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ON TWO CONJECTURES REGARDING THE NEIGHBOR-LOCATING CHROMATIC NUMBER

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Abstract

In the paper (Discussiones Mathematicae Graph Theory 43 (2023) 659–675) has been posed two conjectures on neighbor-locating coloring of graphs. In this paper, we disprove one of them by presenting a family of counterexamples and prove the another one.

Keywords: coloring, neighbor-locating coloring, conjectures.

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1. Introduction

Let $G = (V, E)$ be a simple, finite, connected and undirected graph, with the set of vertices $V = V(G)$ and the set of edges $E = E(G)$. The open neighborhood of a vertex v in G, denoted by $N_G(v) = N(v)$, is the set of vertices adjacent to

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v and the closed neighborhood of v is $N[v] = N(v) \cup \{v\}$. The minimum and the maximum degree of G is the smallest and largest number of neighbors of a vertex in G and denoted by $\delta = \delta(G)$ and $\Delta = \Delta(G)$, respectively. A tree T is a connected graph with no cycle. A vertex of degree 1 is called a leaf or an end-vertex and a support vertex is a vertex adjacent to a leaf. We use P_n and C_n for the path and cycle graphs of order n, and the notations $[n]$ and ℓ show the set $\{1, 2, \dots, n\}$ and the number of leaves in G, respectively. For the terminologies and notation not herein, see [11].

A proper k-coloring of a graph G, $(k \in \mathbb{N})$, is a function f defined from $V(G)$ to a set of colors $[k]$ in which every two adjacent vertices have different colors. Minimum k for coloring of G is called the chromatic number of G denoted by $\chi(G) = k$, and the set of color classes is denoted by $\pi = \{S_1, \dots, S_k\}$ where S_i is the class of vertices with color i. The color-degree of a vertex v is defined to be the number of different colors of π comprising some vertex of $N(v)$. For a connected graph G and two vertices x, y in G, the distance between x and y is denoted by $d(x, y)$. For a vertex $v \in V(G)$ and a set of vertices $S \subseteq V(G)$, we use $d(v, S) = \min\{d(v, w) : w \in S\}$ for the distance between v and S.

A k-coloring $\pi = \{S_1, S_2, \dots, S_k\}$ is called a metric locating (ML-coloring or L-coloring) if for every $i \in \{1, 2, \dots, k\}$ and for any two different vertices $u, v \in S_i$, there exists $j \in \{1, 2, \dots, k\}$ in which $d(u, S_j) \neq d(v, S_j)$. Minimum k for ML-coloring of a graph G, is called metric-locating chromatic number (ML-chromatic number) of G and is denoted by $\chi_L(G) = k$, [3, 5, 6]. Recently, Korivand *et al.* worked on the edge-locating coloring of a graph, which is not without pleasure to see this concept in [9].

A k-neighbor-locating coloring of a graph G is a partition of $V(G)$ to $\pi = \{S_1, S_2, \dots, S_k\}$ in which for any pair of distinct vertices $u_1, u_2 \in S_i$, the set of colors of the neighborhood of u_1 is different from the set of colors of the neighborhood of u_2. Minimum k for a neighbor-locating coloring (an NL-coloring) of a graph G is called the neighbor-locating chromatic number (NL-chromatic number) of G and is denoted by $\chi_{NL}(G) = k$. This concept was defined for the first time, under the name of adjacency locating coloring (L_2-coloring) of G by Behtoei and Anbarloei in [4] and worked on. Also, its chromatic number was named L_2-chromatic number of G and denoted by $\chi_{L_2(G)}$ of G. For more information on this area, see [1, 2, 3, 4, 7, 8, 10].

Alcón *et al.* [2] have been posed two conjectures as follows.

Conjecture 1 ([2] Conjecture 14). *Let G be a graph of diameter d. Then $\chi_{NL}(G) \geq \chi_{NL}(P_{d+1})$.*

Conjecture 2 ([2] Conjecture 13). *Let $k \geq 2$. If T is a tree with $\chi_{NL}(T) = k$, then $\Delta(T) \leq (k-1)^2$, and this bound is tight for every integer $k \geq 2$.*

The paper is organized as follows. In the next section, we shall see the preliminary results. We disprove Conjecture 1 by presenting a family of counterexamples in Section 3, and we prove Conjecture 2 in Section 4.

2. Preliminary Results

In this section, we state some preliminary results related to NL-coloring of graphs, which are necessary to prove the results of the sections 3 and 4.

