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EQUITABLE CLUSTER PARTITION OF GRAPHS WITH SMALL MAXIMUM AVERAGE DEGREE [1]

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Abstract

An equitable $\underbrace{(\mathcal{O}_k, \mathcal{O}_k, \ldots, \mathcal{O}_k)}_{m}$-partition of a graph G, which is also called an equitable k cluster m-partition, is the partition of $V(G)$ into m non-empty subsets V_1, V_2, $\ldots$, V_m such that for every integer i in $\{1, 2, \ldots, m\}$, $G[V_i]$ is a graph with components of order at most k, and for each pair of distinct i, j in $\{1, \ldots, m\}$, there is $-1 \leq |V_i| - |V_j| \leq 1$. In this paper, we proved that every graph G with minimum degree $\delta(G) \geq 2$ and maximum average degree $mad(G) < \frac{8}{3}$ admits an equitable $\underbrace{(\mathcal{O}_6, \mathcal{O}_6, \ldots, \mathcal{O}_6)}_{m}$-partition, for any integer $m \geq 3$.

Keywords: equitable cluster partition, maximum average degree, discharging.

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1. Introduction

All graphs considered in this paper are finite, simple and undirected. For a graph G, we use $V(G)$ to denote the vertex set. An *equitable k-partition* of a graph G is a partition of $V(G)$ into $(V_1, \ldots, V_k)$ such that $-1 \leq |V_i| - |V_j| \leq 1$ for all

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$1 \leq i < j \leq k$, a k-partition is *ascending equitable* if $|V_1| \leq |V_2| \leq \cdots \leq |V_k| \leq |V_1| + 1$ and *descending equitable* if $|V_1| \geq |V_2| \geq \cdots \geq |V_k| \geq |V_1| - 1$. Let $\mathcal{G}_i$ be a class of graphs for $1 \leq i \leq k$. Given a graph G, an *equitable* $(\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_k)$-*partition* of graph G is an equitable k-partition of G such that for all $1 \leq i \leq k$, the induced subgraph $G[V_i]$ belongs to $\mathcal{G}_i$.

The $\mathcal{G}$-*equitable partition number* of a graph G, denoted by $\chi_{e\mathcal{G}}(G)$, is the smallest integer k such that G has an equitable $(\mathcal{G}_1, \ldots, \mathcal{G}_k)$-partition with $\mathcal{G}_1 = \mathcal{G}_2 = \cdots = \mathcal{G}_k = \mathcal{G}$. In contrast to the ordinary vertex partition, a graph may have an equitable $(\mathcal{G}_1, \ldots, \mathcal{G}_k)$-partition, but no equitable $(\mathcal{G}_1, \ldots, \mathcal{G}_k, \mathcal{G}_{k+1})$-partition with $\mathcal{G}_1 = \cdots = \mathcal{G}_k = \mathcal{G}_{k+1} = \mathcal{G}$. The $\mathcal{G}$-*equitable partition threshold* of G, denoted by $\chi^*_{e\mathcal{G}}(G)$, is the smallest integer k such that G has an equitable $(\mathcal{G}_1, \ldots, \mathcal{G}_m)$-partition for all $m \geq k$ with $\mathcal{G}_1 = \mathcal{G}_2 = \cdots = \mathcal{G}_m = \mathcal{G}$.

It is clear that $\chi_{e\mathcal{G}}(G) \leq \chi^*_{e\mathcal{G}}(G)$. In fact, the gap between the two parameters can be arbitrarily large. Let $\mathcal{I}$, $\mathcal{O}_k$, $\mathcal{T}_k$ denote the class of edge-less graphs, the class of graphs each of whose components has order at most k, the class of trees whose components have order at most k, respectively. $\mathcal{I}$-partition is equivalent to proper vertex coloring of graphs. And $\mathcal{O}_k$-partition relax the requirement that adjacent vertices are put in distinct vertex sets. Compared with vertex coloring, we say that a partition of G has cluster k if each of $V_1, \ldots, V_m$ induces a graph each of whose components has at most k vertices, and denote such a partition by $\underbrace{(\mathcal{O}_k, \mathcal{O}_k, \ldots, \mathcal{O}_k)}_{m}$. *Equitable tree partition* is an equitable partition of $V(G)$ into $(V_1, \ldots, V_m)$, where $G[V_i] \in \mathcal{T}_k$. Let $mad(G)$ denote the maximum average degree of G, which is defined as $mad(G) = \max\{\frac{2|E(H)|}{|V(H)|} \big| H \subseteq G\}$.

