $|[V(e_2), V(e_i)]_G| \leq 2$ with $1 \leq i \leq t+1$ and $i \neq 2$. So $|E(G[V(H_1)])| \leq \binom{2(t+1)}{2} - 2t = 2t^2 + t + 1$. Then we have

$$\begin{aligned}
|E(G)| &= |E(G[V(H_1)])| + |E(G[V(H_2)])| + |[V(H_2), V(H_1)]_G| \\
&\quad + |[V(D), V(H)]_G| + |E(G[V(D)])| \\
&\leq 2t^2 + t + 1 + 3 + 2|V(D)| + 1 = 2t^2 + t + 5 + 2|V(D)| \\
&\leq \begin{cases} 2t^2 + 2t + 1, & \text{if } t \text{ is even and } t \geq 6, \\ 2t^2 + 2t + 2, & \text{if } t \text{ is odd and } t \geq 5. \end{cases}
\end{aligned}$$

By $|c(K_n)| = f(n,t) + 1 = 2t^2 + 3t + 3$, we can see that $|E(G)| < |c(K_n)|$ for all $t \geq 5$. Thus we get a contradiction. This implies that $|[v, V(H_2)]_G| \leq 2$ for each vertex $v \in V(D)$ and so the claim holds. □

Claim 5. $[V(D), V(H_1)]_G = \emptyset$.

Proof. Take a vertex $v \in V(D)$. Without loss of generality, let $vx_1 \in E(G)$. Now consider the color of edge y_1v_3. Suppose that $c(y_1v_3) \in c(H_2)$. Let $c(y_1v_3) = c(v_1v_2)$, there is a rainbow $P_5 = vx_1y_1v_3v_2$, the union of this P_5 and $H_1 - x_1y_1$ forms a rainbow $P_5 \cup tP_2$, a contradiction.

So we can assume $c(y_1v_3) = c(v_2v_3)$. Then we can obtain that $v_1v_3 \notin E(G)$ and $[V(e_1), V(H_1 - e_1)]_G = \emptyset$. Otherwise we can obtain a rainbow $P_5 \cup tP_2$. So $|E(G[V(H_1)])| \le \binom{2t}{2} + 1 = 2t^2 - t + 1$. Then we have

$$\begin{aligned}
|E(G)| &= |E(G[V(H_1)])| + |E(G[V(H_2)])| + |[V(H_2), V(H_1)]_G| \\
&\quad + |[V(D), V(H)]_G| + |E(G[V(D)])| \\
&\le 2t^2 - t + 1 + 2 + 2|V(D)| + \frac{3|V(D)|}{2} = 2t^2 - t + 3 + \frac{7|V(D)|}{2} \\
&\le \begin{cases} 2t^2 + \frac{3t}{4} - 4, & \text{if } t \text{ is even and } t \ge 6, \\ 2t^2 + \frac{3t}{4} - \frac{9}{4}, & \text{if } t \text{ is odd and } t \ge 5. \end{cases}
\end{aligned}$$

By $|c(K_n)| = f(n,t) + 1 = 2t^2 + 3t + 3$, we can see that $|E(G)| < |c(K_n)|$ for all $t \ge 5$, a contradiction. Hence $c(y_1v_3) \in c(H_1)$.

Suppose that $c(y_1v_3) = c(e_1)$. Then we get that $|[V(e_1, V(e_i)]_G| \le 2$ with $2 \le i \le t+1$. So $|E(G[V(H_1)])| \le \binom{2(t+1)}{2} - 2t = 2t^2 + t + 1$. Then we have

$$\begin{aligned}
|E(G)| &= |E(G[V(H_1)])| + |E(G[V(H_2)])| + |[V(H_2), V(H_1)]_G| \\
&\quad + |[V(D), V(H)]_G| + |E(G[V(D)])| \\
&\le 2t^2 + t + 1 + 3 + 2|V(D)| + \frac{3|V(D)|}{2} = 2t^2 + t + 4 + \frac{7|V(D)|}{2} \\
&\le \begin{cases} 2t^2 + \frac{11t}{4} - 3, & \text{if } t \text{ is even and } t \ge 6, \\ 2t^2 + \frac{11t}{4} - \frac{5}{4}, & \text{if } t \text{ is odd and } t \ge 5. \end{cases}
\end{aligned}$$

By $|c(K_n)| = f(n,t) + 1 = 2t^2 + 3t + 3$, we can see that $|E(G)| < |c(K_n)|$ for all $t \ge 5$, a contradiction. Hence, without loss of generality, let $c(y_1v_3) = c(e_2)$. Then we have $|[V(e_2), V(e_i)]_G| \le 2$ with $1 \le i \le t+1$ and $i \ne 2$. So $|E(G[V(H_1)])| \le \binom{2(t+1)}{2} - 2t = 2t^2 + t + 1$. Then we have

$$\begin{aligned}
|E(G)| &= |E(G[V(H_1)])| + |E(G[V(H_2)])| + |[V(H_2), V(H_1)]_G| \\
&\quad + |[V(D), V(H)]_G| + |E(G[V(D)])| \\
&\le 2t^2 + t + 1 + 3 + 2|V(D)| + \frac{3|V(D)|}{2} = 2t^2 + t + 4 + \frac{7|V(D)|}{2} \\
&\le \begin{cases} 2t^2 + \frac{11t}{4} - 3, & \text{if } t \text{ is even and } t \ge 6, \\ 2t^2 + \frac{11t}{4} - \frac{5}{4}, & \text{if } t \text{ is odd and } t \ge 5. \end{cases}
\end{aligned}$$

By $|c(K_n)| = 2t^2 + 3t + 3$, we have $|E(G)| < |c(K_n)|$ for all $t \geq 5$, a contradiction. This implies that $|[V(D), V(H_1)]_G| = \emptyset$ and so the claim holds. $\square$

From Claims 4 and 5, we can deduce that $|[v, V(H)]_G| \leq 2$ for each vertex $v \in V(D)$.

Claim 6. $|E(G[V(H_1)])| \leq 2t^2 + t + 2$.

