Discussiones Mathematicae
Graph Theory xx (xxxx) 1–22
https://doi.org/10.7151/dmgt.2578

NON-PATH RESULTS ON THE CONNECTIVITY KEEPING PROBLEM

Meng Ji

College of Mathematical Science, and Institute of Mathematics and Interdisciplinary Sciences, Tianjin Normal University, Tianjin, China and Extremal Combinatorics and Probability Group (ECOPRO) Institute for Basic Science (IBS), Daejeon, South Korea

e-mail: mji@tjnu.edu.cn

Abstract

Fujita and Kawarabayashi conjectured that for all positive integers k, m, there is a (least) non-negative integer $f_k(m)$ such that every k-connected graph G with $\delta(G) \geq \left\lfloor \frac{3k}{2} \right\rfloor + f_k(m) - 1$ contains a connected subgraph W of order m such that $G - V(W)$ is still k-connected. Mader confirmed that Fujita-Kawarabayashi's conjecture is true when W is a path, if $f_k(m) = m$. Mader conjectured that for every positive integer k and finite tree T of order m, every k-connected finite graph G with minimum degree $\delta(G) \geq \left\lfloor \frac{3k}{2} \right\rfloor + m - 1$ contains a subgraph $T' \cong T$ such that $G - V(T')$ remains k-connected. Till now, there is hardly a result on high connectivity G and a non-path. Luo, Tian and Wu proposed a stronger conjecture, that is, for any tree T with bipartition (X, Y), every k-connected bipartite graph G with $\delta(G) \geq k + w$, where $w = \max\{|X|, |Y|\}$, contains a tree $T' \cong T$ such that $\kappa(G - V(T')) \geq k$. In this paper, we develop Mader's method and give a result on high connectivity G and a non-path T. Firstly, the author proves that for positive integers $k \geq 1$ and $m \neq 4$ or 5, every k-connected bipartite graph G with $\delta(G) \geq k + \left\lceil \frac{m+1}{2} \right\rceil$ contains a star-path T^3_{m-3} such that $\kappa(G - V(T^3_{m-3})) \geq k$, where T^3_{m-3} is a tree constructed by connecting one leaf of $K_{1,3}$ and one end-vertex of a path P on $m-4$ vertices. Secondly, we prove Mader's conjecture is true when T is a star-path under condition of $\Delta(G) = |G| - 1$.

Keywords: connectivity, bipartite graph, fragment, end.

2020 Mathematics Subject Classification: 05C05, 05C40.

1. Introduction

In this paper, all the graphs are finite, undirected and simple. For graph-theoretical terminology and notations not defined here, we follow [1]. Let $G = (V(G), E(G))$ be a graph. The minimum degree and the connectivity number of a graph G is denoted by $\delta(G)$ and $\kappa(G)$, respectively. For $H \subseteq G$, both $G - H$ and $G - V(H)$ stands for $G \backslash V(H)$. Let v be a vertex in G, we denote by $d_G(v)$ the degree of v in G. And we denote the connectivity of graph G by $\kappa(G)$.

In 1972, Chartrand, Kaigars and Lick [2] proved that *every k-connected graph G with $\delta(G) \geq \lfloor \frac{3k}{2} \rfloor$ has a vertex x with $\kappa(G-x) \geq k$.* Over 30 years later, Fujita and Kawarabayashi [4] extended this result by showing that every k-connected graph G with $\delta(G) \geq \lfloor \frac{3k}{2} \rfloor + 2$ has an edge xy such that $G - \{x, y\}$ remains k-connected. Furthermore, Fujita and Kawarabayashi posed the following conjecture.

Conjecture 1 (Fujita and Kawarabayashi [4])**.** *For all positive integers k, m, there is a (least) non-negative integer $f_k(m)$ such that every k-connected graph G with $\delta(G) \geq \lfloor \frac{3k}{2} \rfloor + f_k(m) - 1$ contains a connected subgraph W of order m such that $G - V(W)$ is still k-connected.*

In 2010, Mader [11] proved that Conjecture 1 is true if $f_k(m) = m$ and W is a path.

Theorem 2 (Mader [11])**.** *For every positive integer k, every k-connected finite graph G with minimum degree $\delta(G) \geq \lfloor \frac{3k}{2} \rfloor + m - 1$ contains a path P on m vertices such that $G - V(P)$ remains k-connected.*

Mader [11] proposed the following conjecture.

Conjecture 3 (Mader [11])**.** *For every positive integer k and finite tree T of order m, every k-connected finite graph G with minimum degree $\delta(G) \geq \lfloor \frac{3k}{2} \rfloor + m - 1$ contains a subgraph $T' \cong T$ such that $G - V(T')$ remains k-connected.*

Many results indicate that the conjecture is extremely likely to be true. If G is a graph with small connectivity, Diwan and Tholiya [3] confirmed the conjecture for $k = 1$. For 2-connected graphs G, some special trees are studied, such as stars, double stars and path-stars [13, 14], trees with diameter conditions [6, 9], and caterpillars and spider trees [8]. In [5], Conjecture 3 is true if certain conditions on girth in 2-connected graph G are imposed. Hong and Liu [7] confirmed the conjecture for $k = 2, 3$. Moreover, Mader [12] showed that the conjecture is true for k-connected graphs if $\delta \geq 2(k-1+m)^2+m-1$. However, it is unrealistic if one want to use directly Mader's method in [11] to prove more results on Conjecture 3. Because Mader in [12] said that "one can easily construct graphs G with $\kappa(G) = k + 1$ and arbitrarily large $\delta(G)$, where the deletion of any n neighbors of a prescribed vertex p decreases the connectivity number of G by n. On the other side, one cannot expect to find in a k-connected graph 'handy parts' of connectivity much larger than k".

If G is a bipartite graph, Luo, Tian and Wu [10] proved that *every k-connected bipartite graph G with minimum degree at least $k+m$ contains a path P with order m such that $\kappa(G-V(P)) \geq k$.* Based on that, they proposed a stronger conjecture on the bipartite graph.

Conjecture 4. [10] *For any tree T with bipartition (X, Y), every k-connected bipartite graph G with $\delta(G) \geq k+w$, where $w = \max\{|X|, |Y|\}$, contains a tree $T' \cong T$ such that*

$$\kappa(G - V(T')) \geq k.$$

The case that T is caterpillar and $k \leq 2$ has been proved by Zhang [17]. Additionally, Yang and Tian [15] verified Conjecture 4 when T is caterpillar and $k = 3$, or T is a spider and $k \leq 3$. Just as the research of Conjecture 3, there are no results on high connectivity graph to confirm Conjecture 4.

We end up this section by introducing some necessary definitions and notation. Let $G = (V, E)$ be a graph. For simplicity, $|G| := |V(G)|$ and $v \in G$ means $v \in V(G)$. Let x, y-path be a path P from x to y. For $u, v \in P$, let $P[u, v] = P[v, u]$ be the subpath of P between u and v, and $P[u, v)$ means $P[u, v] - v$. If $S \subset V(G)$ in G, we denote the graph $G \cup K(S)$ by $G\langle S\rangle$ and let $\langle S\rangle$ be a complete graph on $|S|$ vertices. For any vertex $x \in V(G)$, we denote the set of neighbors of x in G by $N_G(x)$, and for a subgraph $H \subseteq G$, let $N_G(H) = \bigcup_{x\in H} N_G(x) \setminus V(H)$. A *vertex cut* of a connected graph G is a subset S of $V(G)$ such that $G - S$ has more than one connected component. And a *minimum vertex cut* is a vertex cut with smallest size in a given connected graph G, and the size of a minimum vertex cut S is $\kappa(G)$. For a vertex cut S of a graph G, a union of components F of $G - S$, with $G - S - F \neq \emptyset$, is a *semifragment* F to S, and the *complementary semifragment* $G - (S \cup V(F))$ is denoted by $\bar{F}$. If S is a minimum vertex cut, then F is called a *fragment* F to S. Since a complete graph has no vertex cut, we do not consider the complete graph in this paper. For a fragment F of G to S, it follows that $S = N_G(F)$. If a fragment of G does not contain any other fragments of G, then it is an *end* of G. Clearly, every graph contains an end except for complete graphs. Let F_i be a fragment of G to S_i for $i = 1, 2$, and we denote $(S_1 \cap F_2) \cup (S_1 \cap S_2) \cup (S_2 \cap F_1)$ by $S(F_1, F_2)$.

Let $K_{1,s}$ be a star on $s+1$ vertices with center vertex x and let P be a path on t vertices. Then we connect the vertex x with a vertex of degree one of P by an edge, and we denote by T_t^{s+1} the resulting graph, where we call the vertex x the *root*. The vertices of degree one in $K_{1,s}$ are called *legs*.

In this paper, the author confirms Conjecture 4 when $T = T_{m-3}^3$ by proving that *for $m \neq 4$ or 5, every k-connected bipartite graph G with $\delta(G) \geq k + \lceil \frac{m+1}{2} \rceil$*

contains a tree T^3_{m-3} *such that*

$$\kappa\big(G - V(T^3_{m-3})\big) \geq k,$$

where T^3_{m-3} *is a path* P *on* m *vertices if* $m \leq 3$*.* And then, we prove Mader's conjecture is true when T is $T_m^{t;m-t-1}$ under the condition of $\Delta(G) = |G| - 1$.

2. Preliminaries

In order to prove our theorems, we will use some structural lemmas and theorems. We first define the set $\mathcal{K}^b_k(t)$ containing all pairs (G, C) satisfying the following conditions.

- $\kappa(G) \geq k$;
- $C \subseteq G$ is a complete graph with $|C| = k$;
- $G - C$ is a bipartite graph with $\delta_G(G - C) \geq k + t$;
- every pair of adjacent vertices in $V(G - C)$ have no common neighbors in G;
- we denote by $\mathcal{K}^b_{k^+}(t)$ all the pairs $(G, C) \in \mathcal{K}^b_k(t)$ with $\kappa(G) \geq k + 1$.