Remark 1 ([2] Remark 1)**.** Let G be a graph of order n and maximum degree Δ. Let $\Pi = \{S_1, \ldots, S_k\}$ be a k-NL-coloring of G. There exist at most $\binom{k-1}{j}$ vertices in S_i of color-degree j, for every $1 \leq i \leq k$, where $1 \leq j \leq k-1$ and consequently, $|S_i| \leq \sum_{j=1}^{\Delta} \binom{k-1}{j}$. For $\chi_{NL}(G) = k \geq 3$ and $1 \leq j \leq \Delta$, there are at most $a_j(k) = k\binom{k-1}{j}$ vertices of color-degree j in G. We denote by $\ell(k)$ the maximum number of vertices of color-degree 1 or 2, that is ($\ell(k) = a_1(k) + a_2(k)$). Therefore,

$$a_1(k) = k(k-1) \qquad a_2(k) = \frac{k(k-1)(k-2)}{2} \qquad \ell(k) = k\binom{k}{2} = \frac{k^3 - k^2}{2}.$$

From Remark 1, if $\chi_{NL}(G) = k \geq 2$, then the number of leaves (vertices of degree 1) is at most $k(k-1)$.

Theorem 3 ([1] Theorem 1)**.** *Let G be a non-trivial graph of order n and maximum degree Δ. Let $\chi_{NL}(G) = k$. If G has no isolated vertices and $\Delta \leq k-1$, then*

$$n \leq k \sum_{j=1}^{\Delta} \binom{k-1}{j}.$$

Theorem 4 ([4] Theorem 3.6)**.** *For a positive integer $n \geq 2$, $\chi_{NL}(P_n) = \chi_{L_2(P_n)} = m$, where $m = \min\{k : k \in \mathbb{N},\ n \leq \frac{1}{2}(k^3 - k^2)\}$. More precisely, there exist an adjacency locating m-coloring f_n of the path $P_n = v_1 v_2 \cdots v_n$ with the color set $\{1, 2, \ldots, m\}$, and two specified colors (say "1" and "2") such that f_n satisfies the following properties.*

(a) *$f_n(v_{n-1}) = 2$ and $f_n(v_n) = 1$.*

(b) *If $n \geq 9$, then $f_n(v_{n-2}) = m$.*

(c) *If $n \geq 9$ and $n \neq \frac{1}{2}(m^3 - m^2) - 1$, then $f_n(v_1) = 2$ and $f_n(v_2) = 1$.*

3. Conjecture 1

In this section, we discuss on Conjecture 1. We show that for any $k \geq 4$ there is a graph G of diameter d and a path P_{d+1}, so that $\chi_{NL}(G) = k$ and $\chi_{NL}(P_{d+1}) = k+1$. First, let us see the following example.

Example 1. Consider the path $P_{\ell(4)+1}$ with the NL-chromatic number 5 so that the vertices of $P_{\ell(4)}$ can be NL-colored with colors $1, 2, 3, 4$ and the vertex $v_{\ell(4)+1=25}$ is assigned with color 5. The graph G has diameter 24 and $\chi_{NL}(G) = 4$, (see Figure 1).

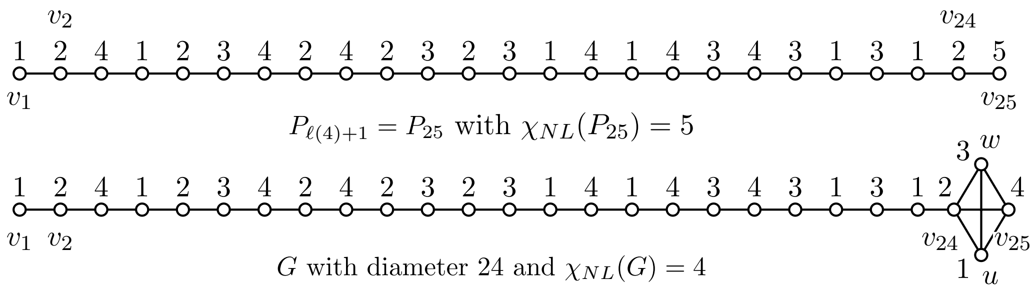


Figure 1. Graph G with $\text{diam}(G) = 24$ and $\chi_{NL}(G) = 4 < 5 = \chi_{NL}(P_{25})$.

Next, we generalize Example 1 for any integer $k \geq 4$.

