

A NOTE ON HAMEED’S CONJECTURE ON THE SEMI-TRANSITIVITY OF MYCIELSKI GRAPHS

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Abstract

An orientation of a graph is semi-transitive if it contains no directed cycles and has no shortcuts. An undirected graph is semi-transitive if it can be oriented in a semi-transitive manner. The class of semi-transitive graphs includes several important graph classes. The Mycielski graph of an undirected graph is a larger graph constructed in a specific manner, which maintains the property of being triangle-free but increases the chromatic number.

In this note, we prove Hameed’s conjecture, which states that the Mycielski graph of a graph G is semi-transitive if and only if G is a bipartite graph. Notably, our solution to the conjecture provides an alternative and shorter proof of the Hameed’s result on a complete characterization of semi-transitive extended Mycielski graphs.

Keywords: semi-transitive graph, semi-transitive orientation, word-representable graph, Mycielski graph, extended Mycielski graph.

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1. INTRODUCTION

Semi-transitive graphs, also known as *word-representable graphs*, include several fundamental classes of graphs (e.g. *circle graphs*, *3-colourable graphs* and *comparability graphs*), and they have been the subject of much research in the literature [3, 4]. In particular, Hameed [2] provided a complete classification of semi-transitive *extended Mycielski graphs* and also *Mycielski graphs of comparability graphs*. Our goal in this note is to confirm a conjecture of Hammed [2] by giving a complete classification of semi-transitive Mycielski graphs.

1.1. Main definitions

An orientation of a graph is *semi-transitive* if it is acyclic (there are no directed cycles), and for any directed path $v_0 \rightarrow v_1 \rightarrow \dots \rightarrow v_k$ either there is no edge between v_0 and v_k , or $v_i \rightarrow v_j$ is an edge for all $0 \leq i < j \leq k$. An induced subgraph on vertices $\{v_0, v_1, \dots, v_k\}$ of an oriented graph is a *shortcut* if its orientation is acyclic and non-transitive, and there is the directed path $v_0 \rightarrow v_1 \rightarrow \dots \rightarrow v_k$ and the edge $v_0 \rightarrow v_k$ called the *shortcutting edge*. Note that any shortcut must contain at least 4 vertices. A semi-transitive orientation can then be alternatively defined as an acyclic shortcut-free orientation. A non-oriented graph is *semi-transitive* if it admits a semi-transitive orientation.

Consider an arbitrary undirected graph $G = (V, E)$ with the vertex set $V = \{1, 2, \dots, n\}$. Put $V' = \{1', 2', \dots, n'\}$. Define the *Mycielski graph* $\mu(G)$ as follows. Let $V(\mu(G)) = V \cup V' \cup \{x\}$ and $E(\mu(G)) = E \cup \{ij' \mid ij \in E\} \cup \{xi' \mid i = 1, \dots, n\}$. For instance, the graph $\mu(C_5)$ also known as Grötzsch graph is presented in Figure 1 to the left. Here and throughout the note, C_n denotes the cycle graph on n vertices labelled around the cycle $1, 2, \dots, n$.

Note that the Mycielski graph of an undirected graph is a larger graph that preserves the property of being triangle-free but enlarges the chromatic number. These graphs were introduced by Mycielski in 1955 (see [7]) to prove the existence of triangle-free graphs with arbitrarily large chromatic number. Since its introduction, Mycielski graphs have attracted considerable attention in the literature from various perspectives; see, e.g., [1] and references therein.

The *extended Mycielski graph* $\mu'(G)$ is derived from a Mycielski graph $\mu(G)$ by connecting every vertex i' to every vertex j , except for $j = i$ (i.e., $E(\mu'(G)) = E \cup \{ij' \mid i \neq j\} \cup \{xi' \mid i = 1, \dots, n\}$). The graph $\mu'(C_5)$ is presented in Figure 1 to the right. Note that for a complete graph K_n , $\mu'(K_n) = \mu(K_n)$.

A *source* (respectively, *sink*) in a directed graph is a vertex without ingoing (respectively, outgoing) edges. The following result proven in [6] is very helpful.