Hajnal and Szemerédi [3] proved that for any graph G with maximum degree $\Delta(G)$, there is $\chi^*_{e\mathcal{I}}(G) \leq \Delta(G) + 1$. Chen, Lih and Wu [2] conjectured that for any connected graph G different from K_m, C_{2m+1} and $K_{2m+1,2m+1}$, there is $\chi^*_{e\mathcal{I}}(G) \leq \Delta(G)$. For planar graphs, Zhang and Yap [8] proved that for every planar graph with $\Delta(G) \geq 13$, there is $\chi^*_{e\mathcal{I}}(G) \leq \Delta(G)$. Wu and Wang [7] proved that for every planar graph with $\delta(G) \geq 2$ and $g(G) \geq 26$, there is $\chi^*_{e\mathcal{I}}(G) \leq 3$ and, for every planar graph with $\delta(G) \geq 2$ and $g(G) \geq 14$, there is $\chi^*_{e\mathcal{I}}(G) \leq 4$. Later, Luo, Sereni, Stephens and Yu [5] improved the above results by proving that for every planar graph with $\delta(G) \geq 2$ and $g(G) \geq 14$, there is $\chi^*_{e\mathcal{I}}(G) \leq 3$, and for every planar graph with $\delta(G) \geq 2$ and $g(G) \geq 10$, there is $\chi^*_{e\mathcal{I}}(G) \leq 4$.

We are interested in the equitable $\underbrace{(\mathcal{O}_k, \ldots, \mathcal{O}_k)}_{m}$-partition for any integers $m \geq 2$ and $k \geq 2$. Such a partition was first studied in [6], under the name of defective equitable coloring. The following results are rewritten using the notion in the present paper.

Theorem 1 (Williams, Vandenbussche and Yu, [6]). *Every planar graph G with* *minimum degree* $\delta(G) \geq 2$ *and girth* $g(G) \geq 10$ *has an equitable* $\underbrace{(\mathcal{O}_2, \ldots, \mathcal{O}_2)}_{m}$-*partition for any integer* $m \geq 3$*, that is* $\chi^*_{e\mathcal{O}_2}(G) \leq 3$.

Theorem 2 (Li and Zhang, [4])**.** *Every planar graph* G *with minimum degree* $\delta(G) \geq 2$ *and girth* $g(G) \geq 8$ *has an equitable* $\underbrace{(\mathcal{O}_2, \ldots, \mathcal{O}_2)}_{m}$-*partition for any integer* $m \geq 4$*, that is* $\chi^*_{e\mathcal{O}_2}(G) \leq 4$.

In this paper, we extend the equitable cluster partition to general graphs, and obtain the following main result.

Theorem 3. *Every graph* G *with minimum degree* $\delta(G) \geq 2$ *and maximum average degree* $mad(G) < \frac{8}{3}$ *admits an equitable* $\underbrace{(\mathcal{O}_6, \mathcal{O}_6, \ldots, \mathcal{O}_6)}_{m}$-*partition for any integer* $m \geq 3$*, that is* $\chi^*_{e\mathcal{O}_6}(G) \leq 3$.

By Euler's formula, a planar graph G with girth g satisfies $mad(G) < \frac{2g}{g-2}$ [1]. According to Theorem 3, it is natural to infer the following corollary.

