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RAMSEY AND GALLAI-RAMSEY NUMBERS FOR FORESTS

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Abstract

Given two non-empty graphs G, H and a positive integer k, the Gallai-Ramsey number $gr_k(G : H)$ is defined as the minimum integer N such that for all $n \geq N$, every k-edge-coloring of K_n contains either a rainbow copy of G or a monochromatic copy of H. Given a graph H, the k-color Ramsey

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number $\mathrm{R}_k(H)$ is the minimum number n such that every k-edge-coloring of K_n contains a monochromatic H. In this paper, we determine several exact values and bounds for Gallai-Ramsey numbers $gr_k(G : H)$ and Ramsey numbers $\mathrm{R}_3(H)$, where G is a special tree and H is a union of stars.

Keywords: Ramsey number, Gallai-Ramsey number, edge-coloring.

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1. Introduction

All graphs considered in this paper are undirected, finite and simple. Any undefined concepts or notations can be found in [5]. Let $G = (V(G), E(G))$ be a graph. For a vertex subset V of G, $G \setminus V$ is obtained by deleting all vertices of V. Let $G[V]$ be the subgraph of G induced by V. For any vertex $u \in V(G)$, we abbreviate $G \setminus \{u\}$ as $G - u$. We denote the minimum degree of G by $\delta(G)$. For a bipartite graph $K_{1,t}$ on $t+1$ vertices, the *center vertex* of the bipartite graph is the vertex adjacent to the other t vertices, and the remaining vertices are called *leaf vertices*. Let $[k] = \{1, 2, \ldots, k\}$. The k-edge-coloring is *exact* if every color is used at least once. Let H be a disconnected graph and $\mathcal{C}(H)$ be the set of connected graphs containing H as a subgraph. In this paper, we solely focus on exact k-edge-colorings of graphs. An edge-colored graph is called *rainbow* if no two edges share the same color. It is referred to as *monochromatic* if every edge has the same color.

As we all know, Ramsey theory was introduced in 1930 (see [28]), and the main subject of the theory are complete graphs whose subgraphs can have some regular properties. Many results about the exact values and bounds for Ramsey numbers have been obtained, which can be referred to a survey [27].

Definition. Given graphs $H_1, H_2, \ldots, H_k$, the multi-color Ramsey number $\mathrm{R}(H_1, H_2, \ldots, H_k)$ is the minimum number n for which every k-edge-coloring of K_n contains a monochromatic H_i for some $1 \leq i \leq n$. If $H_1 = H_2 = \cdots = H_k$, we write the number as $\mathrm{R}_k(H)$.

Definition. Given two graphs G and H, the general k-edge-colored Gallai-Ramsey number $gr_k(G : H)$ is the minimum integer m such that every k-edge-coloring of the complete graph on m vertices contains either a rainbow copy of G or a monochromatic copy of H.

Edge colorings of complete graphs that contain no rainbow triangle possess very interesting and somewhat surprising structures. In 1967, Gallai [12] first examined the edge colorings of complete graphs without rainbow triangle under the guise of transitive orientations of graphs and it can also be traced back to [5].

For this reason, edge-colored complete graphs without rainbow triangle are called *Gallai colorings*. Gallai's result was restated in [17] in the terminology of graphs. Ramsey number has its applications on the fields of communications, information retrieval in computer science, and decision-making; see [29, 30]. For the following statement, a trivial partition is a partition into only one part.

Theorem 1 [5, 12, 17]**.** *For each edge coloring of a complete graph containing no rainbow triangle, there is a nontrivial partition of the vertices* (*called a Gallai partition*) *such that at most two colors are on the edges between the parts and only one color is on the edges between each pair of parts.*

The induced subgraph of a Gallai colored complete graph constructed by selecting a single vertex from each part of a Gallai partition is called the *reduced graph*. By Theorem 1, the reduced graph is a 2-colored complete graph. This kind of restriction on the distribution of colors has led to a variety of interesting results like [14].

With the additional restriction of forbidding rainbow copies of G, we have $gr_k(G : H) \leq R_k(H)$ for any G. Till now, most research focuses on the case of $G = K_3$; see [6, 8, 11, 17, 19, 21, 24–26, 32, 33]. Additionally, Gallai-Ramsey numbers of hypergraphs were also studied by Li *et al.* [18] and Liu [22]. For more details on the Gallai-Ramsey numbers, we refer to the book [23] and a survey paper [9].

Some small trees have been studied. Li *et al.* [20] obtained the structural theorems for complete bipartite graphs without rainbow 3-path and 4-path and some exact values and bounds of $gr_k(P_5 : K_t)$. Fujita and Magnant [10] obtained the structural theorem for $G = S_3^+$. Zou *et al.* [34] studied $gr_k(P_5 : H)$. Zhou *et al.* [35] got exact values or bounds of $gr_k(K_{1,3} : P_3 \cup K_{1,q})$ for m, $q \geq 3$, and the Gallai-Ramsey number $gr_k(P_5 : P_3 \cup K_{1,q})$ for $q \geq 3$. Thomason and Wagner [31] obtained construct theorems without rainbow 3-path, rainbow P_4^+, rainbow 4-path or rainbow $K_{1,3}$. The subject has also been expanded through several other publications including but certainly not limited to [11, 16]. In particular, in [16], the following general behavior of Gallai-Ramsey numbers was established.

Theorem 2 [16]**.** *Let H be a fixed graph with no isolated vertices. If H is bipartite and not a star, then $gr_k(K_3 : H)$ is linear in k. If H is not bipartite, then $gr_k(K_3 : H)$ is exponential in k.*

By Theorem 2, it is clear that $gr_k(K_3 : H)$ is dependent on the variable k. Motivated by the phenomenon, we will determine several exact values and bounds for Gallai-Ramsey numbers $gr_k(G : H)$ which is independent on the variable k, where G is a special tree and H is a union of stars.

Theorem 3. *For positive integers n, t, k with $n \geq 2$, $t = 3, 4$, $k > 4$, we have*

$$(1)\quad gr_k(P_5 : nK_{1,t}) \begin{cases} = tn + n + 1, & \text{if } 2 \leq n \leq 4,\ 4 < k \leq tn + n, \\ = tn + n + 1, & \text{if } n \geq 5,\ \left\lfloor \frac{n+3}{2} \right\rfloor + 1 \leq k \leq tn + n, \\ = 5n - 1, & \text{if } t = 3,\ n \geq 5,\ 4 < k \leq \left\lfloor \frac{n+3}{2} \right\rfloor, \\ \in [6n - 1, 9n + 20], & \text{if } t = 4,\ n \text{ is sufficiently large and} \\ & 4 < k \leq \left\lfloor \frac{n+3}{2} \right\rfloor, \\ = \left\lceil \frac{1+\sqrt{1+8k}}{2} \right\rceil, & \text{if } n \geq 2,\ k \geq tn + n + 1. \end{cases}$$

Theorem 4. *For positive integers n, t, k with $n \geq 2$, $t = 3, 4$, $k \geq 3$, we have*

$$(2)\quad gr_k(K_{1,3} : nK_{1,t}) \begin{cases} = 6n, & \text{if } k = 3,\ n = odd, \\ = 6n - 1, & \text{if } k = 3,\ n = even, \\ \in [3\left\lfloor \frac{5n}{2} \right\rfloor + \beta, 9n + 18], & \text{if } t=4,\ n \text{ is sufficiently large and} \\ & n \geq 7, k = 3, where\ \beta = -1\ for \\ & even\ n, otherwise, \beta = 2, \\ = tn + n, & \text{if } 2 \leq n \leq 4,\ 4 \leq k \leq \left\lfloor \frac{tn+n+1}{2} \right\rfloor, \\ = tn + n, & \text{if } n \geq 5,\ \left\lfloor \frac{n+3}{2} \right\rfloor + 1 \leq k \leq \left\lfloor \frac{tn+n+1}{2} \right\rfloor, \\ = 5n - 1, & \text{if } t = 3,\ n \geq 5,\ 4 \leq k \leq \left\lfloor \frac{n+3}{2} \right\rfloor, \\ \in [6n - 1, 9n + 20], & \text{for } t=4, \text{ sufficiently large } n \geq 13, \\ & 4 \leq k \leq \left\lfloor \frac{n+3}{2} \right\rfloor, \\ = \left\lceil \frac{1+\sqrt{1+8k}}{2} \right\rceil, & \text{if } n \geq 2,\ k \geq \left\lfloor \frac{tn+n+1}{2} \right\rfloor + 1. \end{cases}$$

Theorem 5. *For positive integers n, t, k with $n \geq 2$, $t = 3, 4$, $k \geq 5$, we have*

$$(3)\quad gr_k(P_4^+ : nK_{1,t}) \begin{cases} = tn + n, & \text{if } 2 \leq n \leq 4,\ 5 \leq k \leq \left\lfloor \frac{tn+n+1}{2} \right\rfloor, \\ = tn + n, & \text{if } n \geq 5,\ \left\lfloor \frac{n+3}{2} \right\rfloor + 1 \leq k \leq \left\lfloor \frac{tn+n+1}{2} \right\rfloor, \\ = 5n - 1, & \text{if } t = 3,\ n \geq 5,\ 5 \leq k \leq \left\lfloor \frac{n+3}{2} \right\rfloor, \\ \in [6n - 1, 9n + 20], & \text{if } t = 4,\ n \text{ is sufficiently large}, \\ & 5 \leq k \leq \left\lfloor \frac{n+3}{2} \right\rfloor, \\ = \left\lceil \frac{1+\sqrt{1+8k}}{2} \right\rceil, & \text{if } n \geq 2,\ k \geq \left\lfloor \frac{tn+n+1}{2} \right\rfloor + 1. \end{cases}$$

2. Preliminaries

In this section, we will give several structure theorems and lemmas that will be used in our main theorems.

Theorem 6 [31]**.** *Let K_n, $n \geq 4$, be edge colored so that it contains no rainbow 3-path P_4. Then one of the following holds.*

(a) *At most two colors are used;*

(b) $n = 4$ *and three colors are used, each color forming a* 1*-factor.*

For $n \geq 1$, let $G_1(n)$ be a 3-edge-coloring of K_n satisfying that the vertices of K_n are partitioned into three pairwise disjoint sets V_1, V_2 and V_3 such that for $1 \leq i \leq 3$ (with indices modulo 3), all edges between V_i and V_{i+1} have color i, and all edges connecting pairs of vertices within V_{i+1} have color i or $i+1$. Note that one of V_1, V_2 and V_3 is allowed to be empty, but at least two of them are non-empty; otherwise, at most two colors can appear. For convenience, for an k-edge-colored graph, we let E^i be the set of all edges with color i for $i \in \{1, 2, \ldots, k\}$. We denote the set of all vertices, each of which is incident with at least one edge of E^i, by V^i.