Theorem 1 [6]. *Suppose that a graph G is semi-transitive, and v is a vertex in G . Then there exists a semi-transitive orientation of G where v is a source (or a sink).*

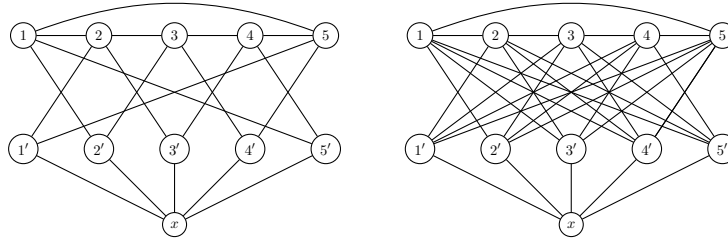


Figure 1. The graphs $\mu(C_5)$ (to the left) and $\mu'(C_5)$ (to the right).

1.2. Hameed's results and our contribution

Hameed [2] provided a complete classification of semi-transitive extended Mycielski graphs.

Theorem 2 [2]. *The graph $\mu'(G)$ is semi-transitive if and only if G is a bipartite graph.*

Regarding Mycielski graphs, Hameed [2] proved the following result and formulated the following conjecture.

Theorem 3 [2]. *Let G be a comparability graph. Then $\mu(G)$ is semi-transitive if and only if G is bipartite.*

Conjecture 4 [2]. *For every graph G the graph $\mu(G)$ is semi-transitive if and only if G is a bipartite graph.*

In this note, we prove Conjecture 4 and consequently generalize Theorem 3. Note that the proof of Theorem 2 is based on proving that $\mu'(C_{2k+1})$ is non-semi-transitive for all $k \geq 1$. Similarly, our proof of Conjecture 4 relies on demonstrating the non-semi-transitivity of $\mu(C_{2k+1})$ for all $k \geq 1$, which was also conjectured in [2]. Notably, our relatively concise proof can be directly applied to establish the non-semi-transitivity of $\mu'(C_{2k+1})$, which originally required a lengthier consideration involving four cases.

2. CLASSIFICATION OF SEMI-TRANSITIVE MYCIELSKI GRAPHS

The following theorem is crucial in proving our main result.

Theorem 5. *The graph $\mu(C_{2k+1})$ is non-semi-transitive for all $k \geq 1$.*

Proof. Recall that the vertices of the subgraph C_{2k+1} are labelled $1, 2, \dots, 2k+1$ around the cycle. Suppose $\mu(C_{2k+1})$ admits a semi-transitive orientation. Then, by Theorem 1, we can assume that the vertex x is a source. Then, to avoid

shortcuts involving x , for each vertex $i \in \{1, \dots, 2k+1\}$, either $(i-1)' \rightarrow i \leftarrow (i+1)'$ or $(i-1)' \leftarrow i \rightarrow (i+1)'$, where $0' = (2k+1)'$ and $(2k+2)' = 1'$. Colour a vertex i red in the former case and blue in the latter case.

Let us show the following properties of this colouring.

(1) If a blue vertex b is adjacent to a red vertex r in the subgraph C_{2k+1} , then the edge is $b \rightarrow r$, i.e. it goes from b to r .

Indeed, if, say, $b = i$ and $r = (i+1)$ and $(i+1) \rightarrow i$, then $xi'(i+1)i(i+1)'$ is a shortcut contradicting the assumption.

(2) If $i \rightarrow (i+1) \rightarrow (i+2)$ in C_{2k+1} , then the vertex i is blue and the vertex $(i+2)$ is red.

Indeed, in any other case, the vertices in $\{i, (i+1), (i+2), (i+1)'\}$ form either a shortcut or a directed cycle. Clearly, the similar property holds if $(i+2) \rightarrow (i+1) \rightarrow i$ in C_{2k+1} .

(3) If vertices i and $(i+m)$ in C_{2k+1} are of the same colour, and all $(i+1), \dots, (i+m-1)$ are of the other colour, then m is even.

Indeed, let i and $i+m$ be red, and all $(i+1), \dots, (i+m-1)$ be blue (the other case is considered similarly). If $m = 2$, the property holds immediately; so, assume $m \geq 3$. From property (1), we have edges $(i+1) \rightarrow i$ and $(i+m-1) \rightarrow (i+m)$. Since there are edges $(i+2) \rightarrow (i+1)' \rightarrow i$, we must have the edge $(i+2) \rightarrow (i+1)$ to avoid the shortcut $(i+1)(i+2)(i+1)'i$. Similarly, we must have the edge $(i+m-2) \rightarrow (i+m-1)$. Based on property (2), each vertex among the vertices $(i+2), \dots, (i+m-2)$ must be either a source or a sink in C_{2k+1} . In particular, $(i+2)$ is a source. Since sources and sinks must alternate in the cycle, all vertices with even addends $(i+2), (i+4), \dots$ must be sources, and those with odd addends $(i+3), (i+5), \dots$ must be sinks. But since $(i+m-2)$ cannot be a sink, m must be even.