Proof. Notice that $[V(D), V(H_1)]_G = \emptyset$, which means that $c([V(D), V(H_1)]) \subseteq c(H)$. Take a vertex $v \in V(D)$ and below we will consider the color of edge x_1v.

Suppose that $c(vx_1) \in c(H_2)$. Since K_n has no rainbow $P_5 \cup tP_2$. Then we have $|[V(e_1), V(e_i)]_G| \leq 2$ with $2 \leq i \leq t+1$. Otherwise we can easily get a rainbow $P_5 \cup tP_2$, a contradiction. So $|E(G[V(H_1)])| \leq \binom{2(t+1)}{2} - 2t = 2t^2 + t + 1$.

Hence we can assume that $c(vx_1) \in c(H_1)$. Assume that $c(vx_1) = c(e_1)$, we can obtain that $|[V(e_1), V(e_i)]_G| \leq 2$ with $2 \leq i \leq t+1$. So $|E(G[V(H_1)])| \leq \binom{2(t+1)}{2} - 2t = 2t^2 + t + 1$. Hence, without loss of generality, let $c(vx_1) = c(e_2)$. Now consider the color of edge y_1v_3. If $c(y_1v_3) \in c(H_2)$, then let $c(y_1v_3) = c(v_1v_2)$. Hence $|[V(e_2), V(e_1)]_G| \leq 3$ and $|[V(e_2), V(e_i)]_G| \leq 2$ with $3 \leq i \leq t+1$. So $|E(G[V(H_1)])| \leq \binom{2(t+1)}{2} - 2(t-1) - 1 = 2t^2 + t + 2$. Hence let $c(y_1v_3) = c(v_2v_3)$, we have $|[V(e_1), V(e_i)]_G| \leq 2$ with $2 \leq i \leq t+1$. So $|E(G[V(H_1)])| \leq \binom{2(t+1)}{2} - 2t = 2t^2 + t + 1$. Hence $c(y_1v_3) \in c(H_1)$. Suppose that $c(y_1v_3) = c(e_1)$. We have $|[V(e_1), V(e_i)]_G| \leq 2$ with $2 \leq i \leq t+1$. So $|E(G[V(H_1)])| \leq \binom{2(t+1)}{2} - 2t = 2t^2 + t + 1$. Hence, without loss of generality, let $c(y_1v_3) = c(e_2)$. Then we have $|[V(e_2), V(e_i)]_G| \leq 2$ with $1 \leq i \leq t+1$ and $i \neq 2$. So $|E(G[V(H_1)])| \leq \binom{2(t+1)}{2} - 2t = 2t^2 + t + 1$. This implies that $|E(G[V(H_1)])| \leq 2t^2 + t + 2$ and so the claim holds. $\square$

Combined with the above claims, we can obtain that

$$\begin{aligned}
|E(G)| &= |E(G[V(H_1)])| + |E(G[V(H_2)])| + |[V(H_2), V(H_1)]_G| \\
&\quad + |[V(D), V(H)]_G| + |E(G[V(D)])| \\
&\leq 2t^2 + t + 2 + 3 + 2|V(D)| + \frac{3|V(D)|}{2} = 2t^2 + t + 5 + \frac{7|V(D)|}{2} \\
&\leq \begin{cases} 2t^2 + \frac{11t}{4} - 2, & \text{if } t \text{ is even and } t \geq 6, \\ 2t^2 + \frac{11t}{4} - \frac{1}{4}, & \text{if } t \text{ is odd and } t \geq 5. \end{cases}
\end{aligned}$$

By $|c(K_n)| = f(n,t) + 1 = 2t^2 + 3t + 3$, we can see that $|E(G)| < |c(K_n)|$ for all $t \geq 5$, a contradiction.

5.2.2. $n = \lfloor n_0(t) \rfloor + 1$.

We have proved that the theorem holds for $2t + 6 \leq n \leq \lfloor n_0(t) \rfloor$. Since $n = \lfloor n_0(t) \rfloor + 1 \geq \lceil n_0(t) \rceil$, we can obtain that $|c(K_n)| = t(n-t) + \binom{t}{2} + 2$. We can see that any subgraph K_{n-1} does not contain rainbow $P_5 \cup tP_2$. So by the induction on n, we have $|c(K_{n-1})| \leq (t+1)(2t+1)+1$. Thus, the saturated degree of each vertex of K_n satisfies

$$|l(v)| \geq t(n-t) + \binom{t}{2} + 2 - (t+1)(2t+1) - 1 = nt - \frac{5t^2}{2} - \frac{7t}{2}.$$

According to the definition of $n_0(t)$, we have

$$n = \lfloor n_0(t) \rfloor + 1 = \begin{cases} \frac{5t+8}{2}, & \text{if } t \text{ is even and } t \geq 6, \\ \frac{5t+9}{2}, & \text{if } t \text{ is odd and } t \geq 5. \end{cases}$$

Hence

$$\begin{aligned} |l(v)| &\geq \begin{cases} \frac{t}{2}, & \text{if } t \text{ is even and } t \geq 6, \\ t, & \text{if } t \text{ is odd and } t \geq 5, \end{cases} \\ &\geq \begin{cases} 3, & \text{if } t \text{ is even and } t \geq 6, \\ 5, & \text{if } t \text{ is odd and } t \geq 5. \end{cases} \end{aligned}$$

Thus $|l(v)| \geq 3$ for each vertex $v \in V(K_n)$. Take a vertex $u \in V(D)$. Now we consider the value of $|l_3(D,u)|$.

