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SOME RESULTS ON THE k-ALLIANCE AND DOMINATION OF GRAPHS

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Abstract

For a graph G of order n and real number $a \geq -1$, let $\rho_i^a(G)$ be the i-th largest eigenvalue of $A_a(G) := aD(G) + A(G)$, where $A(G)$ and $D(G)$ are the adjacency matrix and the diagonal degree matrix of G. In this paper, we investigate connections of the eigenvalues of $A_a(G)$ to the alliances and domination parameters of G including defensive k-alliance number, global offensive k-alliance number, total domination number and signed domination number. Our results in this paper provide a unified approach to study the alliances and domination in a graph by its eigenvalues of associated matrices. We derive two lower bounds for the defensive k-alliance number of G based on the parameter $\rho_2^a(G)$. Moreover, we establish a lower bound for the global offensive k-alliance number of G in relation to $\rho_n^a(G)$. Additionally, we also deduce the lower bounds for $\rho_2^a(G)$ based on its total and signed domination numbers, as well as upper bounds for $\rho_n^a(G)$, respectively. Those results generalize the corresponding known results on $a = -1$ due to Fernau, Rodríguez-Velázquez and Sigarreta (2004), Rodríguez-Velázquez, Yero and Sigarreta (2009) and Shi, Kang and Wu (2010), respectively.

Keywords: total (signed) domination number, defensive (global offensive) k-alliance number, eigenvalue, bound.

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1. Introduction

All graphs considered in this paper are undirected, connected and simple. Let $G = (V(G), E(G))$ be a graph of order n and size m. For any real number $a \geq -1$, let $A_a(G) = aD(G) + A(G)$, where $A(G)$ and $D(G)$ are the adjacency matrix and the diagonal degree matrix of G, respectively. Especially, $A_0(G) = A(G)$, $A_{-1}(G) = -L(G)$ and $A_1(G) = Q(G)$, where $L(G)$ and $Q(G)$ are the Laplacian matrix and the signless Laplacian matrix of G. We denote the eigenvalues of $A_a(G)$ as $\rho_1^a(G) \geq \rho_2^a(G) \geq \cdots \geq \rho_n^a(G)$. In particular, $\rho_1^0(G) \geq \rho_2^0(G) \geq \cdots \geq \rho_n^0(G)$, $-\rho_n^{-1}(G) \geq -\rho_{n-1}^{-1}(G) \geq \cdots \geq -\rho_2^{-1}(G) \geq -\rho_1^{-1}(G) = 0$ and $\rho_1^1(G) \geq \rho_2^1(G) \geq \cdots \geq \rho_n^1(G)$ are the eigenvalues, the Laplacian eigenvalues and the signless Laplacian eigenvalues of G, respectively. Additionally, $-\rho_n^{-1}(G)$ and $-\rho_2^{-1}(G)$ are the Laplacian spectral radius and the algebraic connectivity of G, respectively. When only one graph G is under consideration, we sometimes write ρ_i^a instead of $\rho_i^a(G)$.

The eigenvalues of a graph contain extensive information about the graph. Many studies on this topic have been conducted, see the book [7]. In particular, the study of the relationship between the eigenvalues of matrices associated with a graph and its diverse structural parameters is of particular interest. Such as independence number [13, 14, 16], matching number [11, 31], *etc.* In this paper, we continue to study the k-alliance and domination of a graph from the view of its eigenvalues. Our primary focus will be on exploring the connections between the eigenvalues of a graph and its diverse domination numbers (or k-alliance numbers, respectively). The subsequent sections will introduce the concepts of k-alliance and domination for a graph, along with relevant results.

Initially, we present essential symbols, terminologies and valuable mathematical tools. For $v \in V(G)$, let $d_G(v)$ (or $d(v)$ for short) be the degree of v and $N(v) = N_G(v) = \{u \in V : uv \in E\}$ (respectively, $N[v] = N_G[v] = N(v) \cup \{v\}$) be the open (respectively, closed) neighborhood of v. For $S \subseteq V(G)$, let $N(S) = \bigcup_{v \in S} N(v)$ (respectively, $N[S] = N(S) \cup S$) be the open (respectively, closed) neighborhood of S. The number $\deg_S(v) = |N_S(v)|$ (or $d_S(v)$ for short) is the degree of v in S, where $N_S(v) = N_G(v) \cap S$. The maximum and the minimum degrees of a graph G are denoted by $\Delta(G) = \max_{v \in V(G)} d(v)$ and $\delta(G) = \min_{v \in V(G)} d(v)$, respectively. The degree sequence of G is denoted by $\Delta(G) = d(v_1) \geq \cdots \geq d(v_n) = \delta(G)$ (or $\Delta = d_1 \geq \cdots \geq d_n = \delta$ for short). For any $S \subseteq V(G)$, let $G[S]$ and $G[V \backslash S]$ (or $G[\bar{S}]$ for short) denote the subgraph of G induced by S and $V(G) \backslash S$, respectively. For two disjoint subsets $S, T \subseteq V(G)$, we define $E(S, T) = \{uv \in E(G) : u \in S, v \in T\}$ and $e(S, T) = |E(S, T)|$. A graph G with a partition $V(G) = V_1 \cup V_2$ is called (r, s)-*local-regular* if $d_{V_2}(u) = r$ for $u \in V_1$ and $d_{V_1}(v) = s$ for $v \in V_2$.

The eigenvalues of an $n \times n$ real symmetric matrix M are denoted by $\lambda_1(M) \geq$

$\lambda_2(M) \geq \cdots \geq \lambda_n(M)$, with the convention that the eigenvalues are arranged in nonincreasing order. Given two nonincreasing real sequences $A : \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ and $B : \eta_1 \geq \eta_2 \geq \cdots \geq \eta_m$ with $n > m$. For $i \in \{1, 2, \ldots, m\}$, if $\lambda_i \geq \eta_i \geq \lambda_{n-m+i}$, then B *interlaces* A. If there exists $\lambda_i = \eta_i$ for $1 \leq i \leq k$ and $\lambda_{n-m+i} = \eta_i$ for $k + 1 \leq i \leq m$, where $k \in [0, m]$ is an integer, then the interlacing is *tight*.

Consider a partition $\pi = (X_1, X_2, \ldots, X_k)$ of the set $\{1, 2, \ldots, n\}$, along with a real symmetric matrix M where both the rows and columns are labeled with elements from $\{1, 2, \ldots, n\}$. Then the matrix M can be represented as the following partitioned matrix

$$M = \begin{pmatrix} M_{1,1} & M_{1,2} & \cdots & M_{1,k} \\ M_{2,1} & M_{2,2} & \cdots & M_{2,k} \\ \vdots & \vdots & \ddots & \vdots \\ M_{k,1} & M_{k,2} & \cdots & M_{k,k} \end{pmatrix}$$

with respect to π. The *quotient matrix* $\mathcal{B}$ of the matrix M with respect to π is the $k \times k$ matrix whose entries are the average row sums of the blocks $M_{i,j}$ of M. The partition π is termed *equitable* if each block $M_{i,j}$ of M possesses a constant row sum.

Lemma 1 [3]**.** *Let M be a real symmetric matrix. Then the following holds.*

(1) *The eigenvalue sequence B of $\mathcal{B}$ interlaces with A of M;*

(2) *the partition is equitable if the interlacing is tight.*

2. Results on Defensive (Global Offensive) k-Alliance Number

For a graph G of order n with degree sequence $d_1 \geq \cdots \geq d_n$, a nonempty subset $S \subseteq V(G)$ is defined as a *defensive k-alliance* of G, where $k \in \{-d_1, \ldots, d_1\}$, if for any $v \in S$,

$$\deg_S(v) \geq \deg_{V \setminus S}(v) + k.$$

The minimum cardinality defensive k-alliances of G is called the *defensive k-alliance number*, denoted by $a_k^d(G)$. In particular, $a_{-1}^d(G)$ and $a_0^d(G)$ are known as the *defensive alliance number* and the *strong defensive alliance number* of G [21].

