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MAJORITY DOMINATOR COLORINGS OF GRAPHS

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Abstract

Let G be a simple graph of order n. A majority dominator coloring of a graph G is proper coloring in which each vertex of the graph dominates at least half of one color class. The majority dominator chromatic number $\chi_{md}(G)$ is the minimum number of color classes in a majority dominator coloring of G. In this paper we study properties of the majority dominator coloring of a graph. We obtain tight upper and lower bounds in terms of chromatic number, dominator chromatic number, maximum degree, domination and independence number. We also study the majority dominator coloring number of selected families of graphs.

Keywords: majority dominator chromatic number, majority dominator coloring, chromatic number, independence number, domination number.

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1. Introduction

Let $G = (V, E)$ be a simple graph. For any vertex $v \in V$, the open neighborhood of v is the set $N(v) = \{u \in V : uv \in E\}$ and the closed neighborhood of v is the set $N[v] = N(v) \cup \{v\}$. For a set $S \subseteq V$, the open neighborhood of S is

$N(S) = \bigcup_{v \in S} N(v)$ and the closed neighborhood of S is $N[S] = N(S) \cup S$. A set $S \subseteq V$ is a dominating set if $N[S] = V$, or equivalently, every vertex in $V \setminus S$ is adjacent to at least one vertex in S. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set in G. A dominating set with cardinality $\gamma(G)$ is called a $\gamma(G)$-set, for more information of these parameters, refer to [13, 14]. A set $I \subseteq V(G)$ is called an independent set if the induced subgraph $G[I]$ has no edge. The size of maximum independent set in G is called the independence number and is denoted by $\alpha(G)$ [23]. Finally, the size of any maximum matching in G is called the matching number and denoted by $\nu(G)$.

Further in this paper we are going to consider some graph products. Let G and H be two graphs. The *strong product* $G \boxtimes H$ is a graph with the vertex set $V(G) \times V(H)$. Two vertices (g, h) and (g', h') are adjacent in $G \boxtimes H$ if either $g = g'$ and h is adjacent with h' in H, or $h = h'$ and g is adjacent with g' in G, or g is adjacent with g' in G and h is adjacent with h' in H (see e.g. [12, p. 36]). The corona product $G \circ H$ is in turn a graph consisting of G and $|V(G)|$ copies of H, where each copy H_j corresponds with a different vertex $v_j \in V(G)$ and all its vertices are adjacent with v_j.

A proper coloring of a graph G is an assignment of colors to its vertices, such that no two adjacent vertices receive the same color. The chromatic number $\chi(G)$ is the minimum number of colors required for a proper coloring of G. A dominator coloring is a proper coloring in which each vertex v of the graph dominates every vertex of some color class, i.e., there is a color class being a subset of the closed neighborhood of v. In particular, this can be the color class of v in the special case when it consists only of v itself. The dominator chromatic number $\chi_d(G)$ is the minimum number of color classes in a dominator coloring of G ([11]). The dominator colorings were introduced by Gera *et al.* [11] and later studied by several authors. In particular, Kavitha and David [15] found dominator chromatic number of central graph of various graph families such as cycles, paths or wheel graphs. They also compared these parameters with the dominator chromatic number of the respective base graph families. Arumugam and Bagga [5] obtained several result on graphs such that $\chi_d(G) = \chi(G)$ and $\chi_d(G) = \gamma(G)$. In particular, they proved that if $\mu(G)$ is the Mycielskian of G, then $\chi_d(G) + 1 \leq \chi_d(\mu(G)) \leq \chi_d(G) + 2$. Other results on the dominator coloring in graphs can be found in the works of Abdolghafurian *et al.* [1] (claw-free graphs), Bagan *et al.* [7] (arbitrary graphs, P_4-sparse graphs, P_5-free graphs, claw-free graphs, graphs with bounded treewidth), Chellali and Maffray [8] (trees, chordal graphs, P_4-free graphs and graphs with dominator coloring number at most 3), Gera [9, 10] (various classes of graphs, in particular bipartite graphs), Merouane and Chellali [19] (trees), Kazemi [16] (total version of the problem) and Ramachandran *et al.* [21] (e.g. windmill graphs). Askari *et al.* [6] studied total global dominator coloring, where every vertex needs not only to dominate some color class other than its own, but also to be not adjacent to any vertex from another color class. Mojdeh *et al.* [20] in turn considered strong dominator colorings, i.e., dominator colorings with an additional degree constraint, of central graphs and trees.

Instead of studying proper colorings, where all the neighbors of every vertex v have color different than the color of v, one can focus on other kinds of coloring, e.g. majority coloring, where only half of the neighbors of every vertex need to have different color (or, which is equivalent in the case of finite graphs, at most half of the neighbors of any vertex v can have the same color as v). It was proved a long time ago by Lovász that the majority coloring for any simple graph requires at most 2 colors [18]. The problem was then considered for infinite graphs (see e.g. [2, 22]). Recently the problem was investigated also for digraphs, where the out-neighbors were taken under consideration instead of all the neighbors. Also the list version was analyzed, where every vertex can have a different list of available colors. The best known results here say that four colors are enough for finite digraphs and five for infinite ones (see e.g. [3, 4, 17]).

Motivated by the above concepts, we decided to combine them. One could do this in several ways, by relaxing the condition of proper coloring or the condition of dominating the entire color class. Thus, one could require e.g. a majority coloring where every vertex dominates some color class. In this paper, however, we are interested in somehow opposite problem, where the coloring is supposed to be proper, but only half of some color class is supposed to be dominated by every vertex. Recall that v domimates u if u is v itself or a neighbor of v.

Definition 1.1. *Majority dominator coloring* of a graph G is a proper coloring in which each vertex of G dominates at least half of some color class. The majority dominator chromatic number $\chi_{md}(G)$ is the minimum number of color classes in a majority dominator coloring of G.

Note that, according to the above definition, the domination condition is satisfied also if some vertex v dominates half of its own color class. Of course, in such case this color class can consist of at most two vertices (v and some $u \notin N(v)$).

The structure of the paper is as follows. In the next section we present some general results. In particular we characterize graphs G for which $\chi_{md}(G)$ takes some specific values. In Section 3 we derive the values of $\chi_{md}(G)$ for graphs G from several families. We conclude the paper in the last section with some final remarks, conjectures and open problems.

2. General Results

Since every majority dominator coloring is proper by definition, and on the other hand every dominator coloring is also a majority dominator coloring, we have that

$$\chi(G) \leq \chi_{md}(G) \leq \chi_d(G) \leq n$$

for every graph G. Moreover, these bounds are sharp, as the example of the complete graph K_n shows. The above implies in particular, that for every graph G we have

$$\chi_{md}(G) \leq \chi(G) + \gamma(G),$$

since the inequality $\chi_d(G) \leq \chi(G) + \gamma(G)$ follows from [10, Theorem 3.3].

A well-known fact (see e.g. [23, Lemma 3.1.33]) is that a set of vertices S is an independent dominating set of G if and only if it is a maximal independent set. This fact immediately implies that $\chi_{md}(G) \leq \chi(G) + \alpha(G)$, but we can use it to prove a stronger inequality.

Theorem 1. *Let G be a connected graph. Then $\chi_{md}(G) \leq \chi(G) + \left\lceil \frac{\alpha(G)}{2} \right\rceil - 1$ and the bound is sharp.*

Proof. Let $c : V(G) \to \{1, \ldots, k\}$, where $k = \chi(G)$, be a proper k-coloring of G and let $V_i = \{v \in V(G) : c(v) = i\}$. Observe that there must be a proper coloring c in which V_k is maximal (not necessarily maximum) independent set (otherwise we can extend it by adding new vertices and recolor them with color k). Since V_k is maximal independent, it is also a dominating set in G. In particular, we have that $|V_k| \leq \alpha(G)$, $\chi(G[V(G) \setminus V_k]) = \chi(G) - 1$ (in fact, $G[V(G) \setminus V_k]$ has a proper coloring with colors $1, \ldots, k-1$ which is a restriction of c) and every vertex from $V(G) \setminus V_k$ is adjacent to at least one vertex from V_k. Now, we leave two vertices from V_k colored with k, and recolor the remaining $|V_k| - 2$ of them arbitrarily with $\lceil (|V_k| - 2)/2 \rceil$ new colors so that each color appears at most twice. Since we are recoloring only vertices of one color class of G, the new coloring is a proper coloring with $k + \lceil (|V_k| - 2)/2 \rceil \leq \chi(G) + \lceil (\alpha(G))/2 \rceil - 1$ colors. Every vertex of V_k has itself in its own closed neighborhood. Similarly, as we already observed, every vertex of $V(G) \setminus V_k$ has a neighbor in V_k. Since every color class in V_k has at most two members, the obtained coloring is a majority dominator coloring. The sharpness of the bound can be observed e.g. for $G = C_n \circ K_1$, where $n \geq 4$ is even (i.e., G consists of an even n-cycle, with n extra pendant vertices, each of them adjacent to another cycle vertex). In these graphs $\chi(G) = 2, \alpha(G) = n$, and $\chi_{md}(G) = \lceil n/2 \rceil + 1$ (see Proposition 3.1 for details and note that this property does not hold for odd n). ■

Gera, Rasmussen and Horton [11, Proposition 2.4] proved that for every connected G of order $n \geq 3$, $\chi_d(G) \leq n + 1 - \alpha(G)$ holds, which implies in particular that $\chi_{md}(G) \leq n + 1 - \alpha(G)$. Also this inequality can be strengthened. Recall that $\overline{G}$ denotes the complement of G, that is the graph for which $V(\overline{G}) = V(G)$ and $e \in E(\overline{G}) \iff e \notin E(G)$. G[X] in turn denotes the induced subgraph of G with vertex set X.

