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THE NUMBER OF DISJOINT PAIRS IN FAMILIES OF k-ELEMENT SUBSETS

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Abstract

We call a family $\mathcal{F}$ of subsets of $[n]$ intersecting if for every $A, B \in \mathcal{F}$ it holds that $A \cap B \neq \emptyset$. The famous theorem of Erdős-Ko-Rado states that the maximal size of an intersecting family of k-element subsets of $[n]$ is $\binom{n-1}{k-1}$, if $k \leqslant \frac{n}{2}$. In this paper, we study the number of disjoint pairs of sets in a family of size greater than this. We provide a bound on the number of disjoint pairs depending on the size of minimum vertex cover of the graph representation of the family. Moreover, we obtained a new, elementary proof for a special case of a theorem of Dan, Gas and Sudakov, which claims that the minimal number of disjoint pairs of sets in set systems of size greater than $\binom{n-1}{k-1}$ can be obtained by considering families consisting of the initial segment of lexicographical order. We prove it only for very small families of size $\binom{n-1}{k-1} + r$, where $r \leqslant \frac{1}{3k!} n^{k-1}$.

Keywords: family of subsets, Erdős-Ko-Rado theorem, disjoint pairs.

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1. Introduction

Let $[n] = \{1, 2, \ldots, n\}$ be an n-element set. $\binom{[n]}{k}$ denotes the family of all k-element subsets of $[n]$. A family $\mathcal{F} \subset \binom{[n]}{k}$ is called *intersecting* if any two members of $\mathcal{F}$ have a non-empty intersection. Let us start with the classic theorem of Erdős, Ko and Rado.

Theorem 1.1 [3]**.** *If $2k \le n$ and $\mathcal{F} \subset \binom{[n]}{k}$ is intersecting, then*

$$|\mathcal{F}| \le \binom{n-1}{k-1}.$$

The upper bound in this theorem is sharp: choose all k-element sets containing the element 1. We call a family *trivially intersecting* if the intersection of all members of the family has a non-empty intersection. Otherwise, if this intersection is empty then the family is called *non-trivially intersecting.* The following theorem determines the largest size of the non-trivially intersecting family.

Theorem 1.2 [4]**.** *If $2k \le n$ and $\mathcal{F} \subset \binom{[n]}{k}$ is non-trivially intersecting, then*

$$|\mathcal{F}| \le \binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1$$

which is smaller than $\binom{n-1}{k-1}$ if $2k < n$.

A trivial consequence of Theorem 1.1 is that if

$$|\mathcal{F}| = \binom{n-1}{k-1} + r \tag{1}$$

for $r > 0$, then $\mathcal{F}$ contains a pair of disjoint members. Define the *disjoint-pair graph* $\mathrm{DP}(\mathcal{F})$ on the vertex set $\mathcal{F}$ where two vertices are adjacent if and only if the corresponding members of $\mathcal{F}$ are disjoint. The number of edges in $\mathrm{DP}(\mathcal{F})$ is denoted by $\mathrm{dp}(\mathcal{F})$. This is the number of disjoint pairs in $\mathcal{F}$. The above mentioned consequence of the Erdős-Ko-Rado theorem is that if $\mathcal{F}$ satisfies (1) then $\mathrm{dp}(\mathcal{F}) \ge 1$. However this is not sharp. Take the largest intersecting family, e.g., all the k-element sets containing the element 1 and add one more set, say $\{2, 3, \ldots, k+1\}$. In this family there are $\binom{n-k-1}{k-1}$ disjoint pairs. It was observed that this is the minimum.

