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THE ROBUST CHROMATIC NUMBER OF CERTAIN GRAPH CLASSES

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Abstract

A 1-selection f of a graph G is a partial function $f : V(G) \to E(G)$ such that $f(v)$ is incident to v for every vertex v, where f is defined. The 1-removed G_f is the graph $(V(G), E(G) \setminus f[V(G)])$. The (1-)robust chromatic number $\chi_1(G)$ is the minimum of $\chi(G_f)$ over all 1-selections f of G.

We determine the robust chromatic number of complete multipartite graphs and Kneser graphs and prove tight lower and upper bounds on the robust chromatic number of chordal graphs and some of their extensively studied subclasses, with respect to their ordinary chromatic number.

Keywords: graph coloring, robust coloring.

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1. Introduction

Graph colorings and independent sets are central notions in graph theory. Various versions of graph colorings have been studied in the past decades. The focus of this paper is on the following recent variant.

Definition 1. For every nonnegative integer s, an *s-selection* on $G = (V, E)$ is an assignment $f : V \to 2^E$ such that $f(v) \subseteq E(v)$ and $|f(v)| \le s$, where $E(v)$ denotes the set of edges incident with v. The graph G_f with vertex set $V(G_f) = V(G)$ and edge set

$$E(G_f) := E(G) \setminus \bigcup_{v \in V(G)} f(v)$$

is termed an *s-removed subgraph* of G. Then

- the *s-robust chromatic number of G is* $\chi_s(G) = \min_f \chi(G_f)$,
- the *s-robust independence number of G is* $\alpha_s(G) = \max_f \alpha(G_f)$,
- the *s-robust clique number of G is* $\omega_s(G) = \min_f \omega(G_f)$,

where min and max are taken over all s-selections of G.

Observe that the ordinary chromatic, independence, and clique numbers of G are $\chi_0(G)$, $\alpha_0(G)$, and $\omega_0(G)$, respectively. The notion of 1-robust chromatic number was introduced in [5] as a tool to investigate specific Turán-type problems. The systematic study of 1-robust parameters was initiated in [1]. In this paper, we still concentrate on the $s = 1$ case and say robust instead of 1-robust. Analogously, a coloring c of $V(G)$ is robust if there exists a 1-selection f such that c is proper on G_f, and a subset $U \subset V(G)$ is robust independent if there exists a 1-selection f such that U is independent in G_f.

We now turn to the graph classes that our paper addresses. A graph is *chordal* if it does not contain any *induced* cycles longer than 3. Two famous subclasses (incomparable to each other) are the classes of interval graphs and split graphs, the latter includes the proper subclass of threshold graphs (see definition in Section 2). Here we establish tight inequalities for these graph classes.

The next theorem shows that the general lower bound $\chi_1(G) \geq \left\lceil \frac{\chi(G)}{3} \right\rceil$, which is valid for all graphs [1], is actually tight for an infinite subclass of threshold graphs, and a slightly weaker upper bound is valid for the more general class of *split graphs.* We say that G is *ω-unique* if it contains only one clique of order $\omega(G)$.

Theorem 1. *For every threshold graph G,*

$$\chi_1(G) = \begin{cases} \frac{\chi(G)}{3} + 1, & \text{if } \chi(G) \equiv 0 \pmod 3 \text{ and } G \text{ is not } \omega\text{-unique}, \\ \left\lceil \frac{\chi(G)}{3} \right\rceil, & \text{otherwise.} \end{cases}$$

For split graphs, the upper bound $\chi_1(G) \leq \left\lceil \frac{\chi(G)-1}{3} \right\rceil + 1$ is valid and tight, except for bipartite G. In particular, if G is a split graph with $\chi(G) \equiv 1 \pmod 3$ then $\chi_1(G) = \frac{\chi(G)+2}{3}$.

It is not true for the more general class of chordal graphs that $\chi(G)/3$ is an asymptotically tight upper bound. Instead, the following holds.

Theorem 2. (i) *If G is a chordal graph, then*

$$\omega_1(G) \leq \chi_1(G) \leq \left\lceil \frac{\chi(G)}{2} \right\rceil .$$

(ii) *For every $k \geq 2$ there exists an interval graph G_k such that $\omega(G_k) = \chi(G_k) = k$ and $\omega_1(G_k) = \chi_1(G_k) = \left\lceil \frac{\chi(G_k)}{2} \right\rceil = \left\lceil \frac{k}{2} \right\rceil$.*

On the other hand, a further restriction on interval graphs drops χ_1 down near $\chi/3$.

Theorem 3. *There exists a constant c such that for every unit interval graph G we have*

$$\frac{\chi(G)}{3} \leq \chi_1(G) \leq \frac{\chi(G)}{3} + c.$$

In Section 3 we solve the problem of determining χ_1 for complete multipartite graphs $K_{n_1,\dots,n_t}$. Throughout, we denote the number of vertex classes by t and write n_i for the size of class V_i for all $1 \leq i \leq t$. It will be assumed that the classes are in increasing order of their size, i.e., $n_1 \leq n_2 \leq \cdots \leq n_t$. Then $\chi_1(K_{n_1,\dots,n_t})$ can be computed on the basis of the following two results, which complement each other.