The study of defensive k-alliances in graphs was originally introduced by Kristiansen *et al.* [21]. They established tight bounds for $a_{-1}^d(G)$ and $a_0^d(G)$. Subsequently, there has been considerable interest in determining the smallest size of a defensive k-alliance. One of the classic problems is called the Defensive k-alliance problem, that is, "*Given a graph G of order n and a positive integer $l < n$.*

Does G have a k-defensive alliance of size at most l?" It has been demonstrated in [29] that the defensive k-alliance problem is NP-complete. Therefore, there is interest in establishing bounds for $a_k^d(G)$ in a graph.

In a recent study, Rodríguez-Velázquez, Yero and Sigarreta [27] presented tight bounds for the defensive k-alliance number of graphs, incorporating its algebraic connectivity $-\rho_2^{-1}(G)$. This extends the finding of Rodríguez-Velázquez and Sigarreta [26] to the cases of $k = -1$ and $k = 0$.

Theorem 2 [27]**.** *Let G be a connected graph of order n with degree sequence $d_1 \geq \cdots \geq d_n$. For every $k \in \{-d_n, \ldots, d_1\}$, we have*

$$a_k^d(G) \geq \left\lceil \frac{n(k+1-\rho_2^{-1}(G))}{n-\rho_2^{-1}(G)} \right\rceil.$$

Moreover, this bound is sharp.

Motivated by their result, we investigate the connection between the eigenvalue $\rho_2^a(G)$ of a graph G and its defensive k-alliance number. This approach offers a unified perspective on the adjacency spectrum, the Laplacian spectrum, and the signless Laplacian spectrum of G.

Prior to presenting the proof of our main result, we require the following lemma. This lemma serves as a tool for substantiating the pertinent conclusions and plays a crucial role in the proof. It also establishes a connection between the eigenvalues of a graph and its structural parameters.

Lemma 3. *Suppose G is a graph of order n, and let $V(G) = V_1 \cup V_2$ be a partition of G with $n_1 = |V_1|$ and $n_2 = |V_2|$ such that $x = \sum_{v \in V_1} d_G(v)/n_1 \leq y = \sum_{v \in V_2} d_G(v)/n_2$. Then*

$$\rho_2^a(G) \geq (a+1)x - \frac{nt}{n_1 n_2}, \tag{1}$$

$$\rho_n^a(G) \leq (a+1)y - \frac{nt}{n_1 n_2}, \tag{2}$$

where $t = e(V_1, V_2)$. If the equality holds in (1) (*respectively,* (2)), *then $x = y$. Moreover, if G is regular, then the equality holds in* (1) (*or respectively,* (2)) *only if G is a $\left(\frac{t}{n_1}, \frac{t}{n_2}\right)$-local-regular graph.*

Proof. Let $\mathcal{B}$ represent the quotient matrix of $aD + A$ associated with the partition $V(G) = V_1 \cup V_2$. Then

$$\mathcal{B} = \begin{pmatrix} (a+1)x - \frac{t}{n_1} & \frac{t}{n_1} \\ \frac{t}{n_2} & (a+1)y - \frac{t}{n_2} \end{pmatrix}.$$

Through direct computation, we have

$$\lambda^2 - \left[(1+a)(x+y) - \frac{nt}{n_1 n_2}\right]\lambda + \left[(1+a)x - \frac{t}{n_1}\right]\left[(1+a)y - \frac{t}{n_2}\right] - \frac{t^2}{n_1 n_2}.$$

Hence

$$\begin{aligned} \lambda_1(\mathcal{B}) &= \frac{1}{2}\left((1+a)(x+y) - \frac{nt}{n_1 n_2} + \sqrt{r}\right), \\ \lambda_2(\mathcal{B}) &= \frac{1}{2}\left((1+a)(x+y) - \frac{nt}{n_1 n_2} - \sqrt{r}\right), \end{aligned} \tag{3}$$

where

$$\begin{aligned} r &= \left[(1+a)(x+y) - \frac{nt}{n_1 n_2}\right]^2 - 4\left[(a+1)x - \frac{t}{n_1}\right]\left[(a+1)y - \frac{t}{n_2}\right] + \frac{4t^2}{n_1 n_2} \\ &= (a+1)^2 (y-x)^2 + \left(\frac{nt}{n_1 n_2}\right)^2 - 2(a+1)(y-x)\left(\frac{t}{n_2} - \frac{t}{n_1}\right). \end{aligned} \tag{4}$$

Note that $x \le y$. Then by (4), we have

$$r \ge \left[\frac{nt}{n_1 n_2} - (a+1)(y-x)\right]^2 \tag{5}$$

and

$$r \le \left[(a+1)(y-x) + \frac{nt}{n_1 n_2}\right]^2. \tag{6}$$

Combining (3) with (5), we have

$$\begin{aligned} \lambda_2(\mathcal{B}) &\le \frac{1}{2}\left\{\left[(a+1)(x+y) - \frac{nt}{n_1 n_2}\right] - \left|\frac{nt}{n_1 n_2} - (a+1)(y-x)\right|\right\} \\ &= \begin{cases} (a+1)y - \frac{nt}{n_1 n_2}, & \text{if } \frac{nt}{n_1 n_2} > (a+1)(y-x), \\ (a+1)x, & \text{if } \frac{nt}{n_1 n_2} \le (a+1)(y-x). \end{cases} \end{aligned} \tag{7}$$

Moreover, if $\frac{nt}{n_1 n_2} \le (a+1)(y-x)$, then $(a+1)x \le (a+1)y - \frac{nt}{n_1 n_2}$. Therefore, (7) implies that $\lambda_2(\mathcal{B}) \le (a+1)\bar{d}_2 - \frac{nt}{n_1 n_2}$.

Similarly, by (3) and (6), we have

$$\lambda_2(\mathcal{B}) \ge (a+1)x - \frac{nt}{n_1 n_2}. \tag{8}$$

Moreover, Lemma 1 implies that $\rho_2^a(G) \geq \lambda_2(\mathcal{B}) \geq \rho_n^a(G)$. Thus, we have

$$\rho_2^a(G) \geq \lambda_2(\mathcal{B}) \geq (a+1)\bar{d}_1 - \frac{nt}{n_1 n_2} \text{ and } \rho_n^a(G) \leq \lambda_2(\mathcal{B}) \leq (a+1)\bar{d}_2 - \frac{nt}{n_1 n_2}.$$

If the equality holds in (1) (or respectively, (2)), then the above inequalities must be equalities. Then $\rho_2^a(G) = \lambda_2(\mathcal{B})$ (or respectively, $\rho_n^a(G) = \lambda_2(\mathcal{B})$), $x = y$ from (5) and (6). Moreover, if G is a d-regular graph, it follows that $\rho_1^a(G) = (1+a)d$ and $x = y = d$. From (3) and (4), we have $\lambda_1(\mathcal{B}) = (1+a)d$. Thus, $\rho_1^a(G) = \lambda_1(\mathcal{B})$, and thus Lemma 1 implies that the partition $V(G) = V_1 \cup V_2$ is equitable. For the case of $\rho_n^a(G) = \lambda_2(\mathcal{B})$, a similar analysis applies. Therefore, $d_{V_2}(u) = \frac{t}{n_1}$ for any $u \in V_1$ and $d_{V_1}(v) = \frac{t}{n_2}$ for any $v \in V_2$. In other words, G is a $\left(\frac{t}{n_1}, \frac{t}{n_2}\right)$-local-regular graph. ■

Theorem 4. *Let G be a connected graph of order n with degree sequence $d_1 \geq \cdots \geq d_n = \delta$. Then for every $k \in \{-d_n, \ldots, d_1\}$, we have*

$$a_k^d(G) \geq \left\lceil \frac{n(k+1+(a+1)\delta - \rho_2^a)}{n+(a+1)\delta - \rho_2^a} \right\rceil. \tag{9}$$

Moreover, this bound is sharp.

