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ON THE RESTRICTED ARC-CONNECTIVITY OF ORIENTED GRAPHS

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Abstract

For a strong digraph D, the restricted arc-connectivity $\lambda'(D)$ is defined as the minimum cardinality over all restricted arc-cuts S satisfying that $D-S$ has a non-trivial strong component D_1 such that $D-V(D_1)$ contains an arc. In this paper, we prove that a strong oriented black graph D with $\mathrm{diam}(D) \leqslant 2l_2 - 2$ is λ'-optimal if $\delta(D) \geqslant 2$ and D is super-λ' if $\delta(D) \geqslant 3$, where l_2 is a parameter related with path lengths of D.

Keywords: oriented graph, line digraph, diameter, λ'-optimal, super-λ'.

2020 Mathematics Subject Classification: 05C20, 05C40.

1. Introduction

The digraphs considered here have neither loops nor multiple arcs. Let $D = (V, A)$ denote a *digraph* with finite set of *vertices* $V = V(D)$ and set of *arcs* $A = A(D)$. For any arc $uv \in A(D)$, the vertex u is its *tail* and the vertex v is its *head*, and we say that u is *adjacent to* v and that v is *adjacent from* u. The sets $N^+(u) = \{v : uv \in A(D)\}$ and $N^-(u) = \{v : vu \in A(D)\}$ are respectively called the *out-neighborhood* and *in-neighborhood* of the vertex u. Their cardinalities are the *out-degree* of u, $d^+(u) = |N^+(u)|$, and the *in-degree* of u, $d^-(u) = |N^-(u)|$. The *minimum out-degree* of D is $\delta^+(D) = \min\{d^+(u) : u \in V(D)\}$ and the *minimum in-degree* of D is $\delta^-(D) = \min\{d^-(u) : u \in V(D)\}$. The *minimum degree* of D is $\delta(D) = \min\{\delta^+(D), \delta^-(D)\}$. A digraph is *symmetric* (respectively, *asymmetric*), if the existence of an arc $uv \in A(D)$ implies that $vu \in A(D)$ (respectively, $vu \notin A(D)$). The set of symmetric arcs of D is denoted by $Sym(D)$. If $F \subset V(D)$, then $D[F]$ is the *subdigraph* induced by F. For any pair of vertices $u, v \in V(D)$, $uu_2u_3 \cdots u_{r-1}v$ represents a $u \to v$ *path* with all its vertices different, where $u_i \in V(D)$ for $2 \leq i \leq r-1$. A $u \to u$ path is a *cycle*. The *girth* of D, represented by $g(D)$ or simply g for unambiguity, is the length of a shortest cycle of D.

A digraph D is said to be *strongly connected* (or, just, *strong*) if for any pair of vertices $u, v \in V(D)$, there exists a $u \to v$ path. A digraph with one vertex is strong. When a digraph D is not strong, each maximally strong subdigraph of D is called a *strong component*. The *distance* from u to v is denoted by $d(u, v)$ that is the number of arcs of a shortest $u \to v$ path. The parameter $\operatorname{diam}(D) = \max\{d(u, v) : u, v \in V(D)\}$ stands for the *diameter* of D. The distance from u to $F \subset V(D)$ is $d(u, F) = \min\{d(u, f)) : f \in F\}$. The distance from F to u, $d(F, u)$, is defined analogously. For any integer $k \geqslant 1$, let $N_k^+(F) = \{u \in V(D) : d(F, u) = k\}$ and $N_k^-(F) = \{u \in V(D) : d(u, F) = k\}$. Observe that $N^+(u) = N_1^+(u)$, $N^-(u) = N_1^-(u)$, $N^+(F) = N_1^+(F)$ and $N^-(F) = N_1^-(F)$.

For a pair F, F' of nonempty vertex sets of a digraph D, we define $[F, F'] = \{uv \in A(D) : u \in F, v \in F'\}$. If $F' = V(D) \backslash F$, we write $\omega^+(F)$ instead of $[F, F']$. For brevity, we denote $\overline{F} = V(D) \backslash F$. Given a proper subset $F \subset V(D)$ of vertices of D such that $\omega^+(F)$ is a restricted arc-cut of D, one can denote by $X \subseteq F$ and $\overline{X} \subseteq \overline{F}$ the set of tails and the set of heads of $\omega^+(F)$, respectively. This means that $\omega^+(F) = [F, \overline{F}] = [X, \overline{X}]$.

We recall here that in the *line digraph* $L(D)$ of a digraph D, each vertex represents an arc of D. Thus, $V(L(D)) = \{uv : uv \in A(D)\}$ and a vertex uv is adjacent to a vertex wz if and only if $v = w$. From the definition it follows that $\delta(L(D)) = \delta(D)$. For any integer $k \geqslant 1$, the *k-iterated line digraph*, $L^k(D)$, is defined recursively by $L^k(D) = L(L^{k-1}(D))$. A vertex u of $L^k(D)$ can be represented as a sequence $u_0u_1u_2 \cdots u_k$ of vertices of D such that $u_iu_{i+1} \in A(D)$, $0 \leqslant i \leqslant k-1$, and u is adjacent to another vertex v of $L^k(D)$ if and only if $v = u_1u_2 \cdots u_ku_{k+1}$. According to Aigne [1], it is well known that the diameter of any strong digraph different from a directed cycle satisfies $\operatorname{diam}(L^k(D)) = \operatorname{diam}(D) + k$. For more information on line digraphs see, for instance, [8, 15, 17, 18].