Theorem 4. *If $n_t \leq 2$, assume that $n_1 = \cdots = n_p = 1$ and $n_{p+1} = \cdots = n_{p+q} = 2$, where $p + q = t$. Then*

$$\chi_1(K_{n_1,\dots,n_t}) = \left\lceil \frac{p + \lceil 3q/2 \rceil}{3} \right\rceil .$$

Theorem 5. *If $n_t \geq 3$, then an optimal 1-selection for determining χ_1 is obtained by enlarging V_t to an independent set with a vertex of V_1. That is,*

$$\chi_1(K_{n_1,\ldots,n_t}) = 1 + \chi_1(K_{n_1-1,\ldots,n_t-1}).$$

In particular, if $n_1 = 1$ and $n_t \geq 3$, then

$$\chi_1(K_{n_1,\ldots,n_t}) = 1 + \chi_1(K_{n_2,\ldots,n_t-1}).$$

In Section 4 we analyze the behavior of α_1 and χ_1 in Kneser graphs. Let $k \geq 2$ and $n \geq 2k$ be integers. The *Kneser graph* $KG(n,k)$ has $\binom{[n]}{k} = \{S \subseteq \{1,2\ldots,n\} : |S| = k\}$ as its vertex set and two vertices are adjacent if and only if the corresponding k-element sets are disjoint. Hence the *intersecting subsystems* of $\binom{[n]}{k}$ (those $\mathcal{F} \subseteq \binom{[n]}{k}$ for which any $F, F' \in \mathcal{F}$ have non-empty intersection) are in one-to-one correspondence with the independent sets of $KG(n,k)$.

Theorem 6. *For any $k \geq 2$ there exists $n_0(k)$ such that if $n \geq n_0(k)$, then we have*

$$\alpha_1(KG(n,k)) = \binom{n-1}{k-1} + 1,$$

and $n_0(k)$ can be chosen to be $8k^2$. Furthermore, if $\mathcal{F}$ is a robust independent family in $V(KG(n,k))$ such that for every $x \in [n]$ there exist at least two sets $F_x, G_x \in \mathcal{F}$ with $x \notin F_x, G_x$, then $|\mathcal{F}| \leq 8k\binom{n-2}{k-2}$ holds.

Theorem 7. *For any fixed $k \geq 2$, we have*

$$\chi_1(KG(n,k)) = n - \Theta\left(n^{1/k}\right)$$

as $n \to \infty$.

We finish the introduction by stating two propositions from [1] that we will use in our proofs. A graph is *quasi-unicyclic* if each of its components contains at most one cycle.

Proposition 8 [1]**.** (i) *The value $\chi_1(G)$ of a graph $G = (V,E)$ is equal to the minimum number k of vertex classes in a partition $V = V_1 \cup \cdots \cup V_k$ such that each V_i induces a quasi-unicyclic subgraph in G.*

(ii) *A graph G satisfies $\chi_1(G) = 1$ if and only if it is quasi-unicyclic. In particular, every tree has $\chi_1(G) = 1$.*

Proposition 9 [1]**.** *For every graph G we have*

$$\left\lceil \frac{\chi(G)}{3} \right\rceil \leq \chi_1(G) \leq \chi(G). \tag{1}$$

Both the bounds are tight, for all possible values of $\chi(G)$.

2. Classes of Chordal Graphs

In this section, we prove Theorem 1, Theorem 2, and Theorem 3. First, we need to define the graph classes in Theorem 1.

A graph is called a *split graph* if its vertex set can be partitioned into two sets, say A and B, such that A induces a complete subgraph and B is independent.

A graph G is called a *threshold graph* if there exists a threshold h and a function $g : V(G) \to \mathbb{R}$ such that $x, y \in V(G)$ are adjacent if and only if $g(x) + g(y) > h$. It follows from the definition (in fact, it is equivalent to the definition) that the vertex set of a threshold graph G can be partitioned into two sets A and B (one of them might be empty) which satisfy the following conditions.

(i) $A = \{a_1, \dots, a_q\}$ induces a maximum clique in G and its vertices can be ordered such that $N(a_1) \supseteq N(a_2) \supseteq \cdots \supseteq N(a_q)$;

(ii) $B = \{b_1, \dots, b_s\}$ is an independent set in G and its vertices can be ordered such that $N(b_1) \supseteq N(b_2) \supseteq \cdots \supseteq N(b_s)$;

(iii) $N(a_q) \cap B = \emptyset$.

A partition (A, B) of $V(G)$ will be called a *threshold partition* of G if it satisfies (i)–(iii).

***Proof of Theorem* 1.** We consider the threshold partition (A, B) of G with the notation introduced in (i)–(iii). By definition, A is a clique, B is an independent set, and there is no clique that contains both a_q and a vertex from B. It follows that $\omega(G) = q$ and, as every threshold graph is perfect, $\chi(G) = q$. Observe that G is ω-unique if and only if $a_{q-1}b_1$ is not an edge in G. Let $k = \lceil q/3 \rceil - 1$.

Suppose first that $q \not\equiv 0 \pmod 3$, and define $V_j = \{a_{3j-2}, a_{3j-1}, a_{3j}\}$ vertex classes for every $j \in [k]\,(= \{1, 2, \dots, k\})$. The remaining vertices form the class $V_{k+1} = V(G) \setminus \bigcup_{j \in [k]} V_j$. Observe that $G[V_j]$ is a unicyclic graph (in fact a 3-cycle) for every $j \in [k]$. If $q \equiv 1 \pmod 3$, then $V_{k+1} = \{a_q\} \cup B$ which is an independent set. If $q \equiv 2 \pmod 3$, then $V_{k+1} = \{a_{q-1}, a_q\} \cup B$ which induces a cycle-free graph as every edge of $G[V_{k+1}]$ is incident to a_{q-1}. By Proposition 8(i), the partition $V_1, \dots, V_{k+1}$ defines a robust coloring for G in both cases. This implies $\chi_1(G) \le k + 1 = \lceil \chi(G)/3 \rceil$ and, by the lower bound in (1), we may conclude $\chi_1(G) = \lceil \chi(G)/3 \rceil$.

