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VERTEX PARTITIONS OF (C_4, C_5, C_{10})-FREE PLANAR GRAPHS

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Abstract

A graph G is improperly $(d_1, d_2, \ldots, d_k)$-colorable or just $(d_1, d_2, \ldots, d_k)$-colorable if its vertices can be partitioned into k subsets $V_1, V_2, \ldots, V_k$ such that $\Delta(G[V_i]) \leq d_i$ for $1 \leq i \leq k$. It is known that every (C_4, C_i, C_j)-free planar graph is $(1, 0, 0)$-colorable whenever $5 \leq i < j \leq 9$. In this paper, we prove that every (C_4, C_5, C_{10})-free planar graph is $(1, 0, 0)$-colorable.

Keywords: planar graph, improper coloring, Steinberg conjecture, cycle.

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1. Introduction

All graphs considered here are simple and undirected. Let $G = (V, E)$ be a finite graph with the vertex set V and the edge set E. Let $i_1, i_2, \ldots, i_k$ be k positive integers such that $3 \leq i_1 < i_2 < \cdots < i_k$. The $(C_{i_1}, C_{i_2}, \ldots, C_{i_k})$-*free graph* is a graph without cycles of lengths $i_1, i_2, \ldots, i_k$. A *proper k-coloring* of a graph G is a mapping $\varphi : V \rightarrow \{1, 2, \ldots, k\}$ such that $\varphi(u) \neq \varphi(v)$ whenever $uv \in E$. The graph G which has a proper k-coloring is called *k-colorable*. The Four Color Theorem (saying that every planar graph is 4-colorable) was proved by Appel, Haken and Koch [2][3]. In 1976, Steinberg conjectured that every (C_4, C_5)-free planar graph is 3-colorable, which was disproved by Cohen-Addad *et al.* in [1].

Let $d_1, d_2, \ldots, d_k$ be k nonnegative integers. A graph is *improperly* $(d_1, d_2, \ldots, d_k)$-*colorable* or just $(d_1, d_2, \ldots, d_k)$-colorable, if the vertex set V can be partitioned into k subsets $V_1, V_2, \ldots, V_k$ such that $\Delta(G[V_i]) \leq d_i$ for $1 \leq i \leq k$. Under this terminology, the Four Color Theorem can be described as: every planar graph is $(0, 0, 0, 0)$-colorable, and the Steinberg Conjecture says that every

(C_4, C_5)-free planar graph is $(0, 0, 0)$-colorable. Chen *et al.* [6] showed that every (C_4, C_5)-free planar graph is $(2, 0, 0)$-colorable. It was shown that every (C_4, C_i, C_j)-free planar graph is $(1, 0, 0)$-colorable for all $5 \leq i < j \leq 9$, see [5] and [7]–[12].

Let $\mathcal{G}$ be the family of (C_4, C_5, C_{10})-free planar graphs. In this paper, we consider the improper coloring of every planar graph without cycles of lengths 4, 5 and 10.

Theorem 1. *Every planar graph in* $\mathcal{G}$ *is* $(1, 0, 0)$*-colorable.*

The rest of this section is devoted to introduce some definitions. The notation and terminology used but undefined in this paper can be found in [4]. Call a graph G *planar* if it can be embedded into the plane so that its edges meet only at their ends. Any such particular embedding of a planar graph is called a *plane* graph. For a plane graph G, we use F to denote its face set. For a vertex $v \in V$, a *neighbor* of v is a vertex adjacent to v, and the set of neighbors of v is denoted by $N(v)$. The *degree* of v, denoted by $d_G(v)$ or simply $d(v)$, is the number of neighbors of v. The minimal degree of the vertices of G is denoted by $\delta(G)$, and the maximum degree of vertices of G is denoted by $\Delta(G)$. Call the vertex v a *k-vertex*, or a *k^+-vertex*, or a *k^--vertex* if $d(v) = k$, or $d(v) \geq k$, or $d(v) \leq k$, respectively. Let $f_k(v)$ be the number of k-faces incident with v. Similarly, we can define $f_{k^+}(v)$ and $f_{k^-}(v)$.

For a face f, the number of edges on the boundary of f (each cut edge is counted twice) is called the *degree* of f, denoted by $d(f)$. Call the face f a k-$face$, or a k^+-$face$, or a k^--$face$ if $d(f) = k$, or $d(f) \geq k$, or $d(f) \leq k$, respectively. We write $f = [v_1 v_2 \cdots v_k]$ if $v_1, v_2, \ldots, v_k$ are consecutive vertices on the boundary of f in a cyclic order, and say that f is a $(d(v_1), d(v_2), \ldots, d(v_k))$-face. A *pendant* 3-face of a vertex v is a 3-face which is not incident with v but is incident with a 3-vertex adjacent to v. Call a vertex or an edge *triangular* if it is incident with a 3-face. If a 3-vertex v is triangular, say v is incident with a 3-face f, then its neighbor not incident with f is called its *outer neighbor*. If the outer neighbor of a 3-vertex v is a k-vertex, then we call it an *outer k-neighbor* of v. Let k be a positive integer, call a vertex v *k-triangular* if it is incident with k non-adjacent 3-faces.

Let C be a cycle of a plane graph G. We use $int(C)$ and $ext(C)$ to denote the sets of vertices located inside and outside C, respectively. The cycle C is *seperating* if $int(C) \neq \emptyset$ and $ext(C) \neq \emptyset$. Here we have some definitions.

2. Reducible Configurations

As usual, to properly color a vertex v means to assign v a color which has not been appeared to any neighbor of v. For a $(1, 0, 0)$-coloring, to color a vertex v

means to color v with 1 such that v has at most one neighbor with color 1, or to properly color v with i, $i \in \{2,3\}$.

Let $G \in \mathcal{G}$ be a planar graph which is not $(1,0,0)$-colorable but every subgraph of G with fewer vertices is. That is, G is the counterexample to Theorem 1 with fewest number of vertices. Clearly, G is connected. Embed G into the plane.

Lemma 2 [5]**.** $\delta(G) \geq 3$.

Lemma 3 [5]**.** *If v is a 3-vertex in G, then v has a 4^+-neighbor.*

Lemma 4 [5]**.** *If v is a 3-vertex incident with a $(3,3,4^-)$-face, then the outer neighbor of v is a 4^+-vertex.*

3. STRUCTURES OF 2-CONNECTED PLANAR GRAPHS IN $\mathcal{G}$

In this section, $G = (V,E,F)$ is a 2-connected plane graph in $\mathcal{G}$. As G is 2-connected, every face is simple, i.e., its boundary is a cycle. Hence G has no k-face for $k \in \{4,5,10\}$. Suppose f and f' are two faces in G. f and f' are *adjacent* if they share an edge. Two adjacent faces are *normal adjacent* if they have only two common vertices. If f and f' are two adjacent faces with common edge xy, then f' can be represented by f_{xy}. Moreover, if $f = [v_1v_2v_3]$ is a 3-face, then $f_{v_1v_2}$ can be abbreviated to f_{12}.

Lemma 5. *Let f_1 be a 9^--face in G and f_2 be a 3-face in G. If f_1 and f_2 are adjacent, then they are normal adjacent.*

Proof. Suppose $f = [v_1v_2\cdots v_k]$ is the 9^--face adjacent to the 3-face $T = [v_1v_2v]$. Obviously, $k \neq 3$; otherwise, there is a 4-cycle, a contradiction. Hence, $k \in \{6,7,8,9\}$. To prove f and T are normal adjacent, by symmetry, we only need to prove that $v \neq v_i$, $i \in \{3,4,5,6\}$. If $v = v_3$, then $d(v_2) = 2$, contradicting to Lemma 2. If $v = v_4$, then $v_1v_2v_3v_4v_1$ is a 4-cycle, a contradiction. If $v = v_5$, then $v_2v_3v_4v_5v_2$ is a 4-cycle, a contradiction. If $v = v_6$, then $v_2v_3v_4v_5v_6v_2$ is a 5-cycle, a contradiction. ■

Since G is 2-connected, by Lemma 5 and because G is (C_4, C_5, C_{10})-free, it is easy to show that the following lemma holds.

Lemma 6. (1) *No two 3-faces in G are adjacent.*

(2) *A 6-face in G is adjacent to at most three 3-faces.*

(3) *A 7-face in G is adjacent to at most two 3-faces.*

(4) *A 8-face in G is adjacent to at most one 3-face.*

(5) *No* 9*-face in* G *is adjacent to a* 3*-face.*

(6) *No two* 6*-faces in* G *are adjacent.*

Lemma 7. *Let* v *be a* 3*-vertex, and* v_1, v_2 *and* v_3 *be the neighbors of* v *in the clockwise order. Let* f_i *be the face with* vv_i *and* vv_{i+1} *as boundary edges, where* $i \in \{1,2,3\}$ *and* $v_4 = v_1$*. If* $d(f_1) = 3$*,* $d(f_2) = 6$ *and* $d(f_3) = 7$*, then* G *must contain a subgraph* G_1 *as shown in Figure* 1*.*

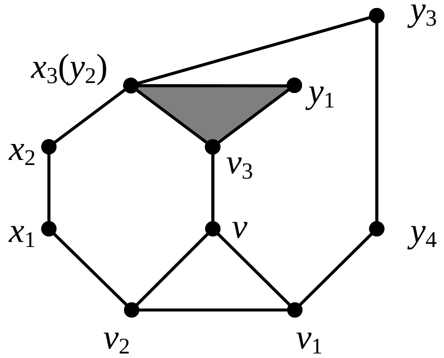

Figure 1. The graph G_1 in Lemma 6. The shadow area might not be a face.

Proof. Let $f_1 = [vv_1v_2]$, $f_2 = [vv_2x_1x_2x_3v_3]$, and $f_3 = [vv_3y_1y_2y_3y_4v_1]$. By Lemma 5, $v_2 \notin \{y_1, y_2, y_3, y_4\}$ and $v_1 \notin \{x_1, x_2, x_3\}$. If $\{x_1, x_2, x_3\} \cap \{y_1, y_2, y_3, y_4\} = \emptyset$, then G has a 10-cycle $v_1v_2x_1x_2x_3v_3y_1y_2y_3y_4v_1$, a contradiction. If $x_1 = y_1$ or $x_1 = y_2$, then G has a 4-cycle $y_1v_3vv_2y_1$ or a 5-cycle $y_2y_1v_3vv_2y_2$, a contradiction. If $x_1 = y_3$ or $x_1 = y_4$, then G has a 4-cycle $y_3y_4v_1v_2y_3$ or a 4-cycle $y_4v_1vv_2y_4$, a contradiction. If $x_2 = y_4$ or $x_2 = y_3$, then G has a 4-cycle $y_4v_1v_2x_1y_4$ or a 5-cycle $y_3y_4v_1v_2x_1y_3$, a contradiction. If $x_3 = y_1$, then $d(v_3) = 2$, contradicting to Lemma 2. If $x_2 = y_2$ or $x_2 = y_1$, then G has a 4-cycle $y_2y_1v_3x_3y_2$ or a 5-cycle $y_1v_3vv_2x_1y_1$, a contradiction. If $x_3 = y_3$ or $x_3 = y_4$, then G has a 4-cycle $v_3y_1y_2y_3v_3$ or a 5-cycle $v_3y_1y_2y_3y_4v_3$, a contradiction. Therefore, $x_3 = y_2$. Hence, G must have a subgraph G_1 as shown in Figure 1. ■

If a 3-vertex v is incident with three faces f_1, f_2 and f_3 such that $d(f_1) \le d(f_2) \le d(f_3)$, then v is called a $3^{(d(f_1),d(f_2),d(f_3))}$-vertex.