Lemma 5 [11]**.** *Let* G *be a graph with* $\kappa(G) = k$ *and let* F *be a fragment of* G *to* S*. Then we have the following properties.*

(i) *If* F *is a fragment of* G *to* S*, then* $G\langle S\rangle[S \cup F]$ *is* k*-connected;*

(ii) *if* F *is an end of* G *with* $|F| \geq 2$*, then* $G\langle S\rangle - V(\bar{F})$ *is* $(k+1)$*-connected.*

Lemma 6 [11]**.** *Let* S *be a vertex cut of* G *with* $|S| = k$ *and let* S_1 *be a vertex cut of* G *with* $|S_1| = k - 1$*. Assume that* F *is a semifragment of* G *to* S *and* F_1 *is a semifragment of* G *to* S_1*. Furthermore, we assume* $G\langle S\rangle - V(F)$ *and* $G\langle S\rangle - V(\bar{F})$ *are* k*-connected. Then we have the following properties.*

(i) *If* $F \cap F_1 \neq \emptyset$*, then* $|S\,(F, F_1)| \geq k$*;*

(ii) *if* $|S\,(F, F_1)| \geq k$*, then* $|S_1 \cap F| \geq \left|S \cap \bar{F}_1\right|$, $|S \cap F_1| > \left|S_1 \cap \bar{F}\right|$ *and* $\bar{F} \cap \bar{F}_1 = \emptyset$*.*

Theorem 7 [16]**.** *For any* $(G, C) \in \mathcal{K}^b_{k^+}\left(\left\lceil \frac{m+1}{2} \right\rceil\right)$ *and* $v_0 \in V(G-C)$*, there exists a path* P *of order* m *starting from* v_0 *in* $G - V(C)$ *such that* $\kappa(G - V(P)) \geq k$*.*

Theorem 8 [16]**.** *Every* k*-connected bipartite graph* G *with* $\delta(G) \geq k + \left\lceil \frac{m+1}{2} \right\rceil$ *contains a path* P *of order* m *such that* $\kappa(G - V(P)) \geq k$*.*

Lemma 9 [16]**.** *Let* $(G, C) \in \mathcal{K}^b_k\left(\left\lceil \frac{m}{2} \right\rceil\right)$ *and* $\kappa(G) = k$*, where* k *and* m *are two positive integers. Assume* S *is a minimum vertex cut of* G *and* F *is a fragment of* G *to* S *such that* $C \subseteq G[S \cup F]$*. If there exists a path* $P \subseteq G - S \cup V(F)$ *of order at most* m *such that* $\kappa(G\langle S\rangle - V(\bar{F} \cup P) \geq k$*, then* $\kappa(G - V(P)) \geq k$*.*

Lemma 10. *Every $(G,C) \in \mathcal{K}_k^b\left(\lceil\frac{m+1}{2}\rceil\right)$ contains a path $P \subseteq G - V(C)$ of order m such that $\kappa(G-P) \geq k$.*

Proof. If $\kappa(G) \geq k+1$, then it holds by Theorem 7. And so we suppose that $\kappa(G) = k$. There is an end F of G with $F \cap C = \emptyset$. Let $S := N_G(F)$ with $|S| = k$ and let $\bar{F}$ denote the complementary fragment to F in G. If $|F| = 1$ holds, then for the vertex x of F, it follows that $k = d_G(x) \geq k + \lceil\frac{m+1}{2}\rceil$, a contradiction. Hence, $|F| \geq 2$. By Lemma 5(ii), we have $\kappa(G\langle S\rangle - V(\bar{F})) \geq k+1$ and $(G\langle S\rangle - V(\bar{F}), \langle S\rangle) \in \mathcal{K}_{k^+}^b\left(\lceil\frac{m+1}{2}\rceil\right)$. By Theorem 7, we get a path P_m such that $G\langle S\rangle - V(\bar{F} \cup P_m)$ remains k-connected. Then by Lemma 9 we have $\kappa(G - P_m) \geq k$. ■

Lemma 11. *Let $G = G[X,Y]$ be a bipartite graph with $\kappa(G) = k$ and $\delta(G) \geq k + \lceil\frac{m+1}{2}\rceil$, where k and m are two positive integers. Assume S is a minimum vertex cut of G and F is a fragment of G to S. If there exists a tree T_{m-3}^3 of order at most m in $G - (S \cup V(F))$ such that $\kappa(G\langle S\rangle - V(F \cup T_{m-3}^3)) \geq k$, then $\kappa(G - V(T_{m-3}^3)) \geq k$.*

Proof. If $k = 1$, then we have the trivial result since $G\langle S\rangle = G$. We consider the case of $k \geq 2$ in the following. For a contradiction, we assume that $\kappa(G - V(T_{m-3}^3)) < k$. Let $G_{T_{m-3}^3} = G - V(T_{m-3}^3)$.

Since G is a bipartite graph and T_{m-3}^3 is a tree, each vertex of $V(G - T_{m-3}^3)$ has at most $\lceil\frac{m+1}{2}\rceil$ neighbors in $V(T_{m-3}^3)$. Since $\delta\left(G_{T_{m-3}^3}\right) \geq k$ and $\left|V\left(G_{T_{m-3}^3}\right)\right| \geq 2k$, $G_{T_{m-3}^3}$ is not a complete graph. Let S_1 be a vertex cut of $G_{T_{m-3}^3}$ with $|S_1| = k-1$ and let F_1 be a semifragment of $G_{T_{m-3}^3}$ to S_1. Let $\bar{F}_{G_{T_{m-3}^3}} = G_{T_{m-3}^3} - (S \cup V(F))$ and $\bar{F}_1 = G_{T_{m-3}^3} - (S_1 \cup V(F_1))$.

If $\bar{F}_{G_{T_{m-3}^3}} = \emptyset$, then, when $|V(T_{m-3}^3)| = 1$, we have $d_G(v) \leq |S| = k$ for the only vertex $v \in V(T_{m-3}^3)$, a contradiction. When $|V(T_{m-3}^3)| \geq 2$, for any $xy \in E(T_{m-3}^3)$ we have $2\left(k + \lceil\frac{m+1}{2}\rceil\right) \leq d_G(x) + d_G(y) \leq |S| + |V(T_{m-3}^3)| \leq k+m$, a contradiction. Hence, $\bar{F}_{G_{T_{m-3}^3}} \neq \emptyset$. Furthermore, S is also a vertex cut of $G_{T_{m-3}^3}$ and F is a semifragment of $G_{T_{m-3}^3}$ to S. Since F is a fragment of G to S, $G\langle S\rangle[S \cup F]$ is k-connected by Lemma 5, that is, $G_{T_{m-3}^3}\langle S\rangle - V\left(\bar{F}_{G_{T_{m-3}^3}}\right)$ is k-connected. Also, $G_{T_{m-3}^3}\langle S\rangle - V(F) = G\langle S\rangle - V(F \cup T_{m-3}^3)$ is k-connected by preassumptions of Lemma 11.

Now we apply Lemma 6 in the remaining of the proof. If $V(F_1) \subseteq S$, then for any $v \in V(F_1)$, we have

$$\left|N_{G_{T_{m-3}^3}}(v) \cap V(F_1)\right| \geq d_{G_{T_{m-3}^3}}(v) - \mid N_{G_{T_{m-3}^3}}(v) \cap S_1 \mid \geq k - (k-1) \geq 1.$$

So $\delta(F_1) \geq 1$ and for any $uv \in E(F_1)$, we have $2k \leq d_{G_{T^3_{m-3}}}(u) + d_{G_{T^3_{m-3}}}(v) \leq |S|+|S_1| \leq 2k-1$, a contradiction. Thus, $F_1 \cap F \neq \emptyset$ or $F_1 \cap \bar{F}_{G_{T^3_{m-3}}} \neq \emptyset$. Without loss of generality, suppose $F_1 \cap F \neq \emptyset$. Applying Lemma 6 to the semifragments F and F_1, the complementary semifragments $\bar{F}_{G_{T^3_{m-3}}}$ and $\bar{F}_1$ in $G_{T^3_{m-3}}$, we have $\bar{F}_1 \cap \bar{F}_{G_{T^3_{m-3}}} = \emptyset$. Hence $V(\bar{F}_1) \subseteq S \cup V(F)$. We consider the following two cases.

Case 1. $V(F) \cap V(\bar{F}_1) = \emptyset$. Since $V(\bar{F}_1) \subseteq S \cup V(F)$, we have $V(\bar{F}_1) \subseteq S$. For any $v \in V(\bar{F}_1)$, we have

$$\left|N_{G_{T^3_{m-3}}}(v) \cap V\left(\bar{F}_1\right)\right| \geq d_{G_{T^3_{m-3}}}(v) - \left|N_{G_{T^3_{m-3}}}(v) \cap S_1\right| \geq k-(k-1) \geq 1.$$

So $\delta(\bar{F}_1) \geq 1$, and thus, for any $uv \in E(\bar{F}_1)$, we have $2k \leq d_{G_{T^3_{m-3}}}(u) + d_{G_{T^3_{m-3}}}(v) \leq |S| + |S_1| \leq 2k-1$ by $\delta\left(G_{T^3_{m-3}}\right) \geq k$, a contradiction.

Case 2. $V(F) \cap V(\bar{F}_1) \neq \emptyset$. By Lemma 6, it follows that $F_1 \cap \bar{F}_{G_{T^3_{m-3}}} = \emptyset$. Since $\bar{F}_1 \cap \bar{F}_{G_{T^3_{m-3}}} = \emptyset$, we have $V\left(\bar{F}_{G_{T^3_{m-3}}}\right) \subseteq S_1$, that is, $\left|V\left(\bar{F}_{G_{T^3_{m-3}}}\right)\right| \leq k-1$. Suppose $\delta\left(\bar{F}_{G_{T^3_{m-3}}}\right) \geq 1$. Then for any $uv \in E\left(\bar{F}_{G_{T^3_{m-3}}}\right)$ we have $2k \leq d_{G_{T^3_{m-3}}}(u) + d_{G_{T^3_{m-3}}}(v) \leq |S| + |S_1| \leq 2k-1$, a contradiction. So $\delta\left(\bar{F}_{G_{T^3_{m-3}}}\right) = 0$. For any $v \in V\left(\bar{F}_{G_{T^3_{m-3}}}\right)$, we have $k + \lceil \frac{m+1}{2} \rceil \leq d_G(v) \leq |N_G(v) \cap S| + |N_G(v) \cap V(T^3_{m-3})| \leq k + \lceil \frac{m+1}{2} \rceil$, hence $N_G(v) \cap S = S$ and $|N_G(v) \cap V(T^3_{m-3})| = \lceil \frac{m+1}{2} \rceil$. Without loss of generality, assume $N_G(v) \cap V(T^3_{m-3}) = V(T^3_{m-3}) \cap X$. This shows $S \subseteq X$ and $N_G(V(T^3_{m-3}) \cap X) \cap S = \emptyset$. For any $x \in V(T^3_{m-3}) \cap X$, we obtain

$$\begin{aligned} k + \left\lceil \frac{m+1}{2} \right\rceil \leq d_G(x) &= \left|N_G(x) \cap V\left(\bar{F}_{G_{T^3_{m-3}}}\right)\right| + \left|N_G(x) \cap V(T^3_{m-3})\right| \\ &\leq \left|N_G(x) \cap V\left(\bar{F}_{G_{T^3_{m-3}}}\right)\right| + \left\lceil \frac{m}{2} \right\rceil. \end{aligned}$$