Theorem 1.3 [5]**.** *If $\mathcal{F}$ satisfies* (1) *with $r = 1$, then*

$$\mathrm{dp}(\mathcal{F}) \ge \binom{n-k-1}{k-1}.$$

For arbitrary r in (1) we need to introduce an ordering in $\binom{[n]}{k}$ to be able to formulate the relevant statements. Take the characteristic vectors of the k-element sets and order them *lexicographically*. In other words, order them by their binary values. Then the first set will be $\{n-k+1,\ldots,n\}$. Let $\mathcal{L}_{n,k}(t)$ be the family of initial t sets of the lexicographical order while $\bar{\mathcal{L}}_{n,k}(t)$ the family of the last t sets. Ahlswede and Katona [1] solved the case $k=2$ showing that $\mathrm{dp}(\mathcal{F})$ is minimized either for $\mathcal{L}_{n,2}(|\mathcal{F}|)$ or for $\bar{\mathcal{L}}_{n,2}(|\mathcal{F}|)$.

A major step toward the full solution for general k was made by Das, Gan and Sudakov who proved that $\mathrm{dp}(\mathcal{F})$ is minimized for $\mathcal{L}_{n,k}(|\mathcal{F}|)$ if n is large enough. They gave an explicit lower bound for n ensuring their result.

Theorem 1.4 [2]. *Suppose that $n > 108k^2\ell(k+\ell)$ and $|\mathcal{F}| = t \le \binom{n}{k} - \binom{n-\ell}{k}$. Then $dp(\mathcal{F})$ is minimized for $\mathcal{L}_{n,k}(t)$.*

The main purpose of the present paper is to investigate how does $\mathrm{dp}(\mathcal{F})$ increase if certain configurations are excluded. There is an equality in Theorem 1.3 when one k-element set is added to the optimal construction for the Erdős-Ko-Rado theorem. But then this new set is involved in all disjoint pairs. What happens if this is excluded? Does this condition increase $\mathrm{dp}(\mathcal{F})$? Our answer will be positive.

But we will prove a more general statement. In general, let $\tau(G)$ be the minimum vertex cover of the graph G, that is the smallest number of vertices covering at least one vertex of all edges. We will determine the lower bound for $\mathrm{dp}(\mathcal{F})$ for families of size (1) with $r=1$ and satisfying the condition $\tau(\mathrm{DP}(\mathcal{F})) \ge s$ for some positive integer s. This bound will be almost a linear function of s for small values.

In the more general case when the size of the family satisfies (1), notice that the members of the family not covered by the vertex cover are mutually intersecting, therefore their number is at most $\binom{n-1}{k-1}$ by the Erdős-Ko-Rado theorem, the total size of the family is at most $\binom{n-1}{k-1} + \tau(G)$. Hence we have

$$\tau(G) \ge r. \tag{2}$$

We noticed that our method also works in the case of Theorem 1.4, but only for relatively small t. Therefore our related theorem below is a small special case of Theorem 1.4, but it is worth showing since its proof is much simpler.

2. RESULTS

Theorem 2.1. *For every k there exists n_0 such that for all n, r, s with $n > n_0$ and $r \leqslant s \leqslant \frac{n^{k-1}}{3k!}$ the following holds. If $\mathcal{F} \subset \binom{[n]}{k}$, $|\mathcal{F}| = \binom{n-1}{k-1} + r$ and*

$\tau(\mathrm{DP}(\mathcal{F})) \geq s$, then

$$\mathrm{dp}(\mathcal{F}) \geq s\left(\binom{n-k-1}{k-1} + r - s\right). \tag{3}$$

If we take $r = s = 1$ we get back Theorem 1.3. However if $\tau(\mathrm{DP}(\mathcal{F}))$ is 1 then $\binom{n-1}{k-1}$ members are pairwise intersecting. By Theorem 1.2 it is clear that in the case $2k < n$ this implies that the members of $\mathcal{F}$ form a trivially intersecting family and one more set. Otherwise $\tau(\mathrm{DP}(\mathcal{F})) \geq 2$. We obtained the following corollary for "the second best value" of $\mathrm{dp}(\mathcal{F})$.