A processor interconnection network or a communications network is conveniently modeled by a graph or a digraph, in which the vertex set corresponds to processors or switching elements, and the edge set or the arc set corresponds to communication links. In order to estimate more precisely the reliability of networks, Esfahanian and Hakimi [7] introduced the concept of restricted edge-connectivity. A set of edges S in a connected graph G is called a *restricted edge cut* if $G - S$ is disconnected and contains no isolated vertex. If such an edge cut exists, then the *restricted edge connectivity* of G, denoted by $\lambda'(G)$, is the minimum number of edges over all restricted edge cuts of G. A connected graph G is called λ'-*connected* if $\lambda'(G)$ exists. Esfahanian and Hakimi [7] obtained that each connected graph G with at least 4 vertices except a star $K_{1,n-1}$ is λ'-connected. Later, Volkmann [19] extended the notion of restricted edge-connectivity to digraphs as follows.

Definition 1 [19]**.** Let D be a strong digraph. An arc-set S of D is a *restricted arc-cut* if $D-S$ has a non-trivial strong component D_1 such that $D-V(D_1)$ contains an arc. The *restricted arc-connectivity* $\lambda'(D)$ is defined as the minimum cardinality over all restricted arc-cuts S. A strong digraph D is called λ'-*connected* if $\lambda'(D)$ exists. A restricted arc-cut S is called a λ'-*cut* if $|S| = \lambda'(D)$.

Volkmann [19] proved that each strong digraph D of order $n \geqslant 4$ and girth $g = 2, 3$ not belonging to some families of digraphs is λ'-connected and satisfies $\lambda(D) \leqslant \lambda'(D) \leqslant \xi(D)$, where $\xi(D)$ is defined as follows. If $C_g = u_1u_2 \cdots u_gu_1$ is a shortest cycle of D, then $\xi(C_g) = \min\{\Sigma_{i=1}^{g} d^+(u_i) - g, \Sigma_{i=1}^{g} d^-(u_i) - g\}$ and $\xi(D) = \min\{\xi(C_g) : C_g$ is a shortest cycle of $D\}$. Wang and Lin [20] introduced the notion of arc-degree which turns to be a better bound for $\lambda'(D)$. For any arc $uv \in A(D)$, the *arc-degree* of uv is defined as

$$\xi'(uv) = \min\left\{|\omega^+(\{u,v\})|, |\omega^-(\{u,v\})|, |\omega^+(u) \cup \omega^-(v)|, |\omega^-(u) \cup \omega^+(v)|\right\}.$$

The *minimum arc-degree* of D is $\xi'(D) = \min\{\xi'(uv) : uv \in A(D)\}$. One can compute the arc-degree of an arc $uv \in A(D)$ in terms of the out-degrees and in-degrees of the vertices u and v. If $uv \notin Sym(D)$, then $\xi'(uv) = \min\{d^+(u) + d^+(v) - 1, d^-(u) + d^-(v) - 1, d^+(u) + d^-(v) - 1, d^-(u) + d^+(v)\}$. If $uv \in Sym(D)$, then $\xi'(uv) = \min\{d^+(u)+d^+(v)-2, d^-(u)+d^-(v)-2, d^+(u)+d^-(v)-1, d^-(u)+d^+(v) - 1\}$. Wang and Lin [20] claimed that a strong digraph D with $\delta^+(D) \geq 3$ or $\delta^-(D) \geq 3$ is λ'-connected and satisfies $\lambda'(D) \leqslant \xi(D)$. It is clear from the definitions that $\xi'(D) \leqslant \xi(D)$ for every strong digraph D with minimum degree at least 3. Balbuena *et al.* [3] deduced the following result about λ'-connected digraphs.

Theorem 2 [3]**.** *Let D be a strong digraph with order $n \geqslant 4$ and minimum degree $\delta \geqslant 2$. Then D is λ'-connected and $\lambda'(D) \leqslant \xi'(D)$.*

Meierling *et al.* [16] characterized all λ'-connected tournaments, multipartite tournaments, local tournaments and in-tournaments. Chen *et al.* [5] proved that the Cartesian product digraph of two strong digraphs is λ'-connected. They also gave the upper and lower bounds for its restricted arc-connectivity. Lin *et al.* [14] presented a sufficient condition for the former upper bound to be attained and gave an example to show the result is best possible. González-Moreno and Hernández Ortiz [9] showed that a family of strong digraphs with girth four is λ'-connected. Moreover, the values of the restricted arc-connectivity of some special digraphs, for example, unidirectional hypercube [13], unidirectional folded hypercube [13] and unidirectional star graph [21], were investigated.

Wang and Lin [20] called a strong digraph λ'-optimal if its restricted arc-connectivity is equal to its minimum arc-degree.

Definition 3 [20]**.** A digraph D is λ'-*optimal* if $\lambda'(D) = \xi'(D)$.