Theorem 7 [31]**.** *For positive integers k and n, if G is a k-edge-coloring of K_n without rainbow P_4^+, then after renumbering the colors, one of the following holds.*

(a) $k \leq 3$ *or* $n \leq 4$*;*

(b) $k = 4$ *and* $G \in \{G_2(n), G_3(n)\}$*;*

(c) $k \geq 4$ *and G contains on rainbow $K_{1,3}$. In particular, item* (b) *in Theorem* 8 *holds.*

Theorem 8 [31]**.** *Let K_n, $n \geq 5$, be edge colored so that it contains no rainbow 4-path P_5. Then, after renumbering the colors, one of the following must hold.*

(a) *At most three colors are used;*

(b) *color* 1 *is dominant, meaning that the sets V^i, $i \geq 2$, are disjoint;*

(c) *$K_n - a$ is monochromatic for some vertex a;*

(d) *there are three vertices a, b, c such that $E^2 = \{ab\}$, $E^3 = \{ac\}$, E^4 contains bc plus perhaps some edges incident with a, and every other edge is in E^1;*

(e) *there are four vertices a, b, c, d such that $\{ab\} \subseteq E^2 \subseteq \{ab, cd\}$, $E^3 = \{ac, bd\}$, $E^4 = \{ad, bc\}$ and every other edge is in E^1;*

(f) *$n = 5$, $V(K_n) = \{a, b, c, d, e\}$, $E^1 = \{ad, ae, bc\}$, $E^2 = \{bd, be, ac\}$, $E^3 = \{cd, ce, ab\}$ and $E^4 = \{de\}$.*

Theorem 9 [2, 31]**.** *For positive integers k and n, if G is an k-edge-coloring of K_n without rainbow $K_{1,3}$, then after renumbering the colors, one of the following holds.*

(a) $k \leq 2$ *or* $n \leq 3$*;*

(b) $k = 3$ *and* $G \simeq G_1(n)$*;*

(c) $k \geq 4$ *and item* (b) *in Theorem* 8 *holds.*

Theorem 10 [3]**.** *Let k and l be fixed and let n be sufficiently large. Then*

$$R(nK_k, nK_l) = n(k + l - 1) + R(K_{k-1}, K_{l-1}) - 2.$$

We obtain the following corollary which will be used to prove Lemma 25.

Corollary 11. *Let n be a sufficiently large integer. Then*

$$R(nK_5, nK_5) = 9n + 16.$$

Proof. By Theorem 10, taking $k = l = 5$, we have $R(nK_5, nK_5) = 9n + R(K_{k-1}, K_{l-1}) - 2$. Since $R(4,4) = 18$ by [13], $R(nK_5, nK_5) = 9n + 16$. ■

Theorem 12 [4]. *Let G and H be graphs. Then for $m, n > 1$ we have that*

$$R(mG, nH) \leq (m+1)V(G) + (n-1)V(H) + R(G,H).$$

Theorem 13 [1, 7]. $43 \leq R(5,5) \leq 48$.

By Theorems 12 and 13, we have the following corollary which will be used to prove Lemma 25.

Corollary 14. *For positive integers $m, n > 1$, we have that*

$$R(mK_5, nK_5) \leq 5(m+1) + 5(n-1) + R(K_5, K_5) \leq 5m + 5n + 48.$$

3. Proof of Theorem 3

Lemma 15 [21]. *Let H be a disconnected graph and $\mathcal{C}(H)$ be the set of connected graphs containing H as a subgraph. If $R_3(H) \geq R_2(\mathcal{C}(H))$, then $gr_k(P_5 : H) = R_3(H)$.*

Lemma 16 [15]. *Suppose that the edges of a graph G with $\delta(G) \geq \frac{3|V(G)|}{4}$ are 2-colored. Then there is a monochromatic connected subgraph with order larger than $\delta(G)$. This estimate is sharp.*

We first give the proof of Theorem 20 using the following two lemmas.

Lemma 17. *For $1 \leq m \leq n$, $R(\mathcal{C}(nK_{1,3}), mK_2) = 4n + m - 1$.*

Proof. Let G be such a red-blue-coloring graph of K_{4n+m-2}: we partition $V(G)$ into V_1 and V_2 such that $G[V_1]$ is a copy of K_{4n-1} and $G[V_2]$ is a copy of K_{m-1}. All edges of $G[V_1]$ are colored by red, and the remaining edges of G are colored by blue. It is easy to check that there is no red $nK_{1,3}$ or blue mK_2 in this red-blue-coloring. So we have that $R(\mathcal{C}(nK_{1,3}), mK_2) > 4n + m - 2$.

Next, we prove the upper bound by induction on m. It is trivial for $m = 1$. Assume that it holds for $m \leq k$. Then we will prove it also holds for $m = k + 1$. Let G be a red-blue coloring graph of K_{4n+k-1}. If there is a red copy of the graph of $\mathcal{C}(nK_{1,3})$ in G, then the red copy of the graph of $\mathcal{C}(nK_{1,3})$ must occur in the edge-colored graph of K_{4n+k}. If there is no red copy of the graph of $\mathcal{C}(nK_{1,3})$ in G, then G contains a blue copy of kK_2 and the edge-colored graph of K_{4n+k} also has it. In the following, we will show that there is a red copy of the graph of $\mathcal{C}(nK_{1,3})$ in the edge-colored graph of K_{4n+k}.

If some edges between each end of $e_i \in E(kK_2)$ and some vertices of X $(= V\backslash V(kK_2))$ are colored by blue, then there will be a blue $(k+1)K_2$ by replacing this e_i with two independent blue edges. And it is easy to check that X induces a red complete subgraph containing a red copy of the graph of $\mathcal{C}(nK_{1,3})$. If there is a vertex, say a_i, incident with e_i for each edge $e_i \in E(kK_2)$ such that all edges between the vertex a_i and all vertices of X $(= V\backslash V(kK_2))$ are colored by red. Owing to $|X| = 4n+k-2k = 4n-k \geq 3k+4$, there are $3k$ vertices of X as leaf vertices and k vertices $a_1, a_2, \ldots, a_k$ as center vertices of a red copy of $kK_{1,3}$. Since there are $4n+k-2k-3k = 4n-4k$ vertices in the remaining vertices of X, we can find a red copy of $(n-k)K_{1,3}$ on those vertices. It is easy to see that the $nK_{1,3}$ are included in a connected red subgraph, which produces the desired red copy of a graph in $\mathcal{C}(nK_{1,3})$. ∎

Burr and Erdős [4] provided the Ramsey number of union of stars, and this theorem played a significant role in the proof of Theorem 19.

Theorem 18 [4]**.** *For $m \geq n, m \geq 2$, we have that*

$$R(mK_{1,3}, nK_{1,3}) = 4m+n-1.$$

Theorem 19. *For $n \geq 2$, $5n-1 \leq R(\mathcal{C}(nK_{1,3}), \mathcal{C}(nK_{1,3})) \leq 6n$.*

Proof. By Theorem 18 with $m = n$, we have that $5n-1 = R(nK_{1,3}, nK_{1,3}) \leq R(\mathcal{C}(nK_{1,3}), \mathcal{C}(nK_{1,3}))$. For the upper bound, let G be an arbitrary red-blue-coloring G of K_N $(N = 6n)$. We assume that there is no monochromatic copy of $\mathcal{C}(nK_{1,3})$ in G. Clearly, there is a monochromatic connected subgraph of order N by Lemma 16. Without loss of generality, suppose that there is a blue connected subgraph of order N. If a minimal red subgraph of G, say H, contains a copy of $nK_{1,3}$, then H must be disconnected.

Let $X_1, X_2, \ldots, X_s$ be all the components of H with $|X_1| \geq |X_2| \geq \cdots \geq |X_s| \geq 1$ for $s \geq 2$. Note that all edges between those components must be blue. If $|X_1| \geq 5n-1$, then there is a monochromatic $\mathcal{C}(nK_{1,3})$ by Theorem 18 with $R(nK_{1,3}, nK_{1,3}) = 5n-1$. So we have that $|X_1| < 5n-1$. Now we consider the following three cases.

Case 1. $|X_1| \geq 4n$ and $|X_2| \geq n$. Let $|X_1| = 4n+k_1$. By Lemma 17, $G[X_1]$ contains either a red $\mathcal{C}(nK_{1,3})$ or a blue $(k_1+1)K_2$. Assume that there is no red copy of $\mathcal{C}(nK_{1,3})$. Then we will obtain a blue matching $M = (k_1+1)K_2$ in $G[X_1]$. We have that

$$|X_1| - |V(M)| = 4n+k_1-2(k_1+1) = 4n-k_1-2 \geq 3n-1,$$

since $k_1 \leq n-1$. If $|X_1| - |V(M)| \geq 3n$, then we choose n vertices as center vertices from X_2 and $3n$ vertices as leaves from $G[X_1] \setminus M$. Thus, we get a blue copy of $\mathcal{C}(nK_{1,3})$ in G. If $|X_1| - |V(M)| = 3n-1$, then $|V(K_N) \setminus X_1| = n+1$. Since $G[X_1]$ has a blue copy of $(k_1+1)K_2 = nK_2$, the two vertices of K_2 from $G[X_1]$ and two leaves from $V(K_N) \setminus X_1$ will induce a blue $K_{1,3}$. Moreover, there is a blue copy of $(n-1)K_{1,3}$ induced by all the remaining $n-1$ vertices in $V(K_N) \setminus X_1$ and $3n-3$ vertices from $G[X_1] \setminus M$. Thus we still have a blue connected subgraph $\mathcal{C}(nK_{1,3})$.