Suppose that $|l_3(D,u)| = 0$ for each vertex $u \in V(D)$. Notice that $|l_1(H_1,v_1)| = 0$ and $|l_2(H_2,v_1)| \leq 2$. So $|l_3(D,v_1)| \geq 1$. Take a vertex $w \in V(D)$ with $w \neq u$. Since K_n has no rainbow $P_5 \cup tP_2$, we can get that $|l(H,w)| \leq 2$. So $3 \leq |l(w)| = |l_3(D,w)| + |l(H,w)| \leq 2$, a contradiction. Suppose that $|l_3(D,u)| \geq 1$ for each vertex $u \in V(D)$. Since K_n does not contain rainbow $P_5 \cup tP_2$, we can obtain that $|l_1(H_1,v_1)| = 0$ and $|l_3(D,v_1)| = 0$. So $3 \leq |l(v_1)| = |l_1(H_1,v_1)| + |l_2(H_2,v_1)| + |l_3(D,v_1)| \leq 2$, a contradiction. Hence we can assume that there are two distinct vertices $v, w \in V(D)$ such that $|l_3(D,v)| = 0$ and $|l_3(D,w)| \geq 1$. If there is a vertex $v \in V(K_n)$ such that $|l(v)| \geq 4$, it is easy to obtain a rainbow $P_5 \cup tP_2$. Hence $|l(v)| = 3$ for each vertex $v \in V(K_n)$. Since K_n has no rainbow $P_5 \cup tP_2$, we can deduce that each component of K_n is K_4. Now, take two distinct K_4, say $K_4[v_i, v_j, v_k, v_l]$ and $K_4[v_i^{'}, v_j^{'}, v_k^{'}, v_l^{'}]$. Then we can consider the color of edge $v_iv_{i^{'}}$. Regardless of the color of edge $v_iv_{i^{'}}$, we can get a rainbow $P_5 \cup tP_2$, a contradiction.

5.2.3. $n > \lfloor n_0(t) \rfloor + 1$.

Now we proved that the theorem holds for $2t+6 \leq n \leq \lfloor n_0(t) \rfloor + 1$. Since $n > \lfloor n_0(t) \rfloor + 1 \geq \lceil n_0(t) \rceil$, we have $|c(K_n)| = t(n-t) + \binom{t}{2} + 2$. By the induction hypothesis on n, we can obtain that $|c(K_{n-1})| \leq t(n-1-t) + \binom{t}{2} + 1$. Thus the saturated degree of each vertex v of K_n satisfies

$$|l(v)| \geq t(n-t) + \binom{t}{2} + 2 - t(n-1-t) - \binom{t}{2} - 1 = t + 1 \geq 6.$$

Thus $|l(v)| \geq 6$ for each vertex $v \in V(K_n)$. Take a vertex $u \in V(D)$. Notice that $|l(H,u)| \leq 3$. By $|l(u)| \geq 6$, we have $|l_3(D,u)| \geq 3$ for each $u \in V(D)$. Since K_n does not contain rainbow $P_5 \cup tP_2$, we have $|l_1(H_1,v_1)| = 0$ and $|l_3(D,v_1)| = 0$. Hence

$$6 \leq |l(v_1)| = |l_1(H_1,v_1)| + |l_2(H_2,v_1)| + |l_3(D,v_1)| \leq 2,$$

a contradiction.

This completes the proof of the theorem. ■

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APPENDIX

***Proof of Lemma* 7.** In order to show that the lower bound, first we construct an edge-coloring of K_7 without rainbow $P_5 \cup P_2$. Take a vertex $v \in V(K_7)$. All edges incident with the vertex v are colored by distinct colors, and all other edges are colored by two additional colors. Then we can see that the number of color is 8 and there is no rainbow $P_5 \cup P_2$. This implies that $AR(K_7, P_5 \cup P_2) \geq 8$.

Next we prove the upper bound, that is, the inequality $AR(K_7, P_5 \cup P_2) \leq 8$. Suppose that there is a 9-edge-coloring of the graph K_7 such that the graph K_7 does not contain rainbow $P_5 \cup P_2$. According to Theorem 4, we know that $AR(K_7, P_3 \cup 2P_2) = 7$. Thus, K_7 contains a rainbow $P_3 \cup 2P_2$. Let G be a rainbow spanning subgraph of size 9 which contains a subgraph isomorphic to the graph $P_3 \cup 2P_2$. Let $H_1 = P_3$ and $H_2 = 2P_2$ be vertex disjoint subgraph of G and let $H = H_1 \cup H_2$. Let $H_1 = v_1v_2v_3, V(H_2) = \{v_4, v_5, v_6, v_7\}$ and $E(H_2) = \{v_4v_5, v_6v_7\}$. Then we can get $|c(G)\backslash c(H)| = 5$. For convenience, we use the set $\{1, 2, \ldots, 9\}$ to represent the set of colors of $E(G)$. Let $c(v_1v_2) = 1$, $c(v_2v_3) = 2$, $c(v_4v_5) = 3$ and $c(v_6v_7) = 4$. Since $c(K_7)$ does not contain rainbow $P_5 \cup P_2$, we can deduce that $c([\{v_1, v_3\}, V(H_2)]) \subseteq \{1, 2, 3, 4\}$. Let $C_1 = [\{v_4, v_5\}, \{v_6, v_7\}]_G$ and let $C_2 = \{v_1v_3\} \cap E(G)$. Let $d^*(v_2) = [v_2, V(H_2)]_G$. Then we have $|C_1| + |C_2| + |d^*(v_2)| = |c(G)\backslash c(H)| = 5$. Note that when $|d^*(v_2)| \geq 1$, we can obtain that $|C_2| = 0$, otherwise we can easily get a rainbow $P_5 \cup P_2$. Next we distinguish the cases on $|d^*(v_2)|$.

Case 1. $|d^*(v_2)| = 0$. In this case, we have $|C_1| + |C_2| = 5$, i.e., $\{v_1v_3, v_4v_6, v_4v_7, v_5v_6, v_5v_7\} \subseteq E(G)$, see Figure 2. Now consider the color of edge v_4v_2. It is that no matter which color $c(v_4v_2)$ takes, we can get a rainbow $P_5 \cup P_2$, where $P_5 = v_2v_4v_7v_5v_6$ and $P_2 = v_1v_3$, a contradiction.

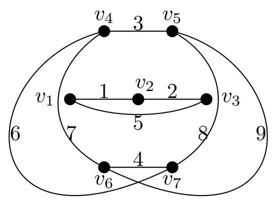

Figure 2. Edge-coloring of G when $|d^*(v_2)| = 0$.