Balbuena *et al.* [4] showed that a generalized p-cycle D is λ'-optimal if $\mathrm{diam}(D) \leq 2l + p - 2$, where l is the semigirth of D and $p \geq 3$, and showed that the k-iterated line digraph of it is λ'-optimal if $\mathrm{diam}(D) \leq 2l+p-2+k$ for $p \geq 3$. Chen *et al.* [6] studied the restricted arc-connectivity of bipartite digraphs and gave sufficient conditions for a bipartite digraph to be λ'-optimal. Balbuena *et al.* [2] gave a sufficient condition for a s-geodetic strong digraph to be λ'-optimal: a strong s-geodetic digraph D with $\delta^+(D) \geq 3$ or $\delta^-(D) \geq 3$ is λ'-optimal if $\mathrm{diam}(D) \leq 2s - 1$. Furthermore, in the same publication, we can see that the h-iterated line digraph of an s-geodetic digraph is λ'-optimal for certain iteration h. Grüter *et al.* [10] characterized all strong tournaments T with $\lambda'(T) \leqslant \xi'(T)$ and they proved that all tournaments with minimum degree $\delta(T) \geq (n+1)/4$ are λ'-optimal. In addition, Grüter *et al.* [11] showed that $\lambda'(T) \leqslant \xi'(T)$ and T is λ'-optimal if $\delta(T) \geq (n+3)/8$ for all strong bipartite tournaments T except for a family.

Given an arc $uv \in A(D)$, define $\Omega_{uv} = \{\omega^+(\{u,v\}), \omega^-(\{u,v\}), \omega^+(u) \cup \omega^-(v), \omega^-(u) \cup \omega^+(v)\}$. Note that for every arc $uv \in A(D)$ of a strong digraph D, the elements of Ω_{uv} are arc-cuts, but they are not necessarily restricted arc-cuts. Balbuena *et al.* [3] introduced the definition of super restricted arc-connected digraphs.

Definition 4 [3]**.** A λ'-connected digraph D is said to be *super*-λ' if for every λ'-cut S there exists an arc $uv \in A(D)$ such that $S \in \Omega_{uv}$.

Clearly, if D is super-λ', then $\lambda'(D) = \xi'(D)$, but the converse is not true. The following result is useful in studying super-λ' digraphs.

Theorem 5 [3]**.** *Let D be a λ'-connected digraph and let S be a λ'-cut of D. If D is not super-λ', then there exists a subset of vertices $F \subset V(D)$ such that $S = \omega^+(F) = [F, \overline{F}]$ and both the induced subdigraphs $D[F]$ and $D[\overline{F}]$ contain an arc.*

Balbuena *et al.* [3] provided a sufficient condition for an s-geodetic digraph to be super-λ' and showed that the h-iterated line digraph of an s-geodetic digraph is super-λ' for a particular h. Lin *et al.* [12] presented some minimum degree conditions for oriented graphs to be super-λ' and gave examples to show the conditions are sharp.

Investigations on the restricted edge-connectivity of graphs were made by many researchers. However, related results on restricted arc-connectivity have received little attention. This paper shows sufficient conditions for a digraph to be λ'-optimal or super-λ' based on a parameter related with path lengths of digraphs in Section 2.

2. Main Results

2.1. λ'-optimal oriented graphs

In this subsection we characterize λ'-optimal oriented graphs in terms of their diameter and a parameter l_2 defined as follows.

Definition 6 [4]**.** For a given digraph D, let $l_2 = l_2(D)$, $1 \leqslant l_2 \leqslant \mathrm{diam}(D)$, be the greatest integer such that for any (not necessarily different) $x, y \in V(D)$,

(a) if $d(x, y) \leqslant l_2$, the shortest $x \to y$ path is unique;

(b) if $d(x, y) \leqslant l_2 - t$, there is no $x \to y$ path of length $d(x, y) + t$ for $1 \leqslant t \leqslant 2$.

Observe that l_2 is well defined in oriented graphs, that is in digraphs with girth at least 3. The parameter l_2 satisfies an equality as follows.

Lemma 7 [4]**.** *Let D be a strong oriented graph other than a directed cycle. Then $l_2(L^k(D)) = l_2(D) + k$.*

Lemma 8. *Let D be an oriented graph and $uv \in A(D)$. If $l_2 \geqslant 2$, then $N_i^+(u) \cap N_i^+(v) = \emptyset$ and $N_i^+(N^+(u) - v) \cap N_i^+(v) = \emptyset$ for all $i = 1, 2, \ldots, l_2 - 1$; if $l_2 \geqslant 3$, then $N_i^+(u) \cap N_i^+(N^+(v)) = \emptyset$ and $N_i^+(N^+(u) - v) \cap N_i^+(N^+(v)) = \emptyset$ for all $i = 1, 2, \ldots, l_2 - 2$.*

Proof. Assume by contradiction that $N_i^+(u) \cap N_i^+(v) \neq \emptyset$. Let $x \in N_i^+(u) \cap N_i^+(v)$, then $d(u, x) = i$ and there is a $u \to x$ path of length $i + 1$, which is a contradiction to the definition of l_2 as $i \leqslant l_2 - 1$. Analogously, if there is a vertex $x \in N_i^+(N^+(u) - v) \cap N_i^+(v)$, then $d(u, x) = i + 1$ and there are two disjoint shortest $u \to x$ paths of length $i + 1$, contradicting again to the definition of l_2 as $i \leqslant l_2 - 1$. The other two equalities can be proved similarly. ■

Theorem 9. *Let D be a strong oriented graph with minimum degree $\delta \geqslant 2$. Then D is λ'-optimal if* $\mathrm{diam}(D) \leqslant 2l_2(D) - 2$.