Theorem 2. *Let G be a connected graph of order $n \geq 2$, where I is an independent set in G and M is a matching in $\overline{G[V(G) \setminus I]}$. Then $\chi_{md}(G) \leq n - |M| - |I| + 1$ and the bound is sharp.*

Proof. Define the coloring $c : V(G) \to \{1, \ldots, n-|M|-|I|+1\}$ as follows. Color all the vertices in I with color 1, the pairs of vertices in M with distinct colors $2, \ldots, |M|+1$ and the remaining $n-|I|-2|M|$ vertices of $V(G) \setminus (I \cup V(M))$ with distinct colors from the set $|M|+2, \ldots, n-|I|-|M|+1$. This coloring is a majority dominator coloring. It is proper, since the two-element color classes in M are independent sets in G, I is independent by definition and all the remaining color classes are singletons. Moreover, every vertex in the graph dominates some vertex from M (that is, half of the 2-element color class) or a vertex in $V(G) \setminus (I \cup V(M))$ (that is, the entire 1-element class). Indeed, every vertex in $V(G) \setminus I$ dominates itself, while every vertex in I is adjacent to some vertex in $V(G) \setminus I$. The sharpness of the bound can be observed e.g. for $G = K_n$ with any nonempty I, where $|I| = 1$, $|M| = 0$ and $\chi_{md}(G) = n$. ■

This immediately implies the following (recall that $\nu(G)$ denotes the size of a maximum matching in G).

Corollary 2.1. *Let G be a connected graph of order $n \geq 2$, where I is a maximum independent set. Then $\chi_{md}(G) \leq n + 1 - \alpha(G) - \nu(\overline{G \setminus I})$.*

If $\Delta(G) = n - 1$, then $\gamma(G) = 1$ and since $\chi(G) = \chi_d(G)$ by [10, Lemma 3.4], we get the following.

Observation 2.2. *Let G be a connected graph with $\Delta(G) = n - 1$. Then*

$$\chi_{md}(G) = \chi(G).$$

We can show that the latter holds for a wider family of graphs.

Proposition 2.3. *Let G be a connected graph with $\Delta(G) \geq n-2$. Then $\chi_{md}(G) = \chi(G)$.*

Proof. Since by Observation 2.2 we can assume that there is a vertex of degree $n-2$, say v_1, there is also exactly one vertex non-adjacent to v_1, say v_2. Consider any proper coloring c of G with $\chi(G)$ colors. The class of $c(v_1)$ has one or two members (in the latter case $c(v_1) = c(v_2)$). Define the coloring c' (possibly equal to c) as $c'(v_2) = c(v_1)$ and $c'(v) = c(v)$ for $v \neq v_2$. Obviously c' is proper, the class of $c'(v_1)$ has exactly two members and every vertex $v \in V(G) \setminus \{v_1, v_2\}$ is adjacent to v_1. Thus c' is a majority dominator coloring. ■

Let us present a result for disconnected graphs.

Proposition 2.4. *If G is a disconnected graph with components $G_1, G_2, \ldots, G_k$ $(k \geq 2)$, then*

$$\max_{j\in\{1,2,\ldots,k\}} \chi_{md}(G_j) \leq \chi_{md}(G) \leq \sum_{j=1}^{k}(\chi_{md}(G_j)).$$

The bounds are sharp.

Proof. For every j, $j \in \{1,2,\ldots,k\}$, let $c_j : V(G_j) \to \{c_j^1, c_j^1+1, \ldots, c_j^2 = c_j^1 + \chi_{md}(G_j) - 1\}$ be a majority dominator coloring of the component G_j, $j = 1,\ldots,k$. If the color sets of components are pairwise disjoint, we can assume that $c_j^1 = c_{j-1}^2 + 1$ for $j = 2,\ldots,k$. Then obviously $\bigcup_{j=1}^{k}\{c_j\}$ is a majority dominator coloring of G and $\chi_{md}(G) \leq \sum_{j=1}^{k}(\chi_{md}(G_j))$. To see that the bound is sharp, consider $G_j = C_8$ for $1 \leq j \leq k$. Here we have $\chi_{md}(G_j) = 2$ for every $j = 1,\ldots,k$. Indeed, we have $\chi(G_j) = 2$ and there is exactly one proper coloring with 2 colors, up the obvious renaming of vertices. Every color appears exactly four times in the coloring and each vertex dominates exactly one vertex of its own color class (itself) and exactly two vertices of the other color class. We claim that repeating this pattern for every component G_j with disjoint sets of colors is optimal (even if not unique optimal). Indeed, given any vertex v in any of the components and any proper coloring of it, at most 2 vertices in the closed neighborhood of v can be colored with the same color. This means that any vertex v can dominate at most 2 colors from one color class and so any color class can consist of at most 4 colors.

To prove the lower bound note the following. By definition, for any majority dominator coloring c of G, the restriction of c to G_j is a majority dominator coloring of G_j, since the adjacency relations in any G_j are exactly the same as in G. This implies that $\chi_{md}(G_j) \leq \chi_{md}(G)$ for every $j = 1,\ldots,k$.

To see that the bound is sharp, let $G_1 = K_n$ for $n \geq 2$ and $G_j = K_2$ for every $j = 2,\ldots,n$. Here we have $\chi_{md}(G) = \chi_{md}(G_1) = n > 2 = \chi_{md}(G_j)$, $j = 2,\ldots,n$ (to see that $\chi_{md}(G) \leq n$, consider coloring $c : V(G) \to \{1,\ldots,n\}$ in which for $j > 1$, the vertices of G_j are colored with 1 and j.). ■

Note that in order to prove the lower bound in the above theorem, we used the fact that the restriction of a majority dominator coloring c of G to G_j is a majority dominator coloring of G_j, because the adjacency relations are preserved. As one can see, it is not true for arbitrary subgraphs. To see it, note that for example $\chi_{md}(C_{14}) = 5$ (see Theorem 5, Lemma 6 and Corollary 3.5 for details). An example of the respective coloring is the one with the colors of consecutive vertices being $1,2,1,2,1,2,5,4,3,4,3,4,3,5$. On the other hand, if we add the edges v_1v_8 and v_7v_{14}, then we obtain a majority dominator coloring with 4 colors by recoloring v_7 with color 1 and v_{14} with color 4. This allows us to state the following.

Observation 2.5. *The majority dominator colorability is not a hereditary property, that is, it is not preserved by subgraphs in the general case.*

In the remainder of this section we discuss the graphs G such that $\chi_{md}(G) \leq 2$ or $\chi_{md}(G) \geq n-1$. Below we use $d_S(v) = |N(v) \cap S|$ to denote the number of neighbors of v being members of a set S. Let us start with graphs G, for which $\chi_{md}(G)$ is close to its minimum possible value.

Observation 2.6. *Let G be a graph. Then $\chi_{md}(G) = 1$ if and only if $G = \overline{K_n}$, $1 \leq n \leq 2$.*

Proof. If $E(G) = \emptyset$, then obviously $\chi_{md}(G) = \lceil \frac{n}{2} \rceil$. If $E(G) \neq \emptyset$, then $\chi_{md}(G) \geq \chi(G) \geq 2$. ■

Note that trivially $\chi(G) = 1$ if and only if $G = \overline{K_n}$ for any $n \geq 1$, while $\chi_d(G) = 1$ if and only if $G = K_1 = \overline{K_1}$.

Proposition 2.7. *A graph $G \notin \{K_1, \overline{K_2}\}$ satisfies $\chi_{md}(G) = 2$ if and only if it is bipartite, where X and Y are partition sets such that at least one of the following holds.*

(i) $1 \leq |X| \leq |Y| \leq 2$,

(ii) $1 \leq |X| \leq 2 < |Y|$ *and for every* $y \in Y$, $d(y) \geq 1$,

(iii) $3 \leq |X| \leq |Y|$, *and for every* $x \in X$ *and* $y \in Y$, $d(x) \geq |Y|/2$ *and* $d(y) \geq |X|/2$, *respectively.*

Proof. By Observation 2.6, for every graph G from the described families we have $\chi_{md}(G) \geq 2$. On the other hand, it is straightforward to check that in every case the coloring assigning color 1 to the vertices of X and color 2 to the vertices of Y is a majority dominator coloring and thus $\chi_{md}(G) \leq 2$.