Case 2. $|d^*(v_2)| = 1$. In this case, we have $|C_1| = 4$, see Figure 3. By symmetry, assume that $v_2v_4 \in E(G)$. Then we have $c(v_1v_6) = 2$, else K_7 contains a rainbow $P_5 \cup P_2$ with $P_5 = v_1v_6v_5v_7v_4$ and $P_2 = v_2v_3$. Similarly, we have $c(v_3v_7) = 1$, else K_7 contains a rainbow $P_5 \cup P_2$ with $P_5 = v_3v_7v_5v_6v_4$ and $P_2 = v_1v_2$. This implies that K_7 has a rainbow $P_5 \cup P_2$ with $P_5 = v_1v_6v_5v_7v_3$ and

$P_2 = v_2v_4$, a contradiction.

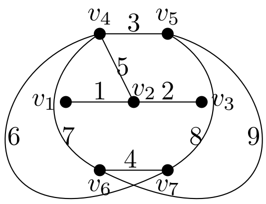

Figure 3. Edge-coloring of G when $|d^*(v_2)| = 1$.

Case 3. $|d^*(v_2)| = 2$. In this case, we have $|C_1| = 3$. By symmetry, we assume that v_2 is adjacent to v_4 and v_i in G, where $i \in \{5, 6\}$.

Suppose first that v_2 is adjacent to v_4 and v_6 in G. Since $|C_1| = 3$, we assume that $v_iv_j \notin E(G)$, where $i \in \{4, 5\}$ and $j \in \{6, 7\}$. Let $\{i, i^{'}\} = \{4, 5\}$ and $\{j, j^{'}\} = \{6, 7\}$. Then we have $c(v_1v_j) = 2$, else we can get a rainbow $P_5 \cup P_2$ in K_7 with $P_5 = v_1v_jv_{i'}v_{j'}v_i$ and $P_2 = v_2v_3$. Similarly, we have $c(v_3v_i) = 1$, else we can get a rainbow $P_5 \cup P_2$ in K_7 with $P_5 = v_3v_iv_{j'}v_{i'}v_j$ and $P_2 = v_1v_2$. This implies that K_7 has a rainbow $P_5 \cup P_2$ with $P_5 = v_3v_iv_{j'}v_jv_1$ and $P_2 = v_2v_{i'}$ when $i = 5$, and $P_5 = v_3v_iv_2v_jv_1$ and $P_2 = v_{i'}v_{j'}$ when $i = 4$ and $j = 6$, and $P_5 = v_3v_iv_{i'}v_jv_1$ and $P_2 = v_2v_{j'}$ when $i = 4$ and $j = 7$, a contradiction.

Suppose that v_2 is adjacent to v_4 and v_5 in G. By symmetry, we assume that $v_5v_6 \notin E(G)$. Then we have $c(v_1v_6) = 2$, else we can get a rainbow $P_5 \cup P_2$ in K_7 with $P_5 = v_1v_6v_4v_7v_5$ and $P_2 = v_2v_3$. Similarly, we have $c(v_3v_5) = 1$, else we can get a rainbow $P_5 \cup P_2$ in K_7 with $P_5 = v_3v_5v_7v_4v_6$ and $P_2 = v_1v_2$. This implies that K_7 has a rainbow $P_5 \cup P_2$ with $P_5 = v_3v_5v_7v_6v_1$ and $P_2 = v_2v_4$, a contradiction.

Case 4. $|d^*(v_2)| = 3$. In this case, we have $|C_1| = 2$. By symmetry, we may assume that v_2 is adjacent to v_4, v_5 and v_6. Let $\{i, i^{'}\} = \{4, 5\}$ and $\{j, j^{'}\} = \{6, 7\}$. Since $|C_1| = 2$, we assume first that $v_iv_j, v_iv_{j'} \in E(G)$. Then we have $c(v_3v_{i'}) = 1$, else we can get a rainbow $P_5 \cup P_2$ in K_7 with $P_5 = v_1v_2v_jv_iv_{j'}$ and $P_2 = v_3v_{i'}$. Similarly, we have $c(v_1v_j) \in \{2, 3\}$, else we can get a rainbow $P_5 \cup P_2$ in K_7 with $P_5 = v_3v_2v_{j'}v_iv_{i'}$ and $P_2 = v_1v_j$. This implies that K_7 contains a rainbow $P_5 \cup P_2$ with $P_5 = v_1v_jv_2v_{j'}v_i$ and $P_2 = v_3v_{i'}$, a contradiction.

Suppose that $v_iv_j, v_{i'}v_{j'} \in E(G)$. Now consider the color of edge v_1v_3. It is that no matter which color $c(v_1v_3)$ takes, we can get a rainbow $P_5 \cup P_2$, where $P_5 = v_iv_jv_2v_{j'}v_{i'}$ and $P_2 = v_1v_3$, a contradiction. Without loss of generality, suppose that $v_iv_j, v_{i'}v_j \in E(G)$. It is that no matter which color $c(v_1v_3)$ takes, we can get a rainbow $P_5 \cup P_2$, where $P_5 = v_{j'}v_2v_iv_jv_{i'}$ and $P_2 = v_1v_3$, a contradiction.

Case 5. $|d^*(v_2)| = 4$. In this case, we have $|C_1| = 1$. By symmetry, we may assume that $v_4v_6 \in E(G)$, see Figure 4. Then we have $c(v_1v_3) \in \{3, 4\}$, else we can get a rainbow $P_5 \cup P_2$ in K_7 with $P_5 = v_5v_4v_6v_7v_2$ and $P_2 = v_1v_3$. However, we can see that there is a rainbow $P_5 \cup P_2$ in K_7 with $P_5 = v_5v_2v_4v_6v_7$ and $P_2 = v_1v_3$ when $c(v_1v_3) = 3$, and $P_5 = v_7v_2v_6v_4v_5$ and $P_2 = v_1v_3$ when $c(v_1v_3) = 4$, a contradiction. This completes the proof of the lemma. ■

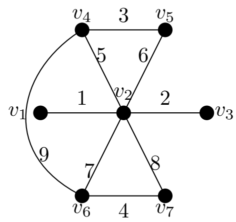

Figure 4. Edge-coloring of G when $|d^*(v_2)| = 4$.