Suppose now that $q \equiv 0 \pmod 3$ and G is ω-unique. For this case, we define $V_{k+1} = \{a_{q-2}, a_{q-1}, a_q\} \cup B$ and keep the notation $V_j = \{a_{3j-2}, a_{3j-1}, a_{3j}\}$ for $j \in [k]$. Property (iii) from the definition ensures $N(a_q) \cap B = \emptyset$. The ω-uniqueness implies $a_{q-1}b_1 \notin E(G)$ that, together with property (ii) gives $N(a_{q-1}) \cap B = \emptyset$. As B is independent, every edge in $G[V_{k+1}]$ except $a_{q-1}a_q$ is incident to a_{q-2} and therefore, V_{k+1} induces a unicyclic graph. Since $G[V_j]$ is also unicyclic for every $j \in [k]$, we infer that $V_1, \dots, V_{k+1}$ gives a robust coloring for G and $\chi_1(G) \le \lceil \chi(G)/3 \rceil$. By inequality (1), we conclude $\chi_1(G) = \lceil \chi(G)/3 \rceil$.

Finally, consider the case, $q \equiv 0 \pmod 3$ and G is not ω-unique. Observe first that $V_j = \{a_{3j-2}, a_{3j-1}, a_{3j}\}$ for $j \in [k+1]$ together with $V_{k+2} = B$ defines a robust coloring for G and therefore, $\chi_1(G) \leq k+2$. Assume now for a contradiction that $W_1, \dots, W_{k+1}$ is a partition of $V(G)$ concerning the color classes of the robust coloring. As W_j induces a quasi-unicyclic graph and A is a clique, $|W_j \cap A| \leq 3$ holds for every $j \in [k+1]$. In fact, $|A| = 3k+3$ implies $|W_j \cap A| = 3$ for every color class. Since G is not ω-unique, $a_{q-1}b_1$ is an edge and b_1 is adjacent to all vertices but a_q from A. Therefore, if b_1 is contained in the color class W_i, then b_1 is adjacent to at least two vertices from $W_i \cap A$ and $G[(W_i \cap A) \cup \{b_1\}]$ is either a complete graph K_4 or a $K_4 - e$. In either case, $G[W_i]$ is not quasi-unicyclic, which is a contradiction. Thus, $\chi_1(G) > k+1$ holds and we conclude $\chi_1(G) = k+2 = \frac{\chi(G)}{3} + 1$.

Split graphs also enjoy the property $\chi(G) = \omega(G)$. Assume that $A \subset V(G)$ induces a complete subgraph of cardinality $\chi(G)$, and $B = V(G) \setminus A$ is an independent set. Then we can have a robust coloring on $G[A]$ with $\left\lceil \frac{\chi(G)}{3} \right\rceil$ colors, and make B monochromatic with a new color. This $\left\lceil \frac{\chi(G)}{3} \right\rceil + 1$ is the same as $\left\lceil \frac{\chi(G)-1}{3} \right\rceil + 1$ unless $\chi(G) = 3k+1$ for some integer k. In that case we can have a robust k-coloring on $G[A] - v$ for a $v \in A$, and since $B \cup \{v\}$ induces a star, all edges from B to v can be omitted by a 1-selection.

The formula $\left\lceil \frac{\chi(G)-1}{3} \right\rceil + 1$ is not tight if G is a bipartite split graph, because in that case G must be a double star, i.e., a particular tree, hence $\chi_1(G) = 1$. For larger $\chi = t \geq 3$, however, we can obtain a tight construction by taking $|A| = |B| = t$ and putting $K_{t,t} - tK_2$ (i.e., all the edges except a complete matching) between A and B. For $t = 3k+1$, we have seen that a robust coloring with $k+1$ colors is possible, and already the set A requires that many colors. For $t = 3k$ and $t = 3k-1$, the verified upper bound is $k+1$, and we argue that k colors do not suffice (except if $3k-1 = 2$).

If $t = 3k$, the only way for a robust k-coloring on A is to select the edges of k disjoint triangles in $G[A]$. This cannot be extended to a robust k-coloring of G, because every $v \in B$ has at least two neighbors in each selected triangle in A, while only one incident edge can be deleted from v.

If $t = 3k-1$, where $k \geq 2$, then a robust k-coloring on A has $k-1$ classes of size 3 (omitted triangles) and one class of size 2, say omitting the edge xy by selecting $f(x) = xy$. As in the previous case, the triangle classes do not admit any extension with vertices from B. The 2-element class $\{x, y\}$ can be extended with the non-neighbor of x, with the non-neighbor of y, and with one further vertex $v \in B$ by defining $f(v) = xv$ and $f(y) = yv$. But there are at least two more vertices in B, hence a further color will necessarily be used. ■

Proof of Theorem 2. (i) Let $G = (V, E)$ be a chordal graph with $\chi(G) = \omega(G) = k \geq 2$. Consider a proper vertex k-coloring of G, with color classes $V_1, \ldots, V_k$. Since G is chordal, the union of any two color classes induces a forest in G. Thus, there exists a 1-selection f such that in the 1-removal G_f all the sets $V_1 \cup V_2$, $V_3 \cup V_4, \ldots, V_{k-1} \cup V_k$ are independent. Hence the chromatic number has been decreased by $\lfloor k/2 \rfloor$, as needed.

(ii) The assertion is trivial for $k = 2$ as shown by K_2, and for $k = 3$ we can take e.g. $K_4 - e$, one edge deleted from the complete graph of order 4, which is an interval graph with $\chi = 3$ and $\omega_1 = 2$ because $|E(K_4 - e)| > 4 = |V(K_4 - e)|$.