Lemma 8. *G contains only nine types of triangular* 3*-vertex.* (1) $3^{(3,6,8)}$*-vertex;* (2) $3^{(3,6,11^+)}$*-vertex;* (3) $3^{(3,7,7)}$*-vertex;* (4) $3^{(3,7,8)}$*-vertex;* (5) $3^{(3,7,11^+)}$*-vertex;* (6) $3^{(3,8,8)}$*-vertex;* (7) $3^{(3,8,11^+)}$*-vertex and* (8) $3^{(3,11^+,11^+)}$*-vertex;* (9) $3^{(3,6,7)}$*-vertex. Moreover, if* G *has* $^{(3,6,7)}$*-vertices, then* G *has a* 2*-connected subgraph* H *so that the outer boundary of* H *is a* 3*-cycle and there is no* $3^{(3,6,7)}$*-vertex in* H*.*

Proof. Suppose a 3-vertex v is incident with faces f_1, f_2 and f_3 so that $d(f_1) = 3$ and $d(f_2) \leq d(f_3)$. By Lemma 6, $d(f_i) \notin \{9, 10\}$ for each $i \in \{2, 3\}$. Since G is (C_4, C_5, C_{10})-free, $d(f_3) \geq d(f_2) \geq 6$. First suppose $d(f_2) = 6$, by Lemma 6(5) and (6), $d(f_3) = 7, 8$ or $d(f_3) \geq 11$. Hence, we get (9), (1) or (2), respectively. Suppose $d(f_2) = 7$. Then $d(f_3) = 7, 8$ or $d(f_3) \geq 11$. Hence, we get (3), (4) or (5), respectively. Suppose $d(f_2) = 8$. If $d(f_3) = 8$, we get (6); otherwise, we get (7). Finally suppose $d(f_2) \geq 11$, we get (8).

Suppose G has a $3^{(3,6,7)}$-vertex v. By Lemma 7, G contains the subgraph G_1 as shown in Figure 1. Then $C = v_3y_1x_3v_3$ is a separating 3-cycle. Assume that $v \in ext(C)$. Let $G' = G[V(C) \cup int(C)]$. The outer boundary of G' is a 3-cycle and $v \notin G'$. G is 2-connected, hence G' is 2-connected. Since G is a finite graph, we can get a 2-connected subgraph H of G by finite induction so that there is no $3^{(3,6,7)}$-vertex in H and the outer boundary of H is a 3-cycle. ■

To simplify notation, the triangular 3-vertex in the Lemma 8 is referred to simply as $3^{(i)}$-vertex. For example, $3^{(1)}$-vertex is the $3^{(3,6,8)}$-vertex, $3^{(3)}$-vertex is the $3^{(3,7,7)}$-vertex.

For convenience, we need to define some notations to indicate the structures around 6^--vertex. A 6^{3g}-vertex stands for the 3-triangular 6-vertex.

For 5-vertex, a 5^{2g}-vertex stands for the 2-triangular 5-vertex which has one pendant 3-face. A 5^{2b}-vertex stands for the 2-triangular 5-vertex which has no pendant 3-face. A 5^{1c}-vertex stands for the 5-vertex which is incident with at most one 3-face. A 5^{1c_1}-vertex stands for the 1-triangular 5-vertex with $f_7(v) = 4$ and three pendant 3-faces. A 5^{1c_2}-vertex stands for the 1-triangular 5-vertex with $f_6(v) = 1$, $f_7(v) = 2$ and three pendant 3-faces.

For 4-vertex, a 4^{2g}-vertex stands for the 2-triangular 4-vertex. A 4^{2g_1}-vertex stands for the 4^{2g}-vertex which is incident with one 6-face and one 7-face. A 4^{1g}-vertex stands for the 1-triangular 4-vertex which has two pendant 3-faces. A 4^{1g_1}-vertex stands for the 4^{1g}-vertex which is incident with three 7-faces. A 4^{1b}-vertex stands for the 1-triangular 4-vertex which has one pendant 3-face. A 4^{1c}-vertex stands for the 1-triangular 4-vertex which has no pendant 3-face. A 4^w-vertex stands for the 4-vertex which is incident with no 3-face and has four pendant 3-faces.

For 3-vertex incident with no 3-face, a 3^w-vertex stands for the 3-vertex whose neighborhoods are all the $3^{(3)}$-vertices. A 3^m-vertex stands for the 3-vertex whose neighborhoods has two $3^{(3)}$-vertices. A 3^s-vertex stands for the 3-vertex whose neighborhoods has two $3^{(1)}$-vertices.

Lemma 9. *If v is a 4^{2g}-vertex, then v cannot be incident with two 6-faces.*

Proof. Suppose v is a 4^{2g}-vertex. Let v_1, v_2, v_3, v_4 be the neighbors of v in clockwise order. Let f_i be the face with vv_i and vv_{i+1} as the boundary edges of f_i, where $i \in \{1,2,3,4\}$ and $v_5 = v_1$. Then $d(f_1) = d(f_3) = 3$, say $f_1 = [vv_1v_2]$ and $f_3 = [vv_3v_4]$. Assume to the contrary that $d(f_2) = d(f_4) = 6$, say $f_2 = [vv_2x_1x_2x_3v_3]$ and $f_4 = [vv_4y_1y_2y_3v_1]$. By Lemma 5, $\{v_1, v_2, v_3, v_4\} \cap \{x_1, x_2, x_3, y_1, y_2, y_3\} = \emptyset$. If $\{x_1, x_2, x_3\} \cap \{y_1, y_2, y_3\} = \emptyset$, then G has a 10-cycle $v_1v_2x_1x_2x_3v_3v_4y_1y_2y_3v_1$, a contradiction. So by symmetry, assume that $x_1 \in \{y_1, y_2, y_3\}$ or $x_2 = y_2$. If $x_1 = y_3$, then G has a 4-cycle $y_3v_1vv_2y_3$, a contradiction. If $x_1 = y_2$ or $x_1 = y_1$, then G has a 4-cycle $y_2y_3v_1v_2y_2$ or a 5-cycle $y_1y_2y_3v_1v_2y_1$, respectively, a contradiction. If $x_2 = y_2$, then G has a 5-cycle $y_2y_3v_1v_2x_1y_2$, a contradiction. ■

For convenience, if $f = [v_1v_2v_3]$ is a $(3^+, 3^+, 4^{2g_1})$-face, say v_3 is a 4^{2g_1}-vertex, and the other 3-face incident with v_3 is a $(3^{(3)}, 4^{2g_1}, 4^{2g_1})$-face, then we call f a *weak* 3-*face*. If $f = [v_1v_2v_3]$ is a $(3^{(2)}, 3^{(2)}, 5^+)$-face say $d(v_3) \geq 5$, such that the outer neighbor of each $3^{(2)}$-vertex is a triangular 3-vertex and $d(f_{12}) \geq 11$, then we call f a *special* 3-*face*.

Lemma 10. *Let $f = [v_1v_2v_3]$ be a weak 3-face so that $d(v_1) \leq d(v_2)$ and v_3 is the 4^{2g_1}-vertex. Then v_i is not a 4^{2g}-vertex, a 4^{1g}-vertex, or a 5^{2g}-vertex for each $i \in \{1,2\}$.*

Proof. Let $f' = [v_3xy]$ be the second 3-face incident with v_3. Then f' is a $(3^{(3)}, 4^{2g_1}, 4^{2g_1})$-face, say x is the 4^{2g_1}-vertex and y is the $3^{(3)}$-vertex. Then f_{13} is a 7-face adjacent to two 3-faces, and f_{23} is a 6-face adjacent to three 3-faces. By Lemma 6(3) and (2), we can deduce that v_i is not a 4^{2g}-vertex, a 4^{1g}-vertex, or a 5^{2g}-vertex for each $i \in \{1,2\}$. Hence, Lemma 10 holds. ■

Lemma 11. *Let $f = [v_1v_2v_3]$ be a $(3^{(3)}, 3^{(3)}, 3^{(3)})$-face, v_i' ($i \in \{1,2\}$) be the outer 4^+-neighbor of v_i. Then at most one of v_1', v_2' and v_3' is a 4^{1g}-vertex, or a 4^w-vertex or a 5^{2g}-vertex. Furthermore, if one of them, say v_1', is a 4^{1g}-vertex, or a 4^w-vertex or a 5^{2g}-vertex, then at most one of v_2' and v_3' is a 4^{1b}-vertex.*

Proof. By the definition of $3^{(3)}$-vertex, $d(f_{12}) = d(f_{13}) = d(f_{23}) = 7$. First we may assume that v_1' is a 4^{1g}-vertex. Then f_{12} and f_{13} are the 7-faces adjacent to two 3-faces. So by Lemma 6(3), v_2' or v_3' cannot be a 4^{1g}-vertex, or a 4^w-vertex or a 5^{2g}-vertex. If v_2' and v_3' are both 4^{1b}-vertex, then one of f_{12}, f_{13} and f_{23} must be a 7-face adjacent to three 3-faces, contradicting to Lemma 6(3). Hence Lemma 13 holds when v_1' is a 4^{1g}-vertex. We can prove that Lemma 11 holds when v_1' is a 4^w-vertex or a 5^{2g}-vertex with a similar discussion as above. ■

4. Discharging Procedure

To complete the proof of Theorem 1, we are going to derive a contradiction by a discharging procedure according to the structures established above.

Let $G = (V, E, F)$ be the counterexample to Theorem 1 with the fewest vertices. First we assume that G is 2-connected. Thus the boundary of every face of G is a cycle, and every vertex v of G is incident with $d(v)$ distinct faces. The initial charge function ch in the discharging procedure is defined as: $ch(v) = d(v) - 4$ for each $v \in V$, and $ch(f) = d(f) - 4$ for each $f \in F$. By Euler's formula $|V| - |E| + |F| = 2$ and Handshaking Theorem $\sum_{v \in V} d(v) = 2|E| = \sum_{f \in F} d(f)$, we can deduce that

$$\sum_{v \in V}(d(v) - 4) + \sum_{f \in F}(d(f) - 4) = -8.$$

If we can define suitable discharging rules to change the initial charge function ch to the final charge function ch' on $V \cup F$ such that $ch'(x) \geq 0$ for all $x \in V \cup F$, then $0 \leq \sum_{x \in V \cup F} ch'(x) = \sum_{x \in V \cup F} ch(x) = -8$, a contradiction completing the proof of Theorem 1 when G is 2-connected.

Now our proof will be divided into two cases depending on the existence of a $3^{(3,6,7)}$-vertex or not. Firstly, assume that G does not contain any $3^{(3,6,7)}$-vertex. We design the following discharging rules.

R1. Every 6^+-face f sends $\frac{ch(f)}{d(f)}$ to each incident vertex.

R2. Let v be a triangular 3-vertex, and f be the 3-face incident with v. Then f sends $\frac{1}{6}$ to v when v is a $3^{(1)}$-vertex, $\frac{1}{33}$ to v when v is a $3^{(2)}$-vertex, $\frac{1}{7}$ to v when v is a $3^{(3)}$-vertex, $\frac{1}{14}$ to v when v is a $3^{(4)}$-vertex.

Suppose that the charge of vertex v is $\alpha(v)$ after applying the rules R1 and R2.

R3. Suppose that v is a 3-vertex.

R3.1. If v is not incident with any 3-face, then v sends $\alpha(v)$ to each pendant 3-face evenly.

R3.2. If v is a $3^{(i)}$-vertex, $i \in \{5, 6, 7, 8\}$, then v sends $\alpha(v)$ to its incident 3-face.

R4. Suppose that v is a 4-vertex.

R4.1. Suppose that v is a 4^{2g}-vertex. If v is incident with a $(3^{(3)}, 4^{2g_1}, 4^{2g_1})$-face f, then v sends $\frac{10}{21}$ to f and sends $\frac{2}{7}$ to another incident 3-face. Otherwise, v sends $\frac{\alpha(v)}{2}$ to each incident 3-face.

R4.2. If v is a 4^{1g}-vertex, then v sends $\frac{4}{7}$ to its incident 3-face and sends $\frac{\alpha(v) - \frac{4}{7}}{2}$ to each pendant 3-face.

R4.3. If v is a 4^{1b}-vertex, then v sends $\frac{2}{3}$ to its incident 3-face and sends $\alpha(v) - \frac{2}{3}$ to pendant 3-face.