***Proof of Lemma* 9.** To show that $AR(n, P_5 \cup tP_2) \geq \frac{3t^2}{2} + \frac{11t}{2} + 1$, first we construct an edge-coloring of K_n without rainbow $P_5 \cup tP_2$. Take a complete subgraph $G = K_t$ of K_n. Color all the edges of G by the same color and then color all the other edges by distinct new colors. Then we get a $(\frac{3t^2}{2} + \frac{11t}{2} + 1)$-edge-colored graph K_n with no rainbow $P_5 \cup tP_2$.

Next we prove the upper bound, that is, the inequality $AR(K_n, P_5 \cup tP_2) \leq \frac{3t^2}{2} + \frac{11t}{2} + 1$. Let c be an edge-coloring of K_n and $|c(K_n)| = \frac{3t^2}{2} + \frac{11t}{2} + 2$. By Theorem 4, we have $|c(K_n)| > AR(K_n, P_3 \cup (t+1)P_2)$. So K_n must contain a rainbow subgraph $P_3 \cup (t+1)P_2$, say H, where $H = H_1 \cup H_2$. Let $H_1 = (t+1)P_2$ with $E(H_1) = \{e_i | e_i = x_iy_i, i \leq 1 \leq t+1\}$ and $H_2 = v_1v_2v_3$. Let $D = K_n - V(H)$. Since $n = 2t+6$, we can obtain that $|V(D)| = 1$ and let $V(D) = \{v\}$.

Let G be a rainbow spanning subgraph of size $|c(K_n)|$ and $H \subseteq G$. Since K_n has no rainbow $P_5 \cup tP_2$, we can obtain that $c([\{v_1, v_3\}, V(H_1)]) \subseteq c(H)$. Moreover, it must be hold that v is adjacent to at most one component of H in G and this implies that $|[v, V(H)]_G| \leq 3$. From $c([\{v_1, v_3\}, V(H_1)]) \subseteq c(H)$, we have $|[V(H_2), V(H_1)]_G| \leq 2(t+1)$, i.e. $|[v_2, V(H_1)]_G| \leq 2(t+1)$. Below, we will discuss the value of $|[v_2, V(H_1)]_G|$.

Case 1. $|[v_2, V(H_1)]_G| \geq 1$. Without loss of generality, let $v_2x_1 \in E(G)$. Then we have $c(v_1v_3) \in c(H) \cup c(v_2x_1)$. And we have the following claims.

Claim 1. $[\{v_1, v_3, y_1\}, v]_G = \emptyset$.

Proof. Suppose that $[\{v_1, v_3, y_1\}, v]_G \neq \emptyset$. Then we can see that there is a rainbow P_5 on the vertex set $V(H_2) \cup \{x_1, y_1, v\}$, the union of this P_5 and $H_1 - x_1y_1$ forms a rainbow $P_5 \cup tP_2$, a contradiction. Thus $[\{v_1, v_3, y_1\}, v]_G = \emptyset$ and so the claim holds. □

Claim 2. $[V(H_1 - e_1), v]_G = \emptyset$.

Proof. Suppose that $[V(H_1 - e_1), v]_G \neq \emptyset$. Without loss of generality, let $vx_2 \in E(G)$. Then we can obtain that $|[V(e_2), V(e_j)]_G| \leq 2$ with $1 \leq j \leq t+1$ and $j \neq 2$. Now consider the color of edge v_1y_2. Suppose that $c(v_1y_2) \in c(H_2)$. Let $c(v_1y_2) = c(v_2v_3)$, it is easy to obtain a rainbow $P_5 = vx_2y_2v_1v_2$, the union of this P_5 and $H_1 - x_2y_2$ forms a rainbow $P_5 \cup tP_2$, a contradiction. Hence $c(v_1y_2) = c(v_1v_2)$. Then we have $|[V(e_1), V(e_i)]_G| = \emptyset$ with $1 \leq i \leq t+1$. So $|E(G[V(H_1)])| \leq \binom{2(t+1)}{2} - 2t - 4t + 2 = 2t^2 - 3t + 3$. Then

$$\begin{aligned}|E(G)| &= |E(G[V(H_1)])| + |E(G[V(H_2)])| + |[V(H_2), V(H_1)]_G| \\ &\quad + |[V(D), V(H)]_G| \\ &\leq 2t^2 - 3t + 3 + 2 + 2(t+1) + 2 = 2t^2 - t + 9.\end{aligned}$$

By $|c(K_n)| = \frac{3t^2}{2} + \frac{11t}{2} + 2$, we can obtain that

$$|c(K_n)| - |E(G)| \geq -\frac{t^2}{2} + \frac{13t}{2} - 7.$$

Let $f(t) = -\frac{t^2}{2} + \frac{13t}{2} - 7$ and then we have $f^{'}(t) < 0$ for all $2 \leq t \leq 4$. Since $f(4) = 11 > 0$, we can get that $f(t) > 0$ for all $2 \leq t \leq 4$. Thus we get a contradiction. Thus $c(v_1y_2) \in c(H_1)$. Suppose that $c(v_1y_2) = c(e_1)$ or $c(e_2)$. Then we have $|[V(e_1), V(e_i)]_G| = \emptyset$ with $1 \leq i \leq t+1$. So $|E(G[V(H_1)])| \leq \binom{2(t+1)}{2} - 2t - 4t + 2 = 2t^2 - 3t + 3$. Then

$$\begin{aligned}|E(G)| &= |E(G[V(H_1)])| + |E(G[V(H_2)])| + |[V(H_2), V(H_1)]_G| \\ &\quad + |[V(D), V(H)]_G| \\ &\leq 2t^2 - 3t + 3 + 2 + 2(t+1) + 2 = 2t^2 - t + 9.\end{aligned}$$