Let $R_1^{even} = K_2$ and $R_1^{odd} = K_4 - e$. Define $G_2 = R_1^{even}$ and $G_3 = R_1^{odd}$. For $k = 2t > 2$, we define $G_k = R_t^{even}$ and for $k = 2t + 1 > 3$ we define $G_k = R_t^{odd}$ as the graph obtained by taking three vertex-disjoint copies of R_{t-1}^{even} (R_{t-1}^{odd}) together with two universal vertices. Formally this means

$$R_t^{even} = 3R_{t-1}^{even} \oplus K_2, \qquad R_t^{odd} = 3R_{t-1}^{odd} \oplus K_2$$

(where $\oplus$ is the complete join operation). As R_1^{even} and R_1^{odd} are interval graphs, and disjoint union of such graphs retains the ω, χ and the structure of an interval graph, while adding a new universal vertex increases ω and χ by one, but keeps the property of being an interval graph, we obtain by induction that R_t^{even} and R_t^{odd} are interval graphs with $\omega(R_t^{even}) = k = 2t$ and $\omega(R_t^{odd}) = k = 2t + 1$. We need to prove that $\omega_1(R_t^{even}) \geq t$ and $\omega_1(R_t^{odd}) \geq t + 1$ hold. As the proofs are almost identical, we only consider the case of $k = 2t + 1$ and omit the superscript *odd* mentioned in R^{odd}.

We set $V_1 = V(R_1)$ and $V_t = \{x_t, y_t\}$, the latter being the set of the two universal vertices in R_t. Consider an arbitrary 1-selection f in R_t. This f defines only two edges in $f(V_t)$, hence there is a copy of R_{t-1} toward which no edge is selected for x_t and y_t. Let $V_{t-1} = \{x_{t-1}, y_{t-1}\}$ denote the set of the two universal vertices in this copy of R_{t-1}. Inside this R_{t-1} subgraph, there is a copy of R_{t-2} toward which no edge is selected for x_{t-1} and y_{t-1}. And so on, finally we obtain t sets $V_1, V_2, \ldots, V_t$ such that $V_i = \{x_i, y_i\}$ for all $2 \leq i \leq t$ and the set $V_1 \cup \cdots \cup V_t$ induces a subgraph, say H, in R_t which is isomorphic to $K_{2t+2} - e$. Further, for every i with $2 \leq i \leq t$, the selected edges $f(x_i)$ and $f(y_i)$ are either contained in $f\big(\bigcup_{j=i}^{t} V_j\big)$ or do not belong to $E(H)$. The proof will be done if we show $\omega(H_f) \geq t + 1$.

Consider the graph F with

$$V(F) = V_1 \cup \cdots \cup V_t \quad \text{and} \quad E(F) = f(V_1 \cup \cdots \cup V_t) \cap E(H).$$

Choose three distinct vertices $x_1, y_1, z_1 \in V_1$ such that

- $x_1 y_1 \notin E(F)$, and

- $x_1z_1 \neq f(z_1)$.

Such x_1, y_1, z_1 exist because f can select at most four of the five edges induced by V_1.

We let F' be the induced subgraph of F obtained by the deletion of the vertex in $V_1 \setminus \{x_1, y_1, z_1\}$. The following procedure implies $\alpha(F') \geq t+1$.

- We put $V^0 = V(F')$ and $I^0 = \emptyset$. As long as the set V^j is not empty, select a vertex v from V^j of degree 0 or 1 in $F[V^j]$, add v to I^j to obtain I^{i+1} and delete v from V^j together with its neighbor if it has one to obtain V^{j+1}. It is easy to see that a vertex v of degree 0 or 1 always exists in $V_m \cap V^j$ where m is the minimum i for which $V_i \cap V^j \neq \emptyset$.

The selected vertices obviously form an independent set in F', hence they induce a complete graph in H_f. F' has $2t+1$ vertices, and in each step, we delete at most two vertices, therefore at least $t+1$ vertices are selected at the end. Vertex x_1 can be selected first, and there is a feasible choice for the next selection until the entire F' is eliminated. ■

We finish this section by proving Theorem 3. A *unit interval graph* is a graph of which the vertices $v_1, v_2, \ldots, v_n$ are labelled with reals $r_1, r_2, \ldots, r_n$ such that v_i is joined to v_j if and only $|r_i - r_j| < 1$. The p^{th} power G^p of graph G has the same vertex set as G, and two vertices are connected by an edge if and only if their distance in G is at most p.

We shall apply the following result proved first in [3]. Later developments and further references are reported in the Introduction of [6].

Theorem 10 (Fine, Harrop [3])**.** *An n-vertex graph G is a unit interval graph if and only if there exist $n' \geq n$ and $p \geq 1$ such that G is an induced subgraph of $P^p_{n'}$.*

As a matter of fact, the exponent p can be chosen to be $\omega(G) - 1$, which is the same as $\chi(G) - 1$.

***Proof of Theorem* 3.** The lower bound follows from Proposition 9. For the upper bound assume that $G \subseteq H = P^p_{n'}$ where $p = \chi(G) - 1$. If $p = 1$, then G is a linear forest and $\chi_1(G) = 1$. Let $\chi(G) = p + 1 = 3k - r \geq 3$ with $r \in \{0, 1, 2\}$, and assume without loss of generality that $n' = 3t$, P being the path $v_1v_2 \cdots v_{3t}$. Then each triplet $S_i = \{v_{3i-2}, v_{3i-1}, v_{3i}\}$ $(1 \leq i \leq t)$ induces a K_3 in H, whose edges can be taken as a 1-selection f. This decomposes H_f into the 3-element independent sets $S_1, \ldots, S_t$. Moreover, if $|i - j| > k$, then there is no edge between S_i and S_j. Consequently, the sets $\bigcup_{i \equiv r (\text{mod } k+1)} S_i$ are independent for each $r = 0, 1, \ldots, k$, so that $\chi_1(G) \leq k + 1$ holds. Since $k \leq \frac{1}{3}(\chi(G) + 2)$, the theorem follows. ■

3. Complete Multipartite Graphs

In this section, we prove Theorem 4 and Theorem 5. Before the proofs of these results let us mention some of their consequences. Recall that we assume $n_1 \leq \cdots \leq n_t$ when we use the notation $K_{n_1,\ldots,n_t}$.