R4.4. If v is a 4^{1c}-vertex, then v sends 1 to its incident 3-face.

R4.5. If v is not incident with any 3-face, then v sends $\alpha(v)$ to each pendant 3-face evenly.

R5. Suppose that v is a 5-vertex.

R5.1. Suppose that v is a 5^{2g}-vertex.

(1) If v is incident with two $(3^{(3)}, 3^{(3)}, 5^{2g})$-faces, then v sends $\frac{20}{21}$ to each incident 3-face and sends $\frac{8}{21}$ to pendant 3-face.

(2) If v is incident with exactly one $(3^{(3)}, 3^{(3)}, 5^{2g})$-face or one $(3^{(1)}, 3^{(1)}, 5^{2g})$-face f, then v sends 1 to f, $\frac{10}{21}$ to pendant 3-face and sends the remaining charge to another incident 3-face;

(3) If v is incident with a special 3-face f, then v sends $\frac{35}{33}$ to f, $\frac{10}{21}$ to the pendant 3-face and sends the remaining charge to another incident 3-face.

(4) Otherwise, v sends $\frac{10}{21}$ to pendant 3-face and sends the remaining charge to each incident 3-face evenly.

R5.2. Suppose that v is a 5^{2b}-vertex. If v is incident with a special 3-face f, then v sends $\frac{35}{33}$ to the special 3-face and the remaining charge to another incident 3-face. Otherwise, v sends $\frac{\alpha(v)}{2}$ to each incident 3-face.

R5.3. Suppose that v is a 5^{1c}-vertex. If v is a 5^{1c_1}-vertex or a 5^{1c_2}-vertex, then v sends 1 to each incident 3-face and $\frac{4}{7}$ to each pendant 3-face. Otherwise, v sends $\frac{8}{7}$ to each incident 3-face and $\frac{4}{7}$ to each pendant 3-face.

R6. Suppose that v is a 6-vertex.

R6.1. Suppose that v is a 6^{3g}-vertex. Then v sends $\frac{23}{21}$ to each incident $(3^{(3)}, 3^{(3)}, 6)$-face, $\frac{35}{33}$ to each incident special 3-face, and sends the remaining charge to other incident 3-faces evenly.

R6.2. Suppose that v is incident with at most two 3-faces. Then v sends $\frac{8}{7}$ to each incident 3-face, and sends $\frac{4}{7}$ to each pendant 3-face.

R7. Every 7^+-vertex sends $\frac{8}{7}$ to each incident 3-face, and $\frac{4}{7}$ to each pendant 3-face.

Now we are going to check that $ch'(x) \geq 0$ for all $x \in V \cup F$.

Claim 12. *Let $v \in V$. Then $ch'(v) \geq 0$.*

Proof. By Lemma 2, $d(v) \geq 3$. Let $d(v) = k$. Set $v_1, v_2, \ldots, v_k$ be the neighbors of v in clockwise order. Let f_i be the face incident with vv_i and vv_{i+1}, where $i \in \{1, 2, \ldots, k\}$ and $v_{k+1} = v_1$. Let t be the number of 3-faces incident with v and m be the number of pendant 3-faces of v. Since G has no 4-cycles, $t \leq \left\lfloor \frac{d(v)}{2} \right\rfloor$ and $m \leq d(v) - 2t$.

Case 1. $d(v) \geq 7$. By R1 and R7, v gets at least $\frac{1}{3}$ from each incident 6^+-face, sends $\frac{8}{7}$ to each incident 3-face, and sends $\frac{4}{7}$ to each pendant 3-face. Hence, $ch'(v) \geq ch(v) + \frac{1}{3}(d(v) - t) - \frac{8}{7}t - \frac{4}{7}m \geq d(v) - 4 + \frac{1}{3}(d(v) - t) - \frac{8}{7}t - \frac{4}{7}(d(v) - 2t) = \frac{16}{21}d(v) - \frac{1}{3}t - 4 \geq \frac{16}{21}d(v) - \frac{1}{6}d(v) - 4 = \frac{25}{42}d(v) - 4 \geq \frac{1}{6}$.

Case 2. $d(v) = 6$. Then $ch(v) = 2$. Note that $\alpha(v) \geq ch(v) + \frac{1}{3}(d(v) - t) \geq 3$ by R1. Suppose that v is a 6^{3g}-vertex. If $f_{7^+}(v) = 3$, then $ch'(v) \geq 2 + \frac{3}{7} \times 3 - \frac{23}{21} \times 3 = 0$ by R1 and R6. So $f_{7^+}(v) \leq 2$. That is, v is incident with at most one $(3^{(3)}, 3^{(3)}, 6)$-face. Moreover, if v is incident with one $(3^{(3)}, 3^{(3)}, 6)$-face, then v is not incident with any special 3-face. Therefore, $ch'(v) = 0$ by R6. So assume that v is not incident with any $(3^{(3)}, 3^{(3)}, 6)$-face. By Lemma 6(2) and the definition of special 3-face, we can show that v is incident with at most one special 3-face. So $ch'(v) = 0$ by R6 and $\alpha(v) \geq 3$.

Suppose that v is not a 6^{3g}-vertex. Then $t \leq 2$. By Lemma 6(6), v is incident with at least one 7^+-face, which sends at least $\frac{3}{7}$ to v. Hence, by R1 and R6, $ch'(v) \geq 2 + \frac{3}{7} + \frac{1}{3}(6 - t - 1) - \frac{8}{7}t - \frac{4}{7}m = \frac{2}{3} - \frac{1}{3}t \geq 0$.

Case 3. $d(v) = 5$. Then $t \leq 2$ and $ch(v) = 1$. By R1, $\alpha(v) \geq 1 + \frac{1}{3}(d(v) - t) \geq 2$.

Suppose that $t = 2$. Suppose that v is a 5^{2g}-vertex, say $d(f_1) = d(f_4) = 3$. If v is incident with two $(3^{(3)}, 3^{(3)}, 5^{2g})$-faces, then $f_7(v) = 3$. Hence, by R1 and R5, $ch'(v) \geq 1 + \frac{3}{7} \times 3 - \frac{20}{21} \times 2 - \frac{8}{21} = 0$. Now assume that v is incident with at most one $(3^{(3)}, 3^{(3)}, 5^{2g})$-face. Let $\mathcal{A} = \{f \mid f\ \textit{is a}\ (3^{(3)},\ 3^{(3)},\ 5^{2g})\textit{-face, or a}\ (3^{(1)},\ 3^{(1)},\ 5^{2g})\textit{-face, or a special}\ 3\textit{-face}\}$. If f_1 is a $(3^{(3)}, 3^{(3)}, 5^{2g})$-face, then $d(f_2) = d(f_5) = 7$, which implies that f_4 is not in $\mathcal{A}$. If f_1 is a $(3^{(1)}, 3^{(1)}, 5^{2g})$-face or a special 3-face, then $d(f_5) = 6$ by Lemma 6(3). Therefore, $d(f_2) = 6$, which implies that $d(f_3) \geq 7$ by Lemma 6(6). Hence, f_4 is not in $\mathcal{A}$. Thus, at most one of f_1 and f_4 belongs to $\mathcal{A}$. Hence, $ch'(v) = 0$ by R5 and $\alpha(v) \geq 2$. If v is a 5^{2b}-vertex. Then $ch'(v) = \alpha(v) - \max\{2 \times \frac{\alpha(v)}{2}, \frac{35}{33} + (\alpha(v) - \frac{35}{33})\} = 0$ by R5.2 and v is incident with at most one special 3-face.

Suppose that $t = 1$, say $d(f_1) = 3$. Then $f_{7^+}(v) \geq 2$ by Lemma 6(6). If $m \leq 2$, then $ch'(v) \geq 1 + \frac{3}{7} \times 2 + \frac{1}{3} \times 2 - 1 - \frac{4}{7} \times 2 = \frac{8}{21}$ by R1 and R5. Now assume that $m = 3$, say v_i is a triangular 3-vertex for all $i \in \{3, 4, 5\}$. Note that $f_6(v) \leq 2$ by Lemma 6(6). If $f_6(v) = 2$, then $f_{11^+}(v) = 2$ by Lemma 6 and Lemma 8. Hence, $ch'(v) \geq 1 + \frac{1}{3} \times 2 + \frac{7}{11} \times 2 - \frac{8}{7} - \frac{4}{7} \times 3 = \frac{19}{231}$ by R1 and R5. If $f_6(v) = 1$, then $f_{11^+}(v) \geq 1$. Hence, $ch'(v) \geq 1 + \frac{1}{3} + \frac{7}{11} + \frac{3}{7} \times 2 - 1 - \frac{4}{7} \times 3 = \frac{26}{231}$ when $f_7(v) = 2$ or $ch'(v) \geq 1 + \frac{1}{3} + \frac{7}{11} + \frac{3}{7} + \frac{7}{11} - \frac{8}{7} - \frac{4}{7} \times 3 = \frac{41}{231}$ when $f_7(v) \leq 1$. If $f_6(v) = 0$ and $f_7(v) = 4$, then $ch'(v) \geq 1 + \frac{3}{7} \times 4 - 1 - \frac{4}{7} \times 3 = 0$. Otherwise, $f_6(v) = 0$ and $f_7(v) \leq 3$, then $ch'(v) \geq 1 + \frac{3}{7} \times 3 + \frac{7}{11} - \frac{8}{7} - \frac{4}{7} \times 3 = \frac{5}{77}$.

Suppose that $t = 0$. Then $f_{7^+}(v) \geq 3$ by Lemma 6(6). Hence, $ch'(v) \geq 1 + \frac{3}{7} \times 3 + \frac{1}{3} \times 2 - \frac{4}{7} \times 5 = \frac{2}{21}$ by R5.

Case 4. $d(v) = 4$. If v is not a 4^{2g}-vertex, then $\alpha(v) \geq \frac{1}{3} \times 3 \geq 1$. Hence, $ch'(v) \geq 0$ by R4. So we may assume that v is a 4^{2g}-vertex. If v is not incident with any $(3^{(3)}, 4^{2g_1}, 4^{2g_1})$-face, then $ch'(v) \geq \alpha(v) - 2 \times \frac{\alpha(v)}{2} = 0$ by R4. If v is incident with a $(3^{(3)}, 4^{2g_1}, 4^{2g_1})$-face, say f_1 is a $(3^{(3)}, 4^{2g_1}, 4^{2g_1})$-face with v_1 is a $3^{(3)}$-vertex and v_2 is a 4^{2g_1}-vertex, then $d(f_4) = 7$ and $d(f_2) = 6$. Now v_3 is not the 4^{2g_1}-vertex by Lemma 6(2), and v_3 is not the $3^{(3)}$-vertex. So f_3 is not the $(3^{(3)}, 4^{2g_1}, 4^{2g_1})$-face. Hence, $ch'(v) \geq ch(v) + \frac{1}{3} + \frac{3}{7} - \frac{10}{21} - \frac{2}{7} = 0$ by R4.

Case 5. $d(v) = 3$. Suppose v is a triangular 3-vertex, then $ch'(v) \geq 0$ by R1 and R2. Otherwise, $ch'(v) \geq ch(v) + \frac{1}{3} \times 3 = 0$ by R1. ∎

By the rules R3–R7 and since G does not contain any $3^{(3,6,7)}$-vertex, we can check that Observation 13 and Observation 14 hold. We use $\tau(x \to y)$ to denote the charge that x sends to y, where $x, y \in F(G) \cup V(G)$.