By $|c(K_n)| = \frac{3t^2}{2} + \frac{11t}{2} + 2$, we can obtain that

$$|c(K_n)| - |E(G)| \geq -\frac{t^2}{2} + \frac{13t}{2} - 7.$$

Let $f(t) = -\frac{t^2}{2} + \frac{13t}{2} - 7$ and then we have $f^{'}(t) < 0$ for all $2 \leq t \leq 4$. Since $f(4) = 11 > 0$, we can get that $f(t) > 0$ for all $2 \leq t \leq 4$. Thus we get a contradiction. Hence without loss of generality, we can assume that $c(v_1y_2) = c(e_3)$, then we have $|[V(e_3), V(e_i)]_G| \leq 2$ with $1 \leq i \leq t+1$. So $|E(G[V(H_1)])| \leq \binom{2(t+1)}{2} - 2t - 2t = 2t^2 - t + 1$. Then

$$\begin{aligned}|E(G)| &= |E(G[V(H_1)])| + |E(G[V(H_2)])| + |[V(H_2), V(H_1)]_G| \\ &\quad + |[V(D), V(H)]_G| \\ &\leq 2t^2 - t + 1 + 2 + 2(t+1) + 2 = 2t^2 + t + 7.\end{aligned}$$

By $|c(K_n)| = \frac{3t^2}{2} + \frac{11t}{2} + 2$, we can obtain that

$$|c(K_n)| - |E(G)| \geq -\frac{t^2}{2} + \frac{9t}{2} - 5.$$

Let $f(t) = -\frac{t^2}{2} + \frac{9t}{2} - 5$ and then we have $f'(t) < 0$ for all $2 \leq t \leq 4$. Since $f(4) = 5 > 0$, we can get that $f(t) > 0$ for all $2 \leq t \leq 4$, a contradiction. Thus $[V(H_1 - e_1), V(D)]_G = \emptyset$ and so the claim holds. $\square$

From Claim 2 above, we have $|[v, V(H)]_G| \leq 1$.

Claim 3. $|E(G[V(H_1)])| \leq 2t^2 + 1$.

Proof. By $[\{v_1, v_3\}, v]_G = \emptyset$, we have $c([\{v_1, v_3\}, v]) \subseteq c(H) \cup c(v_2x_1)$. Next we will consider the color of edge v_1v.

Suppose that $c(v_1v) \in c(H_2)$. Suppose that $c(v_1v) = c(v_2v_3)$. There is a rainbow $P_5 = vv_1v_2x_1y_1$, the union of this P_5 and $H_1 - x_1y_1$ forms a rainbow $P_5 \cup tP_2$, a contradiction. Hence we can assume that $c(v_1v) = c(v_1v_2)$. Then we can get that $[V(e_1), V(H_1 - e_1)]_G = \emptyset$. Otherwise, it is easy to obtain that there is rainbow $P_5 \cup P_2$ on the vertex set $V(H_2) \cup \{x_1, y_1, x_i, y_i, v\}$ with $2 \leq i \leq t+1$, the union of this $P_5 \cup P_2$ and $H_1 - e_1 - e_i$ forms a rainbow $P_5 \cup tP_2$, a contradiction. So $|E(G[V(H_1)])| \leq \binom{2(t+1)}{2} - 4t = 2t^2 - t + 1$.

Suppose that $c(v_1v) = c(v_2x_1)$. Since K_n has no rainbow $P_5 \cup tP_2$, we can obtain that $[V(e_1), V(H_1 - e_1)]_G = \emptyset$. Suppose that $[V(e_1), V(H_1 - e_1)]_G \neq \emptyset$. Without loss of generality, let $x_1x_2 \in E(G)$, then we can get that $[V(e_2), V(e_i)]_G = \emptyset$ with $3 \leq i \leq t+1$. So $|E(G[V(H_1)])| \leq \binom{2(t+1)}{2} - 4(t-1) = 2t^2 - t + 5$. Then we have

$$\begin{aligned}|E(G)| &= |E(G[V(H_1)])| + |E(G[V(H_2)])| + |[V(H_2), V(H_1)]_G| \\ &\quad + |[V(D), V(H)]_G| \\ &\leq 2t^2 - t + 5 + 2 + 2(t+1) + 1 = 2t^2 + t + 10.\end{aligned}$$

By $|c(K_n)| = \frac{3t^2}{2} + \frac{11t}{2} + 2$, we can obtain that

$$|c(K_n)| - |E(G)| \geq -\frac{t^2}{2} + \frac{9t}{2} - 8.$$

Let $f(t) = -\frac{t^2}{2} + \frac{9t}{2} - 8$ and then we have $f'(t) < 0$ for all $2 \leq t \leq 4$. Since $f(4) = 2 > 0$, we can get that $f(t) > 0$ for all $2 \leq t \leq 4$, a contradiction. So $|E(G[V(H_1)])| \leq \binom{2(t+1)}{2} - 4t = 2t^2 - t + 1$.

Hence we ca assume that $c(v_1v) \in c(H_1)$. Suppose that $c(v_1v) = c(e_1)$. Since K_n has no rainbow $P_5 \cup tP_2$, we can obtain that $[V(e_1), V(H_1 - e_1)]_G = \emptyset$. Suppose that $[V(e_1), V(H_1 - e_1)]_G \neq \emptyset$. Clearly, $[x_1, V(H_1 - e_1)]_G = \emptyset$. Hence without loss of generality, let $y_1x_2 \in E(G)$. Then we have $|[V(e_2), V(e_i)]_G| \leq 2$

with $3 \le i \le t+1$. So $|E(G[V(H_1)])| \le \binom{2(t+1)}{2} - 2t - 2(t-1) = 2t^2 - t + 3$. Then

$$\begin{aligned}|E(G)| &= |E(G[V(H_1)])| + |E(G[V(H_2)])| + |[V(H_2), V(H_1)]_G| \\ &\quad + |[V(D), V(H)]_G| \\ &\le 2t^2 - t + 3 + 2 + 2(t+1) + 1 = 2t^2 + t + 8.\end{aligned}$$

By $|c(K_n)| = \frac{3t^2}{2} + \frac{11t}{2} + 2$, we can obtain that

$$|c(K_n)| - |E(G)| \ge -\frac{t^2}{2} + \frac{9t}{2} - 6.$$

Let $f(t) = -\frac{t^2}{2} + \frac{9t}{2} - 6$ and then we have $f'(t) < 0$ for all $2 \le t \le 4$. Since $f(4) = 4 > 0$, we can get that $f(t) > 0$ for all $2 \le t \le 4$, a contradiction. So $|E(G[V(H_1)])| \le \binom{2(t+1)}{2} - 4t = 2t^2 - t + 1$. Hence without loss of generality, let $c(v_1v) = c(e_2)$.