Case 2. $|X_1| \geq 4n$ and $|X_2| < n$. Let $|X_1| = 4n + k_1$ and $|X_2| = n - k_2$. By Lemma 17, we have that $R(\mathcal{C}(nK_{1,3}), (k_1+1)K_2) = 4n + k_1$. Assume that there is a blue matching $M = (k_1+1)K_2$ in $G[X_1]$. Then we have that

$$|X_1| - |V(M)| = 4n + k_1 - 2(k_1+1) = 4n - k_1 - 2 \geq 3n - 1,$$

since $k_1 \leq n-1$. Also, since $|V(K_N) \setminus X_1| \geq n+1$, we have that

$$|V(K_N) \setminus (X_1 \cup X_2)| = 6n - 4n - k_1 - n + k_2 = n - k_1 + k_2 \geq k_2 + 1.$$

Because there is a blue $(k_1+1)K_2$ by our assumption, two vertices of a blue copy of K_2 from $G[X_1]$ and two vertices, say u_1, u_2, of $V(K_N) \setminus X_1$ will induce a blue copy of $K_{1,3}$. Since $|V(K_N) \setminus (X_1 \cup \{u_1, u_2\})| \geq n-1$, we choose $n-1$ vertices from $V(K_N) \setminus (X_1 \cup \{u_1, u_2\})$ as the center vertices and $3n-3$ vertices as leaves from $X_1 \setminus V(M)$ to form a blue copy of $(n-1)K_{1,3}$. Thus we have a connected blue $\mathcal{C}(nK_{1,3})$.

Case 3. $|X_1| \leq 4n-1$. If $3n \leq |X_1| \leq 4n-1$, then $2n+1 \leq |V(G) \setminus X_1| \leq 3n$. Thus we choose $3n$ vertices from X_1 as leaves, n vertices from $V(G) \setminus X_1$ as center vertices to form a blue $nK_{1,3}$. If $2n \leq |X_1| \leq 3n-1$, then $3n+1 \leq |V(G) \setminus X_1| \leq 4n$. We choose $3n$ vertices from $V(G) \setminus X_1$ as leaves, n vertices from X_1 as center vertices to form a blue $nK_{1,3}$. If $|X_1| < 2n$ and $|X_2| < n$, then $n > |X_2| \geq |X_3|$ and $n \leq |X_2 \cup X_3| < 2n$. Then we choose $3n$ vertices from $V(G) \setminus (X_2 \cup X_3)$ as leaves, n vertices from $X_2 \cup X_3$ as center vertices to form a blue copy of $nK_{1,3}$. If $|X_1| < 2n$ and $|X_2| \geq n$, then there is a vertex set X_j for $j \geq 2$ satisfying $\sum_{i=1}^{j} |X_i| \leq 3n-1$ and $\sum_{i=1}^{j+1} |X_i| \geq 3n$. Thus we choose $3n$ vertices from $\bigcup_{i=1}^{j+1} X_i$ as leaves, n vertices from $V(G) \setminus \left(\bigcup_{i=1}^{j+1} X_i\right)$ as center vertices to form a blue copy of $nK_{1,3}$. Thus we still have a connected blue copy of $\mathcal{C}(nK_{1,3})$ in this case. This completes the proof. ■

Theorem 20. *For the positive integers k, n with $n \geq 2$, we have $gr_3(P_5 : nK_{1,3}) = R_3(nK_{1,3}) \geq 6n$ if n is odd; $R_3(nK_{1,3}) \geq 6n-1$ if n is even.*

Proof. To begin with, we prove $R_3(nK_{1,3}) > 6n-1$ (respectively, $R_3(nK_{1,3}) > 6n-2$) for odd (respectively, even) number n. If n is odd, then we arrange such a 3-edge-coloring, denoted by G, for K_{6n-1}: partition $V(K_{6n-1})$ into V_1, V_2, V_3, and let $G[V_1]$ (respectively, $G[V_2]$) be a copy of the complete subgraph K_{2n} of K_{6n-1} colored by 3 (colored by 2), let $G[V_3]$ be a copy of complete subgraph K_{2n-1} of K_{6n-1} colored by at least one color from $\{2, 3\}$ and we color by 3 the edges between V_1 and V_3 and we color by 2 the edges between V_2 and V_3. It suffices to verify whether the edges between $G[V_1]$ and $G[V_2]$ colored by 1 can induce a monochromatic copy of $nK_{1,3}$. We denote by t_1 (respectively, t_2) the number of center vertices contained in V_1 (respectively, V_2). Then, assume that the sum, denoted by n, of t_1 and t_2 is as large as possible. According to the parity of n, we immediately have $t_i \leq \left\lfloor \frac{n}{2} \right\rfloor$ for $i = 1, 2$. It is easy to check that there is a monochromatic copy of $(n-1)K_{1,3}$ with color 1 and two vertices of V_1(respectively, V_2) are left not contained in the monochromatic copy of $(n-1)K_{1,3}$, say $\left\{v_1^1, v_1^2\right\} \in V_1$ (respectively, $\left\{v_2^1, v_2^2\right\} \in V_2$). But the vertices as leaves cannot induce a monochromatic copy of $K_{1,3}$. Thus, $R_3(nK_{1,3}) > 6n - 1$ for the odd n. By Lemma 15 and Theorem 19, we have the required result.

On the other hand, we prove that $R_3(nK_{1,3}) > 6n - 2$ for the even n. We denote by G such a 3-edge-coloring of K_{6n-2}: partition $V(K_{6n-2})$ into V_1, V_2, V_3, and let $G[V_2]$ be a copy of the complete subgraph K_{2n-1} with color 2, and let $G[V_3]$ be a copy of the complete subgraph K_{2n-1} of K_{6n-2} colored by at least one color from $\{2, 3\}$, and let $G[V_1]$ be a copy of complete subgraph K_{2n} of K_{6n-2} with color 3. The edges between V_1 and V_2 are colored by 1, the edges between V_2 and V_3 are colored by 2 and the edges between V_1 and V_3 are colored by 3. This coloring can guarantee the fact that there is no monochromatic copy of $nK_{1,3}$ in K_{6n-2}. It is concluded that $R_3(nK_{1,3}) > 6n-1$ (respectively, $R_3(nK_{1,3}) > 6n-2$) for odd (respectively, even) number n. ∎

Before proving more theorems, we give an observation utilized in some proofs of theorems.

Observation 21. *Let $K_{a,b}$ be a complete bipartite graph with vertex sets V and U, where $|V| = a \geq tn$ and $|U| = b \geq n$. Then $K_{a,b}$ contains a copy of $nK_{1,t}$.*

Theorem 22. *For positive integers $n > 1, k \geq 4$, and $t = 3, 4$, we have that*

$$gr_k(P_5 : nK_{1,t}) = \begin{cases} tn + n + 1, & \text{if } 2 \leq n \leq 4,\ 4 \leq k \leq tn + n, \\ tn + n + 1, & \text{if } n \geq 5,\ \left\lfloor \frac{n+3}{2} \right\rfloor + 1 \leq k \leq tn + n. \end{cases} \tag{4}$$

Proof. We first prove $gr_k(P_5 : nK_{1,t}) > tn + n$. Let G be such a k-edge-coloring graph of K_{tn+n}: we partition $V(G)$ into V_1 and V_2 such that $G[V_1]$ is a copy of K_{tn+n-1} and $G[V_2]$ is an isolated vertex v, where each edge of $G[V_1]$ is colored by color 1 and all edges between V_1 and V_2 are assigned arbitrarily by $k - 1$ colors, say $2, \ldots, k$. It is easy to check that neither a rainbow copy of P_5 nor a monochromatic copy of $nK_{1,t}$ occurs in G. So $gr_k(P_5 : nK_{1,t}) \geq tn + n + 1$.

Next, we prove $gr_k(P_5 : nK_{1,t}) \leq tn+n+1$. Let G be an arbitrarily k-edge-coloring of K_N ($N \geq tn+n+1$) without a rainbow copy of P_5. Then it follows Theorem 8(b). We partition $V(G)$ into $V^1, \ldots, V^k$ with $|V^2| \geq \cdots \geq |V^k| \geq 2$. So it follows that $\lceil N/(k-1) \rceil \leq |V^1 \cup V^2| \leq N-2(k-2)$.

Case 1. $2 \leq n \leq 4$ and $4 \leq k \leq \lceil \frac{tn+n+2}{2} \rceil - 1$, or $n \geq 5$ and $\lfloor \frac{n+3}{2} \rfloor + 1 \leq k \leq \lceil \frac{tn+n+2}{2} \rceil - 1$.

Clearly, we have that $\lceil N/(k-1) \rceil \leq |V^1 \cup V^2| \leq tn+1$. If the size of $V^1 \cup V^2$ is n, then $|V(K_N) \setminus (V^1 \cup V^2)| = 3n+1$ (similarly, if the size of $V^1 \cup V^2$ is $n+1$, $3n$, or $3n+1$, then $|V(K_N) \setminus (V^1 \cup V^2)|$ will be $3n$, $n+1$, or n, respectively). By Observation 21, there is always a monochromatic copy of $nK_{1,t}$ with color 1 in G. If $n+2 \leq |V^1 \cup V^2| \leq tn-1$, then we claim that there is a monochromatic subgraph $nK_{1,t}$ induced by edges between $V^1 \cup V^2$ and $V(K_N) \setminus (V^1 \cup V^2)$. In order to prove it, we construct the monochromatic subgraph $nK_{1,t}$ when $n+2 \leq |V^1 \cup V^2| \leq \lceil \frac{tn+n+2}{2} \rceil$ and $\lceil (tn+n+1)/2 \rceil \leq |V^1 \cup V^2| \leq tn-1$, respectively.

Let $x = |V^1 \cup V^2|$. We denote the remainder of $x/(t-1)$ by r_0 and denote $|V(K_N) \setminus (V^1 \cup V^2)|$ by y. Clearly, $x+y = tn+n+1 \leq N$. Then, apparently, we have that $x = tm + r_0$ for $m = \lfloor x/(t-1) \rfloor \leq n-1$ and $n-1 \leq m+r_0 \leq n+2$. If $m+r_0 = n+1$, then $y = m+tr_0-t$, and regard r_0-1 vertices of $V^1 \cup V^2$ and m vertices of $V(K_N) \setminus (V^1 \cup V^2)$ as center vertices of a copy of $nK_{1,3}$. Clearly, there must exist a monochromatic copy of $nK_{1,t}$ with color 1. If $m+r_0 = n$, then $y = m+tr_0+1$. We regard r_0 vertices of $V^1 \cup V^2$ and m vertices of $V(K_N) \setminus (V^1 \cup V^2)$ as center vertices of a copy of $nK_{1,3}$. So there must exist a monochromatic copy of $nK_{1,t}$ with color 1. If $m+r_0 = n+2$ and $t=4$, then $y = m+3$, and $x = 4m+3$. There at last exists one edge with color 1 of $G[V(K_N) \setminus (V^1 \cup V^2)]$, and we denote the edge by u_1u_2. Select arbitrarily 3 vertices, say $\{v_1, v_2, v_3\}$, from $V^1 \cup V^2$. Clearly, $\{u_1u_2, u_1v_1, u_1v_2, u_1v_2\}$ is a monochromatic copy of $K_{1,4}$ with color 1 in $G[V(K_N) \setminus (V^1 \cup V^2)]$. By Observation 21, G contains a monochromatic copy of $(n-1)K_{1,t}$ with color 1.