Now assume that $\chi_{md}(G) = 2$. Obviously $G \notin \{K_1, \overline{K_2}\}$ by Observation 2.6 and G is bipartite since $\chi(G) \leq 2$. Let X and Y be the partition sets. Each vertex must dominate half of the vertices in at least one partition set: its own (but then the order of this set must be at most 2) or the other one (with at least half of the vertices of this set being its neighbors in this case). Thus, three cases are possible: $|X| \leq |Y| \leq 2$ (covered by (i)), $|X| \leq 2$ and $|Y| \geq 3$, while every vertex in Y dominates at least half of the vertices in X (covered by (ii)) and both $|X|, |Y| \geq 3$, while every vertex in X (and Y) dominates at least half of the vertices in Y (X, respectively). ■

The following is immediate.

Corollary 2.8. *For every $1 \leq m \leq n$, we have $\chi_{md}(K_{m,n}) = 2$.*

The analogous statements for proper and dominator colorings are well-known: $\chi(G) = 2$ if and only if G is bipartite, while $\chi_d(G) = 2$ if and only if G is complete bipartite.

Now we switch to the graphs G, for which $\chi_{md}(G)$ takes maximum possible values.

Proposition 2.9. *A graph G satisfies $\chi_{md}(G) = n$ if and only if $G = K_n$.*

Proof. The only proper coloring of K_n is also its majority dominator coloring. Thus $\chi_{md}(K_n) = \chi(K_n) = n$. On the other hand, assume that for some graph G, every vertex is colored with a different color. If some two vertices x and y are not adjacent, one can recolor y with the color of x and obtain this way a majority dominator coloring (where each vertex dominates at least half of its own color class) with $n-1$ colors, thus $\chi_{md}(G) \leq n-1$. This means that if $\chi_{md}(G) = n$, then there are no non-adjacent vertices and so $G = K_n$. ■

Note that $\chi(G) = n$ if and only if $G = K_n$, which is also equivalent with $\chi_d(G) = n$.

Let $E(v)$ denote the set of edges incident with given vertex $v \in G$.

Proposition 2.10. *Let G be a connected graph of order $n \geq 3$. Then $\chi_{md}(G) = n-1$ if and only if $G = K_n - E'(v)$ for some set $E'(v) \subset E(v)$ such that $v \in V(G)$ and $1 \leq |E'(v)| \leq n-2$.*

Proof. Let $G = K_n - E'(v)$. Observe that G is exactly K_{n-1} with one extra v, where $1 \leq d(v) \leq n-2$. It means that $\Delta(G) = n-1$ and $\chi(G) = n-1$, so by Proposition 2.3, $\chi_{md}(G) = n-1$.

Now, assume that there is a connected graph G such that $\chi_{md}(G) = n-1$. We claim that $\Delta(G) = n-1$. Indeed, if $d(u) \leq n-2$ for every $u \in V(G)$, then there are at least two non-adjacent vertices u_1, u_2 of degree at most $n-2$ in G. If $G[V(G) \setminus \{u_1, u_2\}]$ is not a complete graph, then by Proposition 2.9 there exists a majority dominator $(n-3)$-coloring c of it and we can extend it to a majority dominator $(n-2)$-coloring c' of G by putting $c'(u_1) = c'(u_2) = n-2$, a contradiction. On the other hand, if $G[V(G) \setminus \{u_1, u_2\}] = K_{n-2}$, then $N(u_1) \cap N(u_2) = \emptyset$, because $d(v) \leq n-2$ implies $d_{\{u_1,u_2\}}(v) \leq 1$ for every $v \in V(G \setminus \{u_1, u_2\})$. For any (majority dominator) $(n-2)$-coloring c of $G[V(G) \setminus \{u_1, u_2\}]$, and for any $v_1 \in N(u_1)$, $v_2 \in N(u_2)$ we can extend it to a majority dominator $(n-2)$-coloring c' of G by setting $c'(u_1) = c(v_2)$ and $c'(u_2) = c(v_1)$, a contradiction. By Observation 2.2, $\Delta(G) = n-1$ implies $\chi(G) = \chi_{md}(G) = n-1$.

If there are two disjoint edges $u_1v_1, u_2v_2 \in E(\overline{G})$, then $\chi(G) \leq n-2$, since one can set $c(u_1) = c(v_1) \neq c(u_2) = c(v_2)$ and color the remaining $n-4$ vertices with other $n-4$ colors. Also, there is no triangle in $\overline{G}$ since in such a case all three of its vertices could get the same color and we would obtain again $\chi(G) \leq n-2$.

Thus the edges in $\overline{G}$ must form a star having at least one and at most $n-2$ edges. This ends the proof. ■

Note that if G is connected, then $\chi(G) = n-1$ if and only if $\chi_{md}(G) = n-1$, since $\chi(G) = n-1$ inplies that its clique number $\omega(G) = n-1$. A reasoning similar to the one in the proof of Proposition 2.10 shows that also $\chi_d(G) = n-1$ if and only if $\chi_{md}(G) = n-1$.

3. MAJORITY DOMINATOR COLORING OF CHOSEN GRAPHS

In this section we are going to present the exact values of $\chi_{md}(G)$ for chosen graphs G. From the previous section we already know that, in particular $\chi_{md}(K_n) = n$ and $\chi_{md}(K_{m,n}) = 2$. We also referred to the following fact.

Proposition 3.1. *For $n \geq 3$, $\chi_{md}(C_n \circ K_1) = \lceil \frac{n}{2} \rceil + 1$.*

Proof. Let $n \geq 3$ and let $G = C_n \circ K_1$. From Theorem 1 it comes that for even n, $\chi_{md}(G) \leq \lceil n/2 \rceil + 1$, since $\chi(G) = 2$ and $\alpha(G) = n$. When n is odd, we have $\chi(G) = 3$. The inequality $\chi(G) \geq 3$ follows from the fact that $\chi(C_n) = 3$ in this case, while any proper 3-coloring of C_n can be extended by using any of two admissible colors on every pendant vertex. From Theorem 1 it follows that $\chi_{md}(G) \leq \lceil n/2 \rceil + 2$, but one can reduce this bound by 1. Denote the consecutive vertices of C_n by $v_1, \ldots, v_n$ (so that $v_i v_{i+1} \in E(G)$ for $i = 1, \ldots, n-1$ and $v_1 v_n \in E(G)$) and their neighbors by $u_1, \ldots, u_n$ (so that $v_i u_i \in E(G)$ for $i = 1, \ldots, n$). Let us define the coloring c as: $c(v_{2i+1}) = c(u_{2i}) = i$ for $i = 1, \ldots, (n-1)/2$, $c(v_1) = c(u_n) = (n+1)/2$ and $c(v_{2i}) = c(u_{2i-1}) = (n+3)/2 = \lceil n/2 \rceil + 1$ for $i = 1, \ldots, (n-1)/2$. Clearly, c is an $(\lceil n/2 \rceil + 1)$-majority dominator coloring of G.

Now we are going to show that at least $\lceil n/2 \rceil + 1$ colors are necessary for every n. Consider the pendant vertices. Each of them can (majority) dominate at most one color class — either its own or the one of its only neighbor. In both cases it comes out that at most 2 vertices can be colored with the dominated color. Since there are n pendant vertices and each of them dominates a color class consisting of at most 2 vertices, it follows that at least $\lceil n/2 \rceil$ colors must be dominated by the pendant vertices. If there are more color classes dominated by the pendant vertices, then we are done. So assume that the pendant vertices dominate exactly $\lceil n/2 \rceil$ color classes. Then at most $2\lceil n/2 \rceil \leq n+1$ vertices can be colored. The remaining $2n - 2\lceil n/2 \rceil \geq n-1 \geq 2$ vertices need at least one extra color. ■

In the case of wheel W_n, every proper coloring is also a majority dominator coloring (in particular, each vertex dominates the one-element class of the central vertex). This implies that for every $n \geq 3$, $\chi_{md}(W_n) = \chi(W_n)$ (for the latter one, see e.g. [24, p. 82]).

Observation 3.2. *For $n \geq 3$,*

$$\chi_{md}(W_n) = \begin{cases} 4, & \text{if } n \equiv 1 \pmod 2, \\ 3, & \text{if } n \equiv 0 \pmod 2. \end{cases}$$

Let $S_{a,b}$ be the double star with central vertices u and v with $d(u) = a \geq 2$ and $d(v) = b \geq 2$. Let $X = \{x_1, x_2, \ldots, x_{a-1}\}$, $Y = \{y_1, y_2, \ldots, y_{b-1}\}$, $N(v) = \{u\} \cup Y$ and $N(u) = \{v\} \cup X$.