Hence $\left|N_G(x) \cap V\left(\bar{F}_{G_{T^3_{m-3}}}\right)\right| \geq k$. Furthermore, $\left|V\left(\bar{F}_{G_{T^3_{m-3}}}\right)\right| \geq k$, which contradicts the fact that $\left|V\left(\bar{F}_{G_{T^3_{m-3}}}\right)\right| \leq k-1$. ■

Lemma 12. *Let $(G,C) \in \mathcal{K}_k^b\left(\left\lceil \frac{m+1}{2} \right\rceil\right)$ and $\kappa(G)=k$, where k and m are two positive integers. Assume S is a minimum vertex cut of G and F is a fragment of G to S such that $C \subseteq G[S \cup F]$. If there exists a tree $T^3_{m-3} \subseteq G - S \cup V(F)$ of order at most m such that $\kappa(G\langle S\rangle - V(F \cup T^3_{m-3})) \geq k$, then $\kappa(G - V(T^3_{m-3})) \geq k$.*

Proof. Since every pair of adjacent vertices in $G-V(C)$ have no common neighbors in G, we have $|V(G)| \geq 2k+m+1$. Let X and Y be bipartition of $G-V(C)$. If $k=1$, then we have trivial result since $G\langle S\rangle = G$. We assume $k \geq 2$ in the following. Suppose, to the contrary, that $\kappa(G-V(T^3_{m-3})) < k$. Let $G_{T^3_{m-3}} = G - V(T^3_{m-3})$.

Since $\left|V\left(G_{T^3_{m-3}}\right)\right| \geq 2k+1$ and $\delta_{G_{T^3_{m-3}}}\left(G_{T^3_{m-3}} - V(C)\right) \geq k$, we have $G_{T^3_{m-3}}$ is not a complete graph. In addition, since C is a complete graph, we can choose a vertex cut S_1 of $G_{T^3_{m-3}}$ with $|S_1| = k-1$ and a semifragment F_1 of $G_{T^3_{m-3}}$ to S_1 such that $V(F_1) \cap V(C) \neq \emptyset$. By $C \subseteq G[S \cup F]$, we have $V(C) \subseteq (F \cap F_1) \cup S(F, F_1)$. Let $\bar{F}_{G_{T^3_{m-3}}} = G_{T^3_{m-3}} - (S \cup V(F))$ and $\bar{F}_1 = G_{T^3_{m-3}} - (S_1 \cup V(F_1))$.

If $\bar{F}_{G_{T^3_{m-3}}} = \emptyset$, then, when $|V(T^3_{m-3})| = 1$, we have $d_G(v) \leq |S| = k$ for the only vertex $v \in V(T^3_{m-3})$, which contradicts the condition of the minimum degree. When $|V(T^3_{m-3})| \geq 2$, for any $xy \in E(T^3_{m-3})$, we have $2\left(k + \left\lceil \frac{m+1}{2}\right\rceil\right) \leq d_G(x) + d_G(y) \leq |S| + |V(T^3_{m-3})| \leq k+m$, a contradiction. So we have that $\bar{F}_{G_{T^3_{m-3}}} \neq \emptyset$. And then, S is a vertex cut of $G_{T^3_{m-3}}$ and F is also a semifragment of $G_{T^3_{m-3}}$ to S.

Since F is a fragment of G to S, we have $G\langle S\rangle[F \cup S]$ is k-connected by Lemma 5, that is, $G_{T^3_{m-3}}\langle S\rangle - V\left(\bar{F}_{G_{T^3_{m-3}}}\right)$ is k-connected. And $G_{T^3_{m-3}}\langle S\rangle - V(F) = G\langle S\rangle - V(F \cup T^3_{m-3})$ is k-connected by preassumptions of Lemma 12. So we can use Lemma 6 in the following proof.

If $F \cap F_1 = \emptyset$, then $V(C) \subseteq S(F, F_1)$ by $V(C) \subseteq (F \cap F_1) \cup S(F, F_1)$, and $|S(F,F_1)| \geq |V(C)| = k$. If $F \cap F_1 \neq \emptyset$, then $|S(F,F_1)| \geq k$ by Lemma 6(i). Thus, by Lemma 6(ii), we have $\bar{F}_{G_{T^3_{m-3}}} \cap \bar{F}_1 = \emptyset$ and $\bar{F}_{G_{T^3_{m-3}}} \subseteq S_1 \cup F_1$. Let us consider the following two cases.

Case 1. $V\left(\bar{F}_{G_{T^3_{m-3}}}\right) \cap V(F_1) = \emptyset$. Since $V\left(\bar{F}_{G_{T^3_{m-3}}}\right) \cap V(F_1) = \emptyset$, we have $V\left(\bar{F}_{G_{T^3_{m-3}}}\right) \subseteq S_1$. Hence, $\left|V\left(\bar{F}_{G_{T^3_{m-3}}}\right)\right| \leq k-1$. If $\delta\left(\bar{F}_{G_{T^3_{m-3}}}\right) \geq 1$, then, for any $uv \in E\left(\bar{F}_{G_{T^3_{m-3}}}\right)$, we have $2k \leq d_{G_{T^3_{m-3}}}(u) + d_{G_{T^3_{m-3}}}(v) \leq |S| + |S_1| \leq 2k-1$, a contradiction. Therefore, $\delta\left(\bar{F}_{G_{T^3_{m-3}}}\right) = 0$. For any $v \in V\left(\bar{F}_{G_{T^3_{m-3}}}\right)$, we have

$N_G(v) = S \cup (V(T^3_{m-3}) \cap X)$ or $N_G(v) = S \cup (V(T^3_{m-3}) \cap Y)$ by $k + \lceil \frac{m+1}{2} \rceil \leq d_G(v) \leq |S| + \left\lceil \frac{|V(T^3_{m-3})|+1}{2} \right\rceil = k + \lceil \frac{m+1}{2} \rceil$. If $|V(T^3_{m-3})| = 1$, then for the only vertex $u \in V(T^3_{m-3})$, by $vu \in E(G-C)$, we have $N_G(u) \cap S = \emptyset$ and $d_G(u) \leq \left| V\left(\bar{F}_{G_{T^3_{m-3}}}\right) \right| \leq k-1$, a contradiction. If $|V(T^3_{m-3})| \geq 2$, then we have $2\left(k + \lceil \frac{m+1}{2} \rceil\right) \leq d_G(x) + d_G(y) \leq |S| + |V(T^3_{m-3})| + \left| V\left(\bar{F}_{G_{T^3_{m-3}}}\right) \right| \leq k + m + \left| V\left(\bar{F}_{G_{T^3_{m-3}}}\right) \right| \leq 2k + m - 1$ for any $xy \in V(T^3_{m-3})$, a contradiction.

Case 2. $V\left(\bar{F}_{G_{T^3_{m-3}}}\right) \cap V(F_1) \neq \emptyset$. We have $\left| S\left(\bar{F}_{G_{T^3_{m-3}}} \cup F_1\right) \right| \geq k$ and $F \cap \bar{F}_1 = \emptyset$ by Lemma 6. Thus $V\left(\bar{F}_1\right) \subseteq S$ by $\bar{F}_{G_{T^3_{m-3}}} \cap \bar{F}_1 = \emptyset$. For any $v \in V\left(\bar{F}_1\right)$, we have $\left| N_{G_{T^3_{m-3}}}(v) \cap V\left(\bar{F}_1\right) \right| \geq d_{G_{T^3_{m-3}}}(v) - \left| N_{G_{T^3_{m-3}}}(v) \cap V(S_1) \right| \geq k - (k-1) \geq 1$. Then $\delta\left(\bar{F}_1\right) \geq 1$. However, for any $uv \in E\left(\bar{F}_1\right)$, we have $2k \leq d_{G_{T^3_{m-3}}}(u) + d_{G_{T^3_{m-3}}}(v) \leq |S| + |S_1| \leq 2k-1$, a contradiction. ■

3. k-Connected Bipartite Graphs

In this section, our main theorem is Theorem 17. In order to prove it, we need first to prove the following Theorem 13.

Theorem 13. *For $m > 5$ and every $(G, C) \in \mathcal{K}^b_{k^+}\left(\lceil \frac{m+1}{2} \rceil\right)$, there is a tree $T^3_{m-3} \subseteq G - V(C)$ such that $\kappa(G - V(T^3_{m-3})) \geq k$.*

Proof. We prove the theorem by induction on the order of the graph at the same time for all m. By the definition of $\mathcal{K}^b_{k^+}\left(\lceil \frac{m+1}{2} \rceil\right)$, we have that $V(G) \geq 2\left(k + \lceil \frac{m+1}{2} \rceil\right)$. The smallest graph G is isomorphic to $K_{k+\lceil \frac{m+1}{2} \rceil, k+\lceil \frac{m+1}{2} \rceil}\langle S \rangle$ for some $S \subseteq V\left(K_{k+\lceil \frac{m+1}{2} \rceil, k+\lceil \frac{m+1}{2} \rceil}\right)$ with $|S| = k$. For any tree $T^3_{m-3} \subseteq G\left[V(K_{k+\lceil \frac{m+1}{2} \rceil, k+\lceil \frac{m+1}{2} \rceil}) \setminus S\right]$ with order m, we have $K_{k+\lceil \frac{m+1}{2} \rceil, k+\lceil \frac{m+1}{2} \rceil}\langle S \rangle - V(T^3_{m-3}) \cong K_{k+1,k}\langle S \rangle$ if m is odd, and $K_{k+\lceil \frac{m+1}{2} \rceil, k+\lceil \frac{m+1}{2} \rceil}\langle S \rangle - V(T^3_{m-3}) \cong K_{k+2,k}\langle S \rangle$ if m is even, and thus, it follows that

$$\kappa\left(K_{k+\lceil \frac{m+1}{2} \rceil, k+\lceil \frac{m+1}{2} \rceil}\langle S \rangle - V(T^3_{m-3})\right) \geq k.$$

Now assume that G is a graph with smallest order such that $(G, C) \in \mathcal{K}^b_{k^+}\left(\lceil \frac{m+1}{2} \rceil\right)$ is a counterexample to Theorem 13 for k, m and C. We first prove the following claim.