For $m+r_0 = n-1$ and $t=3$, let w be a positive integer for $w \in \{(n-m-r_0)/2, (n-m-r_0+1)/2\}$. There is $n-m+w$ vertices of $V^1 \cup V^2$ and $m-w$ vertices of $V(K_N) \setminus (V^1 \cup V^2)$ as center vertices of a copy of $nK_{1,3}$. Clearly, there must exist a monochromatic copy of $nK_{1,3}$ with color 1. For $\lceil N/(k-1) \rceil \leq |V^1 \cup V^2| \leq n-1$, let $|V^1 \cup V^2| = c$ and $|V^3| = s+j$ where $c+s = n$. Hence, we have $\lceil (3n+2)/(k-2) \rceil \leq |V^3| \leq n-1$. Since $\sum_{i=4}^{k} |V^i| \geq N-2c \geq 2c+4s+1$, there is a monochromatic copy of $sK_{1,3}$ with vertex set X having color 1 in $G\big[\bigcup_{i=3}^{k} V^i\big]$. Let $B = V(G) \setminus (X \cup V^1 \cup V^2)$. Then there is a monochromatic copy of $cK_{1,3}$ in $G\big[\bigcup_{i=4}^{k} V^i \cup B\big]$. By $c+t = n$, we have constructed the monochromatic copy of $nK_{1,3}$ with color 1 in G.

Case 2. $\lceil \frac{tn+n+2}{2} \rceil \leq k \leq tn+n$. Since at least two vertices are contained in each V^i for $2 \leq i \leq \lceil (tn+n)/2 \rceil$, let $a_{ij} \in V^i$ ($1 \leq j \leq 2$). Since $(tn+n)/2 \equiv 0$

(mod $t+1$) for even n, then the edge set $\{a_{21}a_{i1}, a_{22}a_{i2} \,|\, 3 \leq i \leq t+2\}$ will induce a monochromatic copy of $2K_{1,t}$ with color 1. Furthermore, there is $n/2$ copies of $2K_{1,t}$ with color 1.

For odd n, if $t = 3$ and $n \equiv 1 \pmod 2$, then $n - 3 \equiv 0 \pmod 2$ holds. It follows that $\big(\lceil \frac{tn+n}{2} \rceil - 6\big)/4 = (n-3)/2$, so, there is a monochromatic copy of $(n-3)K_{1,3}$ with color 1 in $G\big[V(G) \setminus \big(\bigcup_{i=2}^{7} V^i\big)\big]$ for even $n-3$. Choosing distinct vertices $a_{ij} \in V^i$ $(2 \leq i \leq 7)$, the subgraph induced by edge set $\{a_{21}a_{i1}, a_{22}a_{32}, a_{22}a_{42}, a_{22}a_{62}, a_{61}a_{52}, a_{61}b_{7j} \,|\, 3 \leq i \leq 5\}$ contains a monochromatic copy of $3K_{1,3}$ with color 1 in $G\big[\bigcup_{i=2}^{7} V^i\big]$. Therefore, there is a monochromatic copy of $nK_{1,3}$ with color 1. If $t = 4$ and $(5n+1)/2 \equiv 3 \pmod 5$, then $(5n+1)/2 - 3 \equiv 0 \pmod 5$ holds. It follows that $\big(\lceil \frac{tn+n}{2} \rceil - 3\big)/5 = (n-1)/2$, then we find a monochromatic copy of $(n-1)K_{1,4}$ with color 1 in $G\big[V(G) \setminus \big(\bigcup_{i=2}^{4} V^i\big)\big]$ for $n-1$. The subgraph induced by edge set $\{a_{31}a_{2j}, a_{31}a_{4j}\}$ contains a monochromatic copy of $K_{1,4}$ with color 1 in $G\big[\bigcup_{i=2}^{4} V^i\big]$. Therefore, we find a monochromatic copy of $nK_{1,4}$ with color 1, as desired.

If Theorem 8(c) holds, it is easy to check that there is a monochromatic copy of K_{4n} with color 1 containing a monochromatic copy of $nK_{1,3}$. If Theorem 8(d) for $k = 4$ holds, the subgraph induced by $\{v_1a, v_1b, v_1c, v_1v_2\}$ contains a $K_{1,t}$ copy with color 1, where vertex $v_i \in V(G) \setminus \{a, b, c\}$ $(1 \leq i \leq 2)$. There is a monochromatic $(n-t)K_{1,t}$ copy with color 1 in $G \setminus K_{1,t}$. Suppose that it follows Theorem 8(e). We choose distinct vertices $v_1 \in V(K_N) \setminus \{a, b, c, d\}$. The subgraph induced by the edge set $\{av_1, bv_1, cv_1, dv_1\}$ will make up a monochromatic copy of $K_{1,t}$ of with color 1. And it is easy to see that there is a monochromatic copy of K_{N-5} containing a monochromatic copy of $(n-1)K_{1,t}$ with color 1. Consequently, a monochromatic copy of $(n-1)K_{1,t}$ in K_{N-5} is constructed, as desired. ∎

At the end of this section, we give the proof of Theorem 3 by Theorem 23 and Lemmas 24, 25.

Theorem 23 [34]**.** *Let H be a graph of order n. For integers $k \geq 7$ and $k \geq n+1$, we have that*

$$gr_k(P_5 : H) = \left\lceil \frac{1 + \sqrt{1 + 8k}}{2} \right\rceil.$$

Lemma 24. *For any positive integers $n \geq 5$ and $4 < k \leq \lceil \frac{n+3}{2} \rceil$, we have that*

$$gr_k(P_5 : nK_{1,3}) = 5n - 1.$$

Proof. We first prove that $gr_k(P_5 : nK_{1,3}) \geq 5n - 2$. Let G be a k-edge-coloring of K_{5n-2}: we partition $V(G)$ into $V_1, \ldots, V_{k-1}$ with $|V_2| \geq \cdots \geq |V_{k-1}| \geq 2$ such that $G[V_1]$ is a monochromatic copy of K_{4n-1} with color 2, $G[V_i]$ is colored by $i+1$ for $2 \leq i \leq k-1$, and all edges between V_i and V_j for all $i \neq j \in [k-1]$ with color 1. It is easy to check that no rainbow P_5 appears in the graph G. Furthermore, according to $\sum_{i=2}^{k-2}|V_i| = n-1$ the graph G contains no monochromatic copy of $nK_{1,3}$. As a consequence, G contains neither a rainbow copy of P_5 nor a monochromatic copy of $nK_{1,3}$. So it admits $gr_k(P_5 : nK_{1,3}) \geq 5n-1$.

For the upper bound, let G be an arbitrary k-edge-coloring of K_N ($N \geq 5n-1$) that contains no rainbow copy of P_5. Suppose that G satisfies Theorem 8(b), we divide $V(G)$ into $V^1, \ldots, V^k$ with $|V^2| \geq \cdots \geq |V^k| \geq 2$ ($2 \leq i \leq k-1$). We immediately have that $|V^1 \cup V^2| \geq \lceil N/(k-1) \rceil$.

Case 1. $4n+1 \leq |V^1 \cup V^2| \leq 5n+3-2k$. If $4n+1 \leq |V^1 \cup V^2| \leq 5n+3-2k$, then it follows from Theorem 18 that $R(nK_{1,3}, (n-j)K_{1,3}) = 5n-j-1$ ($j = |V(K_N) \setminus (V^1 \cup V^2)|$, $n \geq 2$). If there is a monochromatic copy of $(n-j)K_{1,3}$ with color 1 in $G[V^1 \cup V^2]$, then it is clear that j vertices of $V(K_N) \setminus (V^1 \cup V^2)$ and $3j$ vertices of $V^1 \cup V^2$ induce a monochromatic copy of $jK_{1,3}$ with color 1, which will lead to a monochromatic copy of $nK_{1,3}$ with color 1 contained in G. Otherwise, $G[V^1 \cup V^2]$ contains a monochromatic copy of $nK_{1,3}$ with color 2.

Case 2. $|V^1 \cup V^2| = 4n$ and $|V(K_N) \setminus (V^1 \cup V^2)| = n-1$. If $|V^1 \cup V^2| = 4n$ and $|V(K_N) \setminus (V^1 \cup V^2)| = n-1$, then we have that $R(nK_{1,3}, K_{1,3}) = 4n$ by Theorem 18. We assume that there is no monochromatic copy of $nK_{1,3}$ with color 2 in $G[V^1 \cup V^2]$. Then there is a monochromatic copy of $K_{1,3}$ with color 1 in $G[V^1 \cup V^2]$. And then, we further choose all vertices of $V(K_N) \setminus (V^1 \cup V^2)$ and all vertices of $(V^1 \cup V^2) \setminus V(K_{1,3})$. It will form a monochromatic copy of $(n-1)K_{1,3}$ with color 1.