Proposition 3.3. *For $a, b \geq 3$, $\chi_{md}(S_{a,b}) = 3$. Otherwise, $\chi_{md}(S_{a,b}) = 2$.*

Proof. If $a = 2$ or $b = 2$, then the unique (up to renaming of colors) proper 2-coloring defined by $V_1 = X \cup \{v\}$ and $V_2 = Y \cup \{u\}$ is a majority dominator coloring (at least one of the color classes consists of two members and every vertex in the graph dominates at least one of them). On the other hand, if $a, b \geq 3$, then $u \in V_1$ and $v \in V_2$. Assume that these two color classes are enough. Then $V_1 = \{u\} \cup Y$ and $V_2 = \{v\} \cup X$. We have $|V_1|, |V_2| \geq 3$. But every vertex in $X \cup Y$ can dominate only one vertex and thus a color class of at most two members, a contradiction. It follows that at least three colors are necessary. On the other hand, the coloring defined by $V_1 = \{u\}$, $V_2 = \{v\}$, $V_3 = X \cup Y$ is a majority dominator 3-coloring (actually, even a dominator 3-coloring) of $S_{a,b}$. ■

Note that the chromatic number of any double star is $\chi(S_{a,b}) = 2$, as it is a bipartite graph. The dominator chromatic number, in turn, equals to $\chi_d(S_{a,b}) = 3$ if $a + b \geq 4$ [11].

The star $K_{1,n-1}$ has one vertex v of degree $n-1$ and $n-1$ vertices of degree one. It can be generalized to the multistar graph $K_n(a_1, a_2, \ldots, a_n)$, which is formed by attaching $a_i \geq 1$, $(1 \leq i \leq n)$ pendant vertices to each vertex x_i of a complete graph K_n, with $V(K_n) = \{x_1, x_2, \ldots, x_n\}$.

Proposition 3.4. *For the multistar graph $G = K_n(a_1, a_2, \ldots, a_n)$*

$$\chi_{md}(G) = \begin{cases} n, & \text{if } a_i < n \text{ for some } 1 \leq i \leq n, \\ n+1, & \text{otherwise.} \end{cases}$$

Proof. It is clear that $\chi(G) = n$, so by $\chi(G) \leq \chi_{md}(G)$ it follows that $\chi_{md}(G) \geq n$. Let $V(K_n) = \{x_1, x_2, \ldots, x_n\}$. We will use A_i to denote the set of pendant vertices adjacent to x_i. Note that $a_i = |A_i|$.

Assume first that there is a_i such that $a_i < n$ (without loss of generality, assume that $i = n$). Color K_n properly with n colors so that $c(x_j) = c_j, j = 1, \ldots, n$, then all the vertices in $A_j, j < n$ with c_n, and finally the vertices of A_n

with different colors $c_1, \dots, c_{a_n}$. Every vertex in G dominates at least one vertex colored with some $c_j, j < n$, that is at least half of the respective color class, so c is a majority dominator coloring.

Assume now that for each A_i we have $a_i \geq n$. Suppose that $\chi_{md}(G) = n$. Obviously, we have $c(x_i) = c_i$ for $i = 1, \dots, n$. If any of these colors, say c_j, appears at least 3 times in G, then each pendant neighbor of x_j has a color appearing at most twice in the graph, since they cannot dominate the class of x_j. However, each of those colors appears already in K_n, so it can appear at most once in A_j. This means that there would be at least n different colors in A_j, each of them different than c_j, a contradiction to the assumption that there are only n colors in use. Thus at least $n+1$ colors are necessary. On the other hand the coloring defined as $c(x_i) = c_i$ for $i = 1, \dots, n$ and $c(v) = c_{n+1}$ for $v \in \bigcup_{i=1}^{n} A_i$ is a majority dominator $(n+1)$-coloring. ■

Observe that the chromatic number of a generalized star is always equal to $\chi(K_n(a_1, a_2, \dots, a_n)) = n$, since the n-coloring of the complete subgraph K_n can be trivially extended. It is also known that the dominator chromatic number of this kind of graph equals to $\chi_d(K_n(a_1, a_2, \dots, a_n)) = n + 1$ [10].

In the remainder of this section we are going to focus on the majority dominator coloring numbers of paths and cycles. Before we continue, recall that the chromatic number of any path P_n is $\chi(P_n) = 2$, as it is a bipartite graph [24, p. 82], and thus the only optimal coloring is defined by alternate usage of two colors on consecutive vertices. The dominator chromatic number for $n \geq 2$ is in turn equal to [10, 11]

$$\chi_d(P_n) = \begin{cases} 1 + \left\lceil \frac{n}{3} \right\rceil, & \text{if } n = 2, 3, 4, 5, 7, \\ 2 + \left\lceil \frac{n}{3} \right\rceil, & \text{otherwise.} \end{cases}$$

The results for cycles are as follows. The chromatic number depends on the parity. Namely, for $n \geq 3$ we have [24, p. 82]

$$\chi(C_n) = \begin{cases} 3, & \text{if } n \equiv 1 \pmod 2, \\ 2, & \text{if } n \equiv 0 \pmod 2. \end{cases}$$

Note that any alternate coloring of consecutive vertices with two colors is optimal in this case, with one extra color added on one vertex if n is odd. The dominator chromatic number is in turn equal to [10]

$$\chi_d(C_n) = \begin{cases} \left\lceil \frac{n}{3} \right\rceil, & \text{if } n = 4, \\ 1 + \left\lceil \frac{n}{3} \right\rceil, & \text{if } n = 5, \\ 2 + \left\lceil \frac{n}{3} \right\rceil, & \text{otherwise.} \end{cases}$$

Let us present the following technical result.

Lemma 3. *Let $P = P_n$ be a path with vertex set $V(P) = \{v_1, v_2, \ldots, v_n\}$, let $k = \chi_{md}(P)$ and let $c : V(P) \to \{1, \ldots, k\}$ be its k-majority dominator coloring. Then the following statements are true.*

1. *There are at most two colors in c that appear on at least five vertices.*
2. *If $n \geq 11$, then there are at least two colors in c that appear at most twice.*
3. *If $n \neq 2$, then there is at most one color in c that appears only once.*

Proof. In all three parts of the proof we are going to use the fact that k is the minimum number of colors that can be used in a majority dominator coloring of P.

Part 1. If $n < 15$, then the conclusion is trivial. If $n \geq 15$, suppose that there are three colors c_1, c_2 and c_3 such that they occur at least five times in c. Every vertex has degree at most 2, so it can dominate a color class of order at most 4. Thus there are no three consecutive vertices v_i, v_{i+1}, v_{i+2} such that $c(\{v_i, v_{i+1}, v_{i+2}\}) \subseteq \{c_1, c_2, c_3\}$, because in such a case v_{i+1} could not dominate any color class. For that reason the vertices colored with c_1, c_2 and c_3 can have at most one neighbor colored with one of those colors. It means that one can recolor every vertex colored with c_3 using c_1 (by default) or c_2 if one of its neighbors is colored with c_1. The new coloring c' is obviously proper. It is also a majority dominator coloring, since every vertex in $V(P)$ had to dominate a color class from $\{1, \ldots, k\} \setminus \{c_1, c_2, c_3\}$ before the recoloring. Moreover it uses $k - 1$ colors, which contradicts with the minimality of c and concludes the proof.

Part 2. Since v_1 and v_n can dominate only a color class of order at most two, there must be at least one such color class in c. Suppose that there is only one such class, colored with color c_1. Since v_1 and v_n both need to dominate this class, we must have in particular that $c_1 \in c(\{v_1, v_2\})$ and $c_1 \in c(\{v_{n-1}, v_n\})$. No matter which of v_1 and v_2 is colored with c_1, the color of the other one, say c_2, must appear at least three times, since we assumed that only the color class of c_1 can have cardinality at most two. Thus there must be at least one vertex v_i colored with c_2, $3 \leq i \leq n - 2$.

If $n \geq 12$, then two cases are possible. In the first case there exists v_i colored with c_2 having a neighbor v_j, where $j \in \{i - 1, i + 1\}$, such that $c(v_j) = c_3$, $c_3 \notin \{c_1, c_2\}$, with the following properties: (i) v_j has exactly one neighbor colored with c_2 (namely v_i), (ii) v_j has no neighbor colored with c_1. It follows that in such circumstances v_j has exactly one neighbor colored with yet another color c_4. This means that v_j needs to dominate another color class of order at most two that must be c_3 or c_4, a contradiction. In the second case every neighbor v_j of a vertex colored with c_2 $(3 \leq j \leq n - 2)$ has the other neighbor colored with either c_1 or c_2. Moreover, a neighbor can be colored c_1 only if $j \in \{3, n - 2\}$, otherwise both neighbors must be colored with c_2. The assumption $n \geq 12$ implies that there are at least eight vertices $v_j, j \in \{3, n-2\}$, so at least four of them are colored with c_2, $3 \leq i \leq n-2$, and so there are at least five such vertices in total, thus each of their neighbors v_j such that $4 \leq j \leq n-3$ has to dominate its own class, which must be of order at most 2, a contradiction.

If $n = 11$, then there could be only two ways to avoid the multiple occurrence of a color class of order at most 2. Either there are only three colors in use or four colors are used and all the color classes but one have cardinalities 3. In the first case, however, there must be a vertex such that its color (say c_2) appears at least three times and the color of its both neighbors (say c_3) appears at least five times, so it cannot dominate any color class. In the other case there is a vertex v_j colored c_2 having two neighbors colored c_3 and c_4. Indeed, if any of the end vertices is colored with c_2, c_3 or c_4, then its neighbor must be colored with c_1, since otherwise it would not dominate any color class. Thus the 7 consecutive vertices $v_3, v_4, \ldots, v_9$ are colored with colors c_2, c_3 and c_4. The only sequence of colors avoiding a triple of consecutive vertices colored differently is x, y, x, y, x, y, x, where $x, y \in \{c_2, c_3, c_4\}$. However x occurs 4 times in this sequence, which is impossible. Finally, since each of the color classes c_2, c_3, c_4 has exactly three members, v_j does not dominate any color class.

