Discussiones Mathematicae
Graph Theory 45 (2025) 915–925
https://doi.org/10.7151/dmgt.2565

COMPLEMENTARY COALITION GRAPHS: CHARACTERIZATION AND ALGORITHM

Davood Bakhshesh

Department of Computer Science
University of Bojnord
Bojnord, Iran

e-mail: d.bakhshesh@ub.ac.ir

and

Michael A. Henning

Department of Mathematics and Applied Mathematics
University of Johannesburg
Auckland Park, South Africa

e-mail: mahenning@uj.ac.za

Abstract

A set S of vertices in a graph G is a dominating set if every vertex of $V(G) \setminus S$ is adjacent to a vertex in S. A coalition in G consists of two disjoint sets of vertices X and Y of G, neither of which is a dominating set but whose union $X \cup Y$ is a dominating set of G. Such sets X and Y form a coalition in G. A coalition partition, abbreviated c-partition, in G is a partition $\mathfrak{X} = \{X_1, \ldots, X_k\}$ of the vertex set $V(G)$ of G such that for all $i \in [k]$, each set $X_i \in \mathfrak{X}$ satisfies one of the following two conditions: (1) X_i is a dominating set of G with a single vertex, or (2) X_i forms a coalition with some other set $X_j \in \mathfrak{X}$. Given a coalition partition $\mathfrak{X}$ of a graph G, a coalition graph $\mathrm{CG}(G, \mathfrak{X})$ is constructed by representing each member of $\mathfrak{X}$ as a vertex of the graph, and joining two vertices with an edge if and only if the corresponding sets form a coalition in G. If each set in a coalition partition $\mathfrak{X}$ of G contains only one vertex, then $\mathfrak{X}$ is referred to as a singleton coalition partition. A graph G is a complementary coalition graph if $\mathrm{CG}(G, \mathfrak{X})$ is isomorphic to the complement of G. We characterize complementary coalition graphs. This solves an open problem posed by Haynes *et al.* [Commun. Comb. Optim. 8 (2023) 423–430]. Moreover, we provide a polynomial-time algorithm that determines if a given graph is a complementary coalition graph.

Keywords: coalition number, domination number, coalition partition, coalition graphs.

2020 Mathematics Subject Classification: 05C69.

1. Introduction

A set S of vertices in a graph G is a *dominating set* if every vertex in $V(G) \setminus S$ is adjacent to a vertex in S. If $X, Y \subseteq S$, then set X *dominates* the set Y if every vertex $y \in Y$ belongs to X or is adjacent to a vertex of X. The study of domination in graphs is an active area of research in graph theory. A thorough treatment of this topic can be found in recent so-called "domination books" [6–8].

For graph theory notation and terminology, we generally follow [8]. Specifically, let $G = (V(G), E(G))$ be a graph with vertex set $V(G)$ and edge set $E(G)$, and of order $n(G) = |V(G)|$ and size $m(G) = |E(G)|$. Two adjacent vertices in G are *neighbors*. The *open neighborhood*, denoted by $N_G(v)$, of a vertex v in G is the set of all neighbors of v, and the *closed neighborhood of* v is $N_G[v] = \{v\} \cup N_G(v)$. We denote the *degree* of v in G by $\deg_G(v)$, and so $\deg_G(v) = |N_G(v)|$. The minimum and maximum degrees in G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. An *isolated vertex* in G is a vertex of degree 0 in G. A graph is *isolate-free* if it contains no isolated vertex. A vertex v is a *universal vertex* of G, also called a *full vertex* in the literature, if $N_G[v] = V(G)$, that is, $\deg_G(v) = n(G) - 1$. Given a graph G, we let U_G be the set of all universal vertices in G. If the graph G is clear from the context, we simply write V, E, n, m, $\deg(v)$, $N(v)$, and $N[v]$ rather than $V(G)$, $E(G)$, $n(G)$, $m(G)$, $\deg_G(v)$, $N_G(v)$, and $N_G[v]$, respectively.