Observation 13. *Let f be a 3-face and v be a vertex incident with f.*

(1) *Suppose that $d(v) = 3$. If v is a $3^{(5)}$-vertex, then $\tau(v \to f) \geq \frac{5}{77}$. If v is a $3^{(7)}$-vertex, then $\tau(v \to f) \geq \frac{3}{22}$. If v is a $3^{(8)}$-vertex, then $\tau(v \to f) \geq \frac{3}{11}$.*

(2) *Suppose that v is a 4^{2g}-vertex and f is not the $(3^{(3)}, 4^{2g_1}, 4^{2g_1})$-face. Let f' be the second face incident with v. By Lemma 9, $f_{7^+}(v) \geq 1$. First assume that v is a 4^{2g_1}-vertex. If f' is a $(3^{(3)}, 4^{2g_1}, 4^{2g_1})$-face, then $\tau(v \to f) = \frac{2}{7}$; otherwise, $\tau(v \to f) = \frac{8}{21}$. Next assume that v is not a 4^{2g_1}-vertex, then $\tau(v \to f) \geq \frac{3}{7}$.*

(3) *Suppose that v is a 2-triangular 5-vertex. Let f' be the second 3-face incident with v.*

(3.1) *Suppose that v is a 5^{2g}-vertex and f is not the $(3^{(3)}, 3^{(3)}, 5^{2g})$-face, nor the $(3^{(1)}, 3^{(3)}, 5^{2g})$-face, nor the special 3-face. If f' is a special 3-face, then $f_6(v) = 2$ and $f_{11^+}(v) \geq 1$ by Lemma 6. Therefore, $\tau(v \to f) \geq \frac{59}{77}$. If f' is a $(3^{(3)}, 3^{(3)}, 5^{2g})$-face, then $f_7(v) \geq 2$ and $f_{7^+}(v) \geq 3$. Therefore, $\tau(v \to f) \geq \frac{17}{21}$. If f' is a $(3^{(1)}, 3^{(1)}, 5^{2g})$-face, then $f_6(v) = 2$ and $f_{11^+}(v) \geq 1$ by Lemma 6. Therefore, $\tau(v \to f) \geq \frac{191}{231}$. For the other cases, $\tau(v \to f) \geq \frac{6}{7}$.*

(3.2) *Suppose that v is a 5^{2b}-vertex. If f' is a special 3-face, then $f_6(v) = 2$ and $f_{7^+}(v) \geq 1$. Therefore, $\tau(v \to f) \geq \frac{239}{231}$. Otherwise, $\tau(v \to f) \geq \frac{22}{21}$.*

(4) *Suppose that v is a 6^{3g}-vertex. Let f' and f'' be the 3-faces incident with v other than f. If f' or f'' is a special 3-face, say f', then f and f'' are not the the special 3-faces by Lemma 6. Therefore, $\tau(v \to f) \geq \frac{32}{33}$. If exactly one of f' and f'' is a $(3^{(3)}, 3^{(3)}, 6)$-face, then $f_7(v) \geq 2$ and $\tau(v \to f) \geq \frac{22}{21}$. For the other cases, $\tau(v \to f) \geq 1$.*

Observation 14. *Let $v \in V(G)$ and f be the pendant 3-face of v.*

(1) *Let v be a 3-vertex which is not incident with any 3-face.*

(1.1) *Suppose that v has three pendant 3-faces. If v is a 3^w-vertex, then $\tau(v \to f) = \frac{2}{21}$. Otherwise, $\tau(v \to f) \geq \frac{38}{231}$.*

(1.2) *Suppose that v has two pendant 3-faces. If v is a 3^m-vertex, then $\tau(v \to f) = \frac{1}{7}$. If v is a 3^s-vertex, then $\tau(v \to f) = \frac{1}{6}$. If $f_7(v) = 2$ and $f_8(v) = 1$, then $\tau(v \to f) = \frac{5}{28}$. Otherwise, $\tau(v \to f) \geq \frac{46}{231}$.*

(1.3) *Suppose that* v *has one pendant* 3*-face. If* $f_6(v) = 1$ *and* $f_7(v) = 2$*, then* $\tau(v \to f) = \frac{4}{21}$*. Otherwise,* $\tau(v \to f) \geq \frac{11}{42}$*.*

(2) *Let* v *be a* 4*-vertex.*

(2.1) *Suppose that* v *is a* 4^{1g}*-vertex. If* $f_7(v) = 3$*, then* $\tau(v \to f) = \frac{5}{14}$*; if* $f_6(v) = 2$*, then* $f_{11^+}(v) = 1$ *and* $\tau(v \to f) \geq \frac{169}{462}$*. Otherwise,* $\tau(v \to f) \geq \frac{191}{462}$*.*

(2.2) *Suppose that* v *is a* 4^{1b}*-vertex. If* $f_6(v) = 2$*, then* $f_{8^+}(v) = 1$ *and* $\tau(v \to f) \geq \frac{1}{2}$*. Otherwise,* $\tau(v \to f) \geq \frac{11}{21}$*.*

(2.3) *Suppose that* $f_3(v) = 0$*. If* v *is a* 4^w*-vertex, then* $\tau(v \to f) \geq \frac{3}{7}$*. If* v *is not a* 4^w*-vertex, then* $\tau(v \to f) \geq \frac{4}{7}$*.*

Now we are ready to show that $ch'(f) \geq 0$ for each face $f \in F$. Note that $d(f) \notin \{4, 5, 10\}$.

Claim 15. *Let* $f \in F$*. Then* $ch'(f) \geq 0$*.*

Proof. Suppose that $d(f) \geq 6$. Then $ch'(f) \geq ch(f) - \frac{ch(f)}{d(f)} \times d(f) = 0$ by R1. So we assume that $d(f) = 3$. That is $f = [v_1v_2v_3]$ with $d(v_1) \leq d(v_2) \leq d(v_3)$. Note that $ch(f) = -1$. Let v_i' be the outer neighbor of v_i when $d(v_i) = 3$.

Case 1. $d(v_1) \geq 4$. Suppose that f is a weak 3-face, say v_1 is a 4^{2g_1}-vertex, then v_2 and v_3 cannot be a 4^{2g}-vertex, a 4^{1g}-vertex or a 5^{2g}-vertex by Lemma 10. Hence, $ch'(f) \geq -1 + \frac{2}{7} + \frac{2}{3} \times 2 = \frac{13}{21}$ by R4–R7 and Observation 13. Otherwise, $\tau(v_i \to f) \geq \frac{8}{21}$ for each $i \in \{1, 2, 3\}$ by R4–R7 and Observation 13. Hence, $ch'(f) \geq -1 + \frac{8}{21} \times 3 = \frac{1}{7}$.

Case 2. $d(v_1) = 3$ and $d(v_2) \geq 5$. Note that $\tau(f \to v_1) \leq \frac{1}{6}$ by R2, and $\tau(v_i \to f) \geq \frac{59}{77}$ for each $i \in \{2, 3\}$ by R5–R7 and Observation 13. Hence, $ch'(f) \geq -1 - \frac{1}{6} + \frac{59}{77} \times 2 = \frac{169}{462}$.

Case 3. $d(v_1) = 3$, $d(v_2) = 4$ and $d(v_3) \geq 5$. Suppose that v_1 is a $3^{(1)}$-vertex or a $3^{(4)}$-vertex. Then $\tau(f \to v_1) \leq \frac{1}{6}$ by R2 and because one of f_{12} and f_{13} is an 8-face. If $d(f_{12}) = 8$, then v_2 cannot be a 4^{2g}-vertex or a 4^{1g}-vertex by Lemma 6(4). Hence, $\tau(v_2 \to f) \geq \frac{2}{3}$ by R4 and $\tau(v_3 \to f) \geq \frac{59}{77}$ by R5–R7 and Observation 13. Hence, $ch'(f) \geq -1 - \frac{1}{6} + \frac{2}{3} + \frac{59}{77} = \frac{41}{154}$. If $d(f_{13}) = 8$, then v_3 is not a 5^{2g}-vertex by Lemma 6(4). Hence, $\tau(v_2 \to f) \geq \frac{2}{7}$ and $\tau(v_3 \to f) \geq \frac{32}{33}$ by R4–R7 and Observation 13. Hence, $ch'(f) \geq -1 - \frac{1}{6} + \frac{2}{7} + \frac{32}{33} = \frac{41}{462}$.

Suppose that v_1 is a $3^{(3)}$-vertex, then $\tau(f \to v_1) = \frac{1}{7}$ by R2. If v_3 is a 5^{2g}-vertex, then v_2 is not incident with any $(3^{(3)}, 4^{2g_1}, 4^{2g_1})$-face by Lemma 6. Hence, $\tau(v_2 \to f) \geq \frac{8}{21}$ and $\tau(v_3 \to f) \geq \frac{17}{21}$ by R4 and Observation 13. Thus, $ch'(f) \geq -1 - \frac{1}{7} + \frac{8}{21} + \frac{17}{21} = \frac{1}{21}$. If v_3 is not a 5^{2g}-vertex, then $ch'(f) \geq -1 - \frac{1}{7} + \frac{2}{7} + \frac{32}{33} = \frac{26}{231}$ by R5–R7 and Observation 13.

Otherwise, v_1 is a $3^{(i)}$-vertex, $i \in \{2, 5, 6, 7, 8\}$, then $\tau(f \to v_1) \leq \frac{1}{33}$ by R2, and $\tau(v_2 \to f) + \tau(v_3 \to f) \geq \frac{2}{7} + \frac{59}{77} = \frac{81}{77}$ by R4–R7 and Observation 13. Hence, $ch'(f) \geq -1 - \frac{1}{33} + \frac{81}{77} = \frac{5}{231}$.

Case 4. $d(v_1) = 3$ and $d(v_2) = d(v_3) = 4$.

Subcase 4.1. Both v_2 and v_3 are 4^{2g}-vertices. By symmetry, we may assume $d(f_{13}) \geq d(f_{12})$. By Lemma 6, $d(f_{13}) = 7$ or $d(f_{13}) \geq 11$, and $d(f_{23}) = 6$ or $d(f_{23}) \geq 11$.

Suppose that $d(f_{23}) = 6$. Then $d(f_{12}) \neq 6$ by Lemma 9. First suppose that $d(f_{13}) \geq 11$. By R4.1, $\tau(v_3 \to f) \geq \frac{1}{2} \times (\frac{1}{3} + \frac{7}{11}) = \frac{16}{33}$. If $d(f_{12}) = 7$, then f is not a weak 3-face by Lemma 6(2). Hence, $\tau(v_2 \to f) \geq \frac{8}{21}$ by Observation 13. Note that v_1' is not a triangular 3-vertex. Hence, $\tau(v_1' \to f) \geq \frac{2}{21}$ by Observation 14 and R5–R7. Therefore, $ch'(f) \geq -1 + \frac{8}{21} + \frac{16}{33} + \frac{2}{21} = \frac{5}{77}$. If $d(f_{12}) \geq 11$, then $\tau(v_2 \to f) \geq \frac{1}{2} \times (\frac{1}{3} + \frac{7}{11}) = \frac{16}{33}$ by R4.1 and $\tau(v_1 \to f) \geq \frac{3}{11}$ by Observation 13(1). Hence, $ch'(f) \geq -1 + \frac{16}{33} \times 2 + \frac{3}{11} = \frac{8}{33}$. Next suppose that $d(f_{13}) = 7$. Then $d(f_{12}) = 7$ by G contains no $3^{(3,6,7)}$-vertex. Hence, $\tau(f \to v_1) = \frac{1}{7}$ by R2 and $\tau(v_i \to f) \geq \frac{10}{21}$ ($i \in \{2, 3\}$) by R4.1. Note that v_1' is not a triangular 3-vertex and has at most one pendant 3-face when $d(v_1') = 3$. Hence, $\tau(v_1' \to f) \geq \frac{4}{21}$ by Observation 14 and R5–R7. Hence, $ch'(f) \geq -1 - \frac{1}{7} + \frac{10}{21} \times 2 + \frac{4}{21} = 0$.