Proof. Let $S \subseteq V(G)$ be a defensive k-alliance in G. Then for every vertex $v \in S$, we have $\deg_{V\setminus S}(v) + k \leq \deg_S(v) \leq |S|-1$. It follows that $t = e(S, V\setminus S) = \sum_{v \in S} \deg_{V\setminus S}(v) \leq |S|(|S|-k-1)$. By Lemma 3, we have

$$\rho_2^a(G) \geq (1+a)\delta - \frac{nt}{|S|(n-|S|)} \geq (1+a)\delta - \frac{n(|S|-k-1)}{n-|S|}.$$

Hence, we derive the lower bound for $a_k^d(G)$ as

$$a_k^d(G) \geq \left\lceil \frac{n[k+1+(a+1)\delta - \rho_2^a]}{n+(a+1)\delta - \rho_2^a} \right\rceil.$$

On the other hand, note that if G is d-regular graph of order n, then $A_a(G) = adI_n + A(G)$. Hence, there is a linear correspondence between the spectra of $A_a(G)$ and $A(G)$ as $\rho_k^a(G) = ad + \rho_k^0(G)$ for $1 \leq k \leq n$. Note that $a_k^d(K_n) = \lceil \frac{n+k+1}{2} \rceil$ and the eigenvalues of $A_a(K_n)$ are $(1+a)(n-1), a(n-1)-1, \ldots, a(n-1)-1$. That is, $\rho_2^a(K_n) = a(n-1)-1$. Thus

$$a_k^d(K_n) \geq \left\lceil \frac{n(k+1+(a+1)(n-1)-a(n-1)+1)}{n+(a+1)(n-1)-a(n-1)+1} \right\rceil = \left\lceil \frac{n+k+1}{2} \right\rceil,$$

which indicates that (9) is sharp for K_n. ■

Recall that $A_{-1}(G) = -L(G)$. Then Theorem 4 coincides with Theorem 2 when $a = -1$, and so Theorem 4 can be regarded as a more generalization of Theorem 2. On the other hand, let $a = 0$ and $a = 1$, we also derive lower bounds for the defensive k-alliance number $a_k^d(G)$ in relation to $\rho_2^0(G)$ and $\rho_2^1(G)$, respectively.

Corollary 5. *Let G be a connected graph of order n with degree sequence $d_1 \geq \cdots \geq d_n = \delta$. Then for every $k \in \{-d_n, \ldots, d_1\}$, we have*

$$a_k^d(G) \geq \left\lceil \frac{n(k+1+\delta-\rho_2^0)}{n+\delta-\rho_2^0} \right\rceil \quad \text{and} \quad a_k^d(G) \geq \left\lceil \frac{n(k+1+2\delta-\rho_2^1)}{n+2\delta-\rho_2^1} \right\rceil.$$

Moreover, both bounds are sharp.

Furthermore, we also establish the following lower bound on $a_k^d(G)$ involving its maximum degree.

Theorem 6. *Let G be a connected graph of order n with degree sequence $\Delta = d_1 \geq \cdots \geq d_n = \delta$. Then for every $k \in \{-d_n, \ldots, d_1\}$, we have*

$$a_k^d(G) \geq \left\lceil \frac{n((a+1)\delta - \rho_2^a - \lfloor \frac{\Delta-k}{2} \rfloor)}{(a+1)\delta - \rho_2^a} \right\rceil. \tag{10}$$

Moreover, this bound is sharp.

Proof. Let $S \subseteq V(G)$ be a defensive k-alliance in G. Then for any $v \in S$, we have

$$\Delta(G) \geq d(v) = \deg_{V\backslash S}(v) + \deg_S(v) \geq \deg_{V\backslash S}(v) + \deg_{V\backslash S}(v) + k \geq 2\deg_{V\backslash S}(v) + k,$$

that is

$$\left\lfloor \frac{\Delta-k}{2} \right\rfloor \geq \deg_{V\backslash S}(v).$$

It follows that

$$t = e(S, V\backslash S) = \sum_{v\in S} \deg_{V\backslash S}(v) \leq |S| \left\lfloor \frac{\Delta-k}{2} \right\rfloor.$$

Moreover, by Lemma 3, we have

$$\rho_2^a(G) \geq (1+a)\delta - \frac{nt}{|S|(n-|S|)} \geq (1+a)\delta - \frac{n\lfloor \frac{\Delta-k}{2} \rfloor}{n-|S|}.$$

This yields the following lower bound for $a_k^d(G)$

$$a_k^d(G) \geq \left\lceil \frac{n((a+1)\delta - \rho_2^a - \lfloor \frac{\Delta-k}{2} \rfloor)}{(a+1)\delta - \rho_2^a} \right\rceil.$$

Recall that $a_k^d(K_n) = \lceil \frac{n+k+1}{2} \rceil$ and $\rho_2^a(K_n) = a(n-1) - 1$. Thus

$$a_k^d(K_n) \geq \left\lceil \frac{n((a+1)(n-1) - a(n-1) + 1 - \lfloor \frac{n-1-k}{2} \rfloor)}{(a+1)(n-1) - a(n-1) + 1} \right\rceil = \left\lceil \frac{n+k+1}{2} \right\rceil,$$

which shows that the lower bound in Theorem 6 is sharp for K_n. ■

Similarly, We have the following corollary.

Corollary 7. *Let G be a connected graph of order n with degree sequence $\Delta = d_1 \geq \cdots \geq d_n = \delta$. Then for every $k \in \{-d_n, \ldots, d_1\}$, we have*

$$a_k^d(G) \geq \left\lceil \frac{n(\delta - \rho_2^0 - \lfloor \frac{\Delta-k}{2} \rfloor)}{\delta - \rho_2^0} \right\rceil \quad \textit{and} \quad a_k^d(G) \geq \left\lceil \frac{n(2\delta - \rho_2^1 - \lfloor \frac{\Delta-k}{2} \rfloor)}{2\delta - \rho_2^1} \right\rceil.$$

Moreover, both bounds are sharp.

Recall that $A_{-1}(G) = -L(G)$. Then we also have the following corollary, which coincides with a result of Rodríguez-Velázquez, Yero and Sigarreta in [27].

Corollary 8 [27]**.** *Let G be a connected graph of order n with degree sequence $\Delta = d_1 \geq \cdots \geq d_n$. Then for every $k \in \{-d_n, \ldots, d_1\}$, we have*

$$a_k^d(G) \geq \left\lceil \frac{n(-\rho_2^{-1}(G) - \lfloor \frac{\Delta-k}{2} \rfloor)}{-\rho_2^{-1}(G)} \right\rceil.$$

Moreover, this bound is sharp.

Remark 9. For the cases of $k = -1$ and $k = 0$, it is easy to obtain lower bounds for $a_{-1}^d(G)$ and $a_0^d(G)$ in terms of ρ_2^0, ρ_2^1 and $-\rho_2^{-1}$ for a connected graph G, which covers a result concerning the algebraic connectivity $-\rho_2^{-1}$ due to Rodríguez-Velázquez and Sigarreta in [26].