We denote a *path* and *cycle* on n vertices by P_n and C_n, respectively, and we denote a *complete graph* on n vertices by K_n. A *complete bipartite graph* with partite sets of cardinalities r and s we denote by $K_{r,s}$. A *star* is a complete bipartite graph $K_{1,s}$ where $s \geq 2$. A *nontrivial tree* is a tree of order at least 2. A partition of a set is a grouping of its elements into non-empty subsets, in such a way that every element of the set is included in exactly one subset. For a set $S \subseteq V(G)$, the subgraph induced by S is denoted by $G[S]$. The *join* of two graphs G and H, denoted by $G + H$, is a graph formed by taking the disjoint union of G and H and adding an edge between every vertex in G and every vertex in H. The *union* of two graphs G and H, denoted by $G \cup H$, is the graph formed by taking the disjoint union of G and H.

A *coalition* in a graph G consists of two disjoint sets of vertices X and Y of G, neither of which is a dominating set but whose union $X \cup Y$ is a dominating set of G. Such sets X and Y *form a coalition* in G. A *coalition partition*, called a *c-partition*, in G is a partition $\mathfrak{X} = \{X_1, \ldots, X_k\}$ of $V(G)$ such that for all $i \in [k]$, the set X_i is either a singleton dominating set (that is, a dominating set that consists of a single vertex) or forms a coalition with another set X_j for some j, where $j \in [k] \setminus \{i\}$. The *coalition number*, $C(G)$, in G is the maximum cardinality of a c-partition of G.

If each set in a coalition partition $\mathfrak{X}$ of a graph G contains only one vertex, then $\mathfrak{X}$ is referred to as a *singleton coalition partition*, denoted by $\mathfrak{X}_1$-*partition*. Given a coalition partition $\mathfrak{X}$ of a graph G, a coalition graph $\mathrm{CG}(G, \mathfrak{X})$ is constructed by representing each member of $\mathfrak{X}$ as a vertex of the graph, and joining two vertices with an edge if and only if the corresponding sets form a coalition in G.

Coalitions in graphs were introduced and first studied by Haynes, Hedetniemi, Hedetniemi, McRae, and Mohan [2], and have subsequently been studied, for example, in [1, 3–5]. In [2], the authors determined the coalition number of paths and cycles, and in [4] they presented some upper bounds on the coalition number of graphs. Isolate-free graphs G of order n that satisfy $C(G) = n$ are characterized in [1]. Further, all trees T of order n with $C(T) = n - 1$ are characterized in [1]. In [5], Haynes *et al.* showed that any graph can be a coalition graph. Moreover, they defined a graph G to be a *complementary coalition graph*, abbreviated CC-*graph*, if $\mathrm{CG}(G, \mathfrak{X})$ is isomorphic to the complement of G, denoted by $\overline{G}$. Haynes *et al.* [5] posed the following open problem.

Problem 1 [5]**.** Characterize complementary coalition graphs.

In this paper, we provide a complete characterization of CC-graphs. Moreover, we present a cubic-time algorithm to determine if a given graph is a CC-graph.

2. CC-Graphs

In this section, we characterize all CC-graphs. Suppose G is a CC-graph, and let x be a vertex in G. We use $\overline{x}$ to denote the vertex in $\mathrm{CC}(G, \mathfrak{X}_1)$ that corresponds to the set $\{x\}$ in $\mathfrak{X}_1$-partition of G.

Now, we present the following proposition.

Proposition 2. *A graph G of order $n \geq 2$ and with no universal vertex is a* CC*-graph if and only if for every two distinct vertices $x, y \in V(G)$, the following conditions hold.*

(a) *If $xy \in E(G)$, then $\{x, y\}$ is not a dominating set of G.*

(b) *If $xy \notin E(G)$, then $\{x, y\}$ is a dominating set of G.*

Proof. Let G be a graph of order $n \geq 2$ and with no universal vertex. Suppose firstly that G is a CC-graph, and so $\mathrm{CG}(G, \mathfrak{X})$ is isomorphic to the complement, $\overline{G}$, of G, that is, $\mathrm{CG}(G, \mathfrak{X}_1) \cong \overline{G}$. Since G has order n we infer that $C(G) = n$.