Case 3. $3n \leq |V^1 \cup V^2| \leq 4n-1$ and $n \leq |V(K_N) \setminus (V^1 \cup V^2)| \leq 2n-1$. If $3n \leq |V^1 \cup V^2| \leq 4n-1$ and $n \leq |V(K_N) \setminus (V^1 \cup V^2)| \leq 2n-1$, then by Observation 21 we have $|V(K_N) \setminus (V^1 \cup V^2)| \geq n$ and $|V^1 \cup V^2| \geq 3n$. Clearly, we find a monochromatic copy of $nK_{1,3}$ with color 1. If $2n \leq |V^1 \cup V^2| \leq 3n-1$ and $2n \leq |V(K_N) \setminus (V^1 \cup V^2)| \leq 3n-1$, then let $X = \max\{|V^1 \cup V^2|, |V(K_N) \setminus (V^1 \cup V^2)|\}$ and $Y = \min\{|V^1 \cup V^2|, |V(K_N) \setminus (V^1 \cup V^2)|\}$ with $|Y| \geq 2n-1$ and $|X| \geq 2n+1$. Now we construct a monochromatic copy of $nK_{1,3}$ with color 1. We select $\lceil n/2 \rceil$ vertices from Y and $\lfloor n/2 \rfloor$ vertices from X, and $3\lceil n/2 \rceil$ vertices from X and $3\lfloor n/2 \rfloor$ vertices from Y. It is easy to check that the vertices selected above form a monochromatic copy of $nK_{1,3}$ with color 1. If $n \leq |V^1 \cup V^2| \leq 2n-1$ and $3n \leq |V(K_N) \setminus (V^1 \cup V^2)| \leq 4n-1$, then, by Observation 21, we find a monochromatic copy of $nK_{1,3}$ with color 1. If $|V^1 \cup V^2| \leq n-1$ and $4n \leq |V(K_N) \setminus (V^1 \cup V^2)| \leq 5n-1-\lceil N/(k-1) \rceil$, then there is a vertex set V^j for $2 \leq j \leq k-1$ satisfying $\sum_{i=1}^{j+1}|V^i| \leq n-1$ and $\sum_{i=1}^{j+1}|V^i| \geq n$. Let $s = \sum_{i=1}^{j+1}|V^i|$ and $t = \sum_{i=j+2}^{k-1}|V^i|$. Since $|V^{j+1}| \leq |V^j| \leq n-1$ and by Observation 21, we find a monochromatic copy of $nK_{1,3}$ with color 1 in $K_{s,t}$. If $k=4$, then it is easy to see that we find a monochromatic copy of K_{4n} with color 1 containing a monochromatic copy of $nK_{1,3}$, as desired. ∎

Lemma 25. *For sufficiently large n with $4 < k \leq \lceil \frac{n+3}{2} \rceil$, we have that*

$$6n - 1 \leq gr_k(P_5 : nK_{1,4}) \leq 9n + 20.$$

Proof. For the lower bound, let G be a k-edge-coloring of K_{6n-2}. Then partition $V(G)$ into $V_1, V_2, \ldots, V_{k-1}$ with $|V_i| \geq |V_{i+1}| \geq 2$ $(1 \leq i \leq k-1)$. Let $G[V_1]$ be a monochromatic K_{5n-1} with color 2. And we color $G[V_i]$ by $i+1$ for $2 \leq i \leq k-2$ and the remaining edges of G are colored by color 1. If $\Sigma_{i=2}^{k-1}|V_i| = n-1$, then G contains no monochromatic copy of $nK_{1,4}$. Then there is neither a rainbow copy of P_5 nor a monochromatic copy of $nK_{1,4}$ in G. So $gr_k(P_5 : nK_{1,4}) \geq 6n-1$.

For the upper bound, let G be an arbitrary k-edge-coloring of K_N $(N \geq 9n+20)$ without rainbow copy of P_5. For the case of Theorem 8(b), we divide $V(G)$ into $V^1, \ldots, V^k$ with $|V^2| \geq \cdots \geq |V^k| \geq 2$. We have that $|V^1 \cup V^2| \geq \lceil N/(k-1) \rceil$. If $|V^1 \cup V^2| \geq 9n+16$, then it follows from Corollary 11 that $R(nK_5) = 9n+16$. It is easy to check that there exists a monochromatic nK_5 with color 1.

If $8n+21 \leq |V^1 \cup V^2| \leq 9n+15$, then $5 \leq |V(K_N) \setminus (V^1 \cup V^2)| \leq 9n+15$. Let $|V(K_N) \setminus (V^1 \cup V^2)| = z$. By Corollary 14, we have $R(nK_5, (n-z)K_5) \geq 10n + 38 - 5z$. If there is a monochromatic copy of $(n-z)K_{1,4}$ with color 1 in $G[V^1 \cup V^2]$, then it is clear that z vertices of $V(K_N) \setminus (V^1 \cup V^2)$ and $4z$ vertices of $V^1 \cup V^2$ induce a monochromatic copy of $zK_{1,4}$ with color 1, which lead to a monochromatic copy of $nK_{1,4}$ with color 1 contained in G. Otherwise, in $G[V^1 \cup V^2]$ there is a monochromatic copy of $nK_{1,4}$ with color 2.

If $4n \leq |V^1 \cup V^2| \leq 8n+20$ and $n \leq |V^1 \cup V^2| \leq 4n-1$, then by Observation 21 we find a monochromatic copy of $nK_{1,4}$ with color 1. If $\lceil N/(k-1) \rceil \leq |V(K_N) \setminus (V^1 \cup V^2)| \leq n-1$, then there is a vertex set V^j for $2 \leq j \leq k-1$ satisfying $\sum_{i=1}^{j} |V^i| \leq n-1$ and $\sum_{i=1}^{j+1} |V^i| \geq n$. Let $s = \sum_{i=1}^{j+1} |V^i|$ and $t = \sum_{i=j+2}^{k-1} |V^i|$. Since $|V^{j+1}| \leq |V^j| \leq n-1$ and by Observation 21, we find a monochromatic copy of $nK_{1,4}$ with color 1 in $K_{s,t}$. If $k=4$, then it is easy to see that we find a monochromatic copy of K_{5n} with color 1 containing a monochromatic copy of $nK_{1,4}$, as desired. ∎

4. Proof of Theorem 4

We give the proof of Theorem 4 by proving several lemmas in the following.

Lemma 26. *For $n \geq 2$, we have*

$$gr_3(K_{1,3} : nK_{1,3}) = \begin{cases} 6n, & \text{if } n \text{ is odd}, \\ 6n-1, & \text{if } n \text{ is even}. \end{cases} \tag{5}$$

Proof. To begin with, we prove $R_3(K_{1,3} : nK_{1,3}) > 6n-1$ (respectively, $R_3(K_{1,3} : nK_{1,3}) > 6n-2$) for odd (respectively, even) number n. If n is odd, then we arrange such a 3-edge-coloring, denoted by G, for K_{6n-1}: partition $V(K_{6n-1})$ into V_1, V_2, V_3, and let $G[V_1]$ (respectively, $G[V_2]$) be a copy of the complete subgraph K_{2n} of K_{6n-1} colored by 3 (colored by 2), let $G[V_3]$ be a copy of complete subgraph K_{2n-1} of K_{6n-1} colored by at least one color from $\{2, 3\}$ and we color by 3 the edges between V_1 and V_3, we color by 2 the edges between V_2 and V_3 and we color by 1 the edges between V_1 and V_2. We can check that there is no rainbow $K_{1,3}$ for the edge-coloring. It suffices to verify whether the edges between $G[V_1]$ and $G[V_2]$ colored by 1 can induce a monochromatic copy of $nK_{1,3}$. We denote by t_1 (respectively, t_2) the number of center vertices contained in V_1 (respectively, V_2). Then, assume that the sum, denoted by n, of t_1 and t_2 is as large as possible. According to the parity of n, we immediately have $t_i \leq \left\lfloor \frac{n}{2} \right\rfloor$ for $i = 1, 2$. It is easy to check that there is a monochromatic copy of $(n-1)K_{1,3}$ with color 1 and two vertices of V_1(respectively, V_2) are left not contained in the monochromatic copy of $(n-1)K_{1,3}$, say $\left\{v_1^1, v_1^2\right\} \in V_1$ (respectively, $\left\{v_2^1, v_2^2\right\} \in V_2$). But the vertices as leaves cannot induce a monochromatic copy of $K_{1,3}$. Thus, $R_3(K_{1,3} : nK_{1,3}) > 6n-1$ for the odd n.

On the other hand, we prove that $R_3(K_{1,3} : nK_{1,3}) > 6n-2$ for the even n. We denote by G such a 3-edge-coloring of K_{6n-2}: partition $V(K_{6n-2})$ into V_1, V_2, V_3, and let $G[V_2]$ be a copy of the complete subgraph K_{2n-1} with color 2, and let $G[V_3]$ be a copy of the complete subgraph K_{2n-1} of K_{6n-2} colored by at least one color from $\{2, 3\}$, and let $G[V_1]$ be a copy of complete subgraph K_{2n} of K_{6n-2} with color 3. The edges between V_1 and V_2 are colored by 1, the edges between V_2 and V_3 are colored by 2 and the edges between V_1 and V_3 are colored by 3. This coloring can guarantee the fact that there is no rainbow copy of $K_{1,3}$ and monochromatic copy of $nK_{1,3}$ in K_{6n-2}. It is concluded that $R_3(K_{1,3} : nK_{1,3}) > 6n-1$ (respectively, $R_3(K_{1,3} : nK_{1,3}) > 6n-2$) for odd (respectively, even) number n.

For the upper bound, let G be an arbitrary 3-edge-coloring of K_N without a rainbow copy of $K_{1,3}$. We divide $V(G)$ into V_1, V_2, and V_3 with $|V_1| \geq |V_2| \geq |V_3|$. Let $G[V_1]$ be a 2-edge-coloring complete graph colored by at least one color from $\{1, 3\}$, and let $G[V_2]$ be a 2-edge-coloring complete graph colored by at least one color from $\{1, 2\}$, and let $G[V_3]$ be a 2-edge-coloring complete graph colored by at least one color from $\{2, 3\}$. We construct G by coloring all edges between $G[V_1]$ and $G[V_2]$ with color 1, all edges between $G[V_2]$ and $G[V_3]$ with color 2 and all edges between $G[V_1]$ and $G[V_3]$ with color 3. Then the coloring satisfies Theorem 9(b). If n is odd, then $N \geq 6n$. Otherwise $N \geq 6n-1$. We consider the following two cases.

Case 1. $|V_3| = 0$. We have that $|V_1| \geq \lceil N/2 \rceil = 3n$. If $|V_1| \geq 5n-1$, then $R(nK_{1,3}) = 5n-1$ by Theorem 18. So there is a monochromatic copy of $nK_{1,3}$ with color 1 or color 3 in $G[V_1]$. If $3n \leq |V_1| \leq 5n-2$ and $n+1 \leq |V_2| \leq 3n-1$, then there is a monochromatic copy of $nK_{1,3}$ with color 1 in $G[V_1 \cup V_2]$

by Observation 21.