Part 3. If $n \leq 4$, then the statement follows from the fact that every optimal proper coloring is majority dominator coloring. Thus we may assume that $n \geq 5$. Suppose that there are two colors c_1 and c_2 such that $|c^{-1}(c_1)| = |c^{-1}(c_2)| = 1$. Let v_i and v_j, $1 \leq i < j \leq n$ be the two vertices such that $c(v_i) = c_1$ and $c(v_j) = c_2$. If they are not adjacent, then we can set $c(v_j) = c_1$ and obtain this way a new majority dominator coloring with $k-1$ colors, a contradiction. So assume that $j = i+1$. Since $n \geq 5$, without loss of generality we can assume that $i \geq 3$ (the path can be reversed if needed). We define a new coloring c' by $c'(v_{i-1}) = c_1$, $c'(v_i) = c(v_{i-1})$, $c'(v_{i+1}) = c_1$ and $c'(v) = c(v)$ for $v \in V(P) \setminus \{v_{i-1}, v_i, v_{i+1}\}$. Note that the new coloring is a majority dominator coloring, since the vertices v_s, $i-2 \leq s \leq \min\{i+2, n\}$ dominate now the color class c_1 while the other vertices (if there are any) dominate the same classes as before the recoloring. Moreover the new coloring uses $k-1$ colors, which contradicts with the minimality of c. Thus the number of one-member color classes cannot be greater than one. ■

Lemma 4. *Let $P = P_n$ be a path of order $n \geq 11$ with vertices $v_1, v_2, \ldots, v_{n-1}, v_n$, let $k = \chi_{md}(P)$ and let $c : V(P) \to \{1, \ldots, k\}$ be its k-majority dominator coloring. Let $P' = P_{n+6}$ be an extension of P with vertices $v_1, v_2, \ldots, v_{n+5}, v_{n+6}$. Then $\chi_{md}(P') \geq k+1$.*

Proof. Suppose that there exists a k-majority dominator coloring c' of P'. We will show that this would contradict with the minimality of c. We will consider two cases: when v_{n+6} dominates the color class of $c(v_{n+6})$ and when it does not.

Case 1. v_{n+6} dominates its own color class. Since all the colors used in c' must occur in its restriction to P (otherwise it would use less than k colors), the color $c_1 = c'(v_{n+6})$ appears exactly once in P_n. By Lemma 3 it is the only one such color, thus every other color used in $V(P') \setminus V(P)$ must be used at least two times in P, so at least three times in P'. This means in particular that each vertex v_i, $n+2 \leq i \leq n+4$ must dominate a color class with at least three members and thus each of them must have two neighbors with the same color. In particular $c(v_{n+3}) = c(v_{n+5})$ (because of v_{n+4}) and $c(v_{n+1}) = c(v_{n+3})$ (because of v_{n+2}). This means that the color $c_2 = c(v_{n+1})$ is used three times in $V(P') \setminus V(P)$ and at least two times in $V(P)$, so in fact it cannot be dominated, a contradiction.

Case 2. v_{n+6} does not dominate its own color class. In such situation v_{n+6} has to dominate the color class of v_{n+5}. Using similar argument as in case 1 we conclude that the vertices in $V(P') \setminus V(P)$ have to be colored in the following way: $c(v_{n+6}) = c_1$, $c(v_{n+5}) = c_2$, $c(v_{n+2}) = c(v_{n+4}) = c_3$ and $c(v_{n+1}) = c(v_{n+3}) = c_4$, where c_1, c_2, c_3 and c_4 are pairwise distinct. In particular $c_1 \neq c_3$ (and $c_1 \neq c_4$), because otherwise c_1 would appear three times in $V(P') \setminus V(P)$ and according to Lemma 3 at least two times in $V(P)$, so v_{n+3} (or v_{n+2}, respectively) could not dominate any color class. Also, in the restriction of c' to P, c_1 appears at least two times, c_2 exactly one time and c_3 and c_4 exactly two times. We are going to show that we can recolor the vertices colored with c_1, c_3 or c_4 in P and obtain this way a $(k-1)$-majority dominator coloring of P, which contradicts the minimality of c.

We will consider two subcases: when some vertex dominates the color class of c_1 in c' and when there is no such vertex.

Subcase 2.1. There is a vertex $v \in V(P)$ that dominates the color class c_1 in c'. In such a case c_1 appears at most three times in $V(P)$ and moreover there exists a sequence of consecutive vertices v_1, u_1, v, u_2, v_2 in $V(P)$ such that $c(u_1) = c(u_2) = c_1$. Moreover, if $c(v) = c_3$ (or $c(v) = c_4$), then none of v_1 and v_2 can be colored with c_4 (c_3, respectively), since this would mean that u_1 or u_2 do not dominate any color class in c' (recall that c_3 and c_4 appear exactly four times in c'). In particular, if $c(v) = c_3$, then none of v_1 and v_2 can be colored with c_4, so both u_1 and u_2 can be recolored with c_4 and the new coloring of P (call it c'') will be still proper. Moreover, after recoloring we have $|(c'')^{-1}(c_3)| \leq 4$ and $|(c'')^{-1}(c_4)| \leq 4$. It follows from the fact that before the recoloring both c_3 and c_4 were used exactly two times in P, and exactly two vertices (u_1 and u_2) were recolored. This implies that c'' is a majority dominator coloring of P. Indeed, if any vertex dominated the color class c_3 or c_4 in c' (where the cardinalities of both color classes were exactly four), then it will also dominate it in c'' (where these cardinalities are at most four). Also v dominates c_4 in c''. An analogous reasoning applies when $c(v) = c_4$ (then both u_1 and u_2 can be recolored with c_3).

If $c(v) = c_5 \notin \{c_3, c_4\}$, then we can do the same as above (i.e., recolor both

u_1 and u_2 with the same color, either c_3 or c_4) unless $c(v_1) = c_3$ and $c(v_2) = c_4$ (or $c(v_1) = c_4$ and $c(v_2) = c_3$). But then c_5 appears at most twice in c (and c') since otherwise u_1 would not dominate any color class in c' (it cannot dominate the color class of v_1, since it has cardinality four and it cannot dominate the color class of itself, since it has cardinality at least three). This allows us to recolor u_1 with c_4 and u_2 with c_3 (or u_1 with c_3 and u_2 with c_4, respectively). Again, the obtained coloring is a majority dominator coloring of P (with v dominating the class c_5).

Now, either c'' is a $(k-1)$-coloring (if c_1 appears exactly two times in the restriction of c' to $V(P)$) or there is still one vertex colored c_1, which contradicts Lemma 3 (recall that there is another one-member color class in P, namely c_2).

Subcase 2.2. There is no vertex dominating the color class c_1 in c'. If there is no vertex $v \in V(P)$ that dominates c_3 in c', then we proceed as follows. If there is a vertex $u \in V(P)$ colored c_3 not adjacent to any vertex colored c_1, then we can obtain a new coloring c'' of $V(P)$ by putting $c''(u) = c_1$ and preserving the colors of the remaining vertices. This will not change any domination relation and there are exactly two one-element color classes in c'' (c_2 and c_3), a contradiction with Lemma 3. If each of the two vertices colored c_3 is adjacent to a vertex colored c_1, then its other neighbor must be colored with a color $c_5 \notin \{c_1, c_3, c_4\}$ (otherwise the vertex colored c_3 would not dominate any color class in c'). Note that c_5 appears exactly two times in $V(P)$ if $c_5 \neq c_2$.

Now, if there is no vertex $v \in V(P)$ that dominates c_4 in c', then we recolor any vertex u colored c_3 with c_4 and we are done, since the dominance relations did not change and there are two colors appearing exactly once (c_2 and c_3), a contradiction. If there is a vertex $v \in V(P)$ that dominates c_4, then there is a sequence of consecutive vertices v_1, u_1, v, u_2, v_2 with colors $c'(v_1) = c_6$, $c'(u_1) = c_4$, $c'(v)$,$c'(u_2) = c_4$,$c'(v_2) = c_7$, where some of c_6, c_7, $c'(v)$ may be equal. Let us consider few cases.

1. If none of v_1, v, v_2 is colored c_3, then we recolor u_1 and u_2 with c_3 and obtain this way a $(k-1)$-majority dominator coloring of P, a contradiction.

2. If only v_1 (only v_2) is colored with c_3, then we recolor u_2 (u_1, respectively) with c_3 and obtain a k-majority dominator coloring of P with two one-member classes c_2 and c_4, a contradiction. Note that the obtained coloring is indeed a majority domination coloring, since the color class of v has at most two members. Otherwise u_1 (u_2, respectively) would not dominate any color class in c'. This implies that v dominates its own color class, while the remaining dominance relations do not change.