Suppose that $d(f_{23}) \geq 11$. If $d(f_{13}) \geq 11$, then $\tau(f \to v_1) \leq \frac{1}{33}$ by R2, $\tau(v_2 \to f) \geq \frac{1}{2} \times \left(\frac{1}{3} + \frac{7}{11}\right) = \frac{16}{33}$ and $\tau(v_3 \to f) \geq \frac{1}{2} \times \left(\frac{7}{11} + \frac{7}{11}\right) = \frac{7}{11}$ by R4.1. Hence, $ch'(f) \geq -1 - \frac{1}{33} + \frac{16}{33} + \frac{7}{11} = \frac{1}{11}$. So suppose that $d(f_{13}) = 7$. Then $d(f_{12}) = 7$ by G contains no $3^{(3,6,7)}$-vertex. Hence, $\tau(f \to v_1) = \frac{1}{7}$ by R2 and $\tau(v_i \to f) \geq \frac{1}{2} \times (\frac{7}{11} + \frac{3}{7}) = \frac{41}{77}$ for each $i \in \{2, 3\}$ by R4.1. By Lemma 6(3), v_1' is not a triangular 3-vertex and at most has one pendant 3-face when $d(v_1') = 3$. Hence, $\tau(v_1' \to f) \geq \frac{4}{21}$ by Observation 14 and R5–R7. Thus, $ch'(f) \geq -1 - \frac{1}{7} + \frac{41}{77} \times 2 + \frac{4}{21} = \frac{26}{231}$.

Subcase 4.2. Exactly one of v_2 and v_3 is a 4^{2g}-vertex, say v_2. Suppose that v_3 is a 4^{1g}-vertex. Then $\tau(v_3 \to f) = \frac{4}{7}$ by R4.2. By Lemma 6, $d(f_{12}) \notin \{8, 9\}$, $d(f_{13}) \notin \{8, 9\}$ and $d(f_{23}) \notin \{7, 8, 9\}$. If $d(f_{12}) \geq 11$, then $\tau(v_2 \to f) \geq \frac{1}{2} \times (\frac{1}{3} + \frac{7}{11}) = \frac{16}{33}$ by R4 and $\tau(f \to v_1) \leq \frac{1}{33}$ by R2. Hence, $ch'(f) \geq -1 - \frac{1}{33} + \frac{4}{7} + \frac{16}{33} = \frac{2}{77}$. If $d(f_{12}) = 7$, then $d(f_{13}) = 7$ or $d(f_{13}) \geq 11$, and $\tau(v_2 \to f) \geq \frac{8}{21}$ by Observation 13. When $d(f_{13}) = 7$, v_1' is not a triangular 3-vertex and at most has one pendant 3-face when $d(v_1') = 3$ by Lemma 6(3). Then $\tau(v_1' \to f) \geq \frac{4}{21}$ by R5–R7 and Observation 14. Hence, $ch'(f) \geq -1 - \frac{1}{7} + \frac{8}{21} + \frac{4}{7} + \frac{4}{21} = 0$ by R2. When $d(f_{13}) \geq 11$, $\tau(v_1 \to f) \geq \frac{5}{77}$ by Observation 13. Hence, $ch'(f) \geq -1 + \frac{5}{77} + \frac{4}{7} + \frac{8}{21} = \frac{4}{231}$. If $d(f_{12}) = 6$, then $d(f_{23}) \geq 11$ and $d(f_{13}) \geq 11$ by Lemma 9 and G contains no $3^{(3,6,7)}$-vertex. So $\tau(v_2 \to f) \geq \frac{1}{2} \times \left(\frac{1}{3} + \frac{7}{11}\right) = \frac{16}{33}$ by R4 and $\tau(f \to v_1) \leq \frac{1}{33}$ by R2. Hence, $ch'(f) \geq -1 - \frac{1}{33} + \frac{4}{7} + \frac{16}{33} = \frac{2}{77}$.

Suppose that v_3 is a 4^{1b}-vertex. Then, $\tau(v_3 \to f) = \frac{2}{3}$ by R4. If f is a weak 3-face, then $\tau(v_2 \to f) = \frac{2}{7}$ by R4. Now $d(f_{12}) = 7$ and f_{23} is a 6-face adjacent to three 3-faces, or $d(f_{23}) = 7$ and f_{12} is a 6-face adjacent to three 3-faces. Therefore, v_1' is not a triangular 3-vertex and f_{13} is adjacent to two 3-faces by v_3 is a 4^{1b}-vertex. Then $d(f_{13}) = 7$ or $d(f_{13}) \geq 11$, which implies that v_1 is not a

$3^{(1)}$-vertex or a $3^{(4)}$-vertex. If v_1 is not a $3^{(3)}$-vertex, then $\tau(f \to v_1) \le \frac{1}{33}$ by R2. Hence, $ch'(f) \ge -1-\frac{1}{33}+\frac{2}{3}+\frac{2}{7}+\frac{2}{21} = \frac{4}{231}$ by Observation 14 and R5–R7. If v_1 is a $3^{(3)}$-vertex, then v_1' has exactly one pendant 3-face when $d(v_1') = 3$, and $\tau(v_1' \to f) \ge \frac{4}{21}$ by Observation 14 and R5–R7. Hence, $ch'(f) \ge -1-\frac{1}{7}+\frac{2}{3}+\frac{2}{7}+\frac{4}{21} = 0$ by R2. So assume that f is not a weak 3-face. Then $\tau(v_2 \to f) \ge \frac{8}{21}$ by Observation 13. If v_1 is not a $3^{(i)}$-vertex, $i \in \{1,3,4\}$, then $\tau(f \to v_1) \le \frac{1}{33}$ by R2. Hence, $ch'(f) \ge -1 - \frac{1}{33} + \frac{2}{3} + \frac{8}{21} = \frac{4}{231}$. Otherwise, $d(f_{13}) = 8$ or $d(f_{12}) = 7$, which implies that v_1' is not a triangular 3-vertex or a 3^w-vertex. Then $\tau(v_1' \to f) \ge \frac{1}{7}$ by Observation 14. Hence, $ch'(f) \ge -1 - \frac{1}{6} + \frac{2}{3} + \frac{8}{21} + \frac{1}{7} = \frac{1}{42}$ by R2.

Suppose that v_3 is a 4^{1c}-vertex, then $\tau(f \to v_1) \le \frac{1}{6}$ by R2, $\tau(v_2 \to f) \ge \frac{2}{7}$ by Observation 13 and $\tau(v_3 \to f) = 1$ by R4. Hence, $ch'(f) \ge -1 - \frac{1}{6} + \frac{2}{7} + 1 = \frac{5}{42}$.

Subcase 4.3. Neither v_2 nor v_3 is a 4^{2g}-vertex. Then $\tau(v_i \to f) \ge \frac{4}{7}$ by R4 for each $i \in \{2,3\}$. If v_1 is a $3^{(1)}$-vertex, then at most one of v_2 and v_3 is a 4^{1g}-vertex, say v_2. Hence, $\tau(v_3 \to f) \ge \frac{2}{3}$ by R4. Thus, $ch'(f) \ge -1 - \frac{1}{6} + \frac{4}{7} + \frac{2}{3} = \frac{1}{14}$ by R2. Otherwise, $ch'(f) \ge -1 - \frac{1}{7} + \frac{4}{7} \times 2 = 0$ by R2 and R4.

Case 5. $d(v_1) = d(v_2) = 3$ and $d(v_3) \ge 7$. Suppose that both v_1 and v_2 are $3^{(1)}$-vertices. Now v_i' is not a triangular 3-vertex for each $i \in \{1,2\}$. Hence, $\tau(v_i' \to f) \ge \frac{2}{21}$ for each $i \in \{1,2\}$ by Observation 14. Therefore, $ch'(f) \ge -1 - \frac{1}{6} \times 2 + \frac{8}{7} + \frac{2}{21} \times 2 = 0$ by R2 and R7.

Suppose that exactly one of v_1 and v_2 is a $3^{(1)}$-vertex, say v_1. Then v_2 is not a $3^{(3)}$-vertex. Note that v_1' is not a triangular 3-vertex or a 3^w-vertex. Hence, $\tau(v_1' \to f) \ge \frac{1}{7}$ by Observation 14. Therefore, $ch'(f) \ge -1 - \frac{1}{6} - \frac{1}{14} + \frac{8}{7} + \frac{1}{7} = \frac{1}{21}$ by R2 and R7.

Suppose that neither v_1 nor v_2 is a $3^{(1)}$-vertex. If at least one of v_1 and v_2 is a $3^{(3)}$-vertex, then $d(f_{12}) = 7$. Now at most one of v_1' and v_2' is a triangular 3-vertex or a 3^w-vertex, say v_1'. Hence, $\tau(v_2' \to f) \ge \frac{1}{7}$ by Observation 14. Therefore, $ch'(f) \ge -1 - \frac{1}{7} \times 2 + \frac{8}{7} + \frac{1}{7} = 0$ by R2 and R7. Suppose that neither v_1 nor v_2 is a $3^{(3)}$-vertex. Then $ch'(f) \ge -1 - \frac{1}{14} \times 2 + \frac{8}{7} = 0$ by R2 and R7.

Case 6. $d(v_1) = d(v_2) = 3$ and $d(v_3) = 6$. If v_3 is incident with at most two 3-faces, then $\tau(v_3 \to f) = \frac{8}{7}$ by R6. With the similar arguments as Case 5, we can show that $ch'(f) \ge 0$. So we may assume that v_3 is a 6^{3g}-vertex. Then $d(f_{13}) \notin \{8,9\}$ and $d(f_{23}) \notin \{8,9\}$. By symmetry, assume that $d(f_{23}) \ge d(f_{13})$.

Subcase 6.1. $d(f_{23}) \ge 11$. Suppose that $d(f_{13}) \ge 11$. Then $\tau(f \to v_i) \le \frac{1}{33}$ for each $i \in \{1,2\}$ by R2. By R6, $\tau(v_3 \to f) \ge \frac{1}{3} \times (\frac{7}{11} \times 2 + \frac{1}{3} + 2) = \frac{119}{99}$. Hence, $ch'(f) \ge -1 - \frac{1}{33} \times 2 + \frac{119}{99} = \frac{14}{99}$.

Suppose that $d(f_{13}) = 7$. Then $d(f_{12}) \ge 7$ by G contains no $3^{(3,6,7)}$-vertex. Hence, $\tau(f \to v_1) \le \frac{1}{7}$ by R2, $\tau(v_2 \to f) \ge \frac{5}{77}$ by Observation 13, and $\tau(v_3 \to f) \ge \frac{32}{33}$ by Observation 13. Note that v_1' is not a triangular 3-vertex or a 3^w-vertex by Lemma 6(3). Hence, $\tau(v_1' \to f) \ge \frac{1}{7}$ by Observation 14. Therefore,

$ch'(f) \geq -1 - \frac{1}{7} + \frac{32}{33} + \frac{1}{7} + \frac{5}{77} = \frac{8}{231}$.

Suppose that $d(f_{13}) = 6$. Then $d(f_{12}) = 8$ or $d(f_{12}) \geq 11$ by Lemma 6 and G contains no $3^{(3,6,7)}$-vertex. By Observation 13(4), $\tau(v_3 \to f) \geq \frac{32}{33}$. If $d(f_{12}) = 8$, then $\tau(f \to v_1) = \frac{1}{6}$ by R2 and $\tau(v_2 \to f) \geq \frac{3}{22}$ by Observation 13. Note that v_1' is not a triangular 3-vertex by Lemma 6(4). Hence, $\tau(v_1' \to f) \geq \frac{2}{21}$ by Observation 14. Therefore, $ch'(f) \geq -1 - \frac{1}{6} + \frac{32}{33} + \frac{3}{22} + \frac{2}{21} = \frac{8}{231}$. Thus, $d(f_{12}) \geq 11$. Then $\tau(f \to v_1) \leq \frac{1}{33}$ by R2 and $\tau(v_2 \to f) \geq \frac{3}{11}$ by Observation 13. Hence, $ch'(f) \geq -1 - \frac{1}{33} + \frac{32}{33} + \frac{3}{11} = \frac{7}{33}$.