Corollary 11. *If $n_1 \geq t$, then $\chi_1(K_{n_1,\ldots,n_t}) = t$.*

Corollary 12. *If $n_1 < t$, let j denote the largest integer such that $n_1 + \cdots + n_j \leq t - j$. Then $\chi_1(K_{n_1,\ldots,n_t}) \leq t - j$.*

As shown, for instance, by the complete graph K_n (where it is known that $\chi_1(K_n) \leq \frac{n}{3} + 2$), the upper bound $t - j$ is far from being tight.

Before the proofs of Theorems 4 and 5 we observe that in a complete multipartite graph three types of independent sets (and their subsets) can be created by the removal of a 1-selection. These are formed from

(a) three vertices $v_{i_1}, v_{i_2}, v_{i_3}$ from three distinct classes $V_{i_1}, V_{i_2}, V_{i_3}$, hence deleting the edges of a C_3;

(b) four vertices $v'_{i_1}, v''_{i_1}, v'_{i_2}, v''_{i_2}$ from two distinct classes V_{i_1}, V_{i_2}, hence deleting the edges of a C_4;

(c) one vertex v_{i_1} from vertex class V_{i_1} together with another class V_{i_2}, hence deleting the edges of a star.

Proof of Theorem 4. Let us denote by $f(p,q)$ the value of χ_1 under the assumptions of the theorem. It is obvious that $f(p,q) = \left\lceil \frac{p+\lceil 3q/2 \rceil}{3} \right\rceil$ is indeed valid if $p + q \leq 2$, and also if $p = 3$ and $q = 0$. For the remaining cases, we apply induction on the number of vertices. In the next cases, we check how the selection of an independent set S of type (a), (b), or (c) modifies $f(p,q)$ depending on the sizes of vertex classes met by S. Assuming that f correctly expresses the value of χ_1 for all combinations of p', q' with $p' + 2q' < p + 2q$ (as we have p sets of size 1 and q sets of size 2), we obtain the following recursions.

(a.1) $(1,1,1) \longrightarrow 1 + f(p-3, q)$;

(a.2) $(1,1,2) \longrightarrow 1 + f(p-1, q-1)$;

(a.3) $(1,2,2) \longrightarrow 1 + f(p+1, q-2)$;

(a.4) $(2,2,2) \longrightarrow 1 + f(p+3, q-3)$;

(b.1) $(2,2) \longrightarrow 1 + f(p, q-2)$;

(c.1) $(1,1) \longrightarrow 1 + f(p-2, q)$;

(c.2) $(1,2) \longrightarrow 1 + f(p-1, q-1)$;

(c.3) $(2,2) \longrightarrow 1 + f(p+1, q-2)$.

From these formulas the following ones are relevant

$$1 + f(p-3,q)\,, \quad 1 + f(p-1,q-1)\,, \quad 1 + f(p,q-2)\,, \quad 1 + f(p+3,q-3)\,.$$

In this list (a.3), (c.3), and (c.1) do not appear because they are superseded by (b.1) and (a.1), respectively, which are also their alternatives structurally. Note further that

$$f(p-3,q) + 1 = f(p,q) = f(p,q-2) + 1$$

and the reduction (c.3) or (a.1) can always be applied, hence $f(p,q)$ is a general upper bound on χ_1. But it is also a lower bound because, in the other two cases, we have

$$f(p+3,q-3) + 1 \geq f(p-1,q-1) + 1 \geq f(p,q)\,.$$

This can be verified by comparing the numerators, namely

$$(p+3) + \lceil 3(q-3)/2 \rceil = (p-1) + 4 + \lceil 3(q-1)/2 \rceil - 3 > (p-1) + \lceil 3(q-1)/2 \rceil,$$

and

$$(p-1) + \lceil 3(q-1)/2 \rceil + 3 = p + \lceil (3q+1)/2 \rceil \geq p + \lceil 3q/2 \rceil. \qquad \blacksquare$$

Before proving Theorem 5, let us state results from [5] on bipartite and complete tripartite graphs.

Theorem 13. (i) [5, Proposition 2.6.] *The complete tripartite graph* $K_{r,s,t}$ *with* $1 \leq r \leq s \leq t$ *and* $t \geq 2$ *satisfies* $\chi_1(K_{r,s,t}) = 2$ *if and only if* $r \leq 2$*; otherwise* $\chi_1(K_{r,s,t}) = \chi(K_{r,s,t}) = 3$.

(ii) [5] *A bipartite graph* F *has* $\chi_1(F) = 2$ *(i.e.,* $\chi_1(F) = \chi(F)$*) if and only if it contains a component with more edges than vertices.*

***Proof of Theorem* 5.** The assertion is obvious for $t = 2$, and its validity is easily derived from Theorem 13 for $t = 3$. Assuming $t \geq 4$, let $m = \chi_1(K_{n_1,\ldots,n_t})$ and consider a partition $(X_1, \ldots, X_m)$ of the vertex set $V = V_1 \cup \cdots \cup V_t$ into the minimum number of subsets X_i that all become independent after the removal of a suitably chosen 1-selection. The statement of the theorem is $X_m = \{v_1\} \cup V_t$ for a $v_1 \in V_1$. If this is not the case, then we modify $(X_1, \ldots, X_m)$ to another partition $(X'_1, \ldots, X'_m)$ where $X'_m = \{v_1\} \cup V_t$ will hold.

There can be five types of X_i in the partition.

(a) $X_i \subseteq V_j$ for some $1 \leq j \leq t$;

(b) $|X_i \cap V_j| = |X_i \cap V_k| = |X_i \cap V_l| = 1$ for some $1 \leq j < k < l \leq t$;

(c) $|X_i \cap V_j| = |X_i \cap V_k| = 2$ for some $1 \leq j < k \leq t$;

(d) $|X_i \cap V_j| = 1$ and $(X_i \setminus V_j) \subseteq V_k$ for some $1 \leq j < k \leq t$;

(e) $(X_i \setminus V_k) \subseteq V_j$ and $|X_i \cap V_k| = 1$ for some $1 \le j < k \le t$.