Case 2. $|V_3| \geq 1$. In this case, we have that $|V_1| \geq \lceil N/3 \rceil \geq 2n$. If $|V_1| \geq 5n-1$, then it is easy to check that there is a monochromatic copy of $nK_{1,3}$ with color 1 or 3 in $G[V_1]$. If $4n+1 \leq |V_1| \leq 5n-2$ and $|V_2| \geq n$, then there is a monochromatic $nK_{1,3}$ with color 1 by Observation 21. If $4n+1 \leq |V_1| \leq 5n-2$, then $|V_2| \leq n-1$. Let $|V_2| = m$ and $|V_3| \leq m$. By Theorem 18, we have $R(nK_{1,3}, (n-m)K_{1,3}) = 5n-m-1$. It follows that $|V_1| \geq R(nK_{1,3}, (n-m)K_{1,3})$. If there is a monochromatic copy of $(n-m)K_{1,3}$ with color 1, then it is clear that m vertices of V_2 and $3m$ vertices of V_1 induce a monochromatic copy of $mK_{1,3}$ with color 1, which lead to a monochromatic copy of $nK_{1,3}$ with color 1 in G. Otherwise, $G[V_1]$ contains a monochromatic copy of $nK_{1,3}$ with color 3. If $3n \leq |V_1| \leq 4n$ and $|V_2| \geq \lceil (N-|V_1|)/2 \rceil = n$, then there is a monochromatic copy of $nK_{1,3}$ with color 1 by Observation 21 in $G[V_1 \cup V_2]$.

Suppose that $2n+1 \leq |V_1| < 3n-1$. If n is odd, then we denote $|V_1| = 2n+1+2l$ $(l = 0,1,2,\ldots)$. Clearly, $|V_2| \geq \lceil (N-|V_1|)/2 \rceil = 2n-l$. We regard $\lfloor n/2 \rfloor - l$ of V_1 and $\lceil n/2 \rceil + l$ of V_2 as center vertices and the remaining vertices as leaves of the copy of $nK_{1,3}$. Clearly, there is a monochromatic copy of $nK_{1,3}$ with color 1. Suppose that n is even. If $|V_1| = 2n+2+2l$ $(i \geq 1)$, then $|V_2| \geq \lceil (N-|V_1|)/2 \rceil = 2n-1-l$. We look on $n/2-l$ vertices from V_1 and $n/2+l$ vertices from V_2 as center vertices and remaining vertices as corresponding leaf vertices to from a monochromatic copy of $nK_{1,3}$ with color 1. If $l = 0$, then $|V_1| = 2n+2$ and $|V_2| \geq 2n-1$. Hence, there is a monochromatic copy of $nK_{1,3}$ with color 1 in $G[V_1 \cup V_2]$, which contains $n/2+1$ center vertices in V_1 and $n/2-1$ center vertices in V_2.

Suppose that $|V_1| = 3n-1$. If $|V_1| = 3n-1$ and $|V_2| \geq \lceil (N-3n+1)/2 \rceil$, then $\lceil (N-3n+1)/2 \rceil \geq n+2$. Hence, there is a monochromatic copy of $nK_{1,3}$ with color 1 in $G[V_1 \cup V_2]$, which contains one center vertices in V_1 and $n-1$ center vertices in V_2.

Suppose that $|V_1| = 2n$. If n is odd, then we obtain $|V_1| = |V_2| = |V_3| = 2n$. Let $\lfloor n/2 \rfloor$ vertices be center vertices and $3\lfloor n/2 \rfloor$ vertices be the leaves of V_i $(i = 1,2)$ (respectively, $i = 1,3$) of a copy of $(n-1)K_{1,3}$. Clearly, there must exist a monochromatic copy of $(n-1)K_{1,3}$ with color 1 or color 3. It is obvious that there are two vertices remaining in each vertex sets, denote these edges are $u_i^1 u_i^2 \in V_i$ $(i = 1,2,3)$. Clearly, $\{u_1^1 u_1^2, u_1^1 u_1^1\}$ $(i = 2,3)$ is a monochromatic copy of $K_{1,3}$ with color 1 when $u_1^1 u_1^2$ has color 1. On the other hand, there is a monochromatic copy of $K_{1,3}$ with color 3. We have constructed the monochromatic copy of $nK_{1,3}$ with color 1 or 3 in $G[V_1 \cup V_2]$ or $G[V_1 \cup V_3]$. If n is even, then we obtain $|V_1| = |V_2| = 2n$ and $|V_3| = 2n-1$. There is a monochromatic copy of $nK_{1,3}$ with color 1 in $G[V_1 \cup V_2]$. Let t_1 (respectively, t_2) be the number of center vertices of a copy of $nK_{1,3}$ contained in V_1 (respectively, V_2) and $t_1 = t_2 = n/2$. It is easy to check that there is a monochromatic copy of $nK_{1,3}$ with color 1. ■

Lemma 27. *For sufficiently large n, we have*

$$3\left\lfloor \frac{5n}{2} \right\rfloor + \beta \leq gr_3(K_{1,3} : nK_{1,4}) \leq 9n + 18,$$

where $\beta = -1$ if n is even, otherwise, $\beta = 2$.

Proof. On the one hand, we prove $gr_3(K_{1,3} : nK_{1,4}) > 3\lfloor 5n/2 \rfloor + 1$ (respectively, $gr_3(K_{1,3} : nK_{1,4}) > 3\lfloor 5n/2 \rfloor - 2$) for odd (respectively, even) number n. If n is odd, then $3\lfloor 5n/2 \rfloor + 1 = 15(n-1)/2 + 1$. Let G be a 3-edge-coloring of $K_{15(n-1)/2+1}$ as follows. We first partition $V(K_{15(n-1)/2+1})$ into V_1, V_2, and V_3, and let $G[V_1]$ and $G[V_2]$ be the copies of the complete subgraph $K_{(5n-5)/2+3}$ of $K_{15(n-1)/2+1}$ colored by 3 and 2, respectively. Let $G[V_3]$ be a copy of complete subgraph $K_{(5n-1)/2-1}$ of $K_{15(n-1)/2+1}$ with one color from $\{2,3\}$. It is obvious that the edges between V_1 and V_2 colored by 1 induce a monochromatic copy of $nK_{1,4}$. We choose $n-1$ center vertices of a copy of $(n-1)K_{1,4}$ in $V_1 \cup V_2$. Thus, there is a monochromatic copy of $(n-1)K_{1,4}$ with color 1. Two vertices $v_i^1, v_i^2 \in V_i$ $(i = 1,2)$ are not contained in the monochromatic copy of $(n-1)K_{1,4}$. However, the remaining vertices will not induce a monochromatic copy of $K_{1,4}$. Hence, $gr_3(K_{1,3} : nK_{1,4}) \geq 3\lfloor 5n/2 \rfloor + 2$ for the odd n.

On the other hand, we prove that $gr_3(K_{1,3} : nK_{1,4}) \geq 3\lfloor 5n/2 \rfloor - 1$ for the even n. Consider such a 3-edge-coloring of $K_{15n/2-2}$ and denote it by G: partition $V(K_{15n/2-2})$ into V_1, V_2, V_3, and let $G[V_2]$ be the copy of the complete subgraphs $K_{5n/2-1}$ with color 2, and let $G[V_3]$ be the copy of the complete subgraph $K_{5n/2-1}$ of $K_{15n/2-2}$ colored by at least one color from $\{2,3\}$. Let $G[V_1]$ be a copy of complete subgraph $K_{5n/2}$ of $K_{15n/2-2}$ with color 3. There is no monochromatic copy of $nK_{1,4}$ in $K_{15n/2-2}$. We conclude that $K_{3\lfloor 5n/2 \rfloor+\beta}$ contains no a monochromatic copy of $nK_{1,4}$.

For the upper bound of $gr_3(K_{1,3} : nK_{1,4})$, let G be an arbitrary 3-edge-coloring of K_N $(N \geq 9n+18)$ without a rainbow copy of $K_{1,3}$. Firstly, we divide $V(G)$ into V^1, V^2, and V^3 with $|V_1| \geq |V_2| \geq |V_3|$, where $G[V_1]$ is a complete graph colored by at least one color from $\{1,3\}$, and $G[V_2]$ is a complete graph colored by at least one color from $\{1,2\}$, and $G[V_3]$ is a complete graph colored by at least one color from $\{2,3\}$. Then we color all edges between $G[V_1]$ and $G[V_2]$ with color 1, all edges between $G[V_2]$ and $G[V_3]$ with color 2, and all edges between $G[V_1]$ and $G[V_3]$ with color 3. Then the coloring satisfies Theorem 9(b).

Case 1. $|V_3| = 0$. If $|V_1| \geq 9n+16$, then $R(nK_5) = 9n+16$ by Theorem 10. It follows that there is a monochromatic copy of $nK_{1,4}$ with color 1 or color 3. Because of $4n < \lceil N/2 \rceil \leq |V_1| \leq 9n+15$, we obtain $|V_2| \geq n$. Hence, there is a monochromatic copy of $nK_{1,4}$ with color 1 by Observation 21.