Let x and y be two distinct vertices that belong to $V(G)$. If $xy \in E(G)$, then $xy \notin E(\overline{G})$, and so the vertices x and y are not adjacent in $\mathrm{CG}(G, \mathfrak{X}_1)$, implying that $\{x, y\}$ is not a dominating set of G. On the other hand, if $xy \notin E(G)$, then $xy \in E(\overline{G})$, and so the vertices x and y are adjacent in $\mathrm{CG}(G, \mathfrak{X}_1)$, implying that $\{x, y\}$ is a dominating set of G.

Conversely, suppose that the conditions (a) and (b) hold for every two distinct vertices x and y that belong to the graph G. We show that G is a CC-graph. We show firstly that $C(G) = n$. Since G has no universal vertex, every vertex in G is not adjacent to at least one other vertex in G. Let x be an arbitrary vertex of G, and let y be such a vertex distinct from x that is not adjacent to x, that is, $xy \notin E(G)$. By condition (b), $\{x, y\}$ is a dominating set of G, implying that $\{x\}$ forms a coalition with $\{y\}$. From this property we infer that $C(G) = n$. Now let $H = \mathrm{CG}(G, \mathfrak{X}_1)$, and consider any two distinct vertices u and v of G. If $uv \in E(G)$, then by condition (a), $\{u, v\}$ is not a dominating set of G. Therefore, $\{u\}$ and $\{v\}$ do not form a coalition, which means that $\overline{u}\,\overline{v} \notin E(H)$. If $uv \notin E(G)$, then by condition (b), $\{u, v\}$ is a dominating set of G. Therefore, $\{u\}$ and $\{v\}$ form a coalition, which means that $\overline{u}\,\overline{v} \in E(H)$. Thus, we have shown that $H \cong \overline{G}$. Therefore, G is a CC-graph. ■

If G is a CC-graph that contains a universal vertex u, then the set $\{u\}$ is a dominating set of G. Consequently, the vertex $\overline{u}$ is isolated in the complementary coalition graph $\mathrm{CG}(G, \mathfrak{X}_1)$. The following result follows as a consequence of Proposition 2.

Proposition 3. *If G is a graph and $|U_G| \geq 1$, then G is* CC*-graph if and only if $G[V(G) \setminus U_G]$ is a* CC*-graph.*

Now, we define the family $\mathcal{F}$.

Definition 4. Let $\mathcal{F}$ be the family of all graphs G with no universal vertices such that for every vertex w in G, the following conditions hold.

(a) The induced subgraph $G[V(G) \setminus N[w]]$ is a complete graph.

(b) Every vertex in $N(w)$ is not adjacent in G to at least one vertex of $V(G) \setminus N[w]$.

Figure 1 shows two graphs of the family $\mathcal{F}$.

We are now in a position to characterize complementary coalition graphs that do not contain a universal vertex.

Theorem 5. *A graph G with no universal vertex is a* CC*-graph if and only if $G \in \mathcal{F}$.*

Proof. Let $G = (V, E)$ be a graph with no universal vertex. Suppose firstly that $G \in \mathcal{F}$. We show that G is a CC-graph. Let $x, y \in V$ be two distinct vertices of

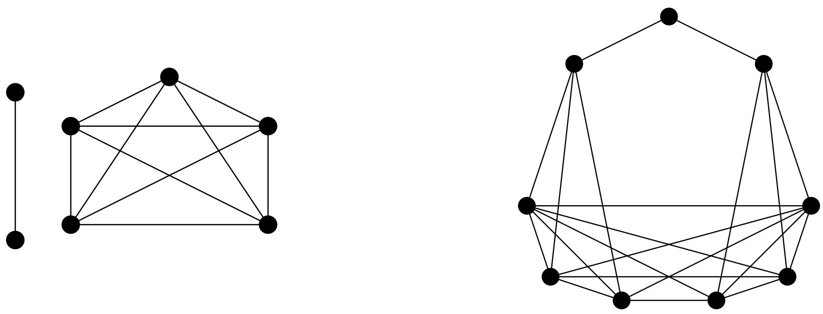

Figure 1. Two graphs of the family $\mathcal{F}$.