For any subset $S \subseteq V(G)$, the boundary of S is defined as

$$\partial(S) = \bigcup_{v \in S} N_{V \setminus S}(v) = N(S) \cap \bar{S}.$$

A nonempty set $S \subseteq V(G)$ is a *global offensive k-alliance* of G, where $k \in \{2 - d_1, \ldots, d_1\}$, if S is a dominating set and for any $v \in \partial(S)$,

$$\deg_S(v) \geq \deg_{V \setminus S}(v) + k.$$

The *global offensive k-alliance number* $\gamma_k^o(G)$ is the minimum cardinality among all global offensive k-alliances of G. In particular, $\gamma_1^o(G)$ (respectively $\gamma_2^o(G)$) are known as the *global* (*respectively strong*) *offensive alliance number* of G [21, 10].

In [12], Fernau *et al.* investigated the complexity of global offensive k-alliances. They demonstrated that determining optimal global offensive k-alliances is an NP-complete. In the same paper, they also derived a tight bound on $\gamma_k^o(G)$ in terms of its $-\rho_n^{-1}(G)$.

Theorem 10 [12]**.** *Let G be a connected graph of order n with degree sequence $d_1 \geq \cdots \geq d_n = \delta$. Then for every $k \in \{2-d_1, \ldots, d_1\}$, we have $\gamma_k^o(G) \geq \left\lceil \frac{n\left\lceil \frac{\delta+k}{2} \right\rceil}{-\rho_n^{-1}(G)} \right\rceil$. Moreover, this bound is sharp.*

Motivated by their result, we investigate the connection between $\rho_n^a(G)$ of a graph G and its global offensive k-alliance number. The main results are as follows.

Theorem 11. *Let G be a connected graph of order n with degree sequence $\Delta = d_1 \geq \cdots \geq d_n = \delta$. Then for every $k \in \{2-d_1, \ldots, d_1\}$, we have $\gamma_k^o(G) \geq \left\lceil \frac{n\left\lceil \frac{\delta+k}{2} \right\rceil}{(1+a)\Delta-\rho_n^a} \right\rceil$. Moreover, this bound is sharp.*

Proof. Let $S \subseteq V(G)$ be a global offensive k-alliance in G. Then for any $v \in \partial(S)$, we have

$$2\deg_S(v) \geq \deg_S(v) + \deg_{V\setminus S}(v) + k = d(v) + k.$$

Therefore,

$$t = e(S,\bar{S}) = \sum_{v\in\bar{S}} \deg_S(v) \geq (n-|S|)\left\lceil \frac{d(v)+k}{2} \right\rceil \geq (n-|S|)\left\lceil \frac{\delta+k}{2} \right\rceil.$$

Since $\bar{d}_i \leq \Delta$, then by Lemma 3, we have

$$\rho_n^a(G) \leq (1+a)\Delta - \frac{nt}{|S|(n-|S|)} \leq (1+a)\Delta - \frac{n}{|S|}\left\lceil \frac{\delta+k}{2} \right\rceil.$$

Hence, we derive the lower bound for $\gamma_k^o(G)$ as

$$\gamma_k^o(G) \geq \left\lceil \frac{n\left\lceil \frac{\delta+k}{2} \right\rceil}{(1+a)\Delta - \rho_n^a} \right\rceil. \tag{11}$$

For $G \cong K_n$, we have $\rho_n^a(K_n) = a(n-1) - 1$. Thus

$$\gamma_k^o(K_n) = \left\lceil \frac{n+k-1}{2} \right\rceil \geq \left\lceil \frac{n\left\lceil \frac{n-1+k}{2} \right\rceil}{(1+a)(n-1) - a(n-1) + 1} \right\rceil = \left\lceil \frac{n+k-1}{2} \right\rceil.$$

This indicates that (11) is sharp for K_n. ■

Recall that $A_{-1}(G) = -L(G)$. Then Theorem 11 coincides with Theorem 10 when $a = -1$, and so Theorem 11 can be regarded as a generalization of Theorem 10. Similarly, We also derive lower bounds for $\gamma_k^o(G)$ based on $\rho_n^0(G)$ and $\rho_n^1(G)$, respectively.

Corollary 12. *Let G be a connected graph of order n with degree sequence $\Delta = d_1 \geq \cdots \geq d_n = \delta$. Then for every $k \in \{2 - d_1, \ldots, d_1\}$, we have*

$$\gamma_k^o(G) \geq \left\lceil \frac{n \left\lceil \frac{\delta+k}{2} \right\rceil}{\Delta - \rho_n^0} \right\rceil \quad \text{and} \quad \gamma_k^o(G) \geq \left\lceil \frac{n \left\lceil \frac{\delta+k}{2} \right\rceil}{2\Delta - \rho_n^1} \right\rceil.$$

Moreover, both bounds are sharp.

Remark 13. For the cases of $k = 1$ and $k = 2$, it is easy to obtain lower bounds for $\gamma_1^o(G)$ and $\gamma_2^o(G)$ in terms of ρ_n^0, ρ_n^1 and $-\rho_n^{-1}$ for a connected graph G, which covers a result concerning the Laplacian spectral radius $-\rho_n^{-1}$ due to Rodríguez-Velázquez and Sigarreta in [26].

3. Results on the Total (Signed) Domination Number

We refer to $S \subseteq V(G)$ as a *total dominating set* of G if $N(S) = V(G)$. Among all total dominating sets of G, the one with the minimum cardinality is referred to as the total domination number of G, denoted by $\gamma_t(G)$. The notion of total domination in graphs was first introduced by Cockayne, Dawes, and Hedetniemi [6] and has subsequently been extensively explored in the field of graph theory (see [17, 18]).

The correlation between the eigenvalues of a graph and its domination number was initially investigated by Lu, Liu and Tian [24]. They provided bounds on the Laplacian spectrum of G involving the domination number, which were later improved by Nikiforov [25] and Har [15], respectively. Furthermore, Li [22] generalized the two results of Lu, Liu and Tian [24] from the domination number to the k-domination number. This result was also extended by Liu and Lu [23] to the signless Laplacian spectrum. Furthermore, Chen, Li, and Shiu [4] further extended the results of Liu and Lu [23] from the signless Laplacian spectrum of a graph to the A_α-spectrum. To date, this topic has received considerable attention, and we refer the interested reader to the book [18] for more details.

Recently, Shi, Kang and Wu [28] studied the relationship between the total domination number of a graph and its algebraic connectivity.

Theorem 14 [28]**.** *Let G be a connected graph of order $n \geq 3$. Then for $G \ncong K_n$, we have*

$$-\rho_2^{-1}(G) \leq \frac{n\left(n - \frac{3}{2}\gamma_t(G) + \frac{1}{2}\right)}{n - \gamma_t(G)}.$$

Motivated by Theorem 14, We explore the relationship between $\rho_2^a(G)$ (respectively $\rho_n^a(G)$) of G and $\gamma_t(G)$. Before presenting the proof of our main result, we need the following results.

Lemma 15 [6]**.** *Let S be a minimal total dominating set of a connected graph G. Then for any $v \in S$, it has at least one of the following two properties.*

$\mathcal{P}(1)$: *There exists a vertex $u \in \bar{S}$ such that $N(u) \cap S = \{v\}$.*

$\mathcal{P}(2)$: *$G[S \setminus \{v\}]$ contains an isolated vertex.*

Lemma 16 [19]**.** *Let $G \ncong K_n$ be a connected graph of order $n \geq 3$. The graph G has a vertex set S with $|S| = \gamma_t(G)$, and for any $v \in S$, it either satisfies property $\mathcal{P}(1)$ or is adjacent to a vertex of degree 1 in $G[S]$ that has property $\mathcal{P}(1)$.*

Theorem 17. *Let $G \ncong K_n$ be a connected graph of order $n \geq 3$ with minimum degree δ. Then*

$$\rho_2^a(G) \geq (1+a)\delta - \frac{n\left(n - \frac{3}{2}\gamma_t(G) + \frac{1}{2}\right)}{n - \gamma_t(G)}. \tag{12}$$

Equality holds only if G is a $\left(\frac{t}{\gamma_t(G)}, \frac{t}{n-\gamma_t(G)}\right)$-local-regular graph, where $t = \gamma_t(G)\left(n - \frac{3}{2}\gamma_t(G) + \frac{1}{2}\right)$.