Claim 14. *For every* $(G, C) \in \mathcal{K}_{k^+}^b\left(\lceil\frac{m+1}{2}\rceil\right)$, *there is a subgraph* T_1^3 *in* $G - C$ *such that* $\kappa(G - V(T_1^3)) \geq k$.

Proof. If G is $(k+4)$-connected graph, then, clearly, there is a subgraph T_1^3 in G such that $\kappa\left(G - V(T_1^3)\right) \geq k$ since $\delta(G) \geq k + \lceil\frac{m+1}{2}\rceil \geq k+4$. Suppose that $\kappa(G) = k+3$. Let $G' = G\langle C'\rangle$ where $C' = C \cup \{s_1, s_2\}$ and $s_1, s_2 \in V(G-C)$ and s_2 is adjacent to one vertex of C in G. Since $\kappa(G) = k+3$ and for each $(G, C) \in \mathcal{K}_{k^+}^b\left(\lceil\frac{m+1}{2}\rceil\right)$ it follows that $\delta_G(G - C) \geq k + \lceil\frac{m+1}{2}\rceil = (k+2) + \left\lceil\frac{(m-4)+1}{2}\right\rceil$, we have $\delta_{G'}(G' - C') \geq (k+2) + \left\lceil\frac{(m-4)+1}{2}\right\rceil$ and $(G', C') \in \mathcal{K}_{(k+2)^+}^b\left(\left\lceil\frac{(m-4)+1}{2}\right\rceil\right)$. Thus, by Theorem 7, there is a path P on $m-4$ vertices starting from x_0 such that $\kappa\left(G' - V(P[x_0, x_{m-5}])\right) \geq k+2$, where x_0 is adjacent to at least one vertex of $\{s_1, s_2\}$. Without loss of generality, let $x_0 s_1 \in E(G)$. Since for each vertex $x_i \in P[x_0, x_{m-5}]$ we have

$$\begin{aligned}
|N_G(x_i) \cap (G' - V(P[x_0, x_{m-5}])| &\geq k + \left\lceil\frac{m+1}{2}\right\rceil - \left\lceil\frac{|P[x_0, x_{m-5}]| - 1}{2}\right\rceil \\
&= k + \left\lceil\frac{m+1}{2}\right\rceil - \left\lceil\frac{m-5}{2}\right\rceil > k+2.
\end{aligned}$$

It follows that $\kappa\left(G' - \{x_0, x_1\}\right) \geq k+2$. Owing to $d_{G'-P[x_0, x_{m-5}]}(x_0) > k+2$, we can choose two neighbors s_1, y_1 of x_0 in $G' - P[x_0, x_{m-5}]$ such that the subgraph T_1^3 on the vertex set $\{s_1, y_1, x_0, x_1\}$ satisfies $\kappa\left(G' - \{s_1, y_1, x_0, x_1\}\right) \geq k$. Clearly, $d_{G'-\{s_1,y_1,x_0,x_1\}}(s_2) \geq d_{G-\{s_1,y_1,x_0,x_1\}}(s_2) \geq k+1$. We then claim that $\kappa\left(G - \{s_1, y_1, x_0, x_1\}\right) \geq k$. Indeed, we can suppose $\kappa\left(G - \{s_1, y_1, x_0, x_1\}\right) < k$. If s_2 is in an $(k-1)$-vertex cut of G, then $\kappa\left(G' - \{s_1, y_1, x_0, x_1\}\right) < k$, a contradiction. Thus, there is $(k-1)$-vertex cut to separate s_2 and C in G. But, since there exists at least an edge between s_2 and C by our setting in G, it follows that there is at least an k vertex cut in G. So $\kappa\left(G - \{s_1, y_1, x_0, x_1\}\right) \geq k$.

If $\kappa(G) = k+2$, then for each $(G, C) \in \mathcal{K}_{k^+}^b\left(\lceil\frac{m+1}{2}\rceil\right)$ it follows that $\delta_G(G - C) \geq k + \lceil\frac{m+1}{2}\rceil = (k+1) + \left\lceil\frac{(m-2)+1}{2}\right\rceil$. Let $G' = G\langle C'\rangle$ where $C' = C \cup \{s_1\}$ and $s_1 \in V(G-C)$ and s_1 is adjacent to at least one vertex of C in C. We have $\delta_{G'}(G' - C') \geq (k+1) + \left\lceil\frac{(m-2)+1}{2}\right\rceil$ and $(G', C') \in \mathcal{K}_{(k+1)^+}^b\left(\left\lceil\frac{(m-2)+1}{2}\right\rceil\right)$. By Theorem 7, there is a path P on $m-2$ vertices starting from x_0 such that $\kappa(G' - V(P[x_0, x_{m-3}])) \geq k+1$. For each vertex $x_i \in P[x_0, x_{m-3}]$ we have that

$$\begin{aligned}
|N_G(x_i) \cap (G' - V(P[x_0, x_{m-3}])| &\geq k + \left\lceil\frac{m+1}{2}\right\rceil - \left\lceil\frac{|P[x_0, x_{m-3}]| - 1}{2}\right\rceil \\
&\geq k + \left\lceil\frac{m+1}{2}\right\rceil - \left\lceil\frac{m-3}{2}\right\rceil = k+2.
\end{aligned}$$

And thus, $\kappa(G' - \{x_0, x_1, x_2\}) \geq k+1$. Furthermore, we can choose arbitrarily one neighbor, say y_1, of x_1 in $G' - P[x_0, x_{m-3}]$ such that the subgraph T_1^3 on the vertex set $\{y_1, x_0, x_1, x_2\}$ satisfies $\kappa(G' - \{y_1, x_0, x_1, x_2\}) \geq k$. Clearly, we also have $\kappa\left(G - \{y_1, x_0, x_1, x_2\}\right) \geq k$.

In the following, we will use Lemma 10 to prove the case of $\kappa(G) = k+1$. Since for each $(G, C) \in \mathcal{K}_{k^+}^b\left(\left\lceil \frac{m+1}{2} \right\rceil\right)$ it follows that $\delta_G(G - C) \geq k + \left\lceil \frac{m+1}{2} \right\rceil = (k+1) + \left\lceil \frac{(m-2)+1}{2} \right\rceil$. Let $G' = G\langle C' \rangle$ where $C' = C \cup \{s_1\}$ and $s_1 \in V(G - C)$ and s_1 is adjacent to at least one vertex of C in G. And then, we have $(G', C') \in \mathcal{K}_{(k+1)}^b\left(\left\lceil \frac{(m-2)+1}{2} \right\rceil\right)$. Thus, by Lemma 10, every $(G', C') \in \mathcal{K}_{(k+1)}^b\left(\left\lceil \frac{(m-2)+1}{2} \right\rceil\right)$ contains a path $P \subseteq G' - V(C)$ of order $m-2$, such that $\kappa(G' - V(P)) \geq k+1$ holds. Let $P = x_0 x_1 \cdots x_{m-3}$. For each vertex $x_i \in P[x_0, x_{m-3}]$ we have that $|N_G(x_i) \cap (G' - V(P[x_0, x_{m-3}])| \geq k + \left\lceil \frac{m+1}{2} \right\rceil - \left\lceil \frac{m-3}{2} \right\rceil = k+2$. And thus, $\kappa(G' - \{x_0, x_1, x_2\}) \geq k+1$. Since $|N_G(x_1) \cap (G' - C' - V(P[x_0, x_{m-3}]))| > 3$, there is a neighbor s of x_1. Consequently, $\kappa(G' - \{s, x_0, x_1, x_2\}) \geq k$. Similarly, we again have $\kappa(G - \{s, x_0, x_1, x_2\}) \geq k$. This completes the proof of Claim 14. $\square$

Subject to above assumption and by Claim 14, we suppose that T_w^3 is maximal on vertices, with root p_0 and legs s_1, s_2, satisfying two conditions.

(i) $4 \leq |T_w^3| = 3 + w < m$;

(ii) $\kappa(G - V(T_w^3)) \geq k$.

We denote $T_w^3 - \{s_1, s_2\}$ by $P[p_0, p_w] = p_0 p_1 \cdots p_w$. Simply, we set $H = G - T_w^3$.

Claim 15. $\kappa(H) = k$.

Proof. We assume that $\kappa(H) > k$. Since

$$|N_G(p_w) \cap (H - C)| \geq \left(k + \left\lceil \frac{m+1}{2} \right\rceil\right) - k - \left\lceil \frac{m-2}{2} \right\rceil \geq 1,$$

it follows that there exists a vertex $s \in H - C$ such that $p_w s \in E(G)$. Note that $T_w^3 \cup \{p_w s\}$ is a tree rooted at p_0 of order $4 + w \leq m$, which contradicts the choice of T_w^3. This completes the proof of Claim 15. $\square$

Since $|V(H)| \geq 2k+2$ and by Claim 15, H is not a complete graph. An end E is contained in H with $E \cap C = \emptyset$. Set $S := N_H(E)$. Then $|S| = k$. Furthermore, let $\bar{E} = H - S - E$.

Claim 16. $|E| \geq 2$.

Proof. If $|E| = 1$, then $k + \left\lceil \frac{m+1}{2} \right\rceil \leq d_G(z) \leq |N_G(z) \cap S| + |N_G(z) \cap V(P)| \leq k + \left\lceil \frac{m-1}{2} \right\rceil$ for the unique vertex $z \in E$, a contradiction. $\square$

By Claim 16 and Lemma 5, it follows that $H\langle S\rangle - \bar{E}$ is $(k+1)$-connected. From Claim 15, we know $\kappa(G) > k = \kappa(H)$. And thus, $N_G(T_w^3) \cap E \neq \emptyset$. Otherwise, S is also a vertex set of G, which contradicts $\kappa(G) > k$. Let y be one of the farthest vertices to p_0 on T_w^3 with $N_G(y) \cap E \neq \emptyset$. Let $q \in N_G(y) \cap E$. We consider the following two cases.

Case 1. $y \in \{p_0, p_1, p_2, \dots, p_w\}$. Let $\bar{P} = P(y, p_w]$. Consider the graph $G - (T_w^3 - \bar{P}) := H \cup \bar{P}$. For any $x \in V(\bar{P})$, we have $|N_G(x) \cap H| \geq k + \lceil \frac{m+1}{2} \rceil - |N_G(x) \cap V(P)| \geq k + \lceil \frac{m+1}{2} \rceil - \lceil \frac{m-2}{2} \rceil \geq k+1$, and then it follows that

$$\kappa\big(G - (T_w^3 - \bar{P})\big) = \kappa(H \cup \bar{P})) \geq k.$$

As y is the farthest vertex to p_0 on T_w^3, we have $N_G(\bar{P}) \cap E = \emptyset$. Virtually, S is also a minimum vertex cut of $H \cup \bar{P}$, and E is an end of $H \cup \bar{P}$. From Lemma 5, $(H \cup \bar{P})\langle S\rangle(E \cup S) = (H \cup \bar{P})\langle S\rangle - \bar{E}$ is $(k+1)$-connected.