G. Assume that $xy \in E$. Since $G \in \mathcal{F}$, there is a vertex $u \in V \setminus N[x]$ which is not adjacent to y. Thus, $uy \notin E$ and $ux \notin E$. Therefore, $\{x, y\}$ is not a dominating set of G. Thus, condition (a) in Proposition 2 is satisfied. Now, assume that $xy \notin E$. Then, $y \in V \setminus N[x]$. Since $G \in \mathcal{F}$, the induced subgraph $G[V \setminus N[x]]$ is a complete graph. Hence the set $\{x, y\}$ is a dominating set of G. Thus, condition (b) in Proposition 2 is satisfied. Since both conditions (a) and (b) in Proposition 2 are satisfied, the graph G is a CC-graph by Proposition 2.

Conversely, suppose next that G is a CC-graph. Let $w \in V$ be an arbitrary vertex, and let $X = V \setminus N[w]$. If $X = \emptyset$, then w is a universal vertex of G, a contradiction. Hence, $X \neq \emptyset$. We show that $G[X]$ is a complete graph. If $|X| = 1$, then the result is immediate since in this case $G[X] \cong K_1$. Thus we assume that $|X| \geq 2$. Let x be an arbitrary vertex of X. Since $x \notin N[w]$, the vertices w and x are distinct and not adjacent in G, and consequently by Proposition 2, $\{w, x\}$ is a dominating set of G. Since w has no neighbor in X, all vertices of $X \setminus \{x\}$ must be adjacent to x. Thus since x is an arbitrary vertex of X, we infer that $G[X]$ is a complete graph. Thus, condition (a) in Definition 4 is satisfied. We show next that condition (b) in Definition 4 is satisfied. Let y be an arbitrary vertex in $N(w)$. By Proposition 2, $\{w, y\}$ is not a dominating set of G, implying that there must exist a vertex z that is adjacent to neither w nor y. Thus, $z \in X$ and the vertex $y \in N(w)$ is not adjacent to z. Thus, condition (b) in Definition 4 is satisfied. Therefore $G \in \mathcal{F}$. ■

2.1. CC-graphs with small minimum degree

As a consequence of Theorem 5 we provide next an exact characterization of all CC-graphs that are trees.

Theorem 6. *A tree T is a* CC*-graph if and only if T is a path of order at most* 3.

Proof. It is immediate to verify that the paths P_1, P_2, and P_3 are CC-graphs.

Let $T = (V, E)$ be a tree of order $n \geq 1$ that is a CC-graph. If $n \leq 3$, then T is a path of order at most 3, as claimed. Hence we may assume that $n \geq 4$, for otherwise the desired result follows. Suppose that T has a universal vertex. In this case, T is isomorphic to a star $K_{1,s}$ where $s = n - 1 \geq 3$. By Proposition 3, the tree $T[V \setminus U_T]$ is also a CC-graph. Since $T[V \setminus U_T]$ is isomorphic to the empty graph $\overline{K}_s$, which is a CC-graph only when $s \leq 2$, we infer that $n = s + 1 \leq 3$, contradicting our assumption that $n \geq 4$. Hence, T has no universal vertex. In this case, applying Theorem 5 we infer that $T \in \mathcal{F}$. Let w be a leaf in T, and so w has degree 1 in T, and let u be the (unique) neighbor of w. Further, let $X = V \setminus \{u, w\}$. Thus, $n = |X| + 2 \geq 4$, and so $|X| \geq 2$. By condition (a) in Definition 4, the subtree $T[X]$ is a complete graph, implying that $|X| \leq 2$. Consequently, $|X| = 2$. Since T is a tree, the vertex u is adjacent to at least one vertex in X. By condition (b) in Definition 4, the vertex u is not adjacent to at least one vertex in X. From these properties, we infer that $T \cong P_4$. However, P_4 is not a CC-graph, which completes the proof. ∎

Moreover as a consequence of Theorem 5, we provide exact characterizations of CC-graphs with small minimum degree.