Proof. Let S be a minimum total dominating set of G. We now consider two cases based on whether all vertices in S satisfy property $\mathcal{P}(1)$ or not.

Case 1. Every vertex $v \in S$ has property $\mathcal{P}(1)$. In this case, $\sum_{v\in S} |N(v) \setminus N[S \setminus v]| \geq |S| = \gamma_t(G)$. Therefore,

$$\begin{aligned}
e(S, \bar{S}) &\leq \sum_{v\in S} |N(v) \setminus N[S \setminus v]| + |S| \left(|\bar{S}| - \sum_{v\in S} |N(v) - N[S - v]|\right) \\
&= |N(S) \cap \bar{S}| + \gamma_t(G) \left(n - \gamma_t(G) - \sum_{v\in S} |N(v) \setminus N[S \setminus v]|\right) \\
&= \gamma_t(G)(n - \gamma_t(G)) - (\gamma_t(G) - 1) \sum_{v\in S} |N(v) \setminus N[S \setminus v]| \\
&\leq \gamma_t(G)(n - 2\gamma_t(G) + 1).
\end{aligned}$$

Let $x = \frac{\sum_{x\in S} d(x)}{|S|}$ and $y = \frac{\sum_{y\in \bar{S}} d(y)}{|\bar{S}|}$. Then $\bar{d}_1 = \min\{x, y\}$. By Lemma 3, we have

$$\rho_2^a(G) \geq (1+a)\bar{d}_1 - \frac{n\left(n - 2\gamma_t(G) + 1\right)}{n - \gamma_t(G)} > (1+a)\delta - \frac{n\left(n - \frac{3}{2}\gamma_t(G) + \frac{1}{2}\right)}{n - \gamma_t(G)}. \tag{13}$$

Case 2. There exists at least one vertex in S which has no property $\mathcal{P}(1)$. In this situation, we first perform a detailed partition of the set S. Let M denote the set of vertices in S that possess property $\mathcal{P}(1)$ and $N = S \setminus M$. Thus, $N \neq \emptyset$. For every $v \in M$, let M_1 denote the set of vertices in M that satisfy $d_S(v) = 1$ and there exists a vertex $u \in B$ such that $uv \in E(G)$. It follows that $|N| \leq |M_1| \leq |M|$ by Lemma 16. Then $\gamma_t(G) = |S| = |M| + |N| \leq 2|M|$. So we have $\frac{1}{2}\gamma_t(G) \leq |M| \leq \sum_{v \in M} |N(v) \setminus N[S \setminus v]|$. Thus

$$
\begin{aligned}
e(S, \bar{S}) &\leq \sum_{v \in M} |N(v) \setminus N[S \setminus v]| + |S| \left(|\bar{S}| - \sum_{v \in M} |N(v) - N[S - v]| \right) \\
&= \gamma_t(G)(n - \gamma_t(G)) - (\gamma_t(G) - 1) \sum_{v \in M} |N(v) \setminus N[S \setminus v]| \\
&\leq \gamma_t(G)(n - \gamma_t(G)) - \frac{1}{2}\gamma_t(G)(\gamma_t(G) - 1) \\
&= \gamma_t(G) \left(n - \frac{3}{2}\gamma_t(G) + \frac{1}{2} \right).
\end{aligned}
$$

Then by Lemma 3, we have

$$
\rho_2^a(G) \geq (1 + a)\delta - \frac{n \left(n - \frac{3}{2}\gamma_t(G) + \frac{1}{2} \right)}{n - \gamma_t(G)}. \tag{14}
$$

If $\rho_2^a(G) = (1 + a)\delta - \frac{n\left(n - \frac{3}{2}\gamma_t(G) + \frac{1}{2}\right)}{n - \gamma_t(G)}$. Then the above all inequalities must be equality, and thus $\bar{d}_1 = \delta$. Let $\bar{d}_2 = \max\{x, y\}$, then we have $\bar{d}_1 = \bar{d}_2 = \delta$ from Lemma 3. That is, G is a δ-regular graph. Also, by (13), (14) and Lemma 3, $d_{\bar{S}}(v) = \frac{t}{|S|}$ for every $v \in S$ and $d_S(u) = \frac{t}{|\bar{S}|}$ for every $u \in \bar{S}$. It follows that G is a $\left(\frac{t}{\gamma_t(G)}, \frac{t}{n - \gamma_t(G)} \right)$-local-regular graph, where $t = \gamma_t(G) \left(n - \frac{3}{2}\gamma_t(G) + \frac{1}{2} \right)$. The proof of Theorem 17 is completed. ■

Theorem 18. *Let G be any graph of order n with maximum degree Δ. Then*

$$
\rho_n^a(G) \leq (1 + a)\Delta - \frac{n}{\gamma_t(G)}. \tag{15}
$$

Equality holds only if G is a $\left(\frac{(n - \gamma_t(G))}{\gamma_t(G)}, 1 \right)$-local-regular graph.

Proof. Let S be a minimum total dominating set of G. Then, by the definition of the total dominating set, we have that

$$
t = e(S, \bar{S}) \geq n - |S| = n - \gamma_t(G). \tag{16}
$$

Moreover, by Lemma 3, we have

$$\rho_n^a(G) \le (1+a)\Delta - \frac{ne(S,\bar{S})}{|S|(n-|S|)} \le (1+a)\Delta - \frac{n}{\gamma_t(G)}. \tag{17}$$

If $\rho_n^a(G) = (1+a)\Delta - \frac{n}{\gamma_t(G)}$, then the above all inequalities must be equality. Similarly to the proof of Theorem 17, it follows that G is a Δ-regular graph. Additionally, $d_{\bar{S}}(v) = \frac{(n-\gamma_t(G))}{\gamma_t(G)}$ for every $v \in S$ and $d_S(u) = 1$ for every $u \in \bar{S}$, as derived from Lemma 3 and (16). Therefore, G is a $\left(\frac{(n-\gamma_t(G))}{\gamma_t(G)}, 1\right)$-local-regular graph. This completes the proof of Theorem 18. ■

For $a = 0, -1, 1$, we then have the following results on $A(G)$, $L(G)$ and $Q(G)$, respectively.

Corollary 19. *For any connected graph G of order $n \ge 3$ and $G \not\cong K_n$, we have*

$$\rho_2^0(G) \ge \delta - \frac{n\left(n - \frac{3}{2}\gamma_t(G) + \frac{1}{2}\right)}{n-\gamma_t(G)} \quad \text{and} \quad \rho_2^1(G) \ge 2\delta - \frac{n\left(n - \frac{3}{2}\gamma_t(G) + \frac{1}{2}\right)}{n-\gamma_t(G)}.$$

Both equalities hold only if G is a $\left(\frac{t}{\gamma_t(G)}, \frac{t}{n-\gamma_t(G)}\right)$-local-regular graph, where $t = \gamma_t(G)\left(n - \frac{3}{2}\gamma_t(G) + \frac{1}{2}\right)$.

Corollary 20. *For any connected graph G of order n with maximum degree Δ, we have*

$$\rho_n^0(G) \le \Delta - \frac{n}{\gamma_t(G)} \quad \text{and} \quad \rho_n^1(G) \le 2\Delta - \frac{n}{\gamma_t(G)}.$$

Both equalities hold only if G is a $\left(\frac{(n-\gamma_t(G))}{\gamma_t(G)}, 1\right)$-local-regular graph.