Since K_n has no rainbow $P_5 \cup tP_2$, we can obtain that $|[V(e_2), V(e_i)]_G| \le 1$ with $1 \le i \le t+1$ and $i \ne 2$. Suppose $|[V(e_2), V(e_i)]_G| \ge 3$, there are two independent edges in $[V(e_2), V(e_i)]_G$, say x_2x_i and y_2y_i. Then $vv_1v_2x_ix_2$, y_2y_i and $H_1 - e_1 - e_2$ form a rainbow $P_5 \cup tP_2$ with $i = 1$ or $vv_1v_2x_1y_1$, x_2x_i, y_2y_i and $H_1 - e_1 - e_2 - e_i$ form a rainbow $P_5 \cup tP_2$ with $i \ge 2$, a contradiction. Hence we can assume that there is an edge e_i such that $|[V(e_2), V(e_i)]_G| = 2$ with $1 \le i \le t+1$ and $i \ne 2$. Without loss of generality, let $x_2x_1, y_2x_1 \in E(G)$, then we have $[V(e_2), V(e_j)]_G = \emptyset$ with $3 \le j \le t+1$. So $|E(G[V(H_1)])| \le \binom{2(t+1)}{2} - 4(t-1) = 2t^2 - t + 5$. Then we have

$$\begin{aligned}|E(G)| &= |E(G[V(H_1)])| + |E(G[V(H_2)])| + |[V(H_2), V(H_1)]_G| \\ &\quad + |[V(D), V(H)]_G| \\ &\le 2t^2 - t + 5 + 2 + 2(t+1) + 1 = 2t^2 + t + 10.\end{aligned}$$

By $|c(K_n)| = \frac{3t^2}{2} + \frac{11t}{2} + 2$, we can obtain that

$$|c(K_n)| - |E(G)| \ge -\frac{t^2}{2} + \frac{9t}{2} - 8.$$

Let $f(t) = -\frac{t^2}{2} + \frac{9t}{2} - 8$ and then we have $f'(t) < 0$ for all $2 \le t \le 4$. Since $f(4) = 2 > 0$, we can get that $f(t) > 0$ for all $2 \le t \le 4$, a contradiction. So $|E(G[V(H_1)])| \le \binom{2(t+1)}{2} - 3t = 2t^2 + 1$. This implies that $|E(G[V(H_1)])| \le 2t^2 + 1$ and so the claim holds. □

Combined with the above claims, we can obtain that

$$\begin{aligned}|E(G)| &= |E(G[V(H_1)])| + |E(G[V(H_2)])| + |[V(H_2), V(H_1)]_G| \\ &\quad + |[V(D), V(H)]_G| + |E(G[V(D)])| \\ &\le 2t^2 + 1 + 2 + 2(t+1) + 1 = 2t^2 + 2t + 6.\end{aligned}$$

By $|c(K_n)| = \frac{3t^2}{2} + \frac{11t}{2} + 2$, we can obtain that

$$|c(K_n)| - |E(G)| \geq -\frac{t^2}{2} + \frac{7t}{2} - 4.$$

Let $f(t) = -\frac{t^2}{2} + \frac{7t}{2} - 4$. Since $f(2) = 1$ and $f(3) = f(4) = 2 > 0$, we have that $f(t) > 0$ for all $2 \leq t \leq 4$, a contradiction.

Case 2. $|[v_2, V(H_1)]_G| = 0$. By $[\{v_1, v_3\}, V(H_1)]_G = \emptyset$ and $|[v_2, V(H_1)]_G| = 0$, we have $[V(H_2), V(H_1)]_G = \emptyset$. And we have the following claims.

Claim 4. $[v, V(H_1)]_G = \emptyset$.

Proof. Without loss of generality, let $vx_1 \in E(G)$. Now consider the color of edge y_1v_3. Suppose that $c(y_1v_3) \in c(H_2)$. Let $c(y_1v_3) = c(v_1v_2)$, there is a rainbow $P_5 = vx_1y_1v_3v_2$, the union of this P_5 and $H_1 - x_1y_1$ forms a rainbow $P_5 \cup tP_2$, a contradiction.

So we can assume that $c(y_1v_3) = c(v_2v_3)$. Then we can obtain that $v_1v_3 \notin E(G)$ and $[V(e_1), V(H_1 - e_1)]_G = \emptyset$. Otherwise we can easily get a rainbow $P_5 \cup tP_2$. So $|E(G[V(H_1)])| \leq \binom{2t}{2} + 1 = 2t^2 - t + 1$. Then we have

$$\begin{aligned}|E(G)| &= |E(G[V(H_1)])| + |E(G[V(H_2)])| + |[V(H_2), V(H_1)]_G| \\ &\quad + |[V(D), V(H)]_G| \\ &\leq 2t^2 - t + 1 + 2 + 2 = 2t^2 - t + 5.\end{aligned}$$

By $|c(K_n)| = \frac{3t^2}{2} + \frac{11t}{2} + 2$, we can obtain that

$$|c(K_n)| - |E(G)| \geq -\frac{t^2}{2} + \frac{13t}{2} - 3.$$

Let $f(t) = -\frac{t^2}{2} + \frac{13t}{2} - 3$ and then we have $f'(t) < 0$ for all $2 \leq t \leq 4$. Since $f(4) = 15 > 0$, we can get that $f(t) > 0$ for all $2 \leq t \leq 4$. Thus we get a contradiction. Hence $c(y_1v_3) \in c(H_1)$.