Proof. Note that the condition $\delta \geqslant 2$ implies that D has order at least 5. Hence, by Theorem 2, D is λ'-connected and $\lambda'(D) \leqslant \xi'(D)$. Assume by contradiction that D is not λ'-optimal, therefore, $\lambda'(D) < \xi'(D)$. Let S be a λ'-cut of D, by Theorem 5, there exists a subset of vertices $F \subset V(D)$ such that $S = \omega^+(F) = [F, \overline{F}] = [X, \overline{X}]$ and both the induced subdigraphs $D[F]$ and $D[\overline{F}]$ contain an arc. Define $\mu = \max\{d(v, X) : v \in V(F)\}$ and $\overline{\mu} = \max\{d(\overline{X}, v) : v \in V(\overline{F})\}$. Also, let the vertices of F and $\overline{F}$ be respectively partitioned into subsets F_i, $0 \leqslant i \leqslant \mu$, and $\overline{F}_j$, $0 \leqslant j \leqslant \overline{\mu}$, according to their distance to X or from $\overline{X}$, that is, $F_i = \{v \in V(F) : d(v, X) = i\}$ and $\overline{F}_j = \{v \in V(\overline{F}) : d(\overline{X}, v) = j\}$. The distance from a vertex in F_μ to one in $\overline{F}_{\overline{\mu}}$ is at least $\mu + \overline{\mu} + 1$, so that $\mu + \overline{\mu} + 1 \leqslant \operatorname{diam}(D)$. Since $\operatorname{diam}(D) \leqslant 2l_2 - 2$, one of them, μ or $\overline{\mu}$, is at most $l_2 - 2$. Without loss of generality, suppose $\mu \leqslant \overline{\mu}$. Then $\mu \leqslant l_2 - 2$.

If $\mu = 0$, then $l_2 \geqslant 2$ and $F = X$. Let $uv \in A(D[X])$, see Figure 1.

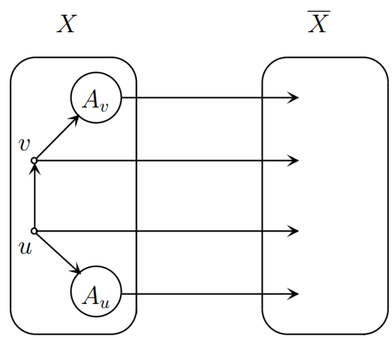

Figure 1.

Define $A_v = N^+(v) \cap X$ and $A_u = (N^+(u) - v) \cap X$. From Lemma 8, it follows that $N^+(u) \cap N^+(v) = \emptyset$ and therefore A_v, A_u and $\{u, v\}$ are pairwise disjoint. Hence,

$$\begin{aligned}\lambda'(D) &= |[X, \overline{X}]| \geqslant |[\{u, v\}, \overline{X}]| + |[A_v, \overline{X}]| + |[A_u, \overline{X}]| \\ &\geqslant |[\{u, v\}, \overline{X}]| + |A_v| + |A_u| = |N^+(v)| + |N^+(u) - v| \\ &= |\omega^+(\{u, v\})| \geqslant \xi'(uv) \geqslant \xi'(D),\end{aligned}$$

which contradicts the hypothesis.

Assume that $1 \leqslant \mu \leqslant l_2 - 2$. We consider the following two cases.

Case 1. There exists an arc uv in $D[F]$ such that $d(u, X) = d(v, X) = \mu$.

Subcase 1.1. $\mu \geqslant 2$, which implies $l_2 \geqslant 4$, see Figure 2. Define $A_v = N^+(v) \cap F_\mu$, $B_v = N^+(v) \cap F_{\mu-1}$, $B_u = N^+(u) \cap F_{\mu-1}$ and $A_u = (N^+(u) - v) \cap F_\mu$. By Lemma 8, one can show that A_v, B_v, B_u and A_u are pairwise disjoint. Set $A_1 = N^+_\mu(A_v) \cap X$, $A_2 = N^+_{\mu-1}(B_v) \cap X$, $A_3 = N^+_{\mu-1}(B_u) \cap X$ and $A_4 = N^+_\mu(A_u) \cap X$.

We will now show that A_1, A_2, A_3 and A_4 are pairwise disjoint. It is clear by Lemma 8 that $A_2 \cap A_3 = \emptyset$, $A_2 \cap A_4 = \emptyset$, $A_1 \cap A_3 = \emptyset$ and $A_1 \cap A_4 = \emptyset$. If there is a vertex $x \in A_1 \cap A_2$, then $d(u,x) = \mu + 1$ and there is a $u \to x$ path of length $\mu + 2$, which contradicts the definition of l_2 as $\mu \leqslant l_2 - 2$. Analogously, it can be deduced that $A_3 \cap A_4 = \emptyset$. We claim that $|A_1| \geqslant |A_v|$, $|A_2| \geqslant |B_v|$, $|A_3| \geqslant |B_u|$ and $|A_4| \geqslant |A_u|$. In fact, if $|A_1| < |A_v|$, then there exists a vertex $x \in A_1$ such that there are two distinct shortest $u \to x$ paths of length $\mu + 2$, contradicting again to the definition of l_2. The other three inequalities can be proved similarly. Consequently,

$$\begin{aligned}\lambda'(D) &= |[X, \overline{X}]| \geqslant |X| \geqslant |A_1| + |A_2| + |A_3| + |A_4| \\ &\geqslant |A_v| + |B_v| + |B_u| + |A_u| = |N^+(v)| + |N^+(u) - v| \\ &= |\omega^+(\{u,v\})| \geqslant \xi'(uv) \geqslant \xi'(D).\end{aligned}$$