Corollary 21. *For a connected graph G of order n, we have $-\rho_n^{-1}(G) \ge \frac{n}{\gamma_t(G)}$, the equality holds only if G is a $\left(\frac{(n-\gamma_t(G))}{\gamma_t(G)}, 1\right)$-local-regular graph.*

In [2], Brešar, Cornet, Dravec and Henning proved an upper bound on the zero forcing number $Z(G)$ of a graph G, expressed in terms of the total domination number $\gamma_t(G)$ of G, where the zero forcing number $Z(G)$ was introduced in [1] and was studied by [20, 5, 32] in terms of various eigenvalues of a graph.

Theorem 22 [2]**.** *For a connected graph G of order n, we have $Z(G) \le n - \gamma_t(G)$.*

By Theorem 18, we have $\gamma_t(G) \ge \left\lceil \frac{n}{(1+a)\Delta - \rho_n^a(G)} \right\rceil$. This together with Theorem 22 implies the following upper bound on $Z(G)$ in terms of $\rho_n^a(G)$.

Corollary 23. *Let G be a connected graph of order n. Then*

$$Z(G) \le n - \left\lceil \frac{n}{(1+a)\Delta - \rho_n^a(G)} \right\rceil.$$

Remark 24. The bound in Corollary 23 is sharp and may be seen as follows. For $G \cong K_n$, we have

$$n-1 = Z(K_n) \leq n - \left\lceil \frac{n}{(1+a)(n-1)-a(n-1)+1} \right\rceil = n-1.$$

On the other hand, for $a=-1$, $a=0$ and $a=1$, we also have the upper bounds on $Z(G)$ in terms of its $-\rho_n^{-1}$, ρ_n^0, and ρ_n^1, respectively.

A function $f : V \to \{-1, 1\}$ is a *signed dominating function* if for every vertex $v \in V(G)$, the closed neighborhood of v contains more vertices with function value 1 than with -1. We will use the symbol $f[v]$ to denote the sum $\sum_{x\in N[v]} f(x)$. Therefore, a function f is termed a signed dominating function if, for every $v \in V$, $f[v] \geq 1$. The weight of f is denoted as $\omega(f) = \sum_{x\in V} f(x)$. For a subset $S \subseteq V$, we define $f(S) = \sum_{v\in S} f(v)$, and then $\omega(f) = f(V)$. The signed domination number $\gamma_s(G)$ of G, is the minimum weight of a signed dominating function on G. This concept was defined in [9] and has been studied by several authors including [28, 8, 30].

Next we study the connection between the eigenvalues $\rho_2^a(G)$ (respectively $\rho_n^a(G)$) and $\gamma_t(G)$ for a connected graph G. We begin with the case when G is regular.

Theorem 25. *Let G be a d-regular graph of order n. Then*

$$\rho_n^a(G) \leq \begin{cases} (1+a)d - \frac{n(d+3)}{\gamma_s(G)+n}, & d \text{ is odd,} \\ (1+a)d - \frac{n(d+2)}{\gamma_s(G)+n}, & d \text{ is even,} \end{cases}$$

the equality holds if and only if $G \cong K_n$.

Proof. Let f be an signed dominating function on G for which $\omega(f) = \gamma_s(G)$. Let P denote the set of vertices assigned the value 1 by f, and M denote the set of vertices assigned the value -1 by f. Clearly, $|P| + |M| = n$. We now consider the following two cases according to the parity of d.

Case 1. d is odd. For any $u \in P$, we may assume that $d_M(u) = s$. Through the definition of the signed dominating function, we have $s \geq 0$, $d_P(u) = d - s$ and $f[u] = d - s - s + 1 = d - 2s + 1 \geq 2$. Therefore, $s \leq \frac{1}{2}(d-1)$. On the other hand, we may assume that $d_P(v) = r$ for any $v \in M$. Through the definition of the signed dominating function, we have $r \geq 2$, $d_M(v) = d - r$ and $f[v] = r - (d-r) - 1 = 2r - d - 1 \geq 2$, which implies that $r \geq \frac{1}{2}(d+3)$. Recall that $d_P(v) = r$, then

$$e(P, M) = \sum_{v\in M} d_P(v) = r|M| \geq \frac{1}{2}m(d+3).$$

Let $|P| = p$ and $|M| = m$. Note that $2p = \gamma_s(G) + n$ and $\bar{d_i} = d$. Hence, by Lemma 3, we have

$$\rho_n^a(G) \leq (1+a)d - \frac{ne(P,M)}{pm} \leq (1+a)d - \frac{n(d+3)}{2p} = (1+a)d - \frac{n(d+3)}{\gamma_s(G)+n}.$$

If $\rho_n^a(G) = (1+a)d - \frac{n(d+3)}{\gamma_s(G)+n}$, then the above all inequalities must be equality. By Lemma 3, we have $d_M(u) = \frac{e(P,M)}{|P|} = s$ for every $u \in P$ and $d_P(v) = \frac{e(P,M)}{|M|} = r$ for every $v \in M$. Then $s|P| = r|M|$. Next, we claim that $d = n-1$. Assume the opposite, namely, that $d < n-1$. It follows that $\rho_n^a(G) < ad - 1$ from the proof of Theorem 28. Hence

$$2p - n = \gamma_s(G) = \frac{n(d+3)}{(1+a)d - \rho_n^a(G)} - n < \frac{n(d+3)}{d+1} - n.$$

Therefore,

$$2p < \frac{n(d+3)}{d+1} = \frac{(d+3)(p+m)}{d+1} = \frac{d+3}{d+1}\left(p + \frac{sp}{r}\right) = \frac{d+3}{d+1}p\left(1 + \frac{2s}{d+3}\right).$$

It follows that $s > \frac{1}{2}(d-1)$. But this is impossible since $s \leq \frac{1}{2}(d-1)$. Therefore, $d = n-1$ and so $G \cong K_n$.

On the other hand, let $G = K_n$ where n is even, and thus $\gamma_s(K_n) = 2$. Then

$$\rho_n^a(G) = ad - 1 \geq (1+a)d - \frac{n(d+3)}{\gamma_s(G)+n} = (1+a)d - \frac{(d+1)(d+3)}{2+d+1} = ad - 1.$$

Case 2. d is even. Similarly to the proof of Case 1, we may assume that $d_M(u) = s$ for any $u \in P$. Then $s \geq 0$, $d_P(u) = d - s$ and $f[u] = d - s - s + 1 = d - 2s + 1 \geq 1$. Therefore, $s \leq \frac{d}{2}$. On the other hand, we may assume that $d_P(v) = r$ for any $v \in M$. Through the definition of the signed dominating function, we have $r \geq 2$, $d_M(v) = d - r$ and $f[v] = r - (d-r) - 1 = 2r - d - 1 \geq 1$, which implies that $r \geq \frac{1}{2}(d+2)$. Recall that $d_P(v) = r$, then $e(P,M) = \sum_{v \in M} d_P(v) = r|M| \geq \frac{1}{2}m(d+2)$. By Lemma 3 and $2p = \gamma_s(G) + n$,

$$\rho_n^a(G) \leq (1+a)d - \frac{ne(P,M)}{pm} \leq (1+a)d - \frac{n(d+2)}{2p} = (1+a)d - \frac{n(d+2)}{\gamma_s(G)+n}.$$

If $\rho_n^a(G) = (1+a)d - \frac{n(d+2)}{\gamma_s(G)+n}$, then the above all inequalities must be equality. By Lemma 3, we have $d_M(u) = \frac{e(P,M)}{|P|} = s$ for every $u \in P$ and $d_P(v) = \frac{e(P,M)}{|M|} = r$ for every $v \in M$. Then $s|P| = r|M|$. Next, we claim that $d = n-1$. Assume the opposite, namely, that $d < n-1$. It follows that $\rho_n^a(G) < ad - 1$ from the proof of Theorem 28. Hence

$$2p - n = \gamma_s(G) = \frac{n(d+2)}{(1+a)d - \rho_n^a(G)} - n < \frac{n(d+2)}{d+1} - n.$$

Therefore,

$$2p < \frac{n(d+2)}{d+1} = \frac{(d+2)(p+m)}{d+1} = \frac{d+2}{d+1}\left(p + \frac{sp}{r}\right) = \frac{d+2}{d+1}p\left(1 + \frac{2s}{d+2}\right).$$

It follows that $s > \frac{d}{2}$. But this is impossible since $s \le \frac{d}{2}$. Therefore, $d = n-1$ and so $G \cong K_n$.