There are several possible immediate simplifications in these types. If $(X_i \setminus V_j) \neq V_k$ in (d), we can extend X_i to contain the entire set V_k and omit the vertices of $V_k \setminus X_i$ from the other sets $X_{i'}$ that meet V_k. A similar step applies to (e), and also to (a) that yields then $X'_i = V_j$. In fact, option (e) can be eliminated because (d) removes V_k while (e) removes V_j—plus one element from each—and we have $|V_j| \le |V_k|$, hence the optimum with (d) is at least as good as the optimum with (e). In the sequel, we analyze further ways of simplifying a partition.

(1) Assume first that $X_m = V_j$ that is of type (a). If $j \neq t$, we modify X_m to V_t, and replace $|V_j|$ vertices in the sets X_i meeting V_t with the vertices of V_j in a way that their sizes remain unchanged. After that, a vertex $v_1 \in V_1$ can be added to the modified X_m and the proof is done.

(2) Assume next that $X_m \cap V_j = \{v_j\}$ and $X_m \setminus \{v_j\} = V_k$ (that is, type (d) occurs). If $k \neq t$, we modify X_m to $(X_m \setminus V_k) \cup V_t (= \{v_j\} \cup V_t)$, and replace $|V_k|$ vertices in the sets X_i meeting V_t with the vertices of V_k, as in case (1). This finishes the proof if $|V_j| = 1$, because in that case V_j can play the role of V_1. Hence suppose that $|V_j| \ge 2$ holds.

(2.1) If an $X_{i'}$ of type (b) or (d) exists that contains a single vertex v_1 from V_1, we switch the positions of v_1 and v_j; then X_m is successfully modified to $X'_m = \{v_1\} \cup V_t$, and the proof is done.

(2.2) Otherwise, all X_is meeting V_1 are of type (c). Say, one of them is $X_i = \{v_1, v'_1, v_i, v'_i\}$, where $v_1, v'_1 \in V_1$ and $v_i, v'_i \in V_i$. All the following subcases will lead to vertex partitions containing a class $\{v_1\} \cup V_t$ with $v_1 \in V_1$.

(2.2.1) If there is a further $v'_j \in V_j$ and $X_{j'} = \{v'_j\} \cup V_k$ of type (d), we replace X_m, X_i, and $X_{j'}$ with $\{v_1\} \cup V_t$, $\{v_i, v'_i, v_j, v'_j\}$, and $\{v'_1\} \cup V_k$, respectively.

(2.2.2) If a $v'_j \in V_j$ is covered with $X_{j'} = \{v'_j, v_k, v_l\}$ of type (b), we replace X_m, X_i, and $X_{j'}$ with $\{v_1\} \cup V_t$, $\{v_i, v'_i, v_j, v'_j\}$, and $\{v'_1, v_k, v_l\}$, respectively.

(2.2.3) If V_j meets an $X_{j'} = \{v'_j, v''_j, v_k, v'_k\}$ of type (c), we replace X_m, X_i, and $X_{j'}$ with $\{v_1\} \cup V_t$, $\{v_i, v'_i, v_k, v'_k\}$, and $\{v'_1\} \cup V_j$, respectively. This completes the proof in case (2).

From now on we can assume that the entire V is partitioned into sets of types (b) and (c) only.

(3) If some V_j meets more than one set of type (c), we can reduce the situation to case (1). Indeed, say $X_i = \{v_j, v'_j, v_k, v'_k\}$ and $X_{i'} = \{v''_j, v'''_j, v_l, v'_l\}$. Then we can replace these sets with $X'_i = V_j$ and $X'_{i'} = \{v_k, v'_k, v_l, v'_l\}$.

(4) If V_t meets a set $X_i = \{v_j, v_k, v_t\}$ of type (b) and a set $X_{i'} = \{v_l, v'_l, v_t, v'_t\}$ of type (c), we can replace them with $X'_i = \{v_j\} \cup V_t$ of type (b) and $X'_{i'} = \{v_k\} \cup V_l$, hence reducing to case (2.2.1).

(5) If V_t only meets sets of type (b), we take three of those sets say X_{m-2}, X_{m-1}, X_m. Then we set $X'_m = V_t$, and split $(X_{m-2} \cup X_{m-1} \cup X_m) \setminus V_t$ into two vertex triplets. This reduces case (5) to case (1) and completes the proof of the theorem. ■

Corollary 14. *The graph invariant χ_1 is computable in polynomial time in the class of complete multipartite graphs.*

4. Kneser Graphs

In this section, we prove Theorem 6 and Theorem 7. Before presenting our results, we quote two fundamental theorems from extremal set theory that will serve as tools in our proofs.

Theorem 15 (Erdős, Ko, Rado [2])**.** *For any integer $k \geq 2$ and any $n \geq 2k$, we have $\alpha(KG(n,k)) = \binom{n-1}{k-1}$.*

Theorem 16 (Hilton, Milner [4])**.** *For any $n \geq 2k+1$, if $\mathcal{F} \subseteq \binom{[n]}{k}$ is intersecting with $\bigcap_{F \in \mathcal{F}} F = \emptyset$, then $|\mathcal{F}| \leq \binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1$.*

***Proof of Theorem* 6.** Let $\mathcal{F}$ be a 1-independent family of sets in $V(KG(n,k))$, where $KG(n,k)$ denotes the Kneser graph. Recall that in $KG(n,k)$, the vertices correspond to all k-subsets of $[n]$, and two vertices are adjacent if and only if the corresponding subsets are disjoint.