If both m and $|V(\bar{P})|$ are odd, it follows that $C \subseteq H \cup \bar{P} - E$ and

$$\begin{aligned}\delta_{(H\cup\bar{P})}(H \cup \bar{P} - C) &\geq k + \left\lceil \frac{m+1}{2} \right\rceil - \left\lceil \frac{3+w-|\bar{P}|}{2} \right\rceil \\ &\geq k + \left\lceil \frac{m+1}{2} \right\rceil - \left\lceil \frac{m-1-|\bar{P}|}{2} \right\rceil \\ &\geq k + \left\lceil \frac{|\bar{P}|+1}{2} \right\rceil.\end{aligned}$$

Hence, we have that

$$(H \cup \bar{P}, C) \in \mathcal{K}_k^b\left(\left\lceil \frac{|\bar{P}|+1}{2} \right\rceil\right)$$

and

$$((H \cup \bar{P})\langle S\rangle[E \cup S], K(S)) \in \mathcal{K}_{k^+}^b\left(\left\lceil \frac{|\bar{P}|+1}{2} \right\rceil\right).$$

By the choice of G and $|S \cup V(E)| < |V(G)|$, there exists a path $Q \subseteq E$ of order $|\bar{P}|$ starting from q such that $(H \cup \bar{P})\langle S\rangle[E \cup S] - V(Q)$ is k-connected, that is, $\kappa\left((H \cup \bar{P})\langle S\rangle[E \cup S] - V(Q)\right) \geq k$. Consider the complementary fragment $\bar{E}_{H\cup\bar{P}} = H \cup \bar{P} - S - E$ in $H \cup \bar{P}$. By Lemma 12, we have $\kappa\left((H \cup \bar{P}) - Q\right) \geq k$. Let q, q' be the end-vertices of Q and $P_1 = (P \setminus \bar{P}) \cup Q \cup \{q, q'\}$. Then $\kappa(G - V(P_1)) \geq k$ and $|V(P_1)| = |V(P)|$. Let $E' = E - V(Q)$ and $G_{P_1} = G - V(P_1)$. Since $|N_G(q') \cap V(E')| = d_G(q') - |N_G(q') \cap V(P_1)| - |N_G(q') \cap S| \geq k + \lceil \frac{m+1}{2} \rceil - \lceil \frac{m-2}{2} \rceil - k \geq 1$, it follows that $V(E') \neq \emptyset$. Consequently, S is also a minimum vertex cut of G_{P_1} and E' is a fragment of G_{P_1} to S. And then we have $|\{s_1, s_2\} \cup V(P_1)| = |\{s_1, s_2\} \cup V(P)|$ and $|V(E)| > |V(E')|$, which contradicts the choice of the smallest fragment E.

If at least one of the integers m and $|\bar{P}|$ is even, then

$$\begin{aligned}\delta_{(H\cup\bar{P})}(H\cup\bar{P}-C) &\geq k+\left\lceil\frac{m+1}{2}\right\rceil-\left\lceil\frac{3+w-|\bar{P}|}{2}\right\rceil\\ &\geq k+\left\lceil\frac{m+1}{2}\right\rceil-\left\lceil\frac{m-1-|\bar{P}|}{2}\right\rceil\\ &\geq k+\left\lceil\frac{|\bar{P}|+2}{2}\right\rceil.\end{aligned}$$

We have $\left(H\cup\bar{P},C\right)\in\mathcal{K}_k^b\left(\frac{|\bar{P}|+2}{2}\right)$. Since $\kappa\left((H\cup\bar{P})\langle S\rangle[S\cup E]\right)\geq k+1$ by Lemma 5, we have $\left((H\cup\bar{P})\langle S\rangle[S\cup E],K(S)\right)\in\mathcal{K}_k^b\left(\frac{|\bar{P}|+2}{2}\right)$. Since $|S\cup V(E)|<|V(G)|$, there is a path Q of order $|\bar{P}|+1$ in E starting from q such that $(H\cup\bar{P})\langle S\rangle[S\cup E]-V(Q)$ is k-connected. that is, $\kappa\left((H\cup\bar{P})\langle S\rangle-V(\bar{E}_{H\cup\bar{P}}\cup Q)\right)\geq k$. By Lemma 12, we have $\kappa\left((H\cup\bar{P})-V(Q)\right)\geq k$. Then there is a tree $T^3_{w+1}:=(T^3_w\setminus\bar{P})\cup yq\cup Q$ rooted at p_0 has order $|T^3_{w+1}|\leq m$ and $G-V(T^3_{w+1})=(H\cup\bar{P})-V(Q)$ is k-connected, a contradiction.

Case 2. $y\in\{s_1,s_2\}$. Let $\bar{P}=P(p_1,p_w]$ and then there is a tree $T^3_{w+1}:=(T^3_w-\bar{P})\cup yp\cup Q$ with order $w+1\leq m$ according to the same way as above case. Consequently, we find out a larger tree T^3_{w+1}. This completes the proof of Theorem 13. ■

Based on Theorem 13, we prove the following main theorem.

Theorem 17. *For $m\neq 4$ or 5 every k-connected bipartite graph G with $\delta(G)\geq k+\left\lceil\frac{m+1}{2}\right\rceil$ contains a tree T^3_{m-3} such that*

$$\kappa(G-V(T^3_{m-3}))\geq k.$$

Proof. If $m\leq 3$, then our Theorem 17 is the main theorem of [16]. In the following, we consider the case of $m\geq 6$. If $\kappa(G)=k$, then let E be an end to S in G and let $\bar{E}=G-S\cup V(E)$. If $|E|=1$, then $k+\left\lceil\frac{m+1}{2}\right\rceil\leq d_G(z)\leq|N_G(z)\cap S|+|N_G(z)\cap V(P)|\leq k+\left\lceil\frac{m-1}{2}\right\rceil$ for the unique vertex $z\in E$, a contradiction. And then we have $|E|\geq 2$. By Lemma 5, $\kappa(G\langle S\rangle-V(\bar{E}))\geq k+1$. Furthermore, $(G\langle S\rangle-V(\bar{E}),K(S))\in\mathcal{K}_{k^+}^b\left(\frac{m+1}{2}\right)$. By Theorem 13, we have $\kappa\left(G\langle S\rangle-V(\bar{E}\cup T^3_{m-3})\right)\geq k$. By Lemma 11, it follows that $\kappa\left(G-V(\bar{E}\cup T^3_{m-3})\right)\geq k$. If $\kappa(G)\geq k+1$, we first prove a claim.

Claim 18. *For every $(k+1)$-connected graph G with $\delta(G)\geq k+\left\lceil\frac{m+1}{2}\right\rceil$, there is a subgraph T^3_1 in G such that $\kappa(G-V(T^3_1))\geq k$.*

Proof. If G is $(k+4)$-connected graph, then, clearly, there is a subgraph T^3_1 in G such that $\kappa\left(G-V(T^3_1)\right)\geq k$ since $\delta(G)\geq k+\left\lceil\frac{m+1}{2}\right\rceil\geq k+4$. Suppose that

$\kappa(G) = k+3$. Since it follows that $\delta(G) \geq k + \lceil \frac{m+1}{2} \rceil = (k+2) + \left\lceil \frac{(m-4)+1}{2} \right\rceil$, by Theorem 8, there is a path P on $m-4$ vertices such that $\kappa\left(G - V(P[x_0, x_{m-5}])\right) \geq k+2$. Since for each vertex $x_i \in P[x_0, x_{m-5}]$ we have

$$\begin{aligned} |N_G(x_i) \cap (G - V(P[x_0, x_{m-5}])| &\geq k + \left\lceil \frac{m+1}{2} \right\rceil - \left\lceil \frac{|P[x_0, x_{m-5}]| - 1}{2} \right\rceil \\ &= k + \left\lceil \frac{m+1}{2} \right\rceil - \left\lceil \frac{m-5}{2} \right\rceil > k+2. \end{aligned}$$

It follows that $\kappa\left(G - \{x_0, x_1\}\right) \geq k+2$. Owing to $d_{G-P[x_0,x_{m-5}]}(x_0) > k+2$, we can choose arbitrarily two neighbors s_1, s_2 of x_0 in $G - P[x_0, x_{m-5}]$ such that the subgraph T_1^3 on the vertex set $\{s_1, s_2, x_0, x_1\}$ satisfies

$$\kappa\left(G - \{s_1, s_2, x_0, x_1\}\right) \geq k.$$

If $\kappa(G) = k+2$, then it follows that $\delta(G) \geq k + \lceil \frac{m+1}{2} \rceil = (k+1) + \left\lceil \frac{(m-2)+1}{2} \right\rceil$. By Theorem 8, there is a path P on $m-2$ vertices such that $\kappa(G - V(P[x_0, x_{m-3}])) \geq k+1$. For each vertex $x_i \in P[x_0, x_{m-3}]$ we have that

$$\begin{aligned} |N_G(x_i) \cap (G - V(P[x_0, x_{m-3}])| &\geq k + \left\lceil \frac{m+1}{2} \right\rceil - \left\lceil \frac{|P[x_0, x_{m-3}]| - 1}{2} \right\rceil \\ &\geq k + \left\lceil \frac{m+1}{2} \right\rceil - \left\lceil \frac{m-3}{2} \right\rceil = k+2. \end{aligned}$$

And thus, $\kappa(G - \{x_0, x_1, x_2\}) \geq k+1$. Furthermore, we can choose arbitrarily one neighbor, say s_1, of x_1 in $G - P[x_0, x_{m-3}]$ such that the subgraph T_1^3 on the vertex set $\{s_1, x_0, x_1, x_2\}$ satisfies $\kappa(G - \{s_1, x_0, x_1, x_2\}) \geq k$.