Theorem 7. *A graph G of order $n \geq 2$ and $\delta(G) = 0$ is a* CC*-graph if and only if $G \cong K_1 \cup K_{n-1}$.*

Proof. Let $G = (V, E)$ be a graph of order $n \geq 2$ with $\delta(G) = 0$. Suppose firstly that G is a CC-graph. By Theorem 5, $G \in \mathcal{F}$. Let x be an isolated vertex in G. By condition (a) in Definition 4, $G[V \setminus x]$ is a complete graph, and so $G[V \setminus x] \cong K_{n-1}$, implying that $G \cong K_1 \cup K_{n-1}$. Conversely, suppose next that $G \cong K_1 \cup K_{n-1}$. Thus, G has no universal vertex. Moreover both conditions (a) and (b) in Definition 4 are satisfied for every vertex in G. Therefore, by Theorem 5 we infer that G is a CC-graph. ∎

Theorem 8. *A graph G of order $n \geq 3$ and $\delta(G) = 1$ that contains a universal vertex is a* CC*-graph if and only if $G \cong (K_1 \cup K_{n-2}) + K_1$.*

Proof. Let $G = (V, E)$ be a graph of order $n \geq 3$ with $\delta(G) = 1$ that contains a universal vertex. Suppose that $G \cong (K_1 \cup K_{n-2}) + K_1$. We show that G is a CC-graph. Let z be the unique universal vertex of G. Let $G' = G[V \setminus z]$, and so $G' \cong K_1 \cup K_{n-2}$. The graph G' has minimum degree $\delta(G') = 0$. According to Proposition 3, G is a CC-graph if and only if G' is a CC-graph. Applying Theorem 7, the graph G' is a CC-graph. Therefore, G is a CC-graph, as claimed.

Conversely, suppose next that G is a CC-graph. Since $\delta(G) = 1$, the graph G of order $n \geq 3$ has a unique universal vertex. Let z be the unique universal vertex of G. Let $G' = G - z$. By Proposition 3, the graph G' is a CC-graph. Moreover, $\delta(G') = 0$. Hence, by Theorem 7, we infer that $G' \cong K_1 \cup K_{n-2}$ noting that the order of G' is $n - 1$. Consequently, $G \cong (K_1 \cup K_{n-2}) + K_1$. ∎

Theorem 9. *A graph G of order $n \geq 3$ and $\delta(G) = 1$ that contains no universal vertex is a* CC*-graph if and only if $G \cong K_2 \cup K_{n-2}$.*

Proof. Let $G = (V, E)$ be a graph of order $n \geq 3$ with $\delta(G) = 1$ that contains no universal vertex. Suppose $G \cong K_2 \cup K_{n-2}$. By Definition 4, the graph G belongs to the family $\mathcal{F}$. Since G has no universal vertices, Theorem 5 implies that G is a CC-graph.

Conversely, suppose that G is a CC-graph. By Theorem 5, $G \in \mathcal{F}$. Let w be a vertex in G with degree $\delta(G) = 1$, and let $N(w) = \{v\}$. Let $X = V \setminus \{v, w\}$ and let $G' = G[X]$. Since $G \in \mathcal{F}$, the subgraph G' is a complete graph of order $n' = n - 2 \geq 1$. Suppose that $\deg_G(v) \geq 2$, and let u be a neighbor of v distinct from w. Let $U = V \setminus N_G[u]$. Since $G \in \mathcal{F}$, the subgraph $G[U]$ is a complete graph. We note that $w \in U$ since $uw \notin E$. Since the vertex v is the only neighbor of w, it follows that $U = \{w\}$. This implies that the vertex $v \in N_G(u)$ is adjacent to every vertex in U, which contradicts condition (b) in Definition 4. Therefore, $\deg_G(v) = 1$. Thus, G is a disconnected graph consisting of two components, namely a K_2-component that contains the edge vw and a component consisting of the subgraph G'. Hence, $G \cong K_2 \cup K_{n-2}$. ■

We note that a graph G with $\delta(G) = 2$ contains at most two universal vertices. We prove next the following theorem.