Let us consider a 1-independent family $\mathcal{F}$. Note that $KG(n,k)[\mathcal{F}]$ is $K_{2,3}$-free. This is true by the fact that the graph $K_{2,3}$ has 5 vertices and 6 edges, and by removing one edge per vertex, it is impossible to make the graph independent. However, since $KG(n,k)[\mathcal{F}]$ must already be an independent set after such deletions, this ensures that $\mathcal{F}$ cannot have certain configurations that induce $K_{2,3}$.

Step 1. Degree Constraints in $\mathcal{F}$.

First, suppose there exists an element $x \in [n]$ such that at most one set $F \in \mathcal{F}$ satisfies $x \notin F$. In this case, the size of $\mathcal{F}$ is bounded by

$$|\mathcal{F}| \leq \binom{n-1}{k-1} + 1,$$

since all sets except possibly one must contain x.

Now, assume that for every $x \in [n]$, there exist at least two sets $F_x, G_x \in \mathcal{F}$ such that $x \notin F_x$ and $x \notin G_x$. Under this assumption, consider the degree of x in $\mathcal{F}$, denoted by

$$d_{\mathcal{F}}(x) := |\{F \in \mathcal{F} : x \in F\}|.$$

Note that all but two sets containing x must intersect $F_x \cup G_x$ using the previously mentioned $K_{2,3}$-freeness. This yields the bound

$$d_{\mathcal{F}}(x) \leq \binom{n-1}{k-1} - \binom{n-2k-1}{k-1} + 2.$$

For $k \geq 3$, we can use the following inequality for this upper bound

$$d_{\mathcal{F}}(x) \leq 2k\binom{n-2}{k-2}.$$

Step 2. Existence of a large intersecting subfamily.

We claim that $\mathcal{F}$ must contain an intersecting subfamily of size at least $\frac{1}{4}|\mathcal{F}|$. This can be established as follows.

1. Since $\mathcal{F}$ is 1-independent, the degree sum in $KG(n,k)[\mathcal{F}]$ satisfies

$$\sum_{x\in[n]} d_{\mathcal{F}}(x) \leq 2|\mathcal{F}|.$$

2. Remove sets from $\mathcal{F}$ iteratively while ensuring that the maximum degree of any set in the remaining subfamily is at most 1. Specifically, remove a maximum-degree set at each step. Every removal decreases the degree sum by at least 4 because each removed set has degree at least 2, and its removal affects at least two other sets.

3. This process stops when all sets in the remaining subfamily, denoted by $\mathcal{F}'$, have degree at most 1. At this stage, at least half of the sets from $\mathcal{F}$ remain (as we decreased the sum of the degrees bf 4 at each step), so

$$|\mathcal{F}'| \geq \frac{1}{2}|\mathcal{F}|.$$

4. Within $\mathcal{F}'$, further select a subset such that no two sets are adjacent (i.e., no two sets are disjoint). This forms an intersecting family, $\mathcal{F}''$, where

$$|\mathcal{F}''| \geq \frac{1}{2}|\mathcal{F}'| \geq \frac{1}{4}|\mathcal{F}|.$$

Let $\mathcal{F}^*$ be a maximum-size intersecting subfamily of $\mathcal{F}$. To complete the proof, we distinguish the following cases.

Case 1. $\bigcap_{F\in\mathcal{F}^*} F = \emptyset$. If the intersection of all sets in $\mathcal{F}^*$ is empty, then by Theorem 16, we have

$$|\mathcal{F}| \leq 4|\mathcal{F}^*|.$$

From earlier estimates, we know that

$$|\mathcal{F}^*| \leq k\binom{n-2}{k-2}.$$

Thus

$$|\mathcal{F}| \leq 4k\binom{n-2}{k-2} + 4.$$

Now, compare this with $\binom{n-1}{k-1} + 1$. For $n \geq 4k^2$, we observe that

$$4k\binom{n-2}{k-2} + 4 < \binom{n-1}{k-1} + 1.$$

Hence, the inequality holds.

Case 2. $\bigcap_{F \in \mathcal{F}^*} F \neq \emptyset$ and $k \geq 3$. If the intersection of all sets in $\mathcal{F}^*$ is non-empty, then the size of $\mathcal{F}^*$ is bounded above by the maximum degree of $\mathcal{F}$, which was shown earlier to satisfy

$$|\mathcal{F}^*| \leq 2k\binom{n-2}{k-2} \; (k \geq 3).$$

Thus

$$|\mathcal{F}| \leq 4|\mathcal{F}^*| \leq 8k\binom{n-2}{k-2}.$$

Now, compare this with $\binom{n-1}{k-1}$. For $n \geq 8k^2$, we observe that

$$8k\binom{n-2}{k-2} \leq \binom{n-1}{k-1}.$$

Case 3. $k = 2$. For $k = 2$, recall from the initial analysis that

$$d_{\mathcal{F}}(x) \leq \binom{n-1}{k-1} - \binom{n-2k-1}{k-1} + 2 = 6.$$

Thus

$$|\mathcal{F}| \leq 4|\mathcal{F}^*| \leq 24.$$

Finally, compare this with $\binom{n-1}{k-1}$. For $n \geq 8 \cdot 2^2 = 32$, we have

$$24 \leq \binom{n-1}{k-1}.$$

In all cases, the upper bounds on $|\mathcal{F}|$ hold, completing the proof. ■

Remark 1. We note that the threshold $n_0(k)$ mentioned in Theorem 6 is at least $3k+1$.

To get his statement we prove that $\alpha_1(KG(3k,k)) \geq \binom{3k-1}{k-1}+2$ holds. Indeed, $\mathcal{F} := \{F \in \binom{[3k]}{k} : 1 \in F\} \cup \{[k+1,2k],[2k+1,2k]\}$ is 1-independent, as all sets in $\{F \in \binom{[3k]}{k} : 1 \in F\}$ but $[k]$ intersect at least one of $[k+1,2k],[2k+1,3k]$, and so $KG(3k,k)[\mathcal{F}]$ is a triangle with pendant edges from two vertices of the triangle.