Subcase 6.2. $d(f_{23}) = 7$. Suppose that $d(f_{13}) = 7$. Then $d(f_{12}) \geq 7$ by G contains no $3^{(3,6,7)}$-vertex. Note that v_i' ($i \in \{1,2\}$) is not a triangular 3-vertex by Lemma 6(3). Then $\tau(v_i' \to f) \geq \frac{2}{21}$ for each $i \in \{1,2\}$ by Observation 14. If $d(f_{12}) = 7$, then $\tau(f \to v_i) = \frac{1}{7}$ for each $i \in \{1,2\}$ by R2, and $\tau(v_3 \to f) = \frac{23}{21}$ by R6.1. Hence, $ch'(f) \geq -1 - \frac{1}{7} \times 2 + \frac{23}{21} + \frac{2}{21} \times 2 = 0$. If $d(f_{12}) \geq 8$, then $\tau(f \to v_i) \leq \frac{1}{14}$ for each $i \in \{1,2\}$ by R2, and $\tau(v_3 \to f) \geq \frac{32}{33}$ by Observation 13(4). Hence, $ch'(f) \geq -1 - \frac{1}{14} \times 2 + \frac{32}{33} + \frac{2}{21} \times 2 = \frac{4}{231}$.

Suppose that $d(f_{13}) = 6$. Then $d(f_{12}) = 8$ or $d(f_{12}) \geq 11$ by Lemma 6 and G contains no $3^{(3,6,7)}$-vertex. By Observation 13(4), $\tau(v_3 \to f) \geq \frac{32}{33}$. If $d(f_{12}) = 8$, then v_i' ($i \in \{1,2\}$) is not a triangular 3-vertex or a 3^w-vertex by Lemma 6(4). Hence, $\tau(v_i' \to f) \geq \frac{1}{7}$ for each $i \in \{1,2\}$. Therefore, $ch'(f) \geq -1 - \frac{1}{6} - \frac{1}{14} + \frac{32}{33} + \frac{1}{7} \times 2 = \frac{4}{231}$ by R2. Thus, $d(f_{12}) \geq 11$, and $\tau(v_2 \to f) \geq \frac{5}{77}$ by Observation 13. Hence, $ch'(f) \geq -1 - \frac{1}{33} + \frac{32}{33} + \frac{5}{77} = \frac{5}{231}$ by R2.

Subcase 6.3. $d(f_{23}) = 6$. Then $d(f_{13}) = 6$. Note that $d(f_{12}) = 8$ or $d(f_{12}) \geq 11$ by Lemma 6 and G contains no $3^{(3,6,7)}$-vertex. By Observation 13(4), $\tau(v_3 \to f) \geq \frac{32}{33}$.

Suppose that $d(f_{12}) = 8$. Note that v_i' ($i \in \{1,2\}$) is not a triangular 3-vertex by Lemma 6(4). Then $\tau(f \to v_i) \leq \frac{1}{6}$ and $\tau(v_i' \to f) \geq \frac{1}{6}$ for each $i \in \{1,2\}$ by R2–R7 and Observation 14. Suppose that v_3 is incident with at least one special 3-face. Note that one of v_1' and v_2', say v_1', has exactly one pendant 3-face f when $d(v_1') = d(v_2') = 3$. Hence, $\tau(v_1' \to f) \geq \frac{11}{42}$ by Observation 14 and R5–R7. Thus, $ch'(f) \geq -1 - \frac{1}{6} \times 2 + \frac{32}{33} + \frac{11}{42} + \frac{1}{6} = \frac{5}{77}$. Suppose that v_3 is not incident with any special 3-face. Then $\tau(v_3 \to f) \geq 1$ by Observation 13(4). Hence, $ch'(f) \geq -1 - \frac{1}{6} \times 2 + 1 + \frac{1}{6} \times 2 = 0$.

Suppose that $d(f_{12}) \geq 11$. Then $\tau(f \to v_i) \leq \frac{1}{33}$ for each $i \in \{1,2\}$ by R2. If f is a special 3-face, then $ch'(f) \geq -1 - \frac{1}{33} \times 2 + \frac{35}{33} = 0$ by R6 and Observation 13(4). Otherwise, at least one of v_1' and v_2' is not a triangular 3-vertex, say v_1', then $\tau(v_1' \to f) \geq \frac{2}{21}$ by Observation 14 and R5–R7. Hence, $ch'(f) \geq -1 - \frac{1}{33} \times 2 + \frac{32}{33} + \frac{2}{21} = \frac{1}{231}$.

Case 7. $d(v_1) = d(v_2) = 3$ and $d(v_3) = 5$. If $\tau(v_3 \to f) \geq \frac{8}{7}$, then $ch'(f) \geq 0$ by the similar argument as Case 5. So we may assume that $\tau(v_3 \to f) < \frac{8}{7}$. That is, v_3 is a 5^{2g}-vertex, or a 5^{2b}-vertex, or a 5^{1c_1}-vertex, or a 5^{1c_2}-vertex. By symmetry, assume that $d(f_{23}) \geq d(f_{13})$.

Subcase 7.1. $d(f_{23}) \geq 11$. Suppose that $d(f_{13}) \geq 11$. Then $\tau(f \to v_i) \leq \frac{1}{33}$ for each $i \in \{1,2\}$ by R2, and v_3 is neither a 5^{1c_1}-vertex nor a 5^{1c_2}-vertex. Note that $\alpha(v_3) \geq 1 + \frac{7}{11} \times 2 + \frac{1}{3} = \frac{86}{33}$ by R1. If v_3 is a 5^{2g}-vertex, then $\tau(v_3 \to f) \geq \frac{1}{2}(\alpha(v_3) - \frac{10}{21}) \geq \frac{82}{77}$ by R5.1. If v_3 is a 5^{2b}-vertex, then $\tau(v_3 \to f) \geq \frac{1}{2}\alpha(v_3) \geq \frac{43}{33}$ by R5.2. Hence, $ch'(f) \geq -1 - \frac{1}{33} \times 2 + \min\{\frac{43}{33}, \frac{82}{77}\} = \frac{1}{231}$.

Suppose that $d(f_{13}) = 8$. Then v_3 is not a 5^{2g}-vertex and v_1' is not a triangular 3-vertex by Lemma 6(4). So $\tau(v_3 \to f) \geq \frac{22}{21}$ by Observation 13(3) and R5.3, and $\tau(v_1' \to f) \geq \frac{1}{6}$ by Observation 14 and R5–R7. Hence, $ch'(f) \geq -1 - \frac{1}{6} - \frac{1}{33} + \frac{22}{21} + \frac{1}{6} = \frac{4}{231}$ by R2.

Suppose that $d(f_{13}) = 7$. Then $d(f_{12}) \geq 7$ since G contains no $3^{(3,6,7)}$-vertex. By R2 and Observation 13, $\tau(f \to v_1) \leq \frac{1}{7}$ and $\tau(v_2 \to f) \geq \frac{5}{77}$. Note that $\alpha(v_3) \geq 1 + \frac{1}{3} + \frac{7}{11} + \frac{3}{7} = \frac{554}{231}$. Suppose that v_3 is a 5^{2g}-vertex. By Lemma 6(3), v_1' is not a triangular 3-vertex or a 3^w-vertex. So $\tau(v_1' \to f) \geq \frac{1}{7}$ by Observation 14 and R5–R7. If v_3 incident with a $(3^{(3)}, 3^{(3)}, 5^{2g})$-face, then $f_7(v_3) = 2$ and $f_{11^+}(v_3) = 1$. Thus, $\tau(v_3 \to f) \geq 1 + \frac{3}{7} + \frac{7}{11} + \frac{3}{7} - (1 + \frac{10}{21}) = \frac{235}{231}$ by R5.1. Hence, $ch'(f) \geq -1 - \frac{1}{7} + \frac{235}{231} + \frac{1}{7} + \frac{5}{77} = \frac{19}{231}$. Otherwise, $\tau(v_3 \to f) \geq \frac{1}{2}(\alpha(v_3) - \frac{10}{21}) \geq \frac{74}{77}$ by R5. Hence, $ch'(f) \geq -1 - \frac{1}{7} + \frac{74}{77} + \frac{1}{7} + \frac{5}{77} = \frac{2}{77}$. If v_3 is a 5^{1c_1}-vertex or a 5^{1c_2}-vertex, then $\tau(v_3 \to f) = 1$ by R5.3. Note that v_1' is not a triangular 3-vertex or a 3^w-vertex. Hence, $ch'(f) \geq -1 - \frac{1}{7} + 1 + \frac{1}{7} = 0$ by Observation 14 and R5–R7. If v_3 is a 5^{2b}-vertex, then $\tau(v_3 \to f) \geq \frac{1}{2}\alpha(v_3) > \frac{8}{7}$ by R5. Hence, $ch'(f) \geq -1 - \frac{1}{7} + \frac{8}{7} = 0$.

Suppose that $d(f_{13}) = 6$. Then $d(f_{12}) = 8$ or $d(f_{12}) \geq 11$ by Lemma 6 and G contains no $3^{(3,6,7)}$-vertex. By Observation 13, $\tau(v_3 \to f) \geq \frac{59}{77}$. If $d(f_{12}) = 8$, then v_i' ($i \in \{1,2\}$) is not a triangular 3-vertex and has at most two pendant 3-faces when $d(v_i') = 3$ by Lemma 6(4). So $\tau(v_1' \to f) \geq \frac{1}{6}$ and $\tau(v_2' \to f) \geq \frac{46}{231}$ by Observation 14 and R5–R7. Hence, $ch'(f) \geq -1 - \frac{1}{6} + \frac{59}{77} + \frac{1}{6} + \frac{46}{231} + \frac{3}{22} = \frac{47}{462}$ by R3. Thus, $d(f_{12}) \geq 11$. Hence, $ch'(f) \geq -1 - \frac{1}{33} + \frac{59}{77} + \frac{3}{11} = \frac{2}{231}$ by R2 and Observation 13.

Subcase 7.2. $d(f_{23}) = 8$. Then v_3 is a 5^{2b}-vertex, and v_2' is not a triangular 3-vertex. Then $\tau(v_2' \to f) \geq \frac{1}{6}$ by Observation 14 and R5–R7, and $\tau(v_3 \to f) \geq \frac{239}{231}$ by Observation 13(3). Hence, $ch'(f) \geq -1 - \frac{1}{6} + \frac{239}{231} + \frac{1}{6} = \frac{8}{231}$ by R2.

Subcase 7.3. $d(f_{23}) = 7$.

Subcase 7.3.1. $d(f_{13}) = 7$. Then $d(f_{12}) \geq 7$ since G contains no $3^{(3,6,7)}$-vertex. Suppose that $d(f_{12}) = 7$. Then v_3 is not a 5^{1c_2}-vertex. Now $\tau(f \to v_i) = \frac{1}{7}$ for each $i \in \{1,2\}$ by R2 and $\tau(v_3 \to f) \geq \frac{20}{21}$ by R5. If $d(v_1') \geq 4$ or $d(v_2') \geq 4$, say v_1', then $\tau(v_1' \to f) \geq \frac{5}{14}$ by Observation 14. Hence, $ch'(f) \geq -1 - \frac{1}{7} \times 2 + \frac{20}{21} + \frac{5}{14} = \frac{1}{42}$. So suppose that $d(v_1') = d(v_2') = 3$. If v_3 is a 5^{2b}-vertex, then $\tau(v_3 \to f) \geq \frac{1}{2} \times (1 + \frac{3}{7} \times 2 + \frac{1}{3}) = \frac{23}{21}$ by R5.2. If both v_1' and v_2' are not the triangular 3-vertices, then at most one of them has three pendant 3-faces, say v_1'. Hence, $\tau(v_1' \to f) \geq \frac{2}{21}$ and $\tau(v_2' \to f) \geq \frac{1}{7}$ by Observation 14. Thus, $ch'(f) \geq -1-\frac{1}{7}\times 2+\frac{23}{21}+\frac{1}{7}+\frac{2}{21}=\frac{1}{21}$. If one of v_1' and v_2' is a triangular 3-vertex, say v_1', then v_2' is not a triangular 3-vertex and has exactly one pendant 3-face. So $\tau(v_2' \to f) \geq \frac{4}{21}$ by Observation 14. Hence, $ch'(f) \geq -1-\frac{1}{7}\times 2+\frac{23}{21}+\frac{4}{21}=0$. Now we may assume that v_3 is a 5^{2g}-vertex or a 5^{1c_1}-vertex. Then v_i' $(i \in \{1,2\})$ is not a triangular 3-vertex and at most one of them has two pendant 3-faces by Lemma 6(3), say v_1'. So $\tau(v_1' \to f) \geq \frac{1}{7}$ and $\tau(v_2' \to f) \geq \frac{4}{21}$ by Observation 14. Hence, $ch'(f) \geq -1-\frac{1}{7}\times 2+\frac{20}{21}+\frac{1}{7}+\frac{4}{21}=0$.