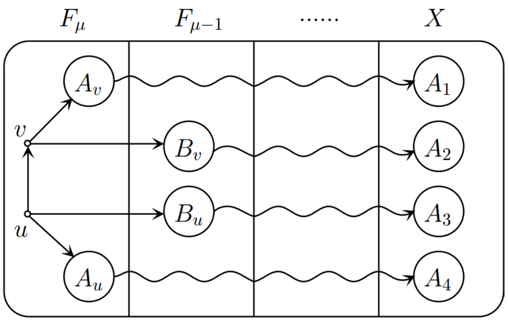

Figure 2.

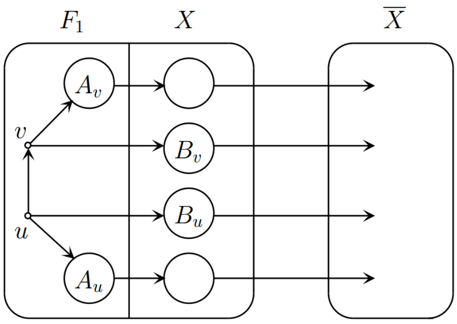

Figure 3.

Subcase 1.2. $\mu = 1$, which implies $l_2 \geqslant 3$, see Figure 3. Define $A_v = N^+(v) \cap F_1$, $B_v = N^+(v) \cap X$, $B_u = N^+(u) \cap X$ and $A_u = (N^+(u) - v) \cap F_1$. Similarly as in the proof of Subcase 1.1, one can show that $N^+(A_v)\cap X$, B_v, B_u and $N^+(A_u)\cap X$ are pairwise disjoint, $|N^+(A_v)\cap X| \geqslant |A_v|$ and $|N^+(A_u)\cap X| \geqslant |A_u|$. Thus,

$$\begin{aligned}
\lambda'(D) &= |[X,\overline{X}]| \\
&\geqslant |[N^+(A_v)\cap X,\overline{X}]| + |[B_v,\overline{X}]| + |[B_u,\overline{X}]| + |[N^+(A_u)\cap X,\overline{X}]| \\
&\geqslant |N^+(A_v)\cap X| + |B_v| + |B_u| + |N^+(A_u)\cap X| \\
&\geqslant |A_v| + |B_v| + |B_u| + |A_u| = |N^+(v)| + |N^+(u)-v| \\
&= |\omega^+(\{u,v\})| \geqslant \xi'(uv) \geqslant \xi'(D).
\end{aligned}$$

Case 2. There is no arc uv in $D[F]$ such that $d(u,X) = d(v,X) = \mu$.

Subcase 2.1. $\mu \geqslant 2$, which implies $l_2 \geqslant 4$, see Figure 4. Define $A_u = (N^+(u)-v)\cap F_{\mu-1}$, $A_v = N^+(v)\cap F_{\mu-1}$, $B_v = N^+(v)\cap F_\mu$, $C = N^+(B_v)\cap F_{\mu-1}$. As $l_2 \geqslant 4$, it is clear that $N^+(u)$, A_v and C are pairwise disjoint. Since the induced subdigraph $D[F_\mu]$ contains no arc, we have $|C| \geqslant \delta|B_v| \geqslant 2|B_v|$. Set $A_1 = N^+_\mu(u)\cap X$, $A_2 = N^+_{\mu-1}(A_v)\cap X$ and $A_3 = N^+_{\mu-1}(C)\cap X$. Similar arguments as in Subcase 1.1 show that A_1, A_2 and A_3 are pairwise disjoint. As $\mu \leqslant l_2 - 2$, it follows that $|A_3| \geqslant |C|$, $|A_2| \geqslant |A_v|$ and $|A_1| = |N^+_\mu(u)\cap X| = |N^+_{\mu-1}(A_u)\cap X| + |N^+_{\mu-1}(v)\cap X| = |N^+_{\mu-1}(A_u)\cap X| + |N^+_{\mu-2}(N^+(v)\cap F_{\mu-2})\cap X| \geqslant |A_u| + |N^+(v)\cap F_{\mu-2}|$. Thereby,

$$\begin{aligned}
\lambda'(D) &= |[X,\overline{X}]| \geqslant |X| \geqslant |A_1| + |A_2| + |A_3| \\
&\geqslant |A_u| + |N^+(v)\cap F_{\mu-2}| + |A_v| + |C| \\
&\geqslant |A_u| + |N^+(v)\cap F_{\mu-2}| + |A_v| + 2|B_v| \\
&= |N^+(u)-v| + |N^+(v)| + |B_v| \\
&= |\omega^+(\{u,v\})| + |B_v| \geqslant \xi'(uv) + |B_v| \geqslant \xi'(D).
\end{aligned}$$

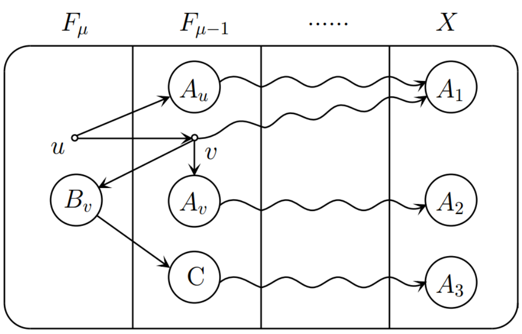

Figure 4.