Case 2. $|V_3| \geq 1$. In this case, we have $|V_1| \geq \lceil N/3 \rceil = 3n+6$. If $|V_1| \geq 9n+16$, then $R(nK_5) = 9n+16$ by Theorem 10. So there is again a monochromatic copy of $nK_{1,4}$ with color 1 or color 3. If $7n + 18 \leq |V_1| \leq 9n + 15$ with $|V_2| \geq n$, then there is a monochromatic copy of $nK_{1,4}$ with color 1 by Observation 21. If $7n+18 \leq |V_1| \leq 9n+15$ with $|V_2| \leq n-1$, then we denote $|V_2|$ by m. By Corollary 14, we have that $R(nK_5, (n-m)K_5) \leq 10n+38-5m$. If there is a monochromatic copy of $(n-m)K_5$ with color 1 in $G[V_1]$, m vertices of V_2 and $4m$ vertices of V_1 induce a monochromatic copy of $mK_{1,4}$ with color 1 in G. Otherwise, $G[V_1]$ contains a monochromatic copy of $nK_{1,4}$ with color 1. If $4n \leq |V_1| \leq 7n + 17$, then $|V_2| \geq \lceil (N - |V_1|)/2 \rceil \geq n$. So, there must exist a monochromatic copy of $nK_{1,4}$ with color 1 by Observation 21. If $3n + 6 \leq |V_1| \leq 4n - 1$, then we have $3n+6 \geq |V_2| \geq \lceil (N-|V_1|)/2 \rceil \geq \lceil (5n+19)/2 \rceil$. Clearly, $\lfloor n/2 \rfloor - 1$ center vertices of V_1 and $\lceil n/2 \rceil + 1$ center vertices of V_2 induce a monochromatic copy of $nK_{1,4}$ with color 1 in $G[V_1 \cup V_2]$. Consequently, we have that $gr_3(K_{1,3} : nK_{1,4}) \leq 9n + 18$. ■

Lemma 28. *For positive integers n, k, and $t = 3, 4$, we have that*

$$gr_k(K_{1,3} : nK_{1,t}) = \begin{cases} tn + n, & \text{if } 2 \leq n \leq 4,\ 4 \leq k \leq \left\lfloor \frac{tn+n+1}{2} \right\rfloor, \\ tn + n, & \text{if } n \geq 5,\ \left\lfloor \frac{n+3}{2} \right\rfloor + 1 \leq k \leq \left\lfloor \frac{tn+n+1}{2} \right\rfloor. \end{cases} \tag{6}$$

Proof. For the lower bound, let G be a k-edge-coloring of K_{tn+n-1}. We divide $V(G)$ into $V_1, \ldots, V_{k-1}$ with $|V_1| = tn + n - 2k + 3$ and $|V_j| = 2$ $(2 \leq j \leq k-1)$. Then the k-edge-coloring follows the rules: each edge of the subgraph $G[V_j]$ for $1 \leq j \leq k-1$ is colored by color j and the remaining edges of G are colored by k. So, there is neither a monochromatic copy of $nK_{1,t}$ nor a rainbow $K_{1,3}$.

For the upper bound, let G be an any k-edge-coloring of K_N $(N \geq tn + n)$ without a rainbow copy of $K_{1,3}$ and we divide $V(G)$ into $V_1, \ldots, V_k$. Then the coloring satisfies Theorem 9(c). Without loss of generality, we assume that $|V^i| \geq |V^{i+1}| \geq 2$ $(2 \leq i \leq k-1)$. It implies that $\lceil N/(k-1) \rceil \leq |V^1 \cup V^2| \leq tn + n - 2(k-2)$. Let $v^i_j \in V^i$ $(2 \leq i \leq k,\ j = 1, 2)$. If $|V^1 \cup V^2| = n$ or tn, then there is a monochromatic copy of $nK_{1,t}$ with color 1 by Observation 21. Now we consider two cases.

Case 1. $|V^1 \cup V^2| \geq n + 1$ and $t = 3$. If $|V^1 \cup V^2| - n = 2m$ for a positive integer m, then we denote $V^1 \cup V^2$ by B_1. Next, we choose m center vertices from $V(K_N) \setminus (V^1 \cup V^2)$ and $n - m$ center vertices from B_1, which will form a monochromatic copy of $nK_{1,3}$ with color 1. If $|V^1 \cup V^2| - n = 2m \pm 1$, we denote $V^1 \cup V^2$ by B_2. Let $v^i_j \in V^i$ for $i = 2, 3, 4$ and $j = 1, 2$. Then the edge set $\{v^3_1 v^2_1, v^3_1 v^2_2, v^3_1 v^4_1\}$ will induce a monochromatic copy of $K_{1,3}$ with color 1. If $|B_2| = |B_1| - 1$, then both of the number of the center vertices in $V^1 \cup V^2$ for a copy of $(n-1)K_{1,3}$ and those in B_1 is $n - m$. If $|B_2| = |B_1| + 1$, then the number of center vertices in $V^1 \cup V^2$ is $n - m - 1$, which is less than that of center vertices of a copy of $(n-1)K_{1,3}$ in B_1. Then there is a monochromatic copy of $nK_{1,3}$ with color 1.

Case 2. $|V^1 \cup V^2| \geq n+1$ and $t = 4$. If $|V^1 \cup V^2| = 4n-3m$ for $m \geq 0$, then m center vertices from $V^1 \cup V^2$ and $n-m$ center vertices from $V(K_N) \setminus (V^1 \cup V^2)$ will induce a monochromatic copy of $nK_{1,4}$ with color 1. If $|V^1 \cup V^2| = 4n-3m+1$ for $m \geq 0$, there is a monochromatic copy of $K_{1,4}$ having color 1 in $G\left[\bigcup_{i=1}^{4} V^i\right]$. Let $v_j^2 \in V^1 \cup V^2$, $v^3 \in V^3$ and $v^4 \in V^4$ for $j = 1,2,3$. Then the edge set $\left\{v^3 v_j^2, v^3 v^4\right\}$ will induce a monochromatic copy of $K_{1,4}$ with color 1. Then m center vertices from $V^1 \cup V^2 - v_j^2$ and $n-m-1$ center vertices from $V(K_N) \setminus (V^1 \cup V^2 \cup \{v^3, v^4\})$ will induce a monochromatic copy of $(n-1)K_{1,4}$ with color 1.

If $|V^1 \cup V^2| = 4n-3m+2$ for $m \geq 0$, there is a monochromatic copy of $K_{1,4}$ having color 1 in $G\left[\bigcup_{i=1}^{4} V^i\right]$. Let $v_j^2 \in V^1 \cup V^2$, $v^3 \in V^3$ and $v_j^4 \in V^4$ for $j = 1,2$. Then the edge set $\left\{v^3 v_j^2, v^3 v_j^4\right\}$ will induce a monochromatic copy of $K_{1,4}$ with color 1. Then choose m center vertices from $V^1 \cup V^2 - v_j^2$ and $n-m-1$ center vertices from $V(K_N) \setminus \left(V^1 \cup V^2 \cup \left\{v^3, v_j^4\right\}\right)$ will induce a monochromatic copy of $(n-1)K_{1,4}$ with color 1. Hence, in this case we have a monochromatic copy of $nK_{1,4}$ with color 1.

If $\lceil N/(k-1) \rceil \leq |V^1 \cup V^2| \leq n-1$, then there is a vertex set V^l for $l \geq 3$ satisfying $\sum_{i=1}^{l-1} |V^i| \leq n-1$ and $\sum_{i=1}^{l} |V^i| > n-1$. When $\sum_{i=1}^{l} |V^i| = n$ and $\left|V(K_N) \setminus \left(\bigcup_{i=1}^{l} V^i\right)\right| = tn$, there is a monochromatic copy of $nK_{1,t}$ with color 1 by Observation 21. If $\sum_{i=1}^{l} |V^i| \geq n+1$, then we consider the following two cases.

Subcase 2.1. $\bigcup_{i=1}^{l} |V^i| \geq n+1$ and $t = 3$. If $\sum_{i=1}^{l} |V^i| - n = 2m$ for a positive integer m, then we denote $\bigcup_{i=1}^{l} V^i$ by B_1. Next, we choose m center vertices from $V(K_N) \setminus \left(\bigcup_{i=1}^{l} V^i\right)$ and $n-m$ center vertices from B_1, which will form a monochromatic copy of $nK_{1,3}$ with color 1. If $\sum_{i=1}^{l} |V^i| - n = 2m \pm 1$, we denote $\bigcup_{i=1}^{l} V^i$ by B_2. Let $v_j^i \in V^i$ for $i = 2, l, l+1$ and $j = 1,2$. Then the edge set $\left\{v_1^l v_j^2, v_1^l v_1^{l+1}\right\}$ will induce a monochromatic copy of $K_{1,3}$ with color 1. If $|B_2| = |B_1| - 1$, then both of the number of the center vertices in $\bigcup_{i=1}^{l} V^i$ for a copy of $(n-1)K_{1,3}$ and those in B_1 is $n-m$. If $|B_2| = |B_1| + 1$, then the number of center vertices in $\bigcup_{i=1}^{l} V^i$ is $n-m-1$, which is less than that of center vertices of a copy of $(n-1)K_{1,3}$ in B_1. Then there is a monochromatic copy of $nK_{1,3}$ with color 1.

Subcase 2.2. $\sum_{i=1}^{l} |V^i| \geq n+1$ and $t = 4$. If $\sum_{i=1}^{l} |V^i| = 4n-3m$ for $m \geq 0$, then m center vertices from $\bigcup_{i=1}^{l} V^i$ and $n-m$ center vertices from $V(K_N) \setminus \left(\bigcup_{i=1}^{l} V^i\right)$ will induce a monochromatic copy of $nK_{1,4}$ with color 1. If $\sum_{i=1}^{l} |V^i| = 4n-3m+1$ for $m \geq 0$, then there is a monochromatic copy of $K_{1,4}$ having color 1 in $G\left[\bigcup_{i=1}^{l+2} V^i\right]$. Let $v_j^2 \in V^1 \cup V^2$, $v^{l+1} \in V^{l+1}$ and $v^{l+2} \in V^{l+2}$ for $j = 1,2,3$. Then the edge set $\left\{v^{l+1} v_j^2, v^{l+1} v^{l+2}\right\}$ will induce a monochromatic copy of $K_{1,4}$ with color 1. Then choose m center vertices from $\bigcup_{i=1}^{l} V^i - v_j^2$ and

$n-m-1$ center vertices from $V(K_N)\setminus\left(\bigcup_{i=1}^{l}V^i\cup\{v^{l+1},v^{l+2}\}\right)$ will induce a monochromatic copy of $(n-1)K_{1,4}$ with color 1.