Corollary 2.2. *Suppose that $\mathcal{F} \subset \binom{[n]}{k}, |\mathcal{F}| = \binom{n-1}{k-1} + 1$. Then either $\mathcal{F}$ is a trivially intersecting family plus one more k-element set, or*

$$\mathrm{dp}(\mathcal{F}) \geq 2\binom{n-k-1}{k-1} - 2$$

if n is large enough.

We will see that the following theorem is a special case of Theorem 2.1.

Theorem 2.3. *For every k there exists n_0 such that for all n, r with $n > n_0$ and $r \leqslant \frac{n^{k-1}}{3k!}$ the following holds. Suppose that $\mathcal{F} \subset \binom{[n]}{k}$ and $|\mathcal{F}| = \binom{n-1}{k-1} + r$. Then*

$$\mathrm{dp}(\mathcal{F}) \geq r\binom{n-k-1}{k-1}.$$

3. TOOLS

In order to prove our main results, we will state and prove some useful lemmas. Firstly, let us define the degree of an element in a family $\mathcal{F}$. For every $x \in [n]$ let $\mathcal{F}(x)$ be the family of sets from $\mathcal{F}$ that contain x. Then $|\mathcal{F}(x)|$ is the degree of x in $\mathcal{F}$. Define $\Delta := \max_{x \in [n]} |\mathcal{F}(x)|$.

Lemma 3.1. *If $\mathcal{F} \subset \binom{[n]}{k}$, $|\mathcal{F}| = \binom{n-1}{k-1} + r$ for $r \in \mathbb{Z}_{\geqslant 1}$ then*

$$\mathrm{dp}(\mathcal{F}) \geqslant \left(\Delta - \binom{n-1}{k-1} + \binom{n-k-1}{k-1}\right)\left(\binom{n-1}{k-1} + r - \Delta\right). \tag{4}$$

Proof. Without loss of generality, we can assume that 1 is an element of the greatest degree, i.e., $\Delta = |\mathcal{F}(1)|$, and let us define

$$\mathcal{F}_1 = \{F \in \mathcal{F} \mid 1 \in F\},$$
$$\mathcal{F}_0 = \{F \in \mathcal{F} \mid 1 \notin F\}.$$

From the definition, it holds that $\mathcal{F}_1 = \mathcal{F}(1)$. Then, $|\mathcal{F}_0| = \binom{n-1}{k-1} + r - \Delta$. Fix a $G \in \mathcal{F}_0$. We will calculate the number of sets from $\mathcal{F}_1$ having a non-empty intersection with G. The number of all possible k-element subsets of $[n]$ containing 1, but not containing any element of G is $\binom{n-k-1}{k-1}$. Thus the total number of all possible k-element subsets of $[n]$ which contain 1 and have a non-empty intersection with G is $\binom{n-1}{k-1} - \binom{n-k-1}{k-1}$. It follows then that

$$\begin{aligned}\left|\{F \in \mathcal{F}_1 \mid F \cap G \neq \emptyset\}\right| &\leqslant \binom{n-1}{k-1} - \binom{n-k-1}{k-1},\\ \left|\{F \in \mathcal{F}_1 \mid F \cap G = \emptyset\}\right| &\geqslant \Delta - \binom{n-1}{k-1} + \binom{n-k-1}{k-1}.\end{aligned}$$

Moreover, the number of all possible choices of G is $\binom{n-1}{k-1} + r - \Delta$. Thus

$$\mathrm{dp}(\mathcal{F}) \geqslant \left(\Delta - \binom{n-1}{k-1} + \binom{n-k-1}{k-1}\right)\left(\binom{n-1}{k-1} + r - \Delta\right).$$

■

To make use of the above lemma, let us introduce the following notation.

Definition. For fixed positive integers n, k and r define

$$f_r(\Delta) = \left(\Delta - \binom{n-1}{k-1} + \binom{n-k-1}{k-1}\right)\left(\binom{n-1}{k-1} + r - \Delta\right).$$

For fixed n, k and r, $f_r(\Delta)$ is a parabola as a function of Δ. We will use it near to its "right hand root" where it will give sharp estimates for us. However it is very weak near the "left hand root". This is why we need another estimate for the small values of Δ.