3. If only $c'(v) = c_3$, then the color classes of both c_6 and c_7 consist of at most two members, since otherwise u_1 or u_2 would not dominate any class in c'. This means that recoloring u_1, v and u_2 with c_3, c_4 and c_3 would produce a

k-majority dominator coloring of P with v_1 and u_1 dominating the class of c_6, v dominating the class of c_3 and u_2 and v_2 dominating the class of c_7. There would be two one-member color classes c_2 and c_4, a contradiction.

4. Finally, if $c_6 = c_7 = c_3$, then the color class of $c'(v)$ must consist of at most two members (otherwise u_1 and u_2 would not dominate any color class in c'). In this case we can recolor both u_1 and u_2 with c_1. The new coloring is a $(k-1)$-majority dominator coloring of P with u_1, v and u_2 dominating the color class of $c'(v)$ and the remaining domination relations unchanged, a contradiction.

Note that other distributions of c_3 among v_1, u_1, v, u_2, v_2 are impossible, since we assumed that no vertex dominates the class c_3 and so the distance betweeen any two vertices of this color must be at least three.

The same reasoning applies if we swap c_3 with c_4. Thus the only case to consider is the situation when there is a sequence of consecutive vertices in $V(P)$, $u_1, v_1, w_1, S, u_2, v_2, w_2$, where S denotes some subsequence of vertices (possibly empty), $c'(u_1) = c'(w_1) = c_3$ and $c'(u_2) = c'(w_2) = c_4$ (or $c'(u_1) = c'(w_1) = c_4$ and $c'(u_2) = c'(w_2) = c_3$, but we can skip this case by symmetry). If S is not empty, then we can define a new $(k-1)$-majority dominator coloring c'' of $V(P)$ by recoloring $c''(u_2) = c''(w_2) = c_3$, a contradiction. If S is empty, then we have a sequence $u_1, v_1, w_1, u_2, v_2, w_2$ such that $c'(u_1) = c'(w_1) = c_3$ and $c'(u_2) = c'(w_2) = c_4$, where w_1 dominates $c'(v_1)$, so $|(c')^{-1}(c'(v_1))| \leq 2$ and $c'(v_1)$ is also dominated by u_1. By symmetry, $|(c')^{-1}(c'(v_2))| \leq 2$ and $c'(v_2)$ is dominated by u_2 and w_2. At least one of $c'(v_1)$, $c'(v_2)$ is not c_2. Without loss of generality assume that $c'(v_1) \neq c_2$. Now, define the new coloring c'' of $V(P)$ by setting $c''(u_1) = c_1$ and $c''(w_1) = c_2$ (or $c''(u_1) = c_2$ and $c''(w_1) = c_1$ if u_1 has a neighbor colored with c_1) and preserving the colors of the remaining vertices. In both cases, after the recoloring, u_1, v_1 and w_1 dominate the color class of $c''(v_1)$ and other domination relations remain intact. It follows that C'' is a $(k-1)$-majority dominator coloring of P, a contradiction. ■

Theorem 5. *Let $P = P_n$ be a path of order n with vertices $v_1, v_2, \ldots, v_{n-1}, v_n$. Then*

$$\chi_{md}(P) = \begin{cases} 1, & \text{if } n = 1, \\ 2, & \text{if } 2 \leq n \leq 5, \\ 3, & \text{if } 6 \leq n \leq 10, \\ 4, & \text{if } 11 \leq n \leq 13, \\ \left\lceil \frac{n}{6} \right\rceil + 2, & \text{if } n \geq 14. \end{cases}$$

Proof. First we are going to show that the given values are minimum possible. For $n \leq 5$ it is obvious, since every coloring must be proper. On the other hand it is impossible to use only two colors for $n \geq 6$, since both colors would appear at least three times and the ends of the path have only one neighbor, so they cannot dominate any color. Assume that three colors are enough for $n = 11$. By Lemma 3 at least two of them occur at most two times, so the third one needs to be used at least 7 times. However, this would force at least one pair of neighboring vertices to be colored with the same (third) color. Thus one needs at least four colors in this case (as well as for every $n \geq 11$).

Assume that in a coloring c of P_{14} one can use four colors. By Lemma 3 there are two or three colors that occur at most 2 times. In the latter case, however, there would be at least eight vertices colored with the same (fourth) color, which would force at least one pair of them to be adjacent. Thus there are exactly two colors (say c_1 and c_2) present on at most two vertices and at least ten vertices colored with two other colors (say c_3 and c_4). Since there are at most four vertices colored c_1 or c_2, in the sequence of colors of the consecutive vertices there can be at most five subsequences consisting of only c_3 and c_4. Either all of them are pairs, or there are some consisting of at least three vertices. The former is impossible, since then each of c_3 and c_4 occurs exactly five times and the ends of the path (colored with c_3 or c_4 and neighboring with vertices colored with c_4 or c_3, respectively) cannot dominate any color class. Thus there must be at least one subsequence consisting of at least three vertices. Without loss of generality assume that it is c_3, c_4, c_3. Because of the middle vertex, $|c^{-1}(c_3)| \leq 4$, so $|c^{-1}(c_4)| \geq 6$. But it means that in at least one of the five subsequences c_4 occurs at least twice, thus there is a subsequence of colors c_4, c_3, c_4 and we deduce that $|c^{-1}(c_4)| \leq 4$, a contradiction. This means that at least five colors are necessary for every $n \geq 14$.

Finally, assume that in a coloring c of P_{19} one can use five colors. According to Lemma 3 the number of colors present on at most two vertices is between two and four. If it is four, then there are at least eleven vertices colored with the fifth color, which forces at least two of them to be adjacent, a contradiction. If there are three such colors, then at least thirteen vertices must be colored with two other colors (say c_1 and c_2) and form at most seven subsequences. If they are distributed in six pairs and one singleton, then there is at least one end of the path that does not dominate any color. Thus there must be at least one triple, say c_1, c_2, c_1. This implies that there are at most four vertices labeled with c_1 and consequently at least nine vertices colored with c_2. Thus at least one of (at most) seven subsequences contains at least two of them, so there is a subsequence c_2, c_1, c_2, which implies that c_2 occurs at most four times, a contradiction. Finally, assume that there are exactly two colors occurring at most two times. Then there are at least fifteen vertices colored with three colors, say c_1, c_2 and c_3. By Lemma 3 there are at most two colors appearing on at least five vertices, so without loss of generality we can assume that c_1 occurs at most four times and c_2 at least six times. Observe that a sequence being a permutation of c_1, c_2, c_3 is impossible since the middle vertex could not dominate any color class. This implies that there is a subsequence c_2, x, c_2, where $x \in \{c_1, c_3\}$. It implies that c_2 can occur at most 4 times, a contradiction. We deduce that at least six colors are necessary in any majority dominator coloring of P_n, where $n \geq 19$.

As we can see, in particular we obtained that $\chi_{md}(P_n) \geq \lceil \frac{n}{6} \rceil + 2$ for $14 \leq n \leq 19$. Using Lemma 4 it follows by induction that $\chi_{md}(P_n) \geq \lceil \frac{n}{6} \rceil + 2$ for every $n \geq 14$.

To conclude the proof, we need to show that there exist majority dominator colorings of P_n using the number of colors given in the statement of the theorem.

As it can be easily verified, for $n \leq 13$ we can use the sequences of colors S_n defined as follows: $S_1 = \{1\}$, $S_2 = \{1,2\}$, $S_3 = \{1,2,1\}$, $S_4 = \{1,2,1,2\}$, $S_5 = \{1,2,1,2,1\}$, $S_6 = \{1,2,1,2,1,3\}$, $S_7 = \{3,1,2,1,2,1,3\}$, $S_8 = \{3,1,2,1,2,1,2,3\}$, $S_9 = \{3,1,2,1,2,1,2,3,1\}$, $S_{10} = \{3,1,2,1,2,1,2,1,3,2\}$, $S_{11} = \{3,4,1,2,1,2,1,2,1,2,4\}$, $S_{12} = \{3,4,1,2,1,2,1,2,1,2,3,4\}$, $S_{13} = \{1,2,1,2,1,3,1,3,1,4,1,4,1\}$.

For $n \geq 14$ the desired coloring can be defined with the following formula.

$$c(v_i) = \begin{cases} 1, & \text{if } i \equiv 0 \pmod 3, \\ 2, & \text{if } i \equiv 1 \pmod 3) \wedge i \neq n, \\ \lceil \frac{n}{6} \rceil + 2, & \text{if } i \equiv 1 \pmod 3) \wedge i = n, \\ \lceil \frac{i}{6} \rceil + 2, & \text{if } i \equiv 2 \pmod 3). \end{cases}$$

Note that according to the above formula for every $1 \leq j \leq n$ we have $c(v_{j-1}) \geq 3$, $c(v_j) \geq 3$ or $c(v_{j+1}) \geq 3$ and each color $x \geq 3$ appears at most two times. So c is indeed a majority dominator coloring of P. ■

In order to present the result for cycles, we will need the following technical lemma.