Furthermore, if $n = 2k+m$ with $1 \leq m < k$ and $f(m)$ denotes the maximum size of an intersecting family $\mathcal{G} \subseteq \binom{[2k+m-1]}{k}$ with $|G \cup G'| > k+m$ for all $G, G' \in \mathcal{G}$, then $\alpha_1(KG(2k+m,k)) \geq \binom{2k+m-1}{k-1} + f(m)$. Indeed, $\mathcal{F} = \{F \in \binom{[2k+m]}{k} : 2k+m \in F\} \cup \mathcal{G}$ is 1-independent if $\mathcal{G}$ is as above. As $\{F \in \binom{[2k+m]}{k} : 2k+m \in F\}$ and $\mathcal{G}$ are intersecting, $KG(2k+m,k)[\mathcal{F}]$ is bipartite and for any $G, G' \in \mathcal{G}$ there does not exist any k-subset of $[2k+m]$ that is disjoint from both G, G' and so $KG(3k,k)[\mathcal{F}]$ is a star forest.

Proof of Theorem 7. For the upper bound, observe that if for some integer c we have $n - c \geq \binom{c}{k} - \binom{2k}{k}$, then $\chi_1(KG(n,k)) \leq n-c+1$ holds. Indeed, if we enumerate $\binom{[c]}{k} \setminus \binom{[2k]}{k}$ as $G_1, G_2, \ldots, G_h$ with $h \leq n-c$, then the families $\mathcal{G}_i = \{G \in \binom{[n]}{k} : \max_{j \in G} j = n+1-i\} \cup \{G_i\}$ are 1-independent as they induce a star in $KG(n,k)$ and they cover the vertices $\binom{[n]}{k} \setminus \binom{[2k]}{k}$. So adding $\binom{[2k]}{k}$ to the $\mathcal{G}_i$s, we obtain a partition of $\binom{[n]}{k}$ into $n-c+1$ 1-independent families. As k is fixed, we have $\binom{c}{k} - \binom{2k}{k} = \Theta(c^k)$, and so the largest value of c for which $n - c \geq \binom{c}{k} - \binom{2k}{k}$ holds is $\Theta(n^{1/k})$. This finishes the proof of the upper bound.

To prove the lower bound, we need the following definition. We say that a family $\mathcal{F}$ of sets is *star-like with center* x if x is contained in all but at most one set F of $\mathcal{F}$. (So all $\mathcal{G}_i$s in the previous paragraph are star-like.) Suppose that a robust coloring of $KG(n,k)$ contains a star-like and b non-star-like color classes. Observe that if $\mathcal{F}$ is star-like with center x, then we can add any set F containing x to still have a star-like and therefore 1-independent family. Therefore, if $x_1, x_2, \ldots, x_a$ are the centers of the star-like color classes, then we can assume that all non-star-like color classes $\mathcal{F}$ are subfamilies of $\binom{[n] \setminus \{x_1, x_2, \ldots, x_a\}}{k}$.

By Theorem 6, all non-star-like color classes have size at most $8k\binom{n-a-2}{k-2}$. Therefore, we must have

$$b \cdot 8k\binom{n-a-2}{k-2} + a \geq \binom{n-a}{k}.$$

If $a \geq n - 2 \cdot k! \cdot n^{1/k}$, then there is nothing to prove. Otherwise, $a \leq \frac{1}{2}\binom{n-a}{k}$ (as $\frac{1}{2}\binom{n-a}{k}$ becomes much larger than n), and thus we must have

$$b \geq \frac{1}{16k} \frac{\binom{n-a}{k}}{\binom{n-a-2}{k-2}} \geq \frac{1}{32k^3}(n-a)^2,$$

if n is large enough compared to k and . (We used also the $(n-a)(n-a-1) \geq (n-a)^2/2$ inequality holds, if $n-a \geq 2$.) Therefore, the number $a+b$ of color classes is at least $a + \frac{1}{32k^3}(n-a)^2$. This expression takes its minimum at $a = n - 16k^3$ with value $n - 8k^3 \geq n - n^{1/k}$. ■

Concluding Remarks

In this article, we introduced some s-robust parameters of graphs for any non-negative integer s. However, our research primarily focused on the $s = 1$ case for certain specific graph classes. A natural progression of this investigation would be to extend the results to larger values of s and other graph classes.

Additionally, we present two problems related to the $s = 1$ case.

Problem 1. Charaterize those graph classes, where parameter χ_1 is around $\frac{\chi}{3}$.

Problem 2. Bound χ_1 with other 1-robust parameters.

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References

[1] G. Bacsó, B. Patkós, Zs. Tuza and M. Vizer, *The robust chromatic number of graphs*, Graphs Combin. **40** (2024) 89.
https://doi.org/10.1007/s00373-024-02817-1

[2] P. Erdős, C. Ko and R. Rado, *Intersection theorems for systems of finite sets*, Quart. J. Math. Oxford **12** (1961) 313–320.
https://doi.org/10.1093/qmath/12.1.313

[3] N.J. Fine and R. Harrop, *Uniformization of linear arrays*, J. Symb. Log. **22** (1957) 130–140.
https://doi.org/10.2307/2964174

[4] A.J.W. Hilton and E.C. Milner, *Some intersection theorems for systems of finite sets*, Quart. J. Math. Oxford **18** (1967) 369–384.
https://doi.org/10.1093/qmath/18.1.369

[5] B. Patkós, Zs. Tuza and M. Vizer, *Extremal graph theoretic questions for q-ary vectors*, Graphs Combin. **40** (2024) 57.
https://doi.org/10.1007/s00373-024-02787-4

[6] F. Soulignac, *Bounded, minimal, and short representations of unit interval and unit circular-arc graphs. Chapter* I: *theory*, J. Graph Algorithms Appl. **21** (2017) 455–489.
https://doi.org/10.7155/jgaa.00425

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