Theorem 5. *Let $k \geq 4$ be an integer. There exist a graph G with diameter $d = \ell(k)$ for which $\chi_{NL}(G) = k$ and $\chi_{NL}(P_{\ell(k)+1}) = k + 1$.*

Proof. Assume that $d = \ell(k)$ and consider the path P_{d+1}. From Theorem 4, $\chi_{NL}(P_{d+1}) = k + 1$ and $\chi_{NL}(P_d) = k$. Without loss of generality, for NL-coloring of P_d, we can use k colors, and for NL-coloring of P_{d+1} we can use $k+1$ colors, for which, we assign color $k + 1$ to v_{d+1} only.

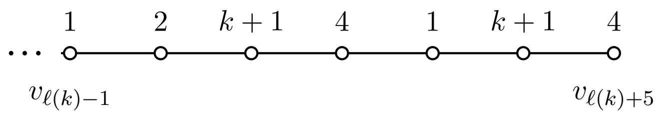
Now, according to Example 1, we construct a graph G with diameter d and $\chi_{NL}(G) = k$. Add two vertices u, w to P_{d+1} and make adjacent to v_d, v_{d+1} for which u and w are adjacent too. We fix the colors of the vertices $v_1, \ldots, v_d$, such as the colors in P_d, we assign three distinct colors from $[k]$ to the vertices u, w, v_{d+1} in which four colors assigned to v_d, u, w, v_{d+1} are distinct in $[k]$. Since any vertex v_i for $1 \leq i \leq d-1$ has color degree at most 2 and each of four vertices v_d, u, w, v_{d+1} in G has color degree 3 so that any two of them have distinct color neighbors. Therefore, G and P_{d+1} have same diameter d with $\chi_{NL}(G) = k$ and $\chi_{NL}(P_{d+1}) = k + 1$. ∎

Remark 2. For $k \geq 5$ one can construct other family of graphs G so that $\text{diam}(G) = d$ and $\chi_{NL}(G) < \chi_{NL}(P_{d+1})$. See the following.

Example 2. Consider the path $P_{\ell(k)+5}$ where $k \geq 5$. Assume without loss of generality that the vertices $v_{\ell(k)-1}$ and $v_{\ell(k)}$ take colors 1 and 2, respectively, in an NL-coloring of path $P_{\ell(k)}$. If we assign the vertices $v_{\ell(k)+1}$, $v_{\ell(k)+2}$, $v_{\ell(k)+3}$,

$v_{\ell(k)+4}$ and $v_{\ell(k)+5}$ the colors $k+1, 4, 1, k+1, 4$, respectively, then it is easy to see that this coloring is an NL-coloring of $P_{\ell(k)+5}$.

Now we create a graph G from $P_{\ell(k)+5}$ as follows. Add three vertices u, v, w to $P_{\ell(k)+5}$, so that we make adjacent u to $v_{\ell(k)}$, $v_{\ell(k)+1}$, $v_{\ell(k)+2}$, and v, w to $v_{\ell(k)+3}$, $v_{\ell(k)+4}$, $v_{\ell(k)+5}$. By the construction we assign color 5 to u, color 3 to $v_{\ell(k)+1}$, color 2 to v, color 5 to w and color 3 to $v_{\ell(k)+4}$. These assignments exist an NL-coloring for G with k colors. Therefore, G and $P_{\ell(k)+5}$ have the same diameter $\ell(k)+4$ but $\chi_{NL}(G) = k < k+1 = \chi_{NL}(P_{\ell(k)+5})$.



$P_{\ell(k)+5}$ with $\chi_{NL}(P_{\ell(k)+5}) = k+1$

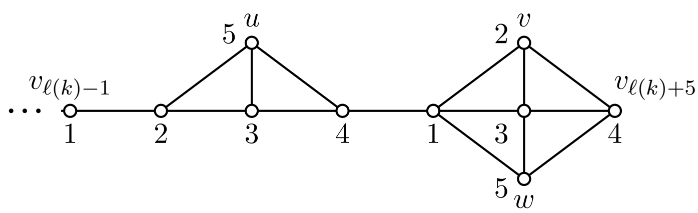


Figure 2. Graph G is obtained from $P_{\ell(k)+5}$, where $\mathrm{diam}(G) = \ell(k)+4$ and $\chi_{NL}(G) = k < k+1$.

As it can be seen, we have disproved Conjecture 1, but our examples show that $\chi_{NL}(G) = k = \chi_{NL}(P_{d+1}) - 1$ while the $\mathrm{diam}(G) = d$.