Theorem 10. *A graph G of order $n \geq 4$ and $\delta(G) = 2$ with $1 \leq |U_G| \leq 2$ is a* CC*-graph if and only if one of the following hold.*

(a) $G \cong K_1 + (K_2 \cup K_{n-3})$, *if* $|U_G| = 1$.

(b) $G \cong K_2 + (K_1 \cup K_{n-3})$, *if* $|U_G| = 2$.

Proof. Let $G = (V, E)$ be a graph of order $n \geq 4$ with $\delta(G) = 2$. Suppose firstly that $|U_G| = 1$. Let $U_G = \{x\}$ and let $G' = G - x$. We note that the graph G' has no universal vertex and satisfies $\delta(G') = 1$. By Theorem 9, G' is a CC-graph if and only if $G' \cong K_2 \cup K_{n-3}$. By Proposition 3, the graph G is a CC-graph if and only if the graph G' is a CC-graph. From this we infer that G is a CC-graph if and only if $G \cong K_1 + (K_2 \cup K_{n-3})$. This proves part (a) in the statement of the theorem.

Suppose next that $|U_G| = 2$. Let $U_G = \{x, y\}$ and in this case, let $G' = G - \{x, y\}$. We note that the graph G' has order $n - 2$. Further, G' has no universal vertex and satisfies $\delta(G') = 0$. By Theorem 7, G' is a CC-graph if and only if $G' \cong K_1 \cup K_{n-3}$. We therefore infer that G is a CC-graph if and only if $G \cong K_2 + (K_1 \cup K_{n-3})$. This proves part (b) in the statement of the theorem.

Conversely, it is straightforward to verify that if $G \cong K_1 + (K_2 \cup K_{n-3})$, then $|U_G| = 1$ and G is a CC-graph, and that if $G \cong K_2 + (K_1 \cup K_{n-3})$, then $|U_G| = 2$ and G is a CC-graph. ■

Let $\mathcal{C}$ be the family of all graphs $G = (V, E)$ such that $V = \{s, p, q\} \cup P \cup Q$, where s is a vertex of degree 2, $N(s) = \{p, q\}$, $N(p) = P \cup \{s\}$, $N(q) = Q \cup \{s\}$, $P \cap Q = \emptyset$, and $G[P \cup Q] \cong K_{n-3}$. A graph in the family $\mathcal{C}$ is illustrated in Figure 2.

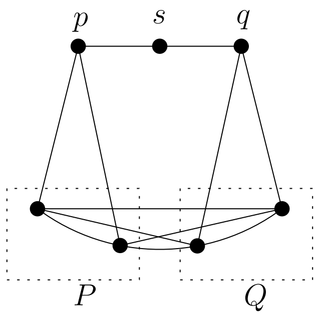

Figure 2. A graph of $\mathcal{C}$.

Theorem 11. *A graph G of order $n \geq 4$ and $\delta(G) = 2$ that contains no universal vertex is a* CC*-graph if and only if $G \cong K_3 \cup K_{n-3}$ or $G \in \mathcal{C}$.*

Proof. Let $G = (V, E)$ be a graph of order $n \geq 4$ with minimum degree $\delta(G) = 2$ that contains no universal vertex. It is straightforward to verify that if $G \cong K_3 \cup K_{n-3}$ or $G \in \mathcal{C}$, then G is a CC-graph. Suppose next that G is a CC-graph. We show that $G \cong K_3 \cup K_{n-3}$ or $G \in \mathcal{C}$. Let s be a vertex in G of minimum degree, and so $\deg(s) = 2$. Further, let $N(s) = \{p, q\}$. Let $X = V \setminus N[s]$ and let $G' = G[X]$. By supposition, G is a CC-graph. Hence by Theorem 5, $G \in \mathcal{F}$, implying that G' is a complete graph K_{n-3}.