On the other hand, let $G = K_n$ where n is odd, and thus $\gamma_s(K_n) = 1$. Then

$$\rho_n^a(G) = ad - 1 \ge (1+a)d - \frac{n(d+2)}{\gamma_s(G)+n} = (1+a)d - \frac{(d+1)(d+2)}{1+d+1} = ad - 1.$$

This completes the proof of Theorem 25. ■

Remark 26. For the cases of $a = 0$, $a = 1$ and $a = -1$, the connection between the signed domination number in terms of ρ_n^0, ρ_n^1 and $-\rho_n^{-1}$ for a d-regular graph G can be easily obtained, which covers a result concerning the Laplacian spectral radius $-\rho_n^{-1}$ due to Shi, Kang and Wu in [28].

Moreover, for general graphs, we have the following result.

Theorem 27. *Let G be a connected graph of order n with minimum degree δ. Then*

$$\rho_2^a(G) \ge (1+a)\delta - \frac{n\,(n+\gamma_s(G)-2)}{n-\gamma_s(G)}. \tag{18}$$

Moreover, this bound is sharp.

Proof. Let f be an signed dominating function on G for which $\omega(f) = \gamma_s(G)$. Let P denote the set of vertices assigned the value 1 by f, and M denote the set of vertices assigned the value -1 by f. Clearly, $|P| + |M| = n$. For every $v \in P$, we may observe that $d_P(v) \ge d_M(v)$ since $f[v] \ge 1$ for every $v \in V(G)$. Let $|P| = p$. Then we have

$$e(P,M) = \sum_{v\in P} d_M(v) \le \sum_{v\in P} d_P(v) \le p(p-1).$$

By Lemma 3 and $2p = \gamma_s(G) + n$, we have

$$\rho_2^0(G) \ge (1+a)\delta - \frac{n\,(n+\gamma_s(G)-2)}{n-\gamma_s(G)}.$$

For $G = K_n$ and n is odd, we have $\gamma_s(G) = 1$ and $\rho_2^a(G) = a(n-1) - 1$. Thus,

$$\rho_2^a(K_n) = a(n-1) - 1 \ge (1+a)(n-1) - \frac{n\,(n+1-2)}{n-1} = a(n-1) - 1.$$

This indicates that (18) is sharp for K_n.

The proof of Theorem 27 is completed. ■

Theorem 28. *Let G be a connected graph of order n with maximum degree Δ. Then*
$$\rho_n^a(G) \leq (1+a)\Delta - \frac{4n}{\gamma_s(G)+n},$$
the equality holds if and only if $G \cong K_3$.

Proof. Let f be an signed dominating function on G for which $\omega(f) = \gamma_s(G)$. Let P denote the set of vertices assigned the value 1 by f, and M denote the set of vertices assigned the value -1 by f. Clearly, $|P| + |M| = n$. For every $v \in M$, we may observe that $d_P(v) \geq 2$ since $f[v] \geq 1$ for every $v \in V(G)$. Let $|P| = p$. Then

$$e(P, M) = \sum_{v \in M} d_P(v) \geq 2(n-p). \tag{19}$$

By Lemma 3 and $2p = \gamma_s(G) + n$, then
$$\rho_n^a(G) \leq (1+a)\Delta - \frac{ne(P,M)}{|P||M|} \leq (1+a)\Delta - \frac{2n}{p} = (1+a)\Delta - \frac{4n}{\gamma_s(G)+n}.$$

If $\rho_n^a(G) = (1+a)\Delta - \frac{4n}{\gamma_s(G)+n}$, then the above all inequalities must be equality. It follows that G is a Δ-regular graph. By Lemma 3, $d_M(u) = r_1$ for every $u \in P$ and $d_P(v) = r_2$ for every $v \in M$, where $r_1 = \frac{e(P,M)}{|P|}$ and $r_2 = \frac{e(P,M)}{|M|}$, respectively. Then by (19), we have $r_2 = 2$. We also have $r_1|P| = r_2|M| = 2|M|$.

Claim 1. $\Delta(G) = n - 1$.

Proof. Assume the opposite, namely, that $\Delta(G) < n-1$. By Wely's inequalities, we have
$$\rho_n^a(G) \leq \lambda_1(aD(G)) + \lambda_n(A(G)) \leq a\Delta - 1,$$
the equality holds if and only if $G \cong K_n$. It follows that $\rho_n^a(G) < a\Delta - 1$. Hence
$$\gamma_s(G) = \frac{4n}{(1+a)\Delta - \rho_n^a(G)} - n < \frac{4n}{\Delta+1} - n.$$

Since $2p = \gamma_s(G) + n$, we have
$$2p - n = \gamma_s(G) < \frac{4n}{\Delta(G)+1} - n = \frac{(4+2r_1)p}{\Delta(G)+1} - n.$$

Therefore,
$$p < \frac{(2+r_1)p}{\Delta(G)+1}.$$

It follows that $\Delta(G) < r_1 + 1$, i.e., $r_1 = \Delta(G)$. But this is impossible. Since
$$2n = 2p + 2|M| = 2p + r_1 p \geq n + 1 + r_1 p \geq n + 1 + \Delta p \geq n + 1 + 2p \geq 2n + 2,$$
a contradiction. □

Since G is a Δ-regular graph, $G \cong K_n$. It follows that $\rho_n^a(G) = a\Delta(G) - 1$. Thus

$$p = \frac{(2 + r_1)p}{\Delta(G) + 1},$$

which implies that $2+r_1 = \Delta(G)+1$, and thus $r_1 = \Delta(G)-1$, i.e., $r_1 = n-2$. Then we have $r_1 p = (n-2)p = 2|M|$. Recall that $n = p + |M|$. So $(n-2)p = 2n - 2p$, which implies that $p = 2$ and $|M| = 1$. Note that $r_2 = 2$, and so $G \cong K_3$.

On the other hand, for $G = K_3$, $\gamma_s(K_3) = 1$. Then we have

$$\rho_n^a(K_3) = 2a - 1 \geq 2(1 + a) - \frac{n\,(n + 1 - 2)}{n - 1} = 2a - 1.$$

This completes the proof of Theorem 28. ■

For $a = 0, -1, 1$, we then have the following results on $A(G)$, $L(G)$ and $Q(G)$, respectively.

Corollary 29. *Let G be a connected graph of order n with minimum degree δ. Then*

$$\rho_2^0(G) \geq \delta - \frac{n\,(n + \gamma_s(G) - 2)}{n - \gamma_s(G)} \quad \textit{and} \quad \rho_2^1(G) \geq 2\delta - \frac{n\,(n + \gamma_s(G) - 2)}{n - \gamma_s(G)}.$$

Moreover, both bounds are sharp.

Corollary 30. *Let G be a connected graph of order n with minimum degree δ. Then*

$$\rho_n^0(G) \leq \Delta - \frac{4n}{\gamma_s(G) + n} \quad \textit{and} \quad \rho_n^1(G) \leq 2\Delta - \frac{4n}{\gamma_s(G) + n},$$

both equalities hold if and only if $G \cong K_3$.