We imagine that for $k \geq 15$, we can find a graph G with $diam(G) = \ell(k)$ and $\chi_{NL}(G) \leq k-1$ where $\chi_{NL}(P_{\ell(k)+1}) = k+1$, and there may be exist a similar method for showing it. However, we pose the following question for the future research.

Question. *Does there exist a graph G such that* $\mathrm{diam}(G) = d$ *and* $\chi_{NL}(G) = k < \chi_{NL}(P_{d+1}) - 1$ *for each* $k \geq 4$?

Remark 3. We should remind that there is a family of graphs G whose diameter is d but $\chi_{NL}(G) \geq \chi_{NL}(P_{d+1})$. For example, consider the broom graph, (see Figure 3.)

Let $\ell(k-1)+1 \leq d \leq \ell(k)-1$. Then $\chi_{NL}(P_{d+1}) = k$. Let G the broom graph, (see Figure 3). Then it is easy to see that G has diameter d and $\chi_{NL}(G) \geq n+k = n+\chi_{NL}(P_{d+1})$.

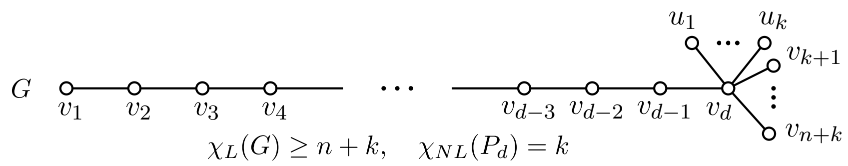


Figure 3. Graph G with $\chi_{NL}(G) > \chi_{NL}(P_d)$.

4. CONJECTURE 2

In this section, we prove Conjecture 2. We state this conjecture as a theorem.

Theorem 6 (Conjecture 2)**.** *Let $k \geq 2$. If T is a tree with $\chi_{NL}(T) = k$, then $\Delta(T) \leq (k-1)^2$, and this bound is tight for every integer $k \geq 2$.*

Proof. The tightness has been shown by Alcon *et al.* in [2]. We show that $\Delta(T) \leq (k-1)^2$ by proving a sequence of results.

Lemma 7. *Let T be a tree and $\chi_{NL}(T) = k$. Let v be a vertex of maximum degree $\Delta(T)$. If $\Delta(T) \geq (k-1)^2 + 1$, then the number of vertices of degree 1 or 2 in $N(v)$ is at least and at most*

$$(k-1)^2 - (k-3) \text{ and } (k-1)^2,$$

respectively.

Proof. Let T be a tree with $\chi_{NL}(T) = k$ and $\Delta(T) \geq (k-1)^2 + 1$.

First, we show the at least. Assume to the contrary, it is at most $(k-1)^2 - (k-3) - 1 = (k-1)^2 - (k-2)$. Then the number of vertices of degree at least 3 in $N(v)$ is at least $k - 1 = (k-1)^2 + 1 - ((k-1)^2 - (k-2))$. Since finally, at least two leaves are issued from each of the neighbors of degree at least 3 in $N(v)$, and since the sum of vertices of degrees of at least 3 and the vertices of degrees 2 or 1 is constant, so if the number of vertices of degree 2 or 1 decreases, then the number of vertices of degrees 3 increases, and the number of leaves on the tree will increase. Therefore, the number of leaves in T is at least $(k-1)^2 - (k-2) + 2(k-1) = k(k-1) + 1$, that is a contradiction.

Second, we show the at most. Let the color of vertex v be 1 in NL-coloring of T. Then the number of vertices of color $i \in [k] \setminus \{1\}$ of degrees 1 or 2 in $N(v)$ is at most $k-1$ (although the vertex of degree 1 in $N(v)$ with color i is not more than 1). If the vertex u of degree 2 in $N(v)$ is assigned with color i, and the vertex in $N(u) \setminus \{v\}$ is w, then whenever there does not exist a vertex of degree 1 of color i in $N(v)$, the colors of (v, u, w) are $(1, i, 1)$, $(1, i, 2), \ldots, (1, i, i-1)$, $(1, i, i+1), \ldots, (1, i, k)$ in NL-coloring of T. On the other hand, since the color i

changes in $[k] \setminus \{1\}$, we deduce that the color of a vertex of degree 1 or 2 adjacent to vertex v is at most $(k-1)^2$. $\square$

Lemma 8. *Let T be a tree and the data in Lemma 7 holds for T. If r is the number of vertices of degree 1 in $N(v)$, then $r \leq k-1$ and these r vertices take r different colors.*