Suppose that p and q are adjacent in G. We show that in this case neither p nor q is adjacent in G to any vertex that belongs to X. Suppose, to the contrary, that there is an edge joining X to a vertex in $\{p, q\}$. Renaming vertices if necessary, we may assume that vertex p is adjacent to a vertex $x \in X$. Since $G \in \mathcal{F}$, for any $u \in N(x)$, there exists at least one vertex in $V \setminus N[x]$ that is not adjacent to u. If $q \notin N(x)$, then we have $V \setminus N[x] = \{s, q\}$. However, since p is adjacent to both s and q while also being in $N(x)$, we have a contradiction. On the other hand, if $q \in N(x)$, then we have $V \setminus N[x] = \{s\}$. But again, this leads to a contradiction since p is adjacent to s and also belongs to $N(x)$. Hence, neither p nor q is adjacent to any vertex that belongs to X. As observed earlier, $G' \cong K_{n-3}$. Therefore, $G \cong K_3 \cup K_{n-3}$.

Suppose next that p and q are not adjacent in G. Suppose that p and q have a common neighbor, z say, different from s, and so $z \in X$ and $\{p, q\} \subseteq N(z)$. By our earlier observations, the vertex z is adjacent to every vertex of G, except for the vertex s, and so $V \setminus N[z] = \{s\}$. Since $G \in \mathcal{F}$, every vertex $u \in N(z)$ is not adjacent to at least one vertex of $V \setminus N[z]$. However, this contradicts the fact that both p and q are adjacent to the vertex s. Therefore, the vertex s is the only common neighbor of p and q in the graph G. Let P and Q be the set of all vertices in X that are adjacent to p and q, respectively. By our earlier observations, $P \cap Q = \emptyset$, $N_G(p) = P \cup \{s\}$, and $N_G(q) = Q \cup \{s\}$. If there is a vertex $u \in X$ that belongs to neither P nor Q, then since G' is a complete graph, we infer that $V \setminus N[u] = \{p, q, s\}$. However, $pq \notin E(G)$, and so $G[V \setminus N[u]]$ is not a complete graph, contradicting condition (a) in Definition 4. Hence, $G[P \cup Q] \cong K_{n-3}$. From these properties of G, we infer that $G \in \mathcal{C}$. ■

3. Algorithm

It seems that the decision problem related to the computation of the coalition number of a given graph is NP-hard. Here, we present a cubic-time algorithm that determines whether a given graph G is a CC-graph. It is remarkable that for any CC-graph G of order n, $C(G) = n$.

For a given graph G of order n, we define an $n^2 \times n$ matrix $\mathcal{D}$ as follows

$$\mathcal{D}(\{a,b\}, x) = \begin{cases} 1 & \text{if the vertex } x \text{ has a neighbor in } \{a,b\}, \\ 0 & \text{otherwise.} \end{cases}$$

Note that if $x \in \{a, b\}$, then we assume that $\mathcal{D}(\{a,b\}, x) = 1$. Now, based on the definition of the matrix $\mathcal{D}$ and Proposition 2, we conclude the following theorem.

Theorem 12. *A graph $G = (V, E)$ of order n with no universal vertex is a* CC*-graph if and only if the following conditions hold.*

(a) *For every pair $xy \in E$, we have $\sum_{v \in V} \mathcal{D}(\{x,y\}, v) < n$, and*

(b) *for every pair $xy \notin E$, we have $\sum_{v \in V} \mathcal{D}(\{x,y\}, v) = n$.*

Now, we present an algorithm that determines whether a given graph G is a CC-graph. The algorithm proceeds as follows: first, it identifies all universal vertices of G and adds them to the set U_G. Next, it considers the induced subgraph $G'(V', E') = G[V \setminus U_G]$ and computes the matrix $\mathcal{D}$ for G'. Then, for every two vertices $x, y \in V$ with $x \neq y$, it applies Theorem 12. For more details, see Algorithm 1.