Corollary 4. *Every planar graph* G *with minimum degree* $\delta(G) \geq 2$ *and girth* $g(G) \geq 8$ *admits an equitable* $\underbrace{(\mathcal{O}_6, \mathcal{O}_6, \ldots, \mathcal{O}_6)}_{m}$-*partition for any integer* $m \geq 3$*, that is* $\chi^*_{e\mathcal{O}_6}(G) \leq 3$.

Though Corollary 4 is not a completely improvement of Theorems 1 and 2, it optimized the condition of $g(G)$ and equitable cluster partition threshold, respectively. And actually Corollary 4 gave an equitable tree partition threshold by the condition $g(G) \geq 8$.

2. The Structure of Minimal Counterexamples

Let G be a counterexample to Theorem 3 with a smallest order. Before discussing the structure of G, we clarify some necessary definitions and notations firstly.

The *degree* of a vertex v in G, written by $d_G(v)$ or simply $d(v)$ when there is no confusion, is the number of edges incident with v in G. A *k-vertex*, k^+-*vertex* and k^--*vertex* is a vertex of degree k, at least k and at most k, respectively. A neighbor of vertex v with degree k, at least k and at most k is called a *k-neighbor*, k^+-*neighbor* and k^--*neighbor* of v, respectively.

A *chain* of G is a maximal induced walk whose internal vertices all have degree 2. A *t-chain* is a chain with t internal vertices. In a chain, the 3^+-vertex is called *endvertex*. Specially, a cycle with exactly one 3^+-vertex and all other vertices of degree 2 is also called a chain, in this case, the endvertices of chain are identical. Let x be an endvertex of a chain P, y be a vertex in P, if the distance between x and y on P is $l+1$, then we say that y is *loosely l-adjacent* to x. Thus "loosely 0-adjacent" is the same as usual "adjacent".

Let x be a vertex with $d(x) \geq 3$, then x is the endvertex of $d(x)$ different chains. Set $T(x) = (a_2, a_1, a_0)$, where a_i is the number of i-chains incident with x, $i \in \{0, 1, 2\}$. Let $A(x)$ be the vertex set consisting of all 2-vertices in its incident chains, then $t(x) = |A(x)| = 2a_2 + a_1$. We define several kinds of 3-vertices that we will use in the following proof. We call a 3-vertex x *bad* 3-*vertex* if $T(x) = (0, 2, 1)$, *terrible* 3-*vertex* if $T(x) = (0, 3, 0)$ or $T(x) = (1, 1, 1)$, *A*-3-*vertex* if $t(x) = 1$ and it is adjacent to a bad 3-vertex, *B*-3-*vertex* if $t(x) = 0$ and it is adjacent to two bad 3-vertices, *C*-3-*vertex* if $t(x) = 1$ and it is adjacent to an *A*-3-*vertex*. If a neighbor of the vertex v is A-3-vertex, B-3-vertex or C-3-vertex, then we call it an *A*-3-*neighbor*, *B*-3-*neighbor* or *C*-3-*neighbor* of v. We use hollow dots to represent 3^+-vertices that the degree is unfixed, use solid dots to represent the vertices that its degree is fixed and the degree is the number of edges incident with it. These kinds of 3-vertices are shown in Figure 1.