Lemma 6. *Let $C = C_n$, $n \geq 10$ be a cycle with vertex set $V(C) = \{v_1, v_2, \ldots, v_n\}$, let $k = \chi_{md}(C)$ and let $c : V(C) \to \{1, \ldots, k\}$ be its k-majority dominator coloring. Let $P = P_n$ be a path of order n. Then $k \geq \chi_{md}(P_n)$. Moreover, if $n = 13$, then $k \geq \chi_{md}(P_n) + 1$.*

Proof. Obviously, there must be a color that appears once or twice in c. Otherwise, every vertex u would dominate a color present on at least three vertices, being the common color of both neighbors of u. However, this could occur only if the cycle were colored alternately with two colors, which is possible only for $n \in \{4,6,8\}$.

If some color occurs exactly once, say on vertex v, then we are done, since we can remove any edge incident with v, say uv, and recolor u with $c(v)$, which defines a majority dominator coloring of P_n.

From now on we will assume that there are no colors appearing only once. First assume that there exist two vertices v_1 and v_2 such that $|c^{-1}(c(v_1))| = 2$,

$|c^{-1}(c(v_2))| = 2$ and distance between v_1 and v_2 is at most 3 (and it can be 1 only if $c(v_1) \neq c(v_2)$). In such a case it is enough to remove an edge from a shortest (v_1, v_2)-path: the edge incident to both v_1 and v_2 (if the distance is 1), either edge (if the distance is 2) or the middle edge (if the distance is 3). This way each of v_1 and v_2 will become either an end vertex of the obtained path or the only neighbor of an end vertex and c will still be a majority dominator coloring of the obtained path.

Let us focus on the remaining case. There are exactly $2p$ vertices colored with colors appearing twice, where p is the number of such colors. The distance between each pair of such vertices is at least four. In other words, for every pair of such vertices for which there is no other vertex of this kind between them, there is a sequence of at least three consecutive vertices colored with some colors that are used at least three times (we will call such sequences *segments*). Note that the number of segments is also $2p$. It is impossible that three distinct colors c_1, c_2, c_3 are used on three consecutive vertices, because in such case the middle vertex would not dominate any color class. This implies in turn that in every segment exactly two colors are used. If it has the form c_1, c_2, c_1, then the color c_1 can appear at most four times in c. If there are at least four vertices in a segment, then each of c_1 and c_2 can appear at most four times. In particular, the length of a segment cannot be more than 8.

If every segment has length 3, then one can use one common color on the middle vertex of every segment and one color on the remaining vertices of every pair of segments. This allows to use $p + 1$ colors in the segments and $k = 2p + 1$ colors in total to color $n = 8p$ vertices. Note that this is the minimum number of colors that can be used. Indeed, since at most four end vertices of segments can be colored with one color, and there are exactly $4p$ end vertices, one needs at least p colors to color them all. If any of these colors is used also for some middle vertex in another segment, then one needs at least $p + 1$ colors to color the end vertices. On the other hand, if these colors are used only to color the end vertices, then one extra color is necessary to color the middle vertices. In any case this gives the number of at least $p + 1$ colors to color the vertices in the segments. Note that since $k = 2p + 1$ and $n = 8p$, we have $k = \frac{n}{4} + 1$ in this case. This bound will be further used to compare the values of χ_{md} of paths and cycles of the same lengths.

Assume that there is a segment of length at least 4. Then there are two sets of segments of length 3, where each of them can be empty. The first one consists of q_0 segments with end vertices colored with colors not used in any longer segment. The other one consists of a number of segments with endpoints colored with some color used also on some longer segment. Since the first set is colored independently from the remaining part of the graph, in order to minimize the total number of colors it is necessary to use the minimum number of colors in this subset, which is at least $\lceil \frac{q_0}{2} \rceil$, according to the considerations above (even $\lceil \frac{q_0}{2} \rceil + 1$ if the middle vertices are colored with an extra color, not used elsewhere). Note that these segments consist of $3q_0$ vertices.

Now, each of the remaining segments of length 3 has end vertices colored with a color used on some longer segment, say c_0. Note that the other segment cannot be longer than 5, since it can have at most two vertices colored c_0. This means that when a segment of length at least 4 (colored with two colors, say c_1 and c_2) is combined with other segments, then one of the following cases must occur.

- C1 Some segment of length 4 is combined with two segments of length 3 (one of them with endpoints colored c_1 and the other with endpoints colored c_2). Let denote the number of such triples by q_4. They use at least $2q_4$ colors and consist of $10q_4$ vertices grouped in $3q_4$ segments.
- C2 Two segments of length 4 have at least one color in common or a segment of length 4 or 5 has a color in common with the end vertices of exactly one segment of length 3. In either case two colors are used to color at most eight vertices.
- C3 The colors c_1 and c_2 of a segment of length between 4 and 8 are not used on any other segment of length between 4 and 8 and they are not used to color the end vertices of any segment of length 3. Again, two colors are used to color at most eight vertices.

Note that the total number of vertices of the segments considered in the cases $C2$ and $C3$ is $n - 2p - 3q_0 - 10q_4$, while the number of colors to be used is at least one-fourth of this number, i.e., it is at least $\left\lceil \frac{n-2p-3q_0-10q_4}{4} \right\rceil$. From the above it follows that in order to color n vertices one needs to use at least

$$k \geq \left\lceil p + \frac{q_0}{2} + 2q_4 + \frac{n - 2p - 3q_0 - 10q_4}{4} \right\rceil = \left\lceil \frac{n}{4} + \frac{p}{2} - \frac{q_0}{4} - \frac{q_4}{2} \right\rceil \geq \left\lceil \frac{n}{4} + \frac{q_4}{4} \right\rceil$$

colors, where the last inequality follows from the fact that the number of segments analyzed above satisfies in particular $q_0 + 3q_4 \leq 2p$. If $q_4 > 0$, this implies $k \geq \lceil \frac{n}{4} + \frac{1}{4} \rceil$. On the other hand, if $q_4 = 0$, then the minimum number of colors that must be used is

$$k \geq \left\lceil p + \frac{q_0}{2} + \frac{n - 2p - 3q_0}{4} \right\rceil = \left\lceil \frac{n}{4} + \frac{p}{2} - \frac{q_0}{4} \right\rceil \geq \left\lceil \frac{n}{4} + \frac{1}{4} \right\rceil,$$

where the last inequality follows from $q_0 \leq 2p - 1$, as the case where all the segments have length 3 was analyzed separately.

From Theorem 5 it follows that for $n \geq 10, n \notin \{11, 14, 15, 19\}$ we have $\chi_{md}(P_n) \leq \lceil \frac{n}{4} + \frac{1}{4} \rceil$. Let us analyze the four mentioned cases and the case $n = 13$ separately.

If $n \leq 15$, then $p = 1$, since for $p \geq 2$ there would be at most seven vertices in at least four segments, which is impossible, since we assume that every segment consists of at least three vertices and if $p \geq 2$ then the total number of vertices would be at least 16. Thus there are exactly two segments.

In the case $n = 11$ they have lengths 3 and 6 or 4 and 5. In both cases if the coloring is optimal, then the longer segment uses two colors, and only one of them can be used also in the other segment, thus at least three colors must be used to color the segments and $\chi_{md}(C_{11}) \geq 4$.

If $n = 13$, then the lengths of the two segments are 3 and 8, 4 and 7 or 5 and 6. In the first case the two colors used in the longer segment cannot be reused (each color is used exactly four times, which is the maximum possible), thus two new colors are necessary for the shorter segment. In the second case one of the colors used in the segment of length 7 could be used in the other segment, but it would not work since each color there must appear twice. Similarly in the third case both colors used in the segment of length 6 could be used in the other segment, but each only once, which contradicts with the fact that both colors in the shorter segment appear at least twice. Thus four colors are necessary in the segments and $\chi_{md}(C_{13}) \geq 5$. Similarly we show that $\chi_{md}(C_{14}) \geq 5$ (the segments must be of lengths 6 and 8 or 7 and 7) and that $\chi_{md}(C_{15}) \geq 5$, where the only possible combination of lengths is 7 and 8.