Lemma 3.2.

$$\mathrm{dp}(\mathcal{F}) \geqslant \binom{|\mathcal{F}|}{2} - \frac{k(\Delta-1)|\mathcal{F}|}{2}. \tag{5}$$

Proof. Let us first prove the following estimate on the number $\mathrm{ip}(\mathcal{F})$ of intersecting pairs in $\mathcal{F}$.

$$\mathrm{ip}(\mathcal{F}) \leqslant \sum_{i=1}^{n} \binom{|\mathcal{F}(i)|}{2}. \tag{6}$$

Fix $i \in [n]$. Every pair of sets containing i form an intersecting pair of sets, so we have exactly $\binom{|\mathcal{F}(i)|}{2}$ pairs of sets containing i. Some pairs of sets have more than one common element, so summing all $\binom{|\mathcal{F}(i)|}{2}$, we count certain pairs of intersecting sets more than once. Thus (6) is proved.

Now, to prove (5), observe that

$$\sum_{i=1}^{n}\binom{|\mathcal{F}(i)|}{2} = \sum_{i=1}^{n}\frac{|\mathcal{F}(i)|\cdot\big(|\mathcal{F}(i)|-1\big)}{2} \leqslant \frac{\Delta-1}{2}\cdot\sum_{i=1}^{n}|\mathcal{F}(i)|$$
$$= \frac{k(\Delta-1)|\mathcal{F}|}{2},$$

where the last equality comes from the fact that adding the degrees of all elements gives the same result as counting every set of $\mathcal{F}$ exactly k times.

The number of disjoint pairs can be bounded from below by subtracting the upper bound for $\mathrm{ip}(\mathcal{F})$ from the total number of pairs of sets from $\mathcal{F}$, thus

$$\mathrm{dp}(\mathcal{F}) \geqslant \binom{|\mathcal{F}|}{2} - \frac{k(\Delta-1)|\mathcal{F}|}{2},$$

which ends the proof of (5). ■

The right hand side of (5) is a linear function of Δ. Let us introduce the following notation.

Definition. For fixed positive integers n, k and r, define

$$g_r(\Delta) = \binom{|\mathcal{F}|}{2} - \frac{k(\Delta-1)|\mathcal{F}|}{2},$$

where $|\mathcal{F}| = \binom{n-1}{k-1} + r$.

Lemma 3.3. *Suppose that $1 \le r \le s$ and let*

$$\Delta_1(r,s) = \binom{n-1}{k-1} + r - s$$

and

$$\Delta_2(s) = \binom{n-1}{k-1} - \binom{n-k-1}{k-1} + s.$$

Suppose that $r = r(n) \le s = s(n) \leqslant c(k)n^{k-1} = \frac{1}{3k!}n^{k-1}$ holds for large n. Then

$$f_r(\Delta_1(r,s)) \le g_r(\Delta_2(s))$$

also holds for large n.

Proof. We actually have to prove

$$\left(\binom{n-k-1}{k-1}+r-s\right)s$$
$$+\frac{k}{2}\left(\binom{n-1}{k-1}-\binom{n-k-1}{k-1}+s-1\right)\left(\binom{n-1}{k-1}+r\right)\leq\binom{\binom{n-1}{k-1}+r}{2}.$$

Since $r\leq s$, we may consider the following stronger inequality:

$$\binom{n-k-1}{k-1}s$$
$$+\frac{k}{2}\left(\binom{n-1}{k-1}-\binom{n-k-1}{k-1}+s\right)\left(\binom{n-1}{k-1}+s\right)\leq\binom{\binom{n-1}{k-1}}{2}.$$