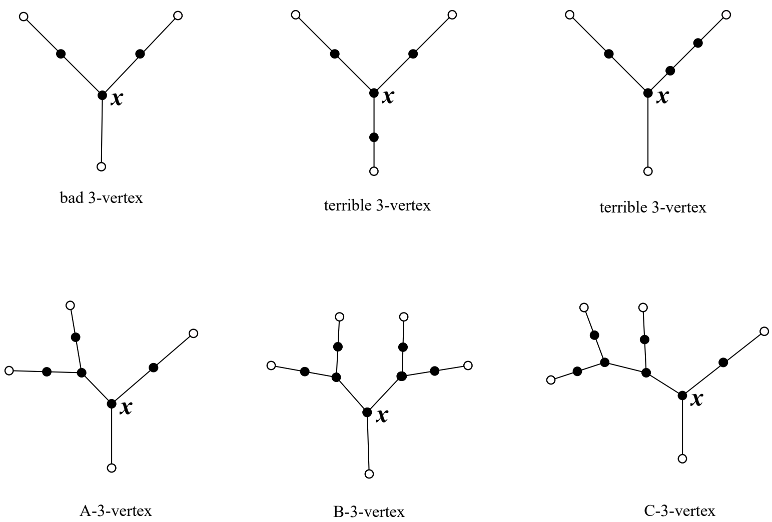

Figure 1. Several kinds of 3-vertices.

For convenience, let $m(\text{mod } m) = m$ for any positive integer m. In the whole paper, all the integers m we mentioned are not less than 3, and there is an obvious fact that if $G_1 \subseteq G$, then $mad(G_1) \leq mad(G)$. For brief, we will not mention them again.

Lemma 5. *The graph G is connected.*

Proof. On the contrary, let $H_1, H_2, \ldots, H_k$ be the components of G, where $k \geq 2$. By the minimality of G, both $H = H_1 \cup H_2 \cup \cdots \cup H_{k-1}$ and H_k admit an equitable 6 cluster m-partition. An ascending equitable 6 cluster m-partition of H with $|V_1(H)| \leq \cdots \leq |V_m(H)|$ and a descending equitable 6 cluster m-partition of H_k with $|V_1(H_k)| \geq \cdots \geq |V_m(H_k)|$ generate an equitable 6 cluster m-partition $(V_1(H) \cup V_1(H_k), \ldots, V_m(H) \cup V_m(H_k))$ of G, which contradicts the choice of G. ■

Lemma 6. *If G has a t-chain, then $t \leq 2$.*

Proof. Suppose to the contrary that G has a t-chain $P = v_0 v_1 \cdots v_t v_{t+1}$ with $t \geq 3$, where $d(v_0), d(v_{t+1}) \geq 3$. Let $G_1 = G - \{v_1, \ldots, v_t\}$.

If $v_0 \neq v_{t+1}$ or $v_0 = v_{t+1}$ and $d(v_0) \geq 4$, then $\delta(G_1) \geq 2$. By the minimality of G, the graph G_1 has an equitable 6 cluster m-partition with $|V_1| \leq \cdots \leq |V_m|$. We can extend the partition of G_1 to an equitable 6 cluster m-partition of G as follows. First put the vertex v_i into the part $V_{i (\text{mod } m)}$ for each $i \in \{1, 2, \ldots, t\}$. Swap the positions of v_1 and v_2 if v_0 and v_1 are put in the same part, and further swap the positions of v_{t-1} and v_t if v_t and v_{t+1} are put in the same part.

Now suppose that $v_0 = v_{t+1}$ and $d(v_0) = 3$. Let x be the neighbor of v_0 in G_1. If $d(x) \geq 3$, consider $G_2 = G - \{v_0, v_1, \ldots, v_t\}$, there is $\delta(G_2) \geq 2$. By the choice of G, the graph G_2 has an equitable 6 cluster m-partition with $|V_1| \leq \cdots \leq |V_m|$. We can extend the partition of G_2 to an equitable 6 cluster m-partition of G as follows. First put the vertex v_i into the part $V_{i+1 (\text{mod } m)}$ for each $i \in \{0, 1, \ldots, t\}$. Swap the positions of v_0 and v_1 if the vertices v_0 and x are put in the same part (the partition of $\{v_0, v_1, \ldots, v_t\}$ generated in this way admits that the order of each component of each part is at most 2). If $d(x) = 2$, then let $Q = x_0 x_1 x_2 \cdots x_q x_{q+1}$ be the chain with $x_0 = v_0$, $x_1 = x$, $q \geq 1$. Consider the graph $G_3 = G - \{x_0, x_1, \ldots, x_q, v_1, \ldots, v_t\}$, there is $\delta(G_3) \geq 2$. By the minimality of G, the graph G_3 has an equitable 6 cluster m-partition with $|V_1| \leq \cdots \leq |V_m|$. We first extend the partition of G_3 to G_1 to obtain an equitable 6 cluster m-partition of $G - \{v_1, \ldots, v_t\}$ as follows. First put the vertex x_i into the part $V_{i+1 (\text{mod } m)}$ for each $i \in \{0, 1, \ldots, q\}$. If x_q and x_{q+1} are put in the same part, swap the positions of x_{q-1} and x_q. Next we put the vertex v_i in the part $V_{q+i+1 (\text{mod } m)}$ for each $i \in \{1, 2, \ldots, t\}$, then swap the vertices similarly to the case that $v_0 = v_{t+1}$ and $d(v_0) \geq 4$ if necessary. In any case, we can always get an equitable 6 cluster m-partition of G. This contradicts the choice of G. Hence, there is no t-chain with $t \geq 3$. In other words, there are only 1-chains and 2-chains in G. ■