Suppose that $\kappa(G) = k+1$. Since $\delta(G) \geq k + \lceil \frac{m+1}{2} \rceil = (k+1) + \left\lceil \frac{(m-2)+1}{2} \right\rceil$. Thus, by Theorem 8, G contains a path $P \subseteq G$ of order $m-2$, such that $\kappa(G - V(P)) \geq k+1$ holds. Let $P = x_0 x_1 \cdots x_{m-3}$. For each vertex $x_i \in P[x_0, x_{m-3}]$ we have that $|N_G(x_i) \cap (G - V(P[x_0, x_{m-3}])| \geq k + \lceil \frac{m+1}{2} \rceil - \lceil \frac{m-3}{2} \rceil = k+2$. And thus, $\kappa(G - \{x_0, x_1, x_2\}) \geq k+1$. Since $|N_G(x_1) \cap (G - C - V(P[x_0, x_{m-3}]))| > 3$, there is a neighbor s of x_1. Consequently, $\kappa(G - \{s, x_0, x_1, x_2\}) \geq k$. This completes the proof of Claim 18. □

Similar to the proof of Theorem 13, we again find a maximal tree T_w^3 with root p_0 and legs s_1, s_2 satisfying the following conditions.

(i) $4 \leq |T_w^3| = 3 + w < m$;

(ii) $\kappa(G - V(T_w^3)) \geq k$.

Let S be a minimum vertex cut of $G - V\left(T_w^3\right)$ and E be an end of $G - V\left(T_w^3\right)$. and thus we again have that $\kappa\left(G - V\left(T_w^3\right)\right) = k$. Let $H = G - T_w^3$. We denote $T_w^3 - \{s_1, s_2\}$ by $P[p_0, p_w] = p_0 p_1 \cdots p_w$.

Clearly, H is not a complete graph. An end E is contained in H with $E \cap C = \emptyset$. Set $S := N_H(E)$. Then $|S| = k$. Furthermore, let $\bar{E} = H - S - E$. If $|E| = 1$, then $k + \lceil \frac{m+1}{2} \rceil \le d_G(z) \le |N_G(z) \cap S| + |N_G(z) \cap V(P)| \le k + \lceil \frac{m-1}{2} \rceil$ for the unique vertex $z \in E$, a contradiction. And then we have $|E| \ge 2$. By Lemma 5, it follows that $H\langle S \rangle - \bar{E}$ is $(k+1)$-connected. We know $\kappa(G) > k = \kappa(H)$. And thus, $N_G(T_w^3) \cap E \ne \emptyset$. Otherwise, S is also a vertex set of G, which contradicts $\kappa(G) > k$. Let y be one of the farthest vertices to p_0 on T_w^3 with $N_G(y) \cap E \ne \emptyset$. Let $q \in N_G(y) \cap E$. We consider the following two cases.

Case 1. $y \in \{p_0, p_1, p_2, \ldots, p_w\}$. Let $\bar{P} = P(y, p_w]$. Consider the graph $G - (T_w^3 - \bar{P}) := H \cup \bar{P}$. For any $x \in V(\bar{P})$, we have $|N_G(x) \cap H| \ge k + \lceil \frac{m+1}{2} \rceil - |N_G(x) \cap V(P)| \ge k + \lceil \frac{m+1}{2} \rceil - \lceil \frac{m-2}{2} \rceil \ge k+1$, and then it follows that

$$\kappa(G - (T_w^3 - \bar{P})) = \kappa(H \cup \bar{P})) \ge k.$$

As y is the farthest vertex to p_0 on T_w^3, we have $N_G(\bar{P}) \cap E = \emptyset$. Virtually, S is also a minimum vertex cut of $H \cup \bar{P}$, and E is an end of $H \cup \bar{P}$. From Lemma 5, $(H \cup \bar{P})\langle S \rangle (E \cup S) = (H \cup \bar{P})\langle S \rangle - \bar{E}$ is $(k+1)$-connected.

If both m and $|V(\bar{P})|$ are odd, it follows that $C \subseteq H \cup \bar{P} - E$ and

$$\begin{aligned} \delta_{(H \cup \bar{P})}(H \cup \bar{P} - C) &\ge k + \left\lceil \frac{m+1}{2} \right\rceil - \left\lceil \frac{3 + w - |\bar{P}|}{2} \right\rceil \\ &\ge k + \left\lceil \frac{m+1}{2} \right\rceil - \left\lceil \frac{m - 1 - |\bar{P}|}{2} \right\rceil \\ &\ge k + \left\lceil \frac{|\bar{P}| + 1}{2} \right\rceil. \end{aligned}$$

Hence, it follows

$$(H \cup \bar{P}, C) \in \mathcal{K}_k^b\left(\left\lceil \frac{|\bar{P}| + 1}{2} \right\rceil\right)$$

and

$$((H \cup \bar{P})\langle S \rangle [E \cup S], \langle S \rangle) \in \mathcal{K}_{k+}^b\left(\left\lceil \frac{|\bar{P}| + 1}{2} \right\rceil\right).$$

By the choice of G and $|S \cup V(E)| < |V(G)|$, there exists a path $Q \subseteq E$ of order $|\bar{P}|$ starting from q such that $(H \cup \bar{P})\langle S \rangle [E \cup S] - V(Q)$ is k-connected, that is, $\kappa\left((H \cup \bar{P})\langle S \rangle [E \cup S] - V(Q)\right) \ge k$. Consider the complementary fragment $\bar{E}_{H \cup \bar{P}} = H \cup \bar{P} - S - E$ in $H \cup \bar{P}$. By Lemma 12, we have $\kappa\left((H \cup \bar{P}) - Q\right) \ge k$. Let q, q' be the end-vertices of Q and $P_1 = (P \setminus \bar{P}) \cup Q \cup \{q, q'\}$. Then

$\kappa(G - V(P_1)) \geq k$ and $|V(P_1)| = |V(P)|$. Let $E' = E - V(Q)$ and $G_{P_1} = G - V(P_1)$. Since $|N_G(q') \cap V(E')| = d_G(q') - |N_G(q') \cap V(P_1)| - |N_G(q') \cap S| \geq k + \lceil \frac{m+1}{2} \rceil - \lceil \frac{m-2}{2} \rceil - k \geq 1$, it follows that $V(E') \neq \emptyset$. Consequently, S is also a minimum vertex cut of G_{P_1} and E' is a fragment of G_{P_1} to S. And then we have $|\{s_1, s_2\} \cup V(P_1)| = |\{s_1, s_2\} \cup V(P)|$ and $|V(E)| > |V(E')|$, which contradicts the choice of the smallest fragment E.

If at least one of the integers m and $|\bar{P}|$ is even, then

$$\begin{aligned}
\delta_{(H\cup\bar{P})}(H \cup \bar{P} - C) &\geq k + \left\lceil \frac{m+1}{2} \right\rceil - \left\lceil \frac{3 + w - |\bar{P}|}{2} \right\rceil \\
&\geq k + \left\lceil \frac{m+1}{2} \right\rceil - \left\lceil \frac{m - 1 - |\bar{P}|}{2} \right\rceil \\
&\geq k + \left\lceil \frac{|\bar{P}| + 2}{2} \right\rceil.
\end{aligned}$$

We have $\left(H \cup \bar{P}, C\right) \in \mathcal{K}_k^b \left(\frac{|\bar{P}|+2}{2}\right)$. Since $\kappa\left((H \cup \bar{P})\langle S\rangle[S \cup E]\right) \geq k + 1$ by Lemma 5, we have $\left((H \cup \bar{P})\langle S\rangle[S \cup E], K(S)\right) \in \mathcal{K}_k^b \left(\frac{|\bar{P}|+2}{2}\right)$. Since $|S \cup V(E)| < |V(G)|$, there is a path Q of order $|\bar{P}| + 1$ in E starting from q such that $(H \cup \bar{P})\langle S\rangle[S \cup E] - V(Q)$ is k-connected. that is, $\kappa\left((H \cup \bar{P})\langle S\rangle - V(\bar{E}_{H\cup\bar{P}} \cup Q)\right) \geq k$. By Lemma 12, we have

$$\kappa\left((H \cup \bar{P}) - V(Q)\right) \geq k.$$

Then there is a tree $T^3_{w+1} := (T^3_w \setminus \bar{P}) \cup yq \cup Q$ rooted at p_0 with order $|T^3_{w+1}| \leq m$ and $G - V(T^3_{w+1}) = (H \cup \bar{P}) - V(Q)$ is k-connected, a contradiction.

Case 2. $y \in \{s_1, s_2\}$. Let $\bar{P} = P(p_1, p_w]$ and then there is a tree $T^3_{w+1} := (T^3_w - \bar{P}) \cup yp \cup Q$ with order $w + 1 \leq m$ according to the same way as above case. Consequently, we find a larger tree T^3_{w+1}. This completes the proof of Theorem 17. ∎

As you can see, our main theorem has the condition of $m \neq 4$ or 5, because the proof of Claim 14 cannot contain the condition. We have tried to over $m = 4$ or 5, but it is hard to construct T^3_1 for us. So we list this problem in this section.

Problem 19. For $m = 4$ or 5 every k-connected bipartite graph G with $\delta(G) \geq k + \lceil \frac{m+1}{2} \rceil$ contains a tree T^3_{m-3} such that

$$\kappa(G - V(T^3_{m-3})) \geq k.$$

4. k-Connected Graphs

We define the set $\mathcal{F}_k(m)$ containing all pairs (G, C) satisfying the following conditions.

- G is a k-connected graph with $|G| \geq k+1$;
- $C \subseteq G$ is a complete subgraph with $|C| = k$ and $\delta_G(G - V(C)) \geq \left\lfloor \frac{3k}{2} \right\rfloor + m - 1$;
- we denote by $\mathcal{F}_k^+(m)$ all the pairs $(G, C) \in \mathcal{F}_k(m)$ with $\kappa(G) \geq k+1$.

Lemma 20 [11]**.** *Let G be a k-connected graph and let S be a vertex cut of G with $|S| = k$. Then the following holds.*

(a) *For every fragment F of G to S, $G[S] - V(F)$ is k-connected.*

(b) *Assume $\delta(G) \geq \left\lfloor \frac{3k}{2} \right\rfloor + m - 1$ and let F be a fragment of G to S. If $W \subseteq G - (S \cup V(F))$ has order at most m and $\kappa(G[S] - V(F \cup W)) \geq k$ holds, then also $\kappa(G - V(W)) \geq k$ holds.*

(c) *Assume $(G, C) \in \mathcal{F}_k(m)$ and let F be a fragment of G to S with $C \subseteq G(F \cup S)$. If $W \subseteq G - (S \cup V(F))$ has order at most m and $\kappa(G[S] - V(F \cup W)) \geq k$ holds, then also $\kappa(G - V(W)) \geq k$ holds.*

Lemma 21 [11]**.** *For all $(G, C) \in \mathcal{F}_k^+(m)$ and $p_0 \in G - V(C)$, there is a path $P \subseteq G - V(C)$ of length $m-1$ starting from p_0, such that $\kappa(G - V(P)) \geq k$ holds.*

The spiders are considered and defined now. For a tree, if there is at most one vertex of degree at least 3, then this tree is called a *spider* (specially, a path is also a spider). Each *leg* of a spider is a path from the vertex adjacent to the *root* x_0 to a vertex of degree 1; if there are z legs, then denote the spider by $T_m^{t_1, t_2, \ldots, t_z}$, where $\left| T_m^{t_1, t_2, \ldots, t_z} \right| = m$ and t_i denotes the order of the ith leg with $t_1 + t_2 + \cdots + t_z + 1 = m$. If there are t legs of order one, then we abbreviate $T_m^{1,1,\ldots,1,m-t-1}$ as $T_m^{t;m-t-1}$.