The running time of Algorithm 1 is $O(n^3)$ when implemented naively. This is because U_G, G', n' consume $O(n)$ time, and calculating $\mathcal{D}$ consumes $O(n^3)$ time. All if-statements takes $O(n)$ time, and consequently the two loops have an overall running time of $O(n^3)$. Hence, the overall running time of the algorithm is $O(n^3)$. We state this formally as follows.

Theorem 13. *The algorithm* CCG *can determine if a given graph G of order n is a* CC*-graph in $O(n^3)$ time.*

Algorithm 1: $\mathrm{CCG}(G, V, E)$

input: A graph G, and with vertex set V and the edge set E.
output: Return "yes" if G is a CC-graph, and return "no" if G is not a CC-graph.

$U_G :=$ the set of all universal vertices of G;
$G'(V', E') := G[V \setminus U_G]$;
$n' := |V| - |U_G|$;
Compute the matrix $\mathcal{D}$ for G';
$flag := 0$;
foreach $x \in G'$ **do**
  foreach $y \in G'$ *with* $x \neq y$ **do**
    if $(x, y) \in E$ **then**
      if $\sum_{v \in V'} \mathcal{D}(\{x, y\}, v) < n'$ **then**
        $flag := 1$;
      end
    end
    else
      if $\sum_{v \in V'} \mathcal{D}(\{x, y\}, v) = n'$ **then**
        $flag := 1$;
      end
    end
    if $flag = 0$ **then**
      return "no";
    end
    $flag = 0$;
  end
end
return "yes";

4. Conclusion

In this paper, we have addressed the problem of characterizing complementary coalition graphs and have provided a cubic-time algorithm to determine if a given graph is a complementary coalition graph. Our main result provides a solution to an open problem that was posed by Haynes *et al.* [5]. As future works, it would be interesting to propose a quadratic or linear time algorithm to determine if a given graph is a CC-graph.

Acknowledgments

Research of Michael A. Henning was supported in part by the South African National Research Foundation (grants 132588, 129265) and the University of Johannesburg.

References

[1] D. Bakhshesh, M.A. Henning and D. Pradhan, *On the coalition number of trees*, Bull. Malays. Math. Sci. Soc. **46** (2023) 95.
https://doi.org/10.1007/s40840-023-01492-4

[2] T.W. Haynes, J.T. Hedetniemi, S.T. Hedetniemi, A.A. McRae and R. Mohan, *Introduction to coalitions in graphs*, AKCE Int. J. Graphs Comb. **17** (2020) 653–659.
https://doi.org/10.1080/09728600.2020.1832874

[3] T.W. Haynes, J.T. Hedetniemi, S.T. Hedetniemi, A.A. McRae and R. Mohan, *Coalition graphs of paths, cycles and trees*, Discuss. Math. Graph Theory **43** (2023) 931–946.
https://doi.org/10.7151/dmgt.2416

[4] T.W. Haynes, J.T. Hedetniemi, S.T. Hedetniemi, A.A. McRae and R. Mohan, *Upper bounds on the coalition number*, Australas. J. Combin. **80** (2021) 442–453.

[5] T.W. Haynes, J.T. Hedetniemi, S.T. Hedetniemi, A.A. McRae and R. Mohan, *Coalition graphs*, Commun. Comb. Optim. **8** (2023) 423–430.
https://doi.org/10.22049/cco.2022.27916.1394

[6] T.W. Haynes, S.T. Hedetniemi and M.A. Henning, Topics in Domination in Graphs, Dev. Math. **64** (Springer, Cham, 2020).
https://doi.org/10.1007/978-3-030-51117-3

[7] T.W. Haynes, S.T. Hedetniemi and M.A. Henning, Structures of Domination in Graphs, Dev. Math. **66** (Springer, Cham, 2021).
https://doi.org/10.1007/978-3-030-58892-2

[8] T.W. Haynes, S.T. Hedetniemi and M.A. Henning, Domination in Graphs: Core Concepts, Springer Monogr. Math. (Springer, Cham, 2023).
https://doi.org/10.1007/978-3-031-09496-5

Received 16 February 2024
Revised 22 September 2024
Accepted 23 September 2024
Available online 11 October 2024