Suppose that $d(f_{12}) = 8$. Then v_i' is not a triangular 3-vertex by Lemma 6(4) and $\tau(f \to v_i) = \frac{1}{14}$ for each $i \in \{1,2\}$ by R1. If v_3 is a 5^{2g}-vertex, then $\tau(v_3 \to f) \geq \frac{17}{21}$ by Observation 13. By Lemma 6(3) and 5(4), v_i' $(i \in \{1,2\})$ has at most one pendant 3-face when $d(v_i') = 3$. Hence, $\tau(v_i' \to f) \geq \frac{4}{21}$ by Observation 14. Thus, $ch'(f) \geq -1-\frac{1}{14}\times 2+\frac{4}{21}\times 2+\frac{17}{21}=\frac{1}{21}$. Otherwise, $\tau(v_3 \to f) \geq \min\{\frac{1}{2}\times(1+\frac{3}{7}\times 2+\frac{1}{3}),1\}=1$ by R5.2 and R5.3. Hence, $ch'(f) \geq -1-\frac{1}{14}\times 2+1+\frac{1}{7}=0$ by Observation 14.

Suppose that $d(f_{12}) \geq 11$. Then $\tau(v_i \to f) \geq \frac{5}{77}$ for each $i \in \{1,2\}$ by Observation 13(1), and $\tau(v_3 \to f) \geq \frac{17}{21}$ by Observation 14 and R5. Note that at most one of v_1' and v_2' is a triangular 3-vertex, say v_1'. So $\tau(v_2' \to f) \geq \frac{2}{21}$ by Observation 14 and R5–R7. Hence, $ch'(f) \geq -1+\frac{5}{77}\times 2+\frac{17}{21}+\frac{2}{21}=\frac{8}{231}$.

Subcase 7.3.2. $d(f_{13}) = 6$. Then $d(f_{12}) = 8$ or $d(f_{12}) \geq 11$ by Lemma 6 and G contains no $3^{(3,6,7)}$-vertex. First suppose that $d(f_{12}) = 8$, then $\tau(f \to v_1)+\tau(f \to v_2)=\frac{1}{6}+\frac{1}{14}=\frac{5}{21}$ by R2. By Lemma 6(4), v_i' $(i \in \{1,2\})$ is not a triangular 3-vertex. If v_3 is a 5^{2g}-vertex, then $\tau(v_3 \to f) \geq \frac{6}{7}$ by Observation 13(3). Note that v_1' in not a 3^w-vertex or a 3^m-vertex and v_2' has exactly one pendant 3-face when $d(v_2') = 3$ by Lemma 6. Then $\tau(v_1' \to f) \geq \frac{1}{6}$ and $\tau(v_2' \to f) \geq \frac{11}{42}$ by Observation 14. Hence, $ch'(f) \geq -1-\frac{5}{21}+\frac{6}{7}+\frac{1}{6}+\frac{11}{42}=\frac{1}{21}$. Otherwise, $\tau(v_3 \to f) \geq 1$ by Observation 13(3) and R5, and $\tau(v_i' \to f) \geq \frac{1}{7}$ for each $i \in \{1,2\}$ by Observation 14. Hence, $ch'(f) \geq -1-\frac{5}{21}+1+\frac{1}{7}\times 2=\frac{1}{21}$. Next suppose that $d(f_{12}) \geq 11$. Then $\tau(f \to v_1) \leq \frac{1}{33}$ by R2 and $\tau(v_2 \to f) \geq \frac{5}{77}$ by Observation 13(1). If v_3 is a 5^{2g}-vertex, then $\tau(v_3 \to f) \geq \frac{6}{7}$ by Observation 13(3). Note that v_2' is not a triangular 3-vertex, or a 3^w-vertex, or a 3^m-vertex. Hence, $\tau(v_2' \to f) \geq \frac{1}{6}$. Thus, $ch'(f) \geq -1-\frac{1}{33}+\frac{1}{6}+\frac{5}{77}+\frac{6}{7}=\frac{9}{154}$. Otherwise, $\tau(v_3 \to f) \geq 1$ by Observation 13(3), $ch'(f) \geq -1-\frac{1}{33}+\frac{5}{77}+1=\frac{8}{231}$.

Subcase 7.4. $d(f_{23}) = 6$. Then $d(f_{13}) = 6$, and v_3 is neither a 5^{1c_1}-vertex nor a 5^{1c_2}-vertex. Note that $d(f_{12}) = 8$ or $d(f_{12}) \geq 11$ by Lemma 6 and G contains no $3^{(3,6,7)}$-vertex. First suppose that $d(f_{12}) = 8$. Then $\tau(v_3 \to f) \geq \min\{1, \frac{239}{231}\} = 1$ by R5 and Observation 13. By Lemma 6(4), v_i' $(i \in \{1,2\})$ is not a triangular 3-vertex, or a 3^w-vertex, or a 3^m-vertex. Hence, $\tau(v_i' \to f) \geq \frac{1}{6}$ for each $i \in \{1,2\}$ by Observation 14. Hence, $ch'(f) \geq -1-\frac{1}{6}\times 2+1+\frac{1}{6}\times 2=0$ by R2. Next suppose that $d(f_{12}) \geq 11$. If f is a special 3-face, then $\tau(v_3 \to f) \geq \frac{35}{33}$ by R5. Hence, $ch'(f) \geq -1-\frac{1}{33}\times 2+\frac{35}{33}=0$ by R2. Otherwise, at least one of

v_i' $(i \in \{1,2\})$ is not a triangular 3-vertex, say v_1'. Hence, $\tau(v_1' \to f) \geq \frac{1}{6}$ by Observation 14. Note that $f_{11^+}(v_3) \geq 11$ when v_3 is a 5^{2g}-vertex. So $\tau(v_3 \to f)$ $\geq$min$\{\frac{1}{2} \times (1 + \frac{1}{3} \times 2 + \frac{7}{11} - \frac{10}{21}), \frac{239}{231})\}$ =min$\{\frac{211}{231}, \frac{239}{231}\} = \frac{211}{231}$ by R5 and Observation 13. Hence, $ch'(f) \geq -1 - \frac{1}{33} \times 2 + \frac{1}{6} + \frac{211}{231} = \frac{3}{154}$ by R2.

Case 8. $d(v_1) = d(v_2) = 3$ and $d(v_3) = 4$. Then $d(v_i') \geq 4$ for each $i \in \{1,2\}$ by Lemma 4. By Observation 14 and R5–R7, $\tau(v_i' \to f) \geq \frac{5}{14}$ for each $i \in \{1,2\}$. By symmetry, assume that $d(f_{23}) \geq d(f_{13})$.

Subcase 8.1. $d(f_{23}) \geq 11$. Suppose that $d(f_{13}) \geq 8$. Then $\tau(f \to v_1) + \tau(f \to v_2) \leq \frac{1}{6} + \frac{1}{33} = \frac{13}{66}$ by R2. Note that $\tau(v_i' \to f) \geq \frac{169}{462}$ for each $i \in \{1,2\}$ by Observation 14 and R5–R7. By R4, $\tau(v_3 \to f)$ $\geq$min$\{\frac{1}{2} \times (\frac{1}{2} + \frac{7}{11}), \frac{4}{7}, \frac{2}{3}, 1\} = \frac{25}{44}$. Hence, $ch'(f) \geq -1 - \frac{13}{66} + \frac{169}{462} \times 2 + \frac{25}{44} = \frac{95}{924}$.

Suppose that $d(f_{13}) \leq 7$. Then $d(f_{12}) \geq 7$ by Lemma 6 and G contains no $3^{(3,6,7)}$-vertex. Hence, $\tau(f \to v_1) \leq \frac{1}{6}$ by R2 and $\tau(v_2 \to f) \geq \frac{5}{77}$ by Observation 13. By Observation 13 and R4, $\tau(v_3 \to f) \geq \frac{3}{7}$. Hence, $ch'(f) \geq -1 - \frac{1}{6} + \frac{5}{77} + \frac{5}{14} \times 2 + \frac{3}{7} = \frac{19}{462}$.

Subcase 8.2. $d(f_{23}) = 8$. By Lemma 6(4), v_3 is not a 4^{2g}-vertex or a 4^{1g}-vertex. Then $\tau(v_3 \to f) \geq \frac{2}{3}$ by R4. Hence, $ch'(f) \geq -1 - \frac{1}{6} \times 2 + \frac{2}{3} + \frac{5}{14} \times 2 = \frac{1}{21}$ by R2.

Subcase 8.3. $d(f_{23}) = 7$. Suppose that $d(f_{13}) = 7$. Then $d(f_{12}) \geq 7$ by G contains no $3^{(3,6,7)}$-vertex. If v_3 is a 4^{2g}-vertex, then $\tau(v_3 \to f) \geq \frac{3}{7}$ by Observation 13(2). By Lemma 6(3), v_i' $(i \in \{1,2\})$ is not a 4^{1g}-vertex or a 2-triangular 5-vertex. Hence, $\tau(v_i' \to f) \geq \frac{1}{2}$ for each $i \in \{1,2\}$ by Observation 14 and R5–R7. Therefore, $ch'(f) \geq -1 - \frac{1}{7} \times 2 + \frac{3}{7} + \frac{3}{7} \times 2 = 0$ by R2. Otherwise, $\tau(v_3 \to f) \geq \frac{4}{7}$ by R4. Hence, $ch'(f) \geq -1 - \frac{1}{7} \times 2 + \frac{4}{7} + \frac{5}{14} \times 2 = 0$ by R2.

Suppose that $d(f_{13}) = 6$. Then $d(f_{12}) = 8$ or $d(f_{12}) \geq 11$ by Lemma 6 and G contains no $3^{(3,6,7)}$-vertex. If $d(f_{12}) = 8$, then $\tau(f \to v_1) + \tau(f \to v_2) = \frac{1}{6} + \frac{1}{14} = \frac{5}{21}$ by R2. By Lemma 6(4), v_i' $(i \in \{1,2\})$ is not a 4^{1g}-vertex, a 4^w-vertex or a 2-triangular 5-vertex. Then $\tau(v_i' \to f) \geq \frac{1}{2}$ for each $i \in \{1,2\}$ by Observation 14 and R5–R7. By Observation 13(2) and R4, $\tau(v_3 \to f) \geq \frac{2}{7}$. Hence, $ch'(f) \geq -1 - \frac{5}{21} + \frac{2}{7} + \frac{1}{2} \times 2 = \frac{1}{21}$. If $d(f_{12}) \geq 11$, then $\tau(f \to v_1) \leq \frac{1}{33}$ by R2 and $\tau(v_2 \to f) \geq \frac{5}{77}$ by Observation 13(1). Note that $\tau(v_3 \to f) \geq \frac{2}{7}$ by Observation 13(1) and R4. Hence, $ch'(f) \geq -1 - \frac{1}{33} + \frac{5}{77} + \frac{2}{7} + \frac{5}{14} \times 2 = \frac{8}{231}$.