Let H be a subgraph of G, for $x \in V(H)$, if x has no neighbors in $G - H$, then we call it *free vertex*, otherwise we call it *non-free vertex*, the neighbors of x in $G - H$ are called *outer neighbors* of x. Let $s(H)$ denote the number of non-free vertices in H, $s_i(H)$ denote the number of non-free vertices with exactly i outer neighbors in $G-H$, $i \geq 1$, $f(H)$ denote the number of free vertices in H, and $n(H)$ denote the number of vertices in H.

Next, we use Hall's Theorem to analyze the reducible structures, allowing us to avoid lengthy case analysis. $[n(H)] = \{1, 2, \ldots, n(H)\}$. Our method is to construct an auxiliary bipartite graph $B(H) = (V(H), [n(H)])$ in such a way that $u \in V(H)$ is adjacent to $i \in [n(H)]$ if and only if no neighbor of u in $G-H$ is put into the part $V_{i(\text{mod } m)}$. A perfect matching in the auxiliary graph corresponds to a descending equitable partition of H. We first show that $B(H)$ has a perfect matching by Hall's Theorem, then modify it to obtain an equitable 6 cluster m-partition of H and further obtain an equitable 6 cluster m-partition of G. This approach may somewhat simplify the proofs in the area of equitable partition.

Let H be a reducible structure of G. By the minimality of G, the graph $G-H$ has an equitable 6 cluster m-partition. Let $c : V(G-H) \to \{V_1, \ldots, V_m\}$ be an ascending equitable 6 cluster m-partition. We claim that c can be extended to an equitable partition of G (but the number of clusters is uncertain) such that the vertices of H that are non-free are put into the part that their outer neighbors are not put in. As long as there is a way of partition such that the number of clusters in H does not exceed 6, we can get that G has an equitable 6 cluster m-partition. Therefore, H cannot appear in G.

We observe a few facts about the graph $B(H)$.

(F1) If $s(H) = s_1(H) + s_2(H) \leq n(H) - \left\lceil \frac{n(H)}{3} \right\rceil \left(\leq n(H) - \left\lceil \frac{n(H)}{m} \right\rceil \right)$ and $s_2(H) \leq 1$, then $B(H)$ has a perfect matching. (By Hall's Theorem, $B(H)$ has a perfect matching if and only if for any $S \subseteq V(H)$, $|N(S)| \geq |S|$. Note that if S contains a free vertex, then $|N(S)| = n(H)$. Thus if $B(H)$ contains no perfect matching, then there exists a set S_0 such that $|S_0| > |N(S_0)|$, obviously, S_0 only contains non-free vertices. If $S_0 = \{x\}$, where x is the vertex which has two outer neighbors, $|S_0| = 1$, $|N(S_0)| \geq n(H) - \left\lceil \frac{n(H)}{3} \right\rceil \times 2 \geq 1$, now $|S_0| \leq |N(S_0)|$, which contradicts the choice of S_0. If S_0 contains the vertex which has one outer neighbor, $s(H) \geq |S_0| > |N(S_0)| \geq n(H) - \left\lceil \frac{n(H)}{3} \right\rceil$, a contradiction. Hence $B(H)$ has a perfect matching.)