Proof. Since the given r vertices are leaves and adjacent to the vertex v, in any NL-coloring of T, their colors should be different. On the other hand, since $\chi_{NL}(T) = k$, there exist $k-1$ colors other than color 1. Thus $r \leq k-1$. $\square$

Lemma 9. *Let T be a tree and the data in Lemma 7 holds for T. Assume that u is a vertex of degree 2 in $N(v)$ with color i. If w is a vertex in $N(u) \setminus \{v\}$, and no vertex of degree 1 in $N(v)$ take the color i, then the vertex w can be NL-colored with the color in $[k] \setminus \{i\}$ and if the vertex of degree 1 in $N(v)$ is assigned with the color i, then the vertex w can be NL-colored with the color in $[k] \setminus \{1, i\}$.*

Proof. If no vertex of degree 1 in $N(v)$ take the color i, then the number of vertices u of degree 2 in $N(v)$ with color i is at most $k-1$. On the other hand, if $w \in N(u) \setminus \{v\}$, and u is assigned with color i, then the color of w is one of the colors in $[k] \setminus \{i\}$. If the vertex of degree 1 in $N(v)$ is assigned with the color i, the proof is clear from the part 1. Therefore, the result is observed. $\square$

Lemma 10. *Let T be a tree and the data in Lemma 7 holds for T. Then, the number of vertices of degree $n \geq 3$ in $N(v)$ is at least 1 and at most $k-2$, and moreover, these vertices can be assigned with same color in an NL-coloring of T.*

Proof. From Lemma 7, the number of vertices of degree $n \geq 3$ is at least 1 and at most $k-2$.

Now, let $x_1, x_2, \ldots, x_s$ for $(k-1)^2 - (k-3) \leq s \leq (k-1)^2$ be the vertices of degree at most 2 in $N(v)$ and let $y_1, y_2, \ldots, y_t$ for $1 \leq t \leq (k-2)$ be the vertices of degree $n \geq 3$ in $N(v)$. Without loss of generality and with the pattern used in Lemma 7, we can use colors $2, 3, \ldots, k-1$ for vertices x_is, and then we can use color k for vertices of y_js in an NL-coloring. $\square$

Observation 11. *Let z, u, x be three successive vertices in T where u is of degree 2 and adjacent to z and x. If z, u, x are assigned with colors j, i, j, respectively, then there does not exist the leaf and its support with colors i and j, respectively, in any NL-coloring of T.*

Observation 12. *Let T be a tree and the data in Lemma 7 holds for T. Then the number of leaves in T is $\ell \geq (k-1)^2 + 2$.*

Proof. Since the number of vertices of degree $r \leq 2$ in $N(v)$ is at most $(k-1)^2$, and the number of vertices of degree $n \geq 3$ in $N(v)$ is at least 1. Therefore, $\ell \geq (k-1)^2 + 2$ in T. □

Theorem 13. *Let T be a tree and the data in Lemma 7 holds for T. Then, in any NL-coloring of T, the support vertices and their leaves cannot be NL-colorable with k colors.*

Proof. Suppose from Lemma 8, there exist $0 \leq r \leq k-1$ leaves in $N(v)$ and without loss of generality, whose be assigned with colors $2, 3, \ldots, r+1$. Thereby, we have at least $(k-1)^2 - (k-3) - r$ vertices of degree 2 in $N(v)$. This result and Observation 12 imply that, for the vertices u and $w \in \{N(u) \setminus \{v\}\}$, the triple (v, u, w) can be NL-colored as follows.

$$[(1, r+2, 1), (1, r+3, 1), \ldots, (1, k, 1);\ \ (1, i, 2), (1, i, 3), \ldots, (1, i, i-1),$$
$$(1, i, i+1), (1, i, k);\ \ (1, k, 2)], \quad \text{where} \ \ i \in [k-1] \setminus \{1\}.$$

On the other hand, from Observation 12, the number of leaves $\ell \geq (k-1)^2+2$. Now, from Observation 11 and the colors assigned to the leaves in $N(v)$ and (v, u, w), we have at most $(r \leq k-1)$ leaves in $N(v)$, and in addition for any color i, there exist at most $k-2$ leaves with color i which can be neighbor to any vertex with color in $[k] \setminus \{1, i\}$. Since $2 \leq i \leq k$, the number of leaves ℓ of T is at most $k - 1 + (k-2)(k-1) = (k-1)^2$, that is a contradiction. □

Now, the proof of $\Delta(T) \leq (k-1)^2$ is deduced from Lemmas 7–10, Observations 11–12 and Theorem 13. ■

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