Take only the terms which can reach the order of magnitude of n^{2k-2}. It is easy to see that $\binom{n-1}{k-1}-\binom{n-k-1}{k-1}=O(n^{k-2})$ and hence this term can be deleted. Supposing that $s(n)\leq c(k)n^{k-1}$ we obtain

$$\frac{n^{k-1}}{(k-1)!}c(k)n^{k-1}+\frac{k}{2}c(k)n^{k-1}\frac{n^{k-1}}{(k-1)!}+\frac{k}{2}c^2(k)n^{2k-2}\leq\frac{n^{2k-2}}{2(k-1)!^2}$$

or

$$c^2(k)+\frac{k+2}{k!}c(k)-\frac{1}{k!(k-1)!}\leq 0.$$

Indeed $c(k)=\frac{1}{3k!}$ satisfies this inequality. ■

4. THE PROOFS OF THE THEOREMS

Proof. Without loss of generality one can suppose that the element 1 has the maximum degree Δ. Choosing all members of $\mathcal{F}$ not containing 1 will be a vertex cover of $\mathrm{DP}(\mathcal{F})$. Hence we have

$$\binom{n-1}{k-1}+r\geq\Delta+\tau(\mathrm{DP}(\mathcal{F}))\geq\Delta+s.$$

In other words, Δ is less than the right hand root of the quadratic function $f_r(\Delta)$ by at least s:

$$\Delta\leq\binom{n-1}{k-1}+r-s.$$

If we substitute this maximum value of Δ into $f_r(\Delta)$ then the desired right hand side of (3) is obtained as a lower bound for $\mathrm{dp}(F)$. Note that $\binom{n-k-1}{k-1}>s$. The value of f_r is greater or equal to this lower bound for the arguments in the interval $\left[\binom{n-1}{k-1} - \binom{n-k-1}{k-1} + s, \binom{n-1}{k-1} + r - s\right]$, namely it holds that

$$f_r(x) \geq s \cdot \left(\binom{n-k-1}{k-1} + r - s\right).$$

When Δ is smaller than the numbers from this interval, then Lemmas 3.2 and 3.3 help us out. Namely, from the first of those lemmas, we know that the number of disjoint pairs can be estimated from below by $g_r(\Delta)$. From the second one it follows that

$$\begin{aligned} f_r\left(\binom{n-1}{k-1} - \binom{n-k-1}{k-1} + s\right) &= f_r\left(\binom{n-1}{k-1} + r - s\right) \\ &\leq g_r\left(\binom{n-1}{k-1} - \binom{n-k-1}{k-1} + s\right). \end{aligned}$$

Since the linear function g_r is decreasing, we deduce that the inequality

$$\begin{aligned} \mathrm{dp}(\mathcal{F}) &\geq g_r(\Delta) \geq g_r\left(\binom{n-1}{k-1} - \binom{n-k-1}{k-1} + s\right) \\ &\geq f_r\left(\binom{n-1}{k-1} - \binom{n-k-1}{k-1} + s\right) = s \cdot \left(\binom{n-k-1}{k-1} + r - s\right) \end{aligned}$$

holds for the values of Δ smaller than $\binom{n-1}{k-1} - \binom{n-k-1}{k-1} + s$. The proof of Theorem 2.1 is completed.

In Theorem 2.3 there is no condition on $\tau(\mathrm{DP}(\mathcal{F}))$. By (2) this is equivalent to the assumption $s = r$, therefore we do not have to do anything else just to replace s by r in Theorem 2.1 to obtain the statement of Theorem 2.3 ■

Finally let us consider the following construction. Choose all the k-element subsets of $[n]$ containing the element 1 and r k-element subsets containing the element 2 but not 1. Here $\mathrm{DP}(\mathcal{F})$ is a bipartite graph with degree $\binom{n-k-1}{k-1}$ in the second class. Hence we have $\tau(\mathrm{DP}(\mathcal{F})) \leq r\binom{n-k-1}{k-1}$ showing that the estimate of Theorem 2.3 is sharp in our range.

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