Since $A_{-1}(G) = -L(G)$, we also have the following corollary, which is a result in [28].

Corollary 31 [28]**.** *Let G be a connected graph of order n. Then the following holds*

$$-\rho_2^{-1}(G) \leq \frac{n\,(n + \gamma_s(G) - 2)}{n - \gamma_s(G)} \quad \textit{and} \quad -\rho_n^{-1}(G) \geq \frac{4n}{\gamma_s(G) + n}.$$

Moreover, both bounds are sharp and the right equality holds if and only if $G \cong K_3$.

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References

[1] AIM Minimum Rank—Special Graphs Work Group, *Zero forcing sets and the minimum rank of graphs*, Linear Algebra Appl. **428** (2008) 1628–1648. https://doi.org/10.1016/j.laa.2007.10.009

[2] B. Brešar, M.G. Cornet, T. Dravec and M.A. Henning, *Bounds on zero forcing using* (*upper*) *total domination and minimum degree*, Bull. Malays. Math. Sci. Soc. **47** (2024) 143. https://doi.org/10.1007/s40840-024-01744-x

[3] A.E. Brouwer and W.H. Haemers, Spectra of Graphs (Springer, New York, NY, 2012). https://doi.org/10.1007/978-1-4614-1939-6

[4] H.-Z. Chen, J. Li and W.C. Shiu, *Some results on the A_α-eigenvalues of a graph*, Linear Multilinear Algebra **71** (2023) 2998–3012. https://doi.org/10.1080/03081087.2022.2135676

[5] H.-Z. Chen, J. Li and S.-J. Xu, *Spectral bounds for the zero forcing number of a graph*, Discuss. Math. Graph Theory **44** (2024) 971–982. https://doi.org/10.7151/dmgt.2482

[6] E.J. Cockayne, R.M. Dawes and S.T. Hedetniemi, *Total domination in graphs*, Networks **10** (1980) 211–219. https://doi.org/10.1002/net.3230100304

[7] D. Cvetković, P. Rowlinson and S. Simić, An Introduction to the Theory of Graph Spectra (Cambridge University Press, 2010). https://doi.org/10.1017/CBO9780511801518

[8] F. Dong, J. Ge and Y. Yang, *Upper bounds on the signed edge domination number of a graph*, Discrete Math. **344** (2021) 112201. https://doi.org/10.1016/j.disc.2020.112201

[9] J.E. Dunbar, S.T. Hedetniemi, M.A. Henning and P.J. Slater, *Signed domination number of a graph*, in: Graph Theory, Combinatorics, and Applications (John Wiley & Sons, New York, 1995) 311–322.

[10] O. Favaron, G. Fricke, W. Goddard, S.M. Hedetniemi, S.T. Hedetniemi, P. Kristiansen, R.C. Laskar and R.D. Skaggs, *Offensive alliances in graphs*, Discuss. Math. Graph Theory **24** (2004) 263–275. https://doi.org/10.7151/dmgt.1230

[11] L. Feng, G. Yu and X.-D. Zhang, *Spectral radius of graphs with given matching number*, Linear Algebra Appl. **422** (2007) 133–138. https://doi.org/10.1016/j.laa.2006.09.014

[12] H. Fernau, J.A. Rodríguez-Velázquez and J.M. Sigarreta, *Offensive r-alliances in graphs*, Discrete Appl. Math. **157** (2009) 177–182. https://doi.org/10.1016/j.dam.2008.06.001

[13] C.D. Godsil and M.W. Newman, *Eigenvalue bounds for independent sets*, J. Combin. Theory Ser. B **98** (2008) 721–734.
https://doi.org/10.1016/j.jctb.2007.10.007

[14] W.H. Haemers, *Hoffman's ratio bound*, Linear Algebra Appl. **617** (2021) 215–219.
https://doi.org/10.1016/j.laa.2021.02.010

[15] J. Har, *A note on Laplacian eigenvalues and domination*, Linear Algebra Appl. **449** (2014) 115-118.
https://doi.org/10.1016/j.laa.2014.02.025

[16] J. Harant and S. Richter, *A new eigenvalue bound for independent sets*, Discrete Math. **338** (2015) 1763–1765.
https://doi.org/10.1016/j.disc.2014.12.008

[17] T.W. Haynes, S.T. Hedetniemi and P.J. Slater, Fundamentals of Domination in Graphs (CRC Press, Boca Raton, 1998).
https://doi.org/10.1201/9781482246582

[18] T.W. Haynes, S.T. Hedetniemi and M.A. Henning, Topics in Domination in Graphs, Dev. Math. **64** (Springer, Cham, 2020).
https://doi.org/10.1007/978-3-030-51117-3

[19] M. A. Henning, *Graphs with large total domination number*, J. Graph Theory **35** (2000) 21–45.
https://doi.org/10.1002/1097-0118(200009)35:1¡21::AID-JGT3¿3.0.CO;2-F

[20] T. Kalinowski, N. Kamčev and B. Sudakov, *The zero forcing number of graphs*, SIAM J. Discrete Math. **33** (2019) 95–115.
https://doi.org/10.1137/17M1133051

[21] P. Kristiansen, S.M. Hedetniemi and S.T. Hedetniemi, *Alliances in graphs*, J. Combin. Math. Combin. Comput. **48** (2004) 157–177.

[22] R. Li, *The k-domination number and bounds for the Laplacian eigenvalues of graphs*, Util. Math. **79** (2009) 189–192.
https://utilitasmathematica.com/index.php/Index/article/view/631

[23] H. Liu and M. Lu, *Bounds of signless Laplacian spectrum of graphs based on the k-domination number*, Linear Algebra Appl. **440** (2014) 83–89.
https://doi.org/10.1016/j.laa.2013.10.020

[24] M. Lu, H. Liu and F. Tian, *Bounds of Laplacian spectrum of graphs based on the domination number*, Linear Algebra Appl. **402** (2005) 390–396.
https://doi.org/10.1016/j.laa.2005.01.006

[25] V. Nikiforov, *Bounds on graph eigenvalues* I, Linear Algebra Appl. **420** (2007) 667–671.
https://doi.org/10.1016/j.laa.2006.08.020

[26] J.A. Rodríguez-Velázquez and J.M. Sigarreta, *Spectral study of alliances in graphs*, Discuss. Math. Graph Theory **27** (2007) 143–157.
https://doi.org/10.7151/dmgt.1351

[27] J.A. Rodríguez-Velázquez, I.G. Yero and J.M. Sigarreta, *Defensive k-alliances in graphs*, Appl. Math. Lett. **22** (2009) 96–100.
https://doi.org/10.1016/j.aml.2008.02.012

[28] W. Shi, L. Kang and S. Wu, *Bounds on Laplacian eigenvalues related to total and signed domination of graphs*, Czechoslovak Math. J. **60** (2010) 315–325.
https://doi.org/10.1007/s10587-010-0035-1

[29] J.M. Sigarreta, S. Bermudo and H. Fernau, *On the complement graph and defensive k-alliances*, Discrete Appl. Math. **157** (2009) 1687–1695.
https://doi.org/10.1016/j.dam.2008.12.006

[30] E. Vatandoost and F. Ramezani, *On the domination and signed domination numbers of zero-divisor graph*, Electron. J. Graph Theory Appl. (EJGTA) **4** (2016) 148–156.
https://doi.org/10.5614/ejgta.2016.4.2.3

[31] W. Zhang, *The maximum spectral radius of t-connected graphs with bounded matching number*, Discrete Math. **345** (2022) 112775.
https://doi.org/10.1016/j.disc.2021.112775

[32] W. Zhang, J. Wang, W. Wang and S. Ji, *On the zero forcing number and spectral radius of graphs*, Electron. J. Combin. **29** (2022) #P1.33.
https://doi.org/10.37236/10638

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