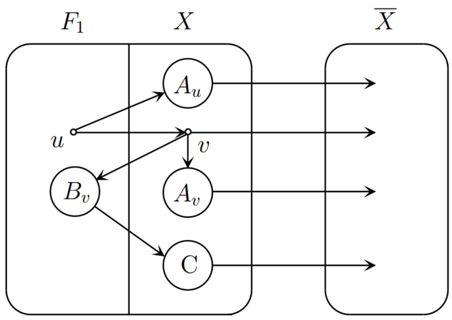

Figure 5.

Subcase 2.2. $\mu = 1$, which implies $l_2 \geqslant 3$, see Figure 5. Consider the sets $A_u = (N^+(u) - v) \cap X$, $A_v = N^+(v) \cap X$, $B_v = N^+(v) \cap F_1$, $C = N^+(B_v) \cap X$. As $l_2 \geqslant 3$, one can show that $N^+(u)$, A_v, and C are pairwise disjoint. Since the induced subdigraph $D[F_1]$ contains no arc, we have $|C| \geqslant \delta|B_v| \geqslant 2|B_v|$. Hence,

$$\begin{aligned}
\lambda'(D) &= |[X,\overline{X}]| \geqslant |[N^+(u),\overline{X}]| + |[A_v,\overline{X}]| + |[C,\overline{X}]| \\
&= |[A_u,\overline{X}]| + |[v,\overline{X}]| + |[A_v,\overline{X}]| + |[C,\overline{X}]| \\
&\geqslant |A_u| + |[v,\overline{X}]| + |A_v| + |C| \geqslant |A_u| + |[v,\overline{X}]| + |A_v| + 2|B_v| \\
&= |N^+(u) - v| + |N^+(v)| + |B_v| = |\omega^+(\{u,v\})| + |B_v| \\
&\geqslant \xi'(uv) + |B_v| \geqslant \xi'(D).
\end{aligned}$$

All cases lead to contradictions. ∎

For tournaments, oriented bipartite graphs and k-iterated line digraphs, the following corollaries follow from Lemma 7 and Theorem 9.

Corollary 10. *Let D be a strong tournament or oriented bipartite graph with minimum degree $\delta \geqslant 2$. Then D is λ'-optimal if* $\operatorname{diam}(D) \leqslant 2l_2(D) - 2$.

Corollary 11. *Let D be a strong oriented graph with minimum degree $\delta \geqslant 2$. Then $L^k(D)$ is λ'-optimal if $k \geqslant$* $\operatorname{diam}(D) - 2l_2(D) + 2$.

2.2. Super-λ' oriented graphs

In this subsection we characterize super-λ' oriented graphs in terms of their diameter and the parameter l_2.

Theorem 12. *Let D be a strong oriented graph with minimum degree $\delta \geqslant 3$. Then D is super-λ' if* $\operatorname{diam}(D) \leqslant 2l_2(D) - 2$.

Proof. Theorem 9 shows that D is λ'-optimal, thus, $\lambda'(D) = \xi'(D)$. Suppose that D is not super-λ'. Let S be a λ'-cut of D such that $S \notin \bigcup_{uv\in A(D)} \Omega_{uv}$. By Theorem 5, there exists a subset of vertices $F \subset V(D)$ such that $S = \omega^+(F) = [F, \overline{F}] = [X, \overline{X}]$ and both the induced subdigraphs $D[F]$ and $D[\overline{F}]$ contain an arc. We use the same notation as in the proof of Theorem 9. Now in this proof we have $\mu + \overline{\mu} + 1 \leqslant 2l_2 - 2$. Without loss of generality, suppose $\mu \leqslant \overline{\mu}$. Then $\mu \leqslant l_2 - 2$.

If $\mu = 0$, then $l_2 \geqslant 2$ and $F = X$, see Figure 1. As shown in the proof of Theorem 9, we have

$$\begin{aligned}\xi'(D) = \lambda'(D) = |[X, \overline{X}]| &\geqslant |[\{u, v\}, \overline{X}]| + |[A_v, \overline{X}]| + |[A_u, \overline{X}]| \\ &\geqslant |[\{u, v\}, \overline{X}]| + |A_v| + |A_u| = |N^+(v)| + |N^+(u) - v| \\ &= |\omega^+(\{u, v\})| \geqslant \xi'(uv) \geqslant \xi'(D),\end{aligned}$$