If $n = 19$, from the fact that the length of a segment is between 3 and 8 it follows that $p = 2$ and the lengths of the four segments are (in the non-decreasing order) $(3, 3, 3, 6)$, $(3, 3, 4, 5)$ or $(3, 4, 4, 4)$. If the sequence of lengths is $(3, 3, 3, 6)$, then the two colors from the longest segment can be reused (each of them once), so they can serve as the colors of two out of three middle vertices in the segments of length 3. The third middle vertex, as well as all the end vertices of the shorter segments need other colors. Since in the case of the end vertices, at most four can get the same color, at least two new colors are needed. Thus at least four colors are necessary to color the segments. In the case of the sequence $(3, 3, 4, 5)$ one of the colors used in the longest segment can be used once and the other twice. The single vertex must be one of the middle vertices of the segments of length 3. The other color can be used on two end vertices in one of the shorter segments, on a pair of vertices in the segment of length 4 or on the middle vertices of two segments of length 3. In any case there will be still at least one middle vertex and three pairs of vertices (supposed to be monochromatic) to color. One needs at least two colors to color them, so at least four colors to color all the segments. Finally if the sequence $(3, 4, 4, 4)$ occurs, the two colors used in one of the segments of length 4 can be reused on at most two vertices, thus at least three colors are necessary to color the segments of this kind. Moreover the use of three colors would be possible only if each of them is used exactly four times, which is maximum possible. At least one more color is necessary to color the shortest segment. In every case we have $\chi_{md}(C_{13}) \geq 6$. This concludes the proof. ∎

Corollary 3.5. *Let $C = C_n$ be a cycle of order $n \geq 3$ with vertices $v_1, v_2, \ldots, v_{n-1}, v_n$. Then*

$$\chi_{md}(C) = \begin{cases} 2, & \text{if } n \in \{4,6,8\}, \\ 3, & \text{if } n \in \{3,5,7,9,10\}, \\ 4, & \text{if } 11 \leq n \leq 12, \\ \lceil \frac{n}{6} \rceil + 2, & \text{if } n \geq 13. \end{cases}$$

Proof. The fact that the given values are minimum possible follows immediately from the relation $\chi_{md}(C) \geq \chi(C)$ (for $n \leq 9$) and from Lemma 6 (for $n \geq 10$).

To see that $\chi_{md}(C)$ does not exceed the given values one can observe that the proper colorings of C_n, $3 \leq n \leq 9$, using two colors for n even and three colors for n odd with the third color used on precisely one vertex, are also majority dominator colorings, a majority dominator coloring of C_{13} is defined by the sequence $S'_{13} = \{1,2,1,2,1,3,1,3,1,4,1,4,5\}$, while for the remaining cycles one can use the sequences of colors used for the paths of respective lengths defined in the proof of Theorem 5. ∎

4. Final Remarks

We introduced a new graph invariant, the majority dominator coloring number $\chi_{md}(G)$ and investigated some of its properties. Although at first glance the concept of majority dominator coloring is similar to the idea of dominator coloring, they are different enough to make it difficult to apply the same proof techniques. For that reasons, we needed to develop new ones.

We presented several general properties of the new parameter, in particular by connecting it with other graph invariants, like chromatic number $\chi(G)$, domination number $\gamma(G)$, independence number $\alpha(G)$ and matching number $\nu(G')$, where G' is a specific subgraph of G. This allowed us to derive the value of $\chi_{md}(G)$ for various classes of graphs and find the families of graphs satisfying chosen constraints imposed on $\chi_{md}(G)$. Obviously, there are many open problems to solve. Below we present those most interesting according to our opinion.

As it was observed in the beginning of Section 2, for every graph G it holds $\chi_{md}(G) \leq \chi_d(G)$. Although one could expect that usually the inequality will be strict, in certain cases equality occurs (see e.g. the complete graphs K_n). For that reason our first question is as follows.

Problem 4.1. Characterize the graphs for which $\chi_{md}(G) = \chi_d(G)$.

A similar problem is connected with Theorem 1. As one can see even in the case of the products $C_n \circ K_1$ the bound is sometimes sharp and sometimes not (see Proposition 3.1). For that reason we would like to know the solution to the following problem.

Problem 4.2. Characterize the graphs for which $\chi_{md}(G) = \chi(G) + \left\lceil \frac{\alpha(G)}{2} \right\rceil - 1$.

Our last problem is connected with the product graphs $G \circ K_1$ for arbitrary G. In such case $\alpha(G \circ K_1) = n$, where n is the order of G, and $\chi(G \circ K_1) = \chi(G)$ if $n \geq 2$. From Theorem 1 it follows that

$$\chi_{md}(G \circ K_1) \leq \chi(G) + \left\lceil \frac{n}{2} \right\rceil - 1.$$

On the other hand, by using a reasoning similar to the one from the proof of Proposition 3.1, we get

$$\chi_{md}(G \circ K_1) \geq \max \left\{ \chi(G), \left\lceil \frac{n}{2} \right\rceil + 1 \right\}.$$

In particular, this allows us to make the following observation.

Observation 4.3. *If G is a bipartite graph of order n, then*

$$\chi_{md}(G \circ K_1) = \left\lceil \frac{n}{2} \right\rceil + 1.$$

However, the value of $\chi_{md}(G \circ K_1)$ remains unknown for general G.

Problem 4.4. Let G be an arbitrary graph of order n. What is the value of $\chi_{md}(G \circ K_1)$? Can it be expressed in terms of n and $\chi(G)$?

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References

[1] A. Abdolghafurian, S. Akbari, S. Hossein Ghorban and S. Qajar, *Dominating coloring number of claw-free graphs*, Electron. Notes Discrete Math. **45** (2014) 91–97. https://doi.org/10.1016/j.endm.2013.11.018

[2] R. Aharoni, E.C. Milner and K. Prikry, *Unfriendly partitions of a graph*, J. Combin. Theory Ser. B **50** (1990) 1–10. https://doi.org/10.1016/0095-8956(90)90092-E

[3] M. Anholcer, B. Bosek and J. Grytczuk, *Majority choosability of digraphs*, Electron. J. Combin. **24** (2017) #P3.57. https://doi.org/10.37236/6923

[4] M. Anholcer, B. Bosek and J. Grytczuk, *Majority coloring of infinite digraphs*, Acta Math. Univ. Comenian. (N.S.) **88(3)** (2019) 371–376.

[5] S. Arumugam, J. Bagga and K.R. Chandrasekar, *On dominator colorings in graphs*, Proc. Indian Acad. Sci. Math. Sci. **122** (2012) 561–571.
https://doi.org/10.1007/s12044-012-0092-5

[6] S. Askari, D.A. Mojdeh and E. Nazari, *Total global dominator chromatic number of graphs*, TWMS J. Appl. and Eng. Math. **12** (2022) 650–661.

[7] G. Bagan, H. Boumediene-Merouane, M. Haddad and H. Kheddouci, *On some domination colorings of graphs*, Discrete Appl. Math. **230** (2017) 34–50.
https://doi.org/10.1016/j.dam.2017.06.013

[8] M. Chellali and F. Maffray, *Dominator colorings in some classes of graphs*, Graphs Combin. **28** (2012) 97–107.
https://doi.org/10.1007/s00373-010-1012-z

[9] R. Gera, *On the dominator colorings in bipartite graphs*, in: Fourth International Conference on Information Technology (ITNG'07) (2007) 947–952.
https://doi.org/10.1109/ITNG.2007.142

[10] R. Gera, *On the dominator colorings in graphs*, Graph Theory Notes N.Y. **52** (2007) 25–30.

[11] R. Gera, C. Rasmussen and S. Horton, *Dominator colorings and safe clique partition*, Congr. Numer. **181** (2006) 19–32.

[12] R. Hammack, W. Imrich and S. Klavžar, Handbook of Product Graphs, Second Edition (CRC Press, Boca Raton, 2011).
https://doi.org/10.1201/b10959

[13] T.W. Haynes, S.T. Hedetniemi and P.J. Slater, Fundamentals of Domination in Graphs (CRC Press, Boca Raton, 1998).
https://doi.org/10.1201/9781482246582

[14] M.A. Henning and A. Yeo, Total Domination in Graphs, Springer Monogr. Math. (Springer, New York, 2013).
https://doi.org/10.1007/978-1-4614-6525-6

[15] K. Kavitha and N.G. David, *Dominator coloring of central graphs*, Int. J. Comput. Appl. **51** (2012) 11–14.
https://doi.org/10.5120/8093-1673

[16] A.P. Kazemi, *Total dominator chromatic number of a graph*, Trans. Comb. **4(2)** (2015) 57–68.
https://doi.org/10.22108/toc.2015.6171

[17] S. Kreutzer, S. Oum, P. Seymour, D. van der Zypen and D.R. Wood, *Majority colourings of digraphs*, Electron. J. Combin **24** (2017) #P2.25.
https://doi.org/10.37236/6410

[18] L. Lovász, *On decomposition of graphs*, Studia Sci. Math. Hungar. **41** (1966) 237–238.

[19] H.B. Merouane and M. Chellali, *On the dominator coloring in trees*, Discuss. Math. Graph Theory **32** (2012) 677–683.
https://doi.org/10.7151/dmgt.1635

[20] D.A. Mojdeh, S. Askari and E. Nazari, *Approach to the central graphs and trees via strong dominator coloring*, Ars Combin. to appear.

[21] T. Ramachandran, D. Udayakumar and A. Naseer Ahmad, *Dominator coloring number of some graphs*, Int. J. Sci. Res. Publ. **5(10)** (2018) 1–4.
http://www.ijsrp.org/research-paper-1015.php?rp=P464625

[22] S. Shelah and E.C. Milner, *Graphs with no unfriendly partitions*, in: A Tribute to Paul Erdős, A.Baker, B. Bollobás and A. Hajnal (Ed(s)) (Cambridge Univ. Press, Cambridge, 1990) 373–384.
https://doi.org/10.1017/CBO9780511983917.031

[23] D.B. West, Introduction to Graph Theory, Second Edition (Prentice Hall, 2001).

[24] R.J. Wilson, Introduction to Graph Theory, Fourth Edition (Prentice Hall, 1996).

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