Lemma 22. *Let $t \geq 0$ be an integer. For any $(G, C) \in \mathcal{F}_k^+(m)$ and any $s_0 \in G - V(C)$ with at least a vertex of degree $|G| - 1$ in $G - C$, there is a spider $T_m^{t;m-t-1} \subseteq G - V(C)$ of order m rooted at s_0 such that*

$$\kappa\big(G - V(T_m^{t;m-t-1})\big) \geq k.$$

Proof. We perform an induction on the order n of the graph G for the lemma. Clearly, the order of the graph G must be no less than $\left\lfloor \frac{3k}{2} \right\rfloor + m$ since $\delta(G - C) \geq \left\lfloor \frac{3k}{2} \right\rfloor + m - 1$. Then it holds for $(G, C) \in \mathcal{F}_k^+(m)$ if G is a complete graph with order at least $\left\lfloor \frac{3k}{2} \right\rfloor + m$. So we just need to consider the case that G is not complete and $|G| \geq \left\lfloor \frac{3k}{2} \right\rfloor + m + 1$. Now assume that G is a graph with smallest order and with $\Delta(G) = |G| - 1$ such that $(G, C) \in \mathcal{F}_k^+(m)$ is a counterexample to Lemma 22 for k, and some $C \subseteq G$, and m. Let $d(s_0) = |G| - 1$ for a vertex $s_0 \in V(G)$.

Subject to above assumption, we find out on the order m of tree $T_m^{t;m-t-1}$ satisfying above assumption. From Lemma 21, there exists a path $P \subseteq G - V(C)$ of length $m-1$ starting from s_0 such that $\kappa(G - V(P)) \geq k$ holds. Let $P = \{s_0p_1\} \cup P[p_1, p]$. Since $|N_G(u) \cap (G - P)| \geq \left\lfloor \frac{3k}{2} \right\rfloor + m - 1 - (|P| - 1) \geq k$ for any vertex $u \in V(P[p_1, p])$, then $\kappa(G - s_0p_1) \geq k$, where $s_0p_1 \in E(G)$ is a subpath of P and also a spider $T_2^{1,0}$ or $T_2^{0,1}$. Now suppose that a spider $T_{t+j+1}^{t;j}$ with the maximal order and root s_0 and legs $s_1, s_2, \ldots, s_t$ satisfies the following conditions.

(i) $2 \leq \left|T_{t+j+1}^{t;j}\right| = t + j + 1 < m$;

(ii) $\kappa(G - T_{t+j+1}^{t;j}) \geq k$.

Note that $s_1, s_2, \ldots, s_t \in V(G)$ and $s_0s_i \in E(G)$, $1 \leq i \leq t$, and $T_{t+j+1}^{t;j} - \{s_i \,|\, 1 \leq i \leq t\}$ is a path of order $j+1$, say $P = s_0p_1 \cup P[p_1, p_j] := p_1p_2 \cdots p_j$. Simply, we set $H = G - T_{t+j+1}^{t;j}$.

Claim 23. $\kappa(H) = k$.

Proof. Assume, to the contrary, that $\kappa(H) > k$. Since

$$|N_G(x) \cap (H - C)| \geq \lfloor 3k/2 \rfloor + m - 1 - k - (m - 2) = \lfloor k/2 \rfloor + 1$$

for $x \in \{s_0, p_j\}$, it follows that there exists a vertex $s \in H - C$ such that $xs \in E(G)$. Note that $T_{t+j+1}^{t;j} \cup xs$ is a spider rooted at s_0 of order $t+j+2 \leq m$, which contradicts the choice of $T_{t+j+1}^{t;j}$. Complete the proof of Claim 23. □

Since H is not a complete graph, it follows that $|V(H)| \geq k + 2$. An end E is contained in H with $E \cap C = \emptyset$. Set $S := N_H(E)$. Then $|S| = k$. Furthermore, let $\bar{E} = H - S - E$.

Claim 24. $|E| \geq 2$.

Proof. Assume, to the contrary, that $|E| = 1$. It satisfies $k = d_H(z) \geq \left\lfloor \frac{3k}{2} \right\rfloor + m - 1 - \left|T_{t+j+1}^{t;j}\right| \geq \left\lfloor \frac{3k}{2} \right\rfloor$ for each $z \in E$, which means $k = 1$ and $\left|T_{t+j+1}^{t;j}\right| = m - 1$. We get $V\left(T_{m-1}^{t;m-t-2}\right) \subseteq N_G(z)$ because of $\delta(G) \geq \left\lfloor \frac{3k}{2} \right\rfloor + m - 1$. Then $\left|T_m^{t;m-t-1}\right| := \left|T_{m-1}^{t;m-t-2} \cup zx\right| = m$ for $x \in \{s_0, p_{m-t-2}\}$ and $G - T_m^{t;m-t-1} = H - z$ is 1-connected. And also, $\left|T_m^{t+1;m-t-2}\right| := \left|T_{m-1}^{t;m-t-2} \cup zx\right| = m$ and $G - T_m^{t+1;m-t-2} = H - z$ is 1-connected, which contradicts the fact that $T_a^{t;j}$ is a maximal spider. Complete the proof of Claim 24. □

From Claim 24, we have $|E| \geq 2$. Then the graph $H[S] - \bar{E}$ is $(k+1)$-connected from Lemma 5. From above assumption, we know $\kappa(G) > k = \kappa(H)$, thus $N_G(T^{t;j}_{t+j+1}) \cap E \neq \emptyset$. Otherwise, S is also a vertex cut of G, which contradicts $\kappa(G) > k$. Let y be one of the farthest vertices to s_0 on $T^{t;j}_{t+j+1}$ with $N_G(y) \cap E \neq \emptyset$. Suppose that q is one vertex in $N_G(y) \cap E$. We distinguish the following two cases to show this lemma. We will construct two lager spiders $T^{t;j+1}_{t+j+2}$ and $T^{t+1;j}_{t+j+2}$ such that $G - T^{t;j+1}_{t+j+2}$ and $G - T^{t+1;j}_{t+j+2}$ remains k-connected, respectively.

Claim 25. *There is a lager spider $T^{t;j+1}_{t+j+2}$ such that $G - T^{t;j+1}_{t+j+2}$ remains k-connected.*

Proof. Suppose that $y \in \{p_1, p_2, \ldots, p_j, s_0\}$. Let $\bar{P} = P[p_j, y)$. Consider the graph $G - \left(T^{t;j}_{t+j+1} - \bar{P}\right) := H \cup \bar{P}$. Since $|N_G(x) \cap H| \geq \lfloor \frac{3k}{2} \rfloor + m - 1 - (t+j) \geq \lfloor \frac{3k}{2} \rfloor + 1 \geq k$ for any $x \in V(\bar{P})$, it follows that

$$\kappa\big(G - (T^{t;j}_{t+j+1} - \bar{P})\big) = \kappa(H \cup \bar{P})) \geq k.$$

As y is the farthest vertex to s_0 on $T^{t;j}_{t+j+1}$, we have $N_G(\bar{P}) \cap E = \emptyset$. Naturally, S is also a minimum vertex cut of $H \cup \bar{P}$, and E is an end of $H \cup \bar{P}$. From Lemma 5, $(H \cup \bar{P})\langle S\rangle[E \cup S] = H\langle S\rangle - \bar{E}$ is $(k+1)$-connected. Furthermore, it follows that $C \subseteq H \cup \bar{P} - E$ and

$$\delta_{(H \cup \bar{P})}(H \cup \bar{P} - C) \geq \left\lfloor \frac{3k}{2} \right\rfloor + m - 1 - (t + j + 1 - |\bar{P}|) \geq \left\lfloor \frac{3k}{2} \right\rfloor + |\bar{P}|.$$

Hence, $(H \cup \bar{P}, C) \in \mathcal{F}_k(|\bar{P}| + 1)$ and $(H \cup \bar{P}\langle S\rangle[E \cup S], \langle S\rangle) \in \mathcal{F}^+_k(|\bar{P}| + 1)$. From Lemma 21, there exists a path $Q \subseteq E$ of order $|\bar{P}| + 1$ starting from q such that $(H \cup \bar{P})\langle S\rangle[E \cup S] - Q$ is k-connected. Now for $(H \cup \bar{P}, C) \in \mathcal{F}_k(|\bar{P}| + 1)$ by Lemma 20(c), then $\kappa((H \cup \bar{P})\langle S\rangle[E \cup S] - Q) \geq k$. The spider $T^{t;j+1}_{t+j+2} := \left(T^{t;j}_{t+j+1} \setminus \bar{P}\right) \cup yq \cup Q$ rooted at s_0 has order $\left|T^{t;j}_{t+j+1}\right| + 1 \leq m$ and $G - V\left(T^{t;j+1}_{t+j+2}\right) = H - Q$ is k-connected, a contradiction.

Suppose that $y \in \{s_1, s_2, \ldots, s_t\}$. Let $\bar{P} = P[p_j, p_1)$ and then there is a spider $T^{t;j+1}_{t+j+2} := \left(T^{t;j}_{t+j+1} - \bar{P}\right) \cup yp \cup Q$ with order $t + j + 2 \leq m$ according to the same way as in the above case. □

Claim 26. *There is a larger spider $T^{t+1;j}_{t+j+2}$ such that $G - T^{t+1;j}_{t+j+2}$ remains k-connected.*

Proof. Since $d_G(s_0) = |G| - 1$, we have $N(s_0) \cap E \neq \emptyset$. Furthermore, from Lemma 21, there exists a vertex $q \subseteq E$ such that $H\langle S\rangle[E \cup S] - q$ is k-connected. Hence, $G - V\left(T^{t+1;j}_{t+j+2}\right) = H - q$ is k-connected, a contradiction. □

By Claim 25 and Claim 26, we completed the proof. ■

Theorem 27. *Every k-connected graph G with $\delta(G) \geq \lfloor \frac{3k}{2} \rfloor + m - 1$ and $\Delta(G) = |G| - 1$ for positive integers k, m, t, contains a spider $T_m^{t;m-t-1}$ such that*

$$\kappa\big(G - V(T_m^{t;m-t-1})\big) \geq k.$$

Proof. Clearly, for the complete graph with order at least $\lfloor \frac{3k}{2} \rfloor + m$ the theorem holds. If $\kappa(G) = k$, clearly we have $|E| \geq 2$ based on the analysis of Lemma 22. Since $G\langle S \rangle - \bar{E}$ is $(k+1)$-connected by Lemma 5 and $\delta_G(E) \geq \lfloor \frac{3k}{2} \rfloor + m - 1$, it follows that $(G\langle S \rangle - \bar{E}, \langle S \rangle) \in \mathcal{K}_k^+(m)$. By Lemma 22, there exists a tree $T_{m-3}^3 \subseteq E$ rooted in p_0 such that $G\langle S \rangle - \bar{E} - T_{m-3}^3$ remains k-connected. From Lemma 20(b), we have $\kappa(G - V(T_{m-3}^3)) \geq k$.