which implies $\lambda'(D) = \xi'(D)$ and all the inequalities are equalities, yielding in particular $|A_v| = |[A_v, \overline{X}]|$, $|A_u| = |[A_u, \overline{X}]|$. This implies $|[z, \overline{X}]| = 1$ for all $z \in A_u \cup A_v$. If $A_u \cup A_v = \emptyset$, then $F = \{u, v\}$ and $S = \omega^+(F) = \omega^+(\{u, v\})$, a contradiction. Thus $A_u \cup A_v \neq \emptyset$. Let $u_0 \in A_u \cup A_v$. Since $|[u_0, \overline{X}]| = 1$ and $\delta \geqslant 3$, we have $|N^+(u_0) \cap X| \geqslant 2$. Let $v_0, z_0 \in N^+(u_0) \cap X$, we consider now the arc u_0v_0. As above, we have $|[z', \overline{X}]| = 1$ for all $z' \in A_{u_0} \cup A_{v_0}$ such that $A_{u_0} = (N^+(u_0) - v_0) \cap X$ and $A_{v_0} = N^+(v_0) \cap X$. It implies the existence of some $t_0 \in N^+(z_0) \cap X$. As $l_2 \geqslant 2$, we have $t_0 \notin A_{u_0} \cup A_{v_0} \cup \{u_0, v_0\}$. Clearly, A_{u_0}, A_{v_0}, $\{u_0, v_0\}$ and $\{t_0\}$ are pairwise disjoint. Thus,

$$\begin{aligned}\xi'(D) = \lambda'(D) &= |[X, \overline{X}]| \\ &\geqslant |[\{u_0, v_0\}, \overline{X}]| + |[A_{v_0}, \overline{X}]| + |[A_{u_0}, \overline{X}]| + |[t_0, \overline{X}]| \\ &\geqslant |[\{u_0, v_0\}, \overline{X}]| + |A_{v_0}| + |A_{u_0}| + |[t_0, \overline{X}]| \\ &= |N^+(v_0)| + |N^+(u_0) - v_0| + |[t_0, \overline{X}]| \\ &> |\omega^+(\{u_0, v_0\})| \geqslant \xi'(u_0v_0) \geqslant \xi'(D),\end{aligned}$$

a contradiction.

Assume that $1 \leqslant \mu \leqslant l_2 - 2$. We distinguish the following two cases.

Case 1. There exists an arc uv in $D[F]$ such that $d(u, X) = d(v, X) = \mu$.

Subcase 1.1. $\mu \geqslant 2$, which implies $l_2 \geqslant 4$, see Figure 2. As mentioned in the proof of Subcase 1.1 of Theorem 9, we deduce that

$$\begin{aligned}\xi'(D) = \lambda'(D) = |[X, \overline{X}]| \geqslant |X| &\geqslant |A_1| + |A_2| + |A_3| + |A_4| \\ &\geqslant |A_v| + |B_v| + |B_u| + |A_u| = |N^+(v)| + |N^+(u) - v| \\ &= |\omega^+(\{u, v\})| \geqslant \xi'(uv) \geqslant \xi'(D),\end{aligned}$$

which implies $\lambda'(D) = \xi'(D)$ and all the inequalities are equalities, which yields in particular $|A_3| = |B_u|$ and $X = A_1 \cup A_2 \cup A_3 \cup A_4$. Since $|A_3| = |B_u|$ and $\delta \geqslant 3$, there exists a vertex $u' \in B_u$ such that $N^+(u') \cap (F_\mu \cup F_{\mu-1}) \neq \emptyset$. Let $z \in N^+(u') \cap (F_\mu \cup F_{\mu-1})$, it can be obtained that $z \notin A_v \cup B_v \cup B_u \cup A_u \cup \{u, v\}$ for the fact that $l_2 \geqslant 4$. Therefore there exists a vertex $x \in N^+_{d(z,X)}(z) \cap X$ with $\mu - 1 \leqslant d(z, x) \leqslant \mu$. If $x \in A_1$, then we find a shortest $u \to x$ path of length $\mu + 1$ or $\mu + 2$ and there is also a $u \to x$ path of length $\mu + 2$, contradicting the definition of l_2 as $\mu \leqslant l_2 - 2$. Similar contradictions will arise when $x \in A_2$ or $x \in A_3$ or $x \in A_4$.

Subcase 1.2. $\mu = 1$, which implies $l_2 \geqslant 3$, see Figure 3. It is recalled from the proof of Subcase 1.2 of Theorem 9 that

$$\begin{aligned}
\xi'(D) = \lambda'(D) &= |[X, \overline{X}]| \\
&\geqslant |[N^+(A_v) \cap X, \overline{X}]| + |[B_v, \overline{X}]| + |[B_u, \overline{X}]| + |[N^+(A_u) \cap X, \overline{X}]| \\
&\geqslant |N^+(A_v) \cap X| + |B_v| + |B_u| + |N^+(A_u) \cap X| \\
&\geqslant |A_v| + |B_v| + |B_u| + |A_u| = |N^+(v)| + |N^+(u) - v| \\
&= |\omega^+(\{u, v\})| \geqslant \xi'(uv) \geqslant \xi'(D),
\end{aligned}$$

which implies $\lambda'(D) = \xi'(D)$ and all the inequalities are equalities, deducing in particular $|B_u| = |[B_u, \overline{X}]|$ and $X = (N^+(A_v) \cap X) \cup B_v \cup B_u \cup (N^+(A_u) \cap X)$. Since $\delta \geqslant 3$, there exists a vertex $u' \in B_u$ such that $N^+(u') \cap (F_1 \cup X) \neq \emptyset$. Let $z \in N^+(u') \cap (F_1 \cup X)$. Then $z \notin A_v \cup B_v \cup B_u \cup A_u \cup (N^+(A_v) \cap X) \cup (N^+(A_u) \cap X) \cup \{u, v\}$ for the fact that $l_2 \geqslant 3$. Thus $z \in F_1$ with $d(z, X) = 1$. Therefore there exists a vertex $x \in N^+(z) \cap X$. If $x \in N^+(A_v) \cap X$, then there are two disjoint $u \to x$ paths of length 3, contradicting the definition of l_2 as $l_2 \geqslant 3$. It will lead to similar contradictions when $x \in B_v$ or $x \in B_u$ or $x \in N^+(A_u) \cap X$.