Subcase 8.4. $d(f_{23}) = 6$. Note that $d(f_{13}) = 6$ and $d(f_{12}) = 8$ or $d(f_{12}) \geq 11$ by Lemma 6 and G contains no $3^{(3,6,7)}$-vertex. By Lemma 9, v_3 is not a 4^{2g}-vertex, then $\tau(v_3 \to f) \geq \frac{4}{7}$ by R4. First suppose that $d(f_{12}) = 8$, then v_i' $(i \in \{1,2\})$ is not a 4^{1g}-vertex by Lemma 6(4). Hence $\tau(v_i' \to f) \geq \frac{8}{21}$ for each $i \in \{1,2\}$ by Observation 14(2) and R5–R7. Hence, $ch'(f) \geq -1 - \frac{1}{6} \times 2 + \frac{4}{7} + \frac{8}{21} \times 2 = 0$ by R2. Next suppose that $d(f_{12}) \geq 11$. Then $ch'(f) \geq -1 - \frac{1}{33} \times 2 + \frac{4}{7} + \frac{5}{14} \times 2 = \frac{52}{231}$ by R2.

Case 9. $d(v_1) = d(v_2) = d(v_3) = 3$. Then $d(v_i') \geq 4$ for each $i\{1,2,3\}$ by Lemma 4. So $\tau(v_i' \to f) \geq \frac{5}{14}$ for each $i \in \{1,2,3\}$ by Observation 14 and R5–R7. By symmetry, we may assume $d(f_{23}) \geq d(f_{13}) \geq d(f_{12})$ and $d(f_{13}) \geq 7$ by Lemma 6.

Suppose that $d(f_{13}) \geq 11$. Then $ch'(f) \geq -1 - \frac{1}{33} \times 2 + \frac{3}{11} + \frac{5}{14} \times 3 = \frac{131}{462}$ by R2 and Observation 13.

Suppose that $d(f_{13}) = 8$. If $d(f_{23}) \geq 11$, then $ch'(f) \geq -1 - \frac{1}{6} - \frac{1}{33} + \frac{3}{22} + \frac{5}{14} \times 3 = \frac{5}{462}$ by R2 and Observation 13. If $d(f_{23}) = 8$, then v_i' ($i \in \{1,2,3\}$) is not a 4^{1g}-vertex, or a 4^{w}-vertex, or a 2-triangular 5-vertex by Lemma 6. Hence, $\tau(v_i' \to f) \geq \frac{1}{2}$ for each $i \in \{1,2,3\}$ by Observation 14 and R5–R7. Therefore, $ch'(f) \geq -1 - \frac{1}{6} \times 2 + \frac{1}{2} \times 3 = \frac{1}{6}$ by R2.

Suppose that $d(f_{13}) = 7$. Then $d(f_{12}) = 7$ since G contains no $3^{(3,6,7)}$-vertex. If $d(f_{23}) \geq 11$, then $\tau(f \to v_1) = \frac{1}{7}$ by R2 and $\tau(v_i \to f) \geq \frac{5}{77}$ for each $i \in \{2,3\}$ by Observation 13(1). Hence, $ch'(f) \geq -1 - \frac{1}{7} + \frac{5}{77} \times 2 + \frac{5}{14} \times 3 = \frac{9}{154}$. If $d(f_{23}) = 8$, then v_i' ($i \in \{2,3\}$) is not a 4^{1g}-vertex, or a 4^{w}-vertex, or a 5^{2g}-vertex by Lemma 6(3). By Observation 14 and R5–R7, $\tau(v_i' \to f) \geq \frac{1}{2}$ for each $i \in \{2,3\}$. Hence, $ch'(f) \geq -1 - \frac{1}{14} \times 2 - \frac{1}{7} + \frac{5}{14} + \frac{1}{2} \times 2 = \frac{1}{14}$ by R2. If $d(f_{23}) = 7$, then $\tau(f \to v_i) = \frac{1}{7}$ for each $i \in \{1,2,3\}$ by R2. By Lemma 11, at most one of v_i' ($i \in \{1,2,3\}$) is a 4^{1g}-vertex, or a 4^{w}-vertex, or a 5^{2g}-vertex, say v_1'. Furthermore, if v_1' is a 4^{1g}-vertex, or a 4^{w}-vertex, or a 5^{2g}-vertex, then at most one of v_2' and v_3' is a 4^{1b}-vertex, say v_2'. Hence, $\tau(v_1' \to f) \geq \frac{5}{14}$, $\tau(v_2' \to f) \geq \frac{1}{2}$ and $\tau(v_3' \to f) \geq \frac{4}{7}$ by Observation 14 and R5–R7. Therefore, $ch'(f) \geq -1 - \frac{1}{7} \times 3 + \frac{5}{14} + \frac{1}{2} + \frac{4}{7} = 0$. ■

So far, it has been proved that if a 2-connected graph G is a minimal counterexample of Theorem 1, then G has at least a $3^{(3,6,7)}$-vertex. By Lemma 8, G has a 2-connected subgraph H such that the outer boundary of H is a 3-cycle and H has no $3^{(3,6,7)}$-vertex. Obviously, H is a planar graph. Let $f_0 = [v_1v_2v_3]$ be the non-interface of H. Now, we can proceed a discharging procedure in $H = (V(H), E(H), F(H))$. Call the edge in $E(H) \setminus E(f_0)$ as the *inner edge*. Let the initial function ch_H of $x \in V(H) \cup F(H)$ be $ch_H(x) = d_H(x) - 4$. Then $d_H(v) \geq 3$ for $v \in V(H) \setminus \{f_0\}$ and $d_H(v) \geq 2$ for $v \in V(f_0)$. We define the following discharging rules.

r1. Let $v \in V(f_0)$. Then v sends $\frac{8}{7}$ to each incident 3-face f, $f \neq f_0$, $\frac{4}{7}$ to pendant 3-face.

r2. For $x \in (V(H) \setminus V(f_0)) \cup (F(H) \setminus \{f_0\})$, we redistribute $ch(x)$ according to the rules R1–R7 defined in the former.

Let the new charge function, obtained by the rules r1 and r2 after discharging be ch_H'. By Observation 13–14, a face $f \in F(H) \setminus V(f_0)$ gets at most $\frac{8}{7}$ from its incident vertices and at most $\frac{4}{7}$ from its outer neighbors. Obviously, following the proofs of Claim 12 and Claim 15, we have $ch'(x) \geq 0$ for all $x \in (V(H) \setminus$

$V(f_0)) \cup (F(H) \setminus \{f_0\})$ after rules r1–r2. That is $\sum_{v\in V(H)\setminus V(f_0)}(d_H(v)-4) + \sum_{f\in F(H)\setminus\{f_0\}}(d_H(f)-4) \geq 0$.

Then we consider the new charge of $v_i \in V(f_0)$. Let t denote the number of incident 3-faces of v_i (except f_0) and m denote the number of pendant 3-faces of v_i. Since H does not contain adjacent 3-faces, we have $t \leq \left\lfloor \frac{d_H(v_i)-2}{2} \right\rfloor$ and $m \leq d(v_i) - 2t - 2$. Since $G \in \mathcal{G}$, $f_{6^+}(v_i) \geq d_H(v_i) - (t+1)$. Let f be the 6^+-face incident with v_i. By R1, $\tau(f \to v_i) \geq \frac{1}{3}$. If $d_H(v_i) = 2$, then $t = m = 0$ and $f_{6^+}(v_i) \geq 1$. So $ch'_H(v_i) \geq d_H(v_i) - 4 + \frac{1}{3} = -\frac{5}{3}$. If $d_H(v_i) = 3$, then $t = 0$, $m \leq 1$ and $f_{6^+}(v_i) \geq 2$. By Lemma 6(6), $f_{7^+}(v_i) \geq 1$. So $ch'_H(v_i) \geq d_H(v_i) - 4 - \frac{4}{7}m + \frac{1}{3} + \frac{3}{7} \geq -\frac{17}{21}$. If $ch'_H(v_i) \geq 4$, then by r1–r2, we can get

$$
\begin{aligned}
ch'_H(v_i) &\geq d_H(v_i) - 4 + \frac{1}{3}(d_H(v_i) - 1 - t) - \frac{8}{7}t - \frac{4}{7}m \\
&\geq \frac{4}{3}d_H(v_i) - \frac{13}{3} - \frac{1}{3}t - \frac{8}{7}t - \frac{4}{7}(d_H(v_i) - 2t - 2) \\
&= \frac{16}{21}d_H(v_i) - \frac{1}{3}t - \frac{67}{21} \geq \frac{16}{21}d_H(v_i) - \frac{1}{3}\left(\frac{d_H(v_i)-2}{2}\right) - \frac{67}{21} \qquad (1)\\
&= \frac{25}{42}d_H(v_i) - \frac{20}{7} \geq -\frac{10}{21}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
-8 &= \sum_{v\in V(H)}(d_H(v) - 4) + \sum_{f\in F(H)}(d_H(f) - 4) \\
&= \sum_{v\in V(H)} ch'(v) + \sum_{f\in F(H)} ch'(f) \\
&= \sum_{v\in V(H)\setminus V\{f_0\}} ch'(v) + \sum_{f\in F(H)\setminus\{f_0\}} ch'(f) + \sum_{v\in V(f_0)} ch'(v) + ch'(f_0) \qquad (2)\\
&\geq \sum_{v\in V(f_0)} ch'(v) + ch'(f_0) \geq 3 \times \left(-\frac{5}{3}\right) - 1 = -6.
\end{aligned}
$$

The proof of Theorem 1 is completed when the minimal counterexample G is 2-connected. Now we assume that G is not 2-connected, i.e., G has some cut vertices. Let B be an end block of G. That is, B contains an unique cut vertex t^*. By Lemma 2, B is 2-connected. Thus the boundary of every face of B is a cycle, and every vertex v of B is incident with $d_B(v)$ distinct faces. Clearly, B has no 4-face, 5-face or 10-face. If B contains a $3^{(3,6,7)}$-vertex, then B has a 2-connected subgraph H' such that the outer boundary of H' is a 3-cycle and H' contains no $3^{(3,6,7)}$-vertex by Lemma 8. In H', we can use the same discharging rules as in H to deduce a contradiction. Suppose B has no $3^{(3,6,7)}$-vertices. Let f_0 be the exterior face of B. Moreover, each structural property established for G in Section 2 and Section 3 fails for B only when t^* is involved. For any $x \in V(B) \cup F(B)$, the initial charge function ch_B in the discharging procedure is defined as $ch_B(x) = d_B(x) - 4$. We proceed the discharging rules R1–R7 in B. For $x \in V(B) \cup F(B)\backslash\{t^*\}$, we can get $ch'_B(x) \geq 0$ after R1–R7. Let t denote the number of incident 3-faces of t^* and m denote the number of pendant 3-faces of t^*. Since B does not contain adjacent 3-faces, we have $t \leq \left\lfloor \frac{d_B(t^*}{2} \right\rfloor$ and $m \leq d_B(t^*) - 2t$. Thus,

$$
\begin{aligned}
ch'_B(t^*) &\geq d_B(t^*) - 4 - \frac{8}{7}t - \frac{4}{7}m + \frac{1}{3}(d_B(t^*) - t) \\
&\geq \frac{4}{3}d_B(t^*) - 4 - \frac{1}{3}t - \frac{8}{7}t - \frac{4}{7}(d_B(t^*) - 2t) \\
&= \frac{16}{21}d_B(t^*) - \frac{1}{3}t - 4 \geq \frac{16}{21}d_B(t^*) - \frac{1}{3} \cdot \frac{d_B(t^*)}{2} - 4 \\
&= \frac{25}{42}d_B(t^*) - 4 > \ -4.
\end{aligned} \tag{3}
$$

Therefore,

$$
\begin{aligned}
-8 &= \sum_{v \in V(B)} (d_B(v) - 4) + \sum_{f \in F(B)} (d_B(f) - 4) \\
&= \sum_{v \in V(B)} ch'_B(v) + \sum_{f \in F(B)} ch'_B(f) \\
&= \sum_{v \in V(B)\backslash\{t^*\}} ch'_B(v) + \sum_{f \in F(B)} ch'_B(f) + ch'_B(t^*) \\
&\geq ch'_B(t^*) > -4.
\end{aligned} \tag{4}
$$

A contradiction completing the proof of Theorem 1 when G is not 2-connected. Hence, we show that every planar graph in $\mathcal{G}$ is $(1, 0, 0)$-colorable. That is, Theorem 1 holds.

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