(F2) A perfect matching in $B(H)$ gives rise to a partition c' of $V(H)$ such that
(i) no vertex is put in the same part with its outer neighbors;
(ii) c' is descending equitable.

(F3) If (i) $s(H) = s_1(H) + s_2(H) \leq n(H) - \left\lceil \frac{n(H)}{3} \right\rceil \left(\leq n(H) - \left\lceil \frac{n(H)}{m} \right\rceil \right)$ and $s_2(H) \leq 1$, (ii) $f(H) \geq \left\lceil \frac{n(H)}{3} \right\rceil \left(\geq \left\lceil \frac{n(H)}{m} \right\rceil \right)$, (iii) $\left\lceil \frac{n(H)}{3} \right\rceil \leq 6$ hold at the same time, then the perfect matching of $B(H)$ induces an equitable 6 cluster m-partition of H that is descending equitable.

Next we use these facts to prove the following structural lemmas. In order to ensure $\delta(G-H)\geq 2$, before the proof, we first provide a remark.

Remark. Let H be one of the reducible structures of G. If $s(H)=s_1(H)+s_2(H)$ and $s_2(H)\leq 1$, then we will proceed as follows. For the non-free vertices u and w in H, if the outer neighbors of them in $G-H$ are identical, three cases on the outer neighbor, say v, to consider. If $d_{G-H}(v)\geq 2$, then the method of constructing H is the same as in the case when the outer neighbors are different. If $d_{G-H}(v)=1$, after deleting the reducible structure, in order to ensure that the minimum degree of the subgraph of G is at least 2, we need to perform the following processing. Let $Q=v_0v_1\cdots v_q$ be the chain in $G-H$, where $v_0=v$, $1\leq q\leq 3$. Let $H'=H\cup\{v_0,v_1,\ldots,v_{q-1}\}$, if $d_{G-H}(v)=1$ and $d_{G-H'}(v)\geq 2$, we will use H' as a new reducible structure; if $d_{G-H}(v)=1$ and $d_{G-H'}(v)=1$, we will continue to take the chains incident with v_q in $G-H'$, and update H' and vertices involved according to the aforementioned construction method until $d_{G-H'}(v)\geq 2$. In each step of processing, record the update reducible structure as H' and the pre updated structure as H, repeat this process until the conditions are met. Due to the fact that the structures in every lemma are explicit, it is evident that H can be obtained within a finite number of steps. Finally, if $d_{G-H}(v)=0$, in other words, the vertex v is isolated in $G-H$, we can always find an equitable partition that meets the requirements. In the process of determining H' at each step, H' compared to the one before updating only increases the number of free vertices without increasing the number of non-free vertices, this operation increases the order of H by at most 3. If $s(H)\leq n(H)-\left\lceil\frac{n(H)}{3}\right\rceil$, $f(H)\geq\left\lceil\frac{n(H)}{3}\right\rceil$, then $s(H')\leq s(H)\leq n(H)-\left\lceil\frac{n(H)}{3}\right\rceil\leq n(H')-\left\lceil\frac{n(H')}{3}\right\rceil$ and $f(H')\geq\left\lceil\frac{n(H')}{3}\right\rceil$. If $\left\lceil\frac{n(H)}{3}\right\rceil\leq 5$, then $\left\lceil\frac{n(H')}{3}\right\rceil\leq 6$. After proving Lemma 7(1), we will know that the case $d(v)=3$ and $t(v)\geq 4$ or $T(v)=(1,0,2)$ is impossible to exist.