If $\left|\bigcup_{i=1}^{l}V^i\right|=4n-3m+2$ for $m\geq 0$, there is a monochromatic copy of $K_{1,4}$ having color 1 in $G\left[\bigcup_{i=1}^{l+2}V^i\right]$. Let $v_j^2\in V^1\cup V^2$, $v^{l+1}\in V^{l+1}$ and $v_j^{l+2}\in V^{l+2}$ for $j=1,2$. Then the edge set $\{v^{l+1}v_j^2,v^{l+1}v_j^{l+2}\}$ will induce a monochromatic copy of $K_{1,4}$ with color 1. Then m center vertices from $\bigcup_{i=1}^{l}V^i-v_j^2$ and $n-m-1$ center vertices from $V(K_N)\setminus\left(\bigcup_{i=1}^{l}V^i\cup\{v^{l+1},v_j^{l+2}\}\right)$ will induce a monochromatic copy of $(n-1)K_{1,4}$ with color 1. Hence, in this case we have a monochromatic copy of $nK_{1,4}$ with color 1. Thus, $gr_k(K_{1,3}:nK_{1,3})\leq tn+n$. ∎

Lemma 29. *For positive integers $n\geq 5$, and $4<k\leq\left\lfloor\frac{n+3}{2}\right\rfloor$, we have*

$$gr_k(K_{1,3}:nK_{1,3})=5n-1.$$

Proof. Basically, the proof is analogous to the proof of $gr_k(P_5:nK_{1,3})=5n-1$. We first prove that $gr_k(K_{1,3}:nK_{1,3})\geq 5n-2$. Let G be a k-edge-coloring of K_{5n-2}: we partition $V(G)$ into $V_1,\dots,V_{k-1}$ with $|V_2|\geq\cdots\geq|V_{k-1}|\geq 2$ such that $G[V_1]$ is a monochromatic copy of K_{4n-1} with color 2, $G[V_i]$ is colored by $i+1$ for $2\leq i\leq k-1$, and all edges between V_i and V_j for all $i\neq j\in[k-1]$ with color 1. It is easy to check that no rainbow $K_{1,3}$ appears in the graph G. Furthermore, according to $\sum_{i=2}^{k-2}|V_i|=n-1$ the graph G contains no monochromatic copy of $nK_{1,3}$. As a consequence, G contains neither a rainbow copy of $K_{1,3}$ nor a monochromatic copy of $nK_{1,3}$. So it admits $gr_k(K_{1,3}:nK_{1,3})\geq 5n-1$.

For the upper bound, let G be an arbitrary k-edge-coloring of K_N ($N\geq 5n-1$) that contains no rainbow copy of $K_{1,3}$. Suppose that G satisfies Theorem 9(c), we divide $V(G)$ into $V^1,\dots,V^k$ with $|V^2|\geq\cdots\geq|V^k|\geq 2$ ($2\leq i\leq k-1$). We immediately have that $|V^1\cup V^2|\geq\lceil N/(k-1)\rceil$.

Case 1. $4n+1\leq|V^1\cup V^2|\leq 5n+3-2k$. If $4n+1\leq|V^1\cup V^2|\leq 5n+3-2k$, then it follows from Theorem 18 that $R(nK_{1,3},(n-j)K_{1,3})=5n-j-1$ ($j=|V(K_N)\setminus(V^1\cup V^2)|$, $n\geq 2$). If there is a monochromatic copy of $(n-j)K_{1,3}$ with color 1 in $G[V^1\cup V^2]$, then it is clear that j vertices of $V(K_N)\setminus(V^1\cup V^2)$ and $3j$ vertices of $V^1\cup V^2$ induce a monochromatic copy of $jK_{1,3}$ with color 1, which will lead to a monochromatic copy of $nK_{1,3}$ with color 1 contained in G. Otherwise, $G[V^1\cup V^2]$ contains a monochromatic copy of $nK_{1,3}$ with color 2.

Case 2. $|V^1\cup V^2|=4n$ and $|V(K_N)\setminus(V^1\cup V^2)|=n-1$. If $|V^1\cup V^2|=4n$ and $|V(K_N)\setminus(V^1\cup V^2)|=n-1$, then we have that $R(nK_{1,3},K_{1,3})=4n$ by Theorem 18. We assume that there is no monochromatic copy of $nK_{1,3}$ with color 2 in $G[V^1\cup V^2]$. Then there is a monochromatic copy of $K_{1,3}$ with color 1 in $G[V^1\cup V^2]$. Next, we further choose all vertices of $V(K_N)\setminus(V^1\cup V^2)$ and all vertices of $(V^1\cup V^2)\setminus V(K_{1,3})$. It will form a monochromatic copy of $(n-1)K_{1,3}$ with color 1.

Case 3. $3n \leq |V^1 \cup V^2| \leq 4n-1$ and $n \leq |V(K_N) \setminus (V^1 \cup V^2)| \leq 2n-1$. If $3n \leq |V^1 \cup V^2| \leq 4n-1$ and $n \leq |V(K_N) \setminus (V^1 \cup V^2)| \leq 2n-1$, then by Observation 21 we have $|V(K_N) \setminus (V^1 \cup V^2)| \geq n$ and $|V^1 \cup V^2| \geq 3n$. Clearly, we find a monochromatic copy of $nK_{1,3}$ with color 1. If $2n \leq |V^1 \cup V^2| \leq 3n-1$ and $2n \leq |V(K_N) \setminus (V^1 \cup V^2)| \leq 3n-1$, then let $X = \max\{|V^1 \cup V^2|, |V(K_N) \setminus (V^1 \cup V^2)|\}$ and $Y = \min\{|V^1 \cup V^2|, |V(K_N) \setminus (V^1 \cup V^2)|\}$ with $|Y| \geq 2n-1$ and $|X| \geq 2n+1$. Now we construct a monochromatic copy of $nK_{1,3}$ with color 1. We select $\lceil n/2 \rceil$ vertices from Y and $\lfloor n/2 \rfloor$ vertices from X, and $3\lceil n/2 \rceil$ vertices from X and $3\lfloor n/2 \rfloor$ vertices from Y. It is easy to check that the selected vertices above form a monochromatic copy of $nK_{1,3}$ with color 1. If $n \leq |V^1 \cup V^2| \leq 2n-1$ and $3n \leq |V(K_N) \setminus (V^1 \cup V^2)| \leq 4n-1$, then, by Observation 21, we find a monochromatic copy of $nK_{1,3}$ with color 1. If $|V^1 \cup V^2| \leq n-1$ and $4n \leq |V(K_N) \setminus (V^1 \cup V^2)| \leq 5n-1-\lceil N/(k-1) \rceil$, then there is a vertex set V^j for $2 \leq j \leq k-1$ satisfying $\sum_{i=1}^{j+1} |V^i| \leq n-1$ and $\sum_{i=1}^{j+1} |V^i| \geq n$. Let $s = \sum_{i=1}^{j+1} |V^i|$ and $t = \sum_{i=j+2}^{k-1} |V^i|$. Since $|V^{j+1}| \leq |V^j| \leq n-1$ and Observation 21, we find a monochromatic copy of $nK_{1,3}$ with color 1 in $K_{s,t}$. If $k=4$, then it is easy to see that we find a monochromatic copy of K_{4n} with color 1 containing a monochromatic copy of $nK_{1,3}$, as desired. ■

Lemma 30. *Let n be a sufficiently large integer. We have*

$$6n-1 \leq gr_k(K_{1,3} : nK_{1,4}) \leq 9n+20,$$

for $4 \leq k \leq \lfloor \frac{n+3}{2} \rfloor$.

Proof. Since $gr_k(K_{1,3} : nK_{1,4}) = gr_k(P_5 : nK_{1,4})$, then we have $6n-1 \leq gr_k(K_{1,3} : nK_{1,4}) \leq 9n+20$ by Lemma 25. ■

Lemma 31. *For positive integers $n \geq 2$ and $k \geq \lfloor \frac{tn+n+1}{2} \rfloor + 1$, we have*

$$gr_k(K_{1,3} : nK_{1,t}) = \left\lceil \frac{1+\sqrt{1+8k}}{2} \right\rceil.$$

Proof. Let $M = \left\lceil \frac{1+\sqrt{1+8k}}{2} \right\rceil$. For the lower bound, we have that $gr_k(K_{1,3} : nK_{1,t}) \geq M$, because every color occurs at least once. Hence, $k \leq |E(K_M - 1)| = \binom{M-1}{2}$, contradicting the fact that $M = \left\lceil \frac{1+\sqrt{1+8k}}{2} \right\rceil$.

For the upper bound, let G be an arbitrary exact k-edge-colored K_M without a rainbow copy of $K_{1,3}$ for $N \geq M$. We divide $V(G)$ into $V_1, \ldots, V_k$. The coloring satisfies Theorem 9(c). Without loss of generality, we assume that $|V^i| \geq |V^{i+1}| \geq 2$ $(2 \leq i \leq k-1)$. Hence, there are only edges of color 1 among the parts. Since $k \geq 2n+1$ and $N \geq 2k-2 \geq tn+n$, the subgraph of $G[V^i]$ $(2 \leq i \leq k-1)$ contains a monochromatic copy of $nK_{1,t}$. ■

5. Proof of Theorem 5

In this section, we give the proof of Theorem 5 by Lemmas 32 and 33.

Lemma 32 [35]. *For integers $k \geq 5$ and $gr_k(K_{1,3} : H) \geq 5$, we have*

$$gr_k(K_{1,3} : H) = gr_k(P_4^+ : H).$$

Lemma 33. *For positive integers $n \geq 2$ and $t = 3, 4$, we have*

$$gr_4(P_4^+ : nK_{1,t}) = tn + n + 2.$$

Proof. Let G_1 be a k-edge-coloring of K_{tn+n-1} with color 1 and x, y be two isolated vertices. Assign color 3 to all edges between $V(G_1)$ and x, color 4 to all edges between $V(G_1)$ and y, and color 2 to xy. We obtain a 4-edge-colored K_{tn+n+1} that contains neither a rainbow copy of P_4^+ nor a monochromatic copy of $nK_{1,t}$.

For the upper bound, let G be a 4-edge-colored K_N where $N \geq tn + n + 2$. If G contains no rainbow copy of P_4^+, then by Theorem 7(b) we get $G \in \{G_2(N), G_3(N)\}$. Clearly, there is a monochromatic copy of $nK_{1,t}$ in G. If $G \cong G_2(N)$, then unlike the lower bound we constructed. There is a monochromatic copy of $nK_{1,t}$ with color 1 of G_1. Let $G_3(N)$ be a 4-edge-coloring of K_{tn+n+2} in which there is a rainbow K_3 having colors $2, 3$ and 4. Suppose that every edge is incident with at most one vertex in the rainbow K_3 with color 1. Define the three vertices of this rainbow K_3 as a, b, c. Let G_2 be a monochromatic copy of K_{tn+n-1} with color 1. Taking the three vertices of a, b, c as the three center vertices of monochromatic $3K_{1,t}$, the edges connected to $V(G_2)$ form a monochromatic copy of $3K_{1,t}$ with color 1. Then there are $|V(G_2)| = tn + n - 3t - 1$ vertices which can form a monochromatic copy of $(n-3)K_{1,t}$ with color 1. Consequently, we have that $gr_4(P_4^+ : nK_{1,t}) \leq tn + n + 2$. ■

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