Note that the proof of property (3) works even in case when $m = 2k+1$, i.e., if i and $(i+m)$ coincide (and since $2k+1$ is odd, property (3) indeed says that this case is impossible). Let us call each maximal sequence of the same coloured vertices in C_{2k+1} a “pattern”. Note that in any orientation of C_{2k+1} there must be a vertex that is neither a source nor a sink. Then by property (2), its neighbours have different colours; so, there are at least two patterns (i.e., not all vertices have the same colour). It follows from property (3) that each pattern contains an odd number of vertices. Since the colours of the patterns alternate, the total number of the patterns in the cycle must be even. But then the total number of vertices in the cycle is also even as the sum of an even number of odd addends — a contradiction, because the cycle C_{2k+1} is odd. Therefore, no semi-transitive orientation of $\mu(C_{2k+1})$ can exist. ■

Remark 6. Our proof of Theorem 5 can be applied without modification if $\mu(C_{2k+1})$ is replaced by $\mu'(C_{2k+1})$, thereby offering an alternative, shorter proof of

the respective result in [2]. Indeed, it is easy to verify that all shortcuts mentioned in the proof of Theorem 5 remain shortcuts also in the graph $\mu'(C_{2k+1})$.

The following theorem confirms Conjecture 4, and its proof follows the steps presented in the proof of Theorem 4 in [2].

Theorem 7. *The graph $\mu(G)$ is semi-transitive if and only if G is a bipartite graph.*

Proof. Suppose that G is not a bipartite graph. Then G must contain an odd cycle. A minimal odd cycle in G is an induced odd cycle C_{2k+1} for some $k \geq 1$. Therefore, $\mu(G)$ contains $\mu(C_{2k+1})$ as an induced subgraph. By Theorem 5, $\mu(G)$ is not semi-transitive.

Now suppose G is a bipartite graph on n vertices $1, \dots, n$. Orient G transitively from one part to the other, ensuring the longest directed path in such an orientation is of length 1. Extend this transitive orientation of G to a semi-transitive orientation of $\mu(G)$ by letting x be a source and orienting edges $z \rightarrow y'$ for all $z, y \in \{1, 2, \dots, n\}$. Clearly, this orientation is acyclic. Since the longest directed path in such an orientation has length 2 there can be no shortcuts. Hence, $\mu(G)$ is semi-transitive in this case. ■

3. CONCLUDING REMARKS

In conclusion, we would like to discuss two natural generalizations of Mycielski graphs, namely, iterated Mycielski graphs and the generalized Mycielski graphs.

Iterated Mycielski graph $\mu^p(G)$ is a result of a sequence of p repeatedly applied Mycielski constructions, starting with a simple graph G . By Theorem 7, since the graph $\mu(C_{2k+1})$ is non-bipartite, its iterated Mycielski graph $\mu^p(C_{2k+1})$ is non-semi-transitive for any $p \geq 1$. Since C_{2k+1} is triangle-free for all $k \geq 2$, $\mu^p(C_{2k+1})$ is a larger non-semi-transitive triangle-free graph for all $p \geq 1$ (see [5] for a discussion of semi-transitive triangle-free graphs). Another (much denser) large class of non-semi-transitive triangle-free graphs can be produced by $\mu^p(G)$, where the initial graph G is a complete bipartite graph $K_{m,n}$ with one subdivided edge (to make it non-bipartite).

Given a positive integer t and a simple graph $G = (V, E)$, a *generalized Mycielski graph* $\mu_t(G)$ is constructed as follows. For every $i = 0, \dots, t$, let $V_i = \{v^i \mid v \in V\}$, where $V_0 = V$. Then $V(\mu_t(G)) = V_0 \cup V_1 \cup \dots \cup V_t \cup \{x\}$ and $E(\mu_t(G)) = E \cup \{u^{i-1}v^i \mid uv \in E, i = 1, \dots, t\} \cup \{v^tx \mid v^t \in V_t\}$. The problem of determining the semi-transitivity of $\mu_t(G)$ remains open for $t \geq 2$. We conclude our note with the following conjecture, which is clearly true in the backward direction, as $\mu_t(G)$ is 3-colourable and therefore semi-transitive [3, 4].

Conjecture 8. *For every $t \geq 2$, the graph $\mu_t(G)$ is semi-transitive if and only if G is a bipartite graph.*

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