Case 2. There is no arc uv in $D[F]$ such that $d(u, X) = d(v, X) = \mu$.

Subcase 2.1. $\mu \geqslant 2$, which implies $l_2 \geqslant 4$, see Figure 4. As the induced subdigraph $D[F_\mu]$ contains no arc, $|C| \geqslant \delta |B_v| \geqslant 3|B_v|$. It is easily obtained from the proof of Subcase 2.1 of Theorem 9 that

$$\begin{aligned}
\xi'(D) = \lambda'(D) = |[X, \overline{X}]| \geqslant |X| &\geqslant |A_1| + |A_2| + |A_3| \\
&\geqslant |A_u| + |N^+(v) \cap F_{\mu-2}| + |A_v| + |C| \\
&\geqslant |A_u| + |N^+(v) \cap F_{\mu-2}| + |A_v| + 3|B_v| \\
&= |N^+(u) - v| + |N^+(v)| + 2|B_v| = |\omega^+(\{u, v\})| + 2|B_v| \\
&\geqslant \xi'(uv) + 2|B_v| \geqslant \xi'(D) + 2|B_v|.
\end{aligned}$$

This is a contradiction unless $B_v = \emptyset$, which yields in this case $C = \emptyset$, and all the inequalities are equalities. Therefore, $|A_1| = |A_u| + |N^+(v) \cap F_{\mu-2}|$ and

$X = A_1 \cup A_2$. As $\delta \geqslant 3$, there exists a vertex $z \in N^+(A_u) \cap (F_\mu \cup F_{\mu-1})$. Observe that $z \notin A_u \cup A_v \cup \{u, v\}$ as $l_2 \geqslant 4$. Thus there exists a vertex $x \in N^+_{d(z,X)}(z) \cap X$ with $\mu - 1 \leqslant d(z, x) \leqslant \mu$. If $x \in A_1$, then $d(u, x) = \mu$ and there is a $u \to x$ path of length $\mu + 1$ or $\mu + 2$, contradicting the definition of l_2 as $\mu \leqslant l_2 - 2$. One can get a similar contradiction if $x \in A_2$.

Subcase 2.2. $\mu = 1$, which implies $l_2 \geqslant 3$, see Figure 5. As the induced subdigraph $D[F_1]$ contains no arc, $|C| \geqslant \delta|B_v| \geqslant 3|B_v|$. The proof of Subcase 2.2 of Theorem 9 shows that

$$\begin{aligned}
\xi'(D) = \lambda'(D) = |[X, \overline{X}]| &\geqslant |[N^+(u), \overline{X}]| + |[A_v, \overline{X}]| + |[C, \overline{X}]| \\
&= |[A_u, \overline{X}]| + |[v, \overline{X}]| + |[A_v, \overline{X}]| + |[C, \overline{X}]| \geqslant |A_u| + |[v, \overline{X}]| + |A_v| + |C| \\
&\geqslant |A_u| + |[v, \overline{X}]| + |A_v| + 3|B_v| = |N^+(u) - v| + |N^+(v)| + 2|B_v| \\
&= |\omega^+(\{u, v\})| + 2|B_v| \geqslant \xi'(uv) + 2|B_v| \geqslant \xi'(D) + 2|B_v|.
\end{aligned}$$

This is a contradiction unless $B_v = \emptyset$, inferring that in this case $C = \emptyset$, and all the inequalities are equalities. Therefore $|A_u| = |[A_u, \overline{X}]|$ and $X = N^+(u) \cup A_v$. As $\delta \geqslant 3$, there exists a vertex $z \in N^+(A_u) \cap (F_1 \cup X)$. It is seen that $z \notin A_u \cup A_v \cup \{u, v\}$ as $l_2 \geqslant 3$. Thus $z \in F_1$ with $d(z, X) = 1$, and there exists a vertex $x \in N^+(z) \cap X$. If $x \in N^+(u)$, then $d(u, x) = 1$ and there is a $u \to x$ path of length 3, contradicting the definition of l_2 as $l_2 \geqslant 3$. There will be a contradiction when $x \in A_v$. ■

For tournaments, oriented bipartite graphs and k-iterated line digraphs, the following corollaries are obtained from Lemma 7 and Theorem 12.

Corollary 13. *Let D be a strong tournament or oriented bipartite graph with minimum degree $\delta \geqslant 3$. Then D is super-λ' if* $\mathrm{diam}(D) \leqslant 2l_2(D) - 2$.

Corollary 14. *Let D be a strong oriented graph with minimum degree $\delta \geqslant 3$. Then $L^k(D)$ is super-λ' if $k \geqslant \mathrm{diam}(D) - 2l_2(D) + 2$.*

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