Assume that $\kappa(G) \geq k + 1$. Using Theorem 2, there exists a path $P \subseteq G$ of length $m - 1$ starting from s_0, where the degree of s_0 is $|G| - 1$ in G, such that $\kappa(G - V(P)) \geq k$ holds. Let $P = \{s_0 p_1\} \cup P[p_1, p]$. Since $|N_G(u) \cap (G - P)| \geq \lfloor \frac{3k}{2} \rfloor + m - 1 - (|P| - 1) \geq k$ for any vertex $u \in V(P[p_1, p])$, then $\kappa(G - s_0 p_1) \geq k$, where $s_0 p_1 \in E(G)$ is a subpath of P and also a spider $T_2^{1,0}$. We again find a maximal spider $T_{t+j+1}^{t;j}$ with root s_0 and legs $s_1, s_2, \ldots, s_t$ satisfying the conditions: (i) $2 \leq |T_{t+j+1}^{t;j}| = t + j + 1 < m$; (ii) $\kappa(G - T_{t+j+1}^{t;j}) \geq k$. Then in the following proof we prove exactly in the same way and symbols as in the proof of Lemma 22. Naturally, we can show

$$\kappa(H) = k \text{ and } |E| \geq 2.$$

Since $H := G - T_{t+j+1}^{t;j}$ is not a complete graph, it follows that $|V(H)| \geq k+2$. An end E is contained in H with $E \cap C = \emptyset$. Set $S := N_H(E)$. Then $|S| = k$. Furthermore, let $\bar{E} = H - S - E$. Then the graph $H\langle S \rangle - \bar{E}$ is $(k+1)$-connected from Lemma 5. From above assumption, we know $\kappa(G) > k = \kappa(H)$, thus $N_G\left(T_{t+j+1}^{t;j}\right) \cap E \neq \emptyset$. Let y be one of farthest vertices to s_0 on $T_{t+j+1}^{t;j}$ with $N_G(y) \cap E \neq \emptyset$. Set $q \in N_G(y) \cap E$.

Claim 28. *There is a lager spider $T_{t+j+2}^{t;j+1}$ such that $G - T_{t+j+2}^{t;j+1}$ remains k-connected.*

Proof. Suppose that $y \in \{p_1, p_2, \ldots, p_j, s_0\}$. Let $\bar{P} = P[p_j, y)$. Consider the graph $G - \left(T_{t+j+1}^{t;j} - \bar{P}\right) := H \cup \bar{P}$. Since $|N_G(x) \cap H| \geq \lfloor \frac{3k}{2} \rfloor + m - 1 - (t + j) \geq \lfloor \frac{3k}{2} \rfloor + 1 \geq k$ for any $x \in V(\bar{P})$, it follows that

$$\kappa(G - (T_{t+j+1}^{t;j} - \bar{P})) = \kappa(H \cup \bar{P})) \geq k.$$

As y is the farthest vertex to s_0 on $T_{t+j+1}^{t;j}$, we have $N_G(\bar{P}) \cap E = \emptyset$. Naturally, S is also a minimum vertex cut of $H \cup \bar{P}$, and E is an end of $H \cup \bar{P}$. From Lemma

5, $(H \cup \bar{P})\langle S\rangle[E \cup S] = H\langle S\rangle - \bar{E}$ is $(k+1)$-connected. Furthermore, it follows that $C \subseteq H \cup \bar{P} - E$ and

$$\delta_{(H\cup\bar{P})}(H \cup \bar{P} - C) \geq \left\lfloor \frac{3k}{2} \right\rfloor + m - 1 - (t + j + 1 - |\bar{P}|) \geq \left\lfloor \frac{3k}{2} \right\rfloor + |\bar{P}|.$$

Hence, $(H \cup \bar{P}, C) \in \mathcal{F}_k(|\bar{P}| + 1)$ and $(H \cup \bar{P}\langle S\rangle[E \cup S], \langle S\rangle) \in \mathcal{F}_k^+(|\bar{P}| + 1)$. From Lemma 21, there exists a path $Q \subseteq E$ of order $|\bar{P}| + 1$ starting from q such that $(H \cup \bar{P})\langle S\rangle[E \cup S] - Q$ is k-connected. Now for $(H \cup \bar{P}, C) \in \mathcal{F}_k(|\bar{P}| + 1)$ by Lemma 20(c), then $\kappa((H \cup \bar{P})\langle S\rangle[E \cup S] - Q) \geq k$. The spider $T^{t;j+1}_{t+j+2} := \left(T^{t;j}_{t+j+1} \setminus \bar{P}\right) \cup yq \cup Q$ rooted at s_0 has order $\left|T^{t;j}_{t+j+1}\right| + 1 \leq m$ and $G - V\left(T^{t;j+1}_{t+j+2}\right) = H - Q$ is k-connected, a contradiction. We can again prove it holds for $y \in \{s_1, s_2, \ldots, s_t\}$ by the above way, where $\bar{P} = P[p_j, y)$. This completes the proof of Claim 28. □

Claim 29. *There is a lager spider $T^{t+1;j}_{t+j+2}$ such that $G - T^{t+1;j}_{t+j+2}$ remains k-connected.*

Proof. Since $d_G(s_0) = |G| - 1$, it follows that $N(s_0) \cap E \neq \emptyset$. Let $N(s_0) \cap E = \{q\}$. Furthermore, from Lemma 21, there exists a vertex $q \subseteq E$ such that $H\langle S\rangle[E \cup S] - q$ is k-connected. Hence, $G - V\left(T^{t+1;j}_{t+j+2}\right) = H - q$ is k-connected, a contradiction. □

By Claims 28 and 29, we completed the proof. ■

Acknowledgement

The author is grateful for the financial support provided by Program of China Scholarship Council (Grant No. 202308120095) and the Institute for Basic Science (IBS-R029-C4). We would like to thank the anonymous reviewers for the valuable suggestions and comments for the paper.

References

[1] J.A. Bondy and U.S.R. Murty, Graph Theory, Grad. Texts in Math. **244** (Springer-Verlag, London, 2008).
https://doi.org/10.1007/978-1-84628-970-5

[2] G. Chartrand, A. Kaigars and D.R. Lick, *Critically n-connected graphs*, Proc. Amer. Math. Soc. **32** (1972) 63–68.

[3] A.A. Diwan and N.P. Tholiya, *Non-separating trees in connected graphs*, Discrete Math. **309** (2009) 5235–5237.
https://doi.org/10.1016/j.disc.2009.03.037

[4] S. Fujita and K.-i. Kawarabayashi, *Connectivity keeping edges in graphs with large minimum degree*, J. Combin. Theory Ser. B **98** (2008) 805–811.
https://doi.org/10.1016/j.jctb.2007.11.001

[5] T. Hasunuma, *Connectivity keeping trees in* 2*-connected graphs with girth conditions*, Algorithmica **83** (2021) 2697–2718.
https://doi.org/10.1007/s00453-021-00833-8

[6] T. Hasunuma and K. Ono, *Connectivity keeping trees in* 2*-connected graphs*, J. Graph Theory **94** (2020) 20–29.
https://doi.org/10.1002/jgt.22504

[7] Y. Hong and Q. Liu, *Mader's conjecture for graphs with small connectivity*, J. Graph Theory **101** (2022) 379–388.
https://doi.org/10.1002/jgt.22831

[8] Y. Hong, Q. Liu, C. Lu and Q. Ye, *Connectivity keeping caterpillars and spiders in* 2*-connected graphs*, Discrete Math. **344(3)** (2021) 112236.
https://doi.org/10.1016/j.disc.2020.112236

[9] C. Lv and P. Zhang, *Connectivity keeping trees in* 2*-connected graphs*, Discrete Math. **343(2)** (2020) 111677.
https://doi.org/10.1016/j.disc.2019.111677

[10] L. Luo, Y. Tian and L. Wu, *Connectivity keeping paths in* k*-connected bipartite graphs*, Discrete Math. **345(4)** (2022) 112788.
https://doi.org/10.1016/j.disc.2021.112788

[11] W. Mader, *Connectivity keeping paths in* k*-connected graphs*, J. Graph Theory **65** (2010) 61–69.
https://doi.org/10.1002/jgt.20465

[12] W. Mader, *Connectivity keeping trees in* k*-connected graphs*, J. Graph Theory **69** (2012) 324–329.
https://doi.org/10.1002/jgt.20585

[13] Y. Tian, H.-J. Lai, L. Xu and J. Meng, *Nonseparating trees in* 2*-connected graphs and oriented trees in strongly connected digraphs*, Discrete Math. **342** (2019) 344–351.
https://doi.org/10.1016/j.disc.2018.10.001

[14] Y. Tian, J. Meng, H.-J. Lai and L. Xu, *Connectivity keeping stars or double-stars in* 2*-connected graphs*, Discrete Math. **341** (2018) 1120–1124.
https://doi.org/10.1016/j.disc.2017.10.017

[15] Q. Yang and Y. Tian, *Connectivity keeping caterpillars and spiders in bipartite graphs with connectivity at most three*, Discrete Math. **346(1)** (2023) 113207.
https://doi.org/10.1016/j.disc.2022.113207

[16] Q. Yang and Y. Tian, *Proof of a conjecture on connectivity keeping odd paths in* k*-connected bipartite graphs* (2024).
arXiv:2209.08373v2

[17] P. Zhang, Research on the Existence of Subgraphs Related to Connectivity, Ph.D. Dissertation (East China Normal University, 2021).

Received 16 August 2024
Revised 24 December 2024
Accepted 6 January 2025
Available online 1 February 2025