Suppose that $c(y_1v_3) = c(e_1)$. Then we can get that $|[V(e_1), V(e_i)]_G| \leq 2$ with $1 \leq i \leq t+1$. So $|E(G[V(H_1)])| \leq \binom{2(t+1)}{2} - 2t = 2t^2 + t + 1$. Then we have

$$\begin{aligned}|E(G)| &= |E(G[V(H_1)])| + |E(G[V(H_2)])| + |[V(H_2), V(H_1)]_G| \\ &\quad + |[V(D), V(H)]_G| \\ &\leq 2t^2 + t + 1 + 3 + 2 = 2t^2 + t + 6.\end{aligned}$$

By $|c(K_n)| = \frac{3t^2}{2} + \frac{11t}{2} + 2$, we can obtain that

$$|c(K_n)| - |E(G)| \geq -\frac{t^2}{2} + \frac{9t}{2} - 4.$$

Let $f(t) = -\frac{t^2}{2} + \frac{9t}{2} - 4$ and then we have $f^{'}(t) < 0$ for all $2 \leq t \leq 4$. Since $f(4) = 6 > 0$, we can get that $f(t) > 0$ for all $2 \leq t \leq 4$, a contradiction. Hence, without loss of generality, let $c(y_1v_3) = c(e_2)$. Then we have $|[V(e_2, V(e_i)]_G| \leq 2$ with $1 \leq i \leq t+1$ and $i \neq 2$. So $|E(G[V(H_1)])| \leq \binom{2(t+1)}{2} - 2t = 2t^2 + t + 1$. Then we have

$$
\begin{aligned}
|E(G)| &= |E(G[V(H_1)])| + |E(G[V(H_2)])| + |[V(H_2), V(H_1)]_G| \\
&\quad + |[V(D), V(H)]_G| \\
&\leq 2t^2 + t + 1 + 3 + 2 = 2t^2 + t + 6.
\end{aligned}
$$

By $|c(K_n)| = \frac{3t^2}{2} + \frac{11t}{2} + 2$, we can obtain that

$$|c(K_n)| - |E(G)| \geq -\frac{t^2}{2} + \frac{9t}{2} - 4.$$

Let $f(t) = -\frac{t^2}{2} + \frac{9t}{2} - 4$ and then we have $f^{'}(t) < 0$ for all $2 \leq t \leq 4$. Since $f(4) = 6 > 0$, we can get that $f(t) > 0$ for all $2 \leq t \leq 4$, a contradiction. This implies that $|[V(D), V(H_1)]_G| = \emptyset$ and so the claim holds. $\square$

From Claim 3, we can deduce that $c([v, V(H)]) \leq 3$.

Claim 5. $|E(G[V(H_1)])| \leq 2t^2 + t + 2$.

Proof. By $[v, V(H_1)]_G = \emptyset$, we have $c([v, V(H_1)]) \subseteq c(H)$. Below we will consider the color of edge vx_1.

Suppose that $c(vx_1) \in c(H_2)$. Since K_n has no rainbow $P_5 \cup tP_2$. Then we have $|[V(e_1), V(e_i)]_G| \leq 2$ with $2 \leq i \leq t+1$. Otherwise we can easily get a rainbow $P_5 \cup tP_2$, a contradiction. So $|E(G[V(H_1)])| \leq \binom{2(t+1)}{2} - 2t = 2t^2 + t + 1$.

Hence we can assume that $c(vx_1) \in c(H_1)$. Assume that $c(vx_1) = c(e_1)$, we can obtain that $|[V(e_1), V(e_i)]_G| \leq 2$ with $2 \leq i \leq t+1$. So $|E(G[V(H_1)])| \leq \binom{2(t+1)}{2} - 2t = 2t^2 + t + 1$. Hence, without loss of generality, let $c(vx_1) = c(e_2)$. Now consider the color of edge y_1v_3. If $c(y_1v_3) \in c(H_2)$, then let $c(y_1v_3) = c(v_1v_2)$. Thus we have $|[V(e_2), V(e_1)]_G| \leq 3$ and $|[V(e_2), V(e_i)]_G| \leq 2$ with $3 \leq i \leq t+1$, So $|E(G[V(H_1)])| \leq \binom{2(t+1)}{2} - 2(t-1) - 1 = 2t^2 + t + 2$. Hence let $c(y_1v_3) = c(v_2v_3)$, we have $|[V(e_1), V(e_i)]_G| \leq 2$ with $2 \leq i \leq t+1$. So $|E(G[V(H_1)])| \leq \binom{2(t+1)}{2} - 2t = 2t^2 + t + 1$. Hence $c(y_1v_3) \in c(H_1)$. Suppose that $c(y_1v_3) = c(e_1)$. We have $|[V(e_1), V(e_i)]_G| \leq 2$ with $2 \leq i \leq t+1$. So $|E(G[V(H_1)])| \leq \binom{2(t+1)}{2} - 2t = 2t^2 + t + 1$. Hence, without loss of generality, let $c(y_1v_3) = c(e_2)$. Then we have $|[V(e_2), V(e_i)]_G| \leq 2$ with $1 \leq i \leq t+1$ and $i \neq 2$. So $|E(G[V(H_1)])| \leq \binom{2(t+1)}{2} - 2t = 2t^2 + t + 1$. This implies that $|E(G[V(H_1)])| \leq 2t^2 + t + 2$ and so the claim holds. $\square$

Combined with the above claims, we can obtain that

$$\begin{aligned}|E(G)| &= |E(G[V(H_1)])| + |E(G[V(H_2)])| + |[V(H_2), V(H_1)]_G| \\ &\quad + |[V(D), V(H)]_G| + |E(G[V(D)])| \\ &\leq 2t^2 + t + 2 + 3 + 3 = 2t^2 + t + 8.\end{aligned}$$

By $|c(K_n)| = \frac{3t^2}{2} + \frac{11t}{2} + 2$, we can obtain that

$$|c(K_n)| - |E(G)| \geq -\frac{t^2}{2} + \frac{9t}{2} - 6.$$

Let $f(t) = -\frac{t^2}{2} + \frac{9t}{2} - 6$. Then we have $f^{'}(t) < 0$ for all $2 \leq t \leq 4$. Since $f(4) = 4 > 0$, we can get that $f(t) > 0$ for all $2 \leq t \leq 4$, a contradiction.

This completes the proof of the lemma. ∎

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