By (F_1), if H has an equitable 6 cluster 3-partition, then H must have an equitable 6 cluster m-partition for any integer $m\geq 4$. Therefore, we only need to prove H admits an equitable 6 cluster 3-partition. And we will find that the structures H in Lemmas 7–26 satisfy $s(H)=s_1(H)+s_2(H)$ and $s_2(H)\leq 1$.

Lemma 7. *The following cases of* 2*-vertex on the chain incident with* x *hold.*

(1) *If* $d(x)=3$*, then* $t(x)\leq 3$ *and* $T(x)\neq(1,0,2)$*;*

(2) *If* $d(x)=4$*, then* $t(x)\leq 4$ *and* $T(x)\neq(2,0,2)$*;*

(3) *If* $d(x)=5$*, then* $t(x)\leq 6$ *and* $T(x)\neq(3,0,2)$*,* $T(x)\neq(2,1,2)$*;*

(4) *If* $d(x)=6$*, then* $t(x)\leq 7$ *and* $T(x)\neq(3,1,2)$*;*

(5) *If* $d(x)=7$*, then* $t(x)\leq 9$ *and* $T(x)\neq(3,2,2)$*,* $T(x)\neq(4,1,2)$.

Proof. For each case of the lemma, we suppose to the contrary as follows: (1) $d(x)=3$, $t(x)\geq 4$ or $T(x)=(1,0,2)$; (2) $d(x)=4$, $t(x)\geq 5$ or $T(x)=(2,0,2)$;

(3) $d(x) = 5$, $t(x) \geq 7$ or $T(x) = (3, 0, 2)$ or $T(x) = (2, 1, 2)$; (4) $d(x) = 6$, $t(x) \geq 8$ or $T(x) = (3, 1, 2)$; (5) $d(x) = 7$, $t(x) \geq 10$ or $T(x) = (3, 2, 2)$ or $T(x) = (4, 1, 2)$. Let H be the graph induced by x and the 2-vertices in its incident chains. Lemma 6 implies that x is not incident with t-chains for any $t \geq 3$. By Remark, $\delta(G - H) \geq 2$ or $\delta(G - H') \geq 2$. Then we will refer to the following tables for some computations.

$n(H)$	3	5	6	7
$\lceil \frac{n(H)}{3} \rceil$	1	2	2	3
$n(H) - \lceil \frac{n(H)}{3} \rceil$	2	3	4	4
$s(H)$	2	3	3	3
$f(H)$	1	2	3	4

Table 1. $d(x) = 3$.

$n(H)$	5	6	7	8	9
$\lceil \frac{n(H)}{3} \rceil$	2	2	3	3	3
$n(H) - \lceil \frac{n(H)}{3} \rceil$	3	4	4	5	6
$s(H)$	3	4	4	4	4
$f(H)$	2	2	3	4	5

Table 2. $d(x) = 4$.

$n(H)$	6	7	8	9	10	11
$\lceil \frac{n(H)}{3} \rceil$	2	3	3	3	4	4
$n(H) - \lceil \frac{n(H)}{3} \rceil$	4	4	5	6	6	7
$s(H)$	4	4	5	5	5	5
$f(H)$	2	3	3	4	5	6

Table 3. $d(x) = 5$.

$n(H)$	8	9	10	11	12	13
$\lceil \frac{n(H)}{3} \rceil$	3	3	4	4	4	5
$n(H) - \lceil \frac{n(H)}{3} \rceil$	5	6	6	7	8	8
$s(H)$	5	5,6	6	6	6	6
$f(H)$	3	4,3	4	5	6	7

Table 4. $d(x) = 6$.

$n(H)$	9	10	11	12	13	14	15
$\left\lceil \frac{n(H)}{3} \right\rceil$	3	4	4	4	5	5	5
$n(H) - \left\lceil \frac{n(H)}{3} \right\rceil$	6	6	7	8	8	9	10
$s(H)$	6	6	6,7	7	7	7	7
$f(H)$	3	4	5,4	5	6	7	8

Table 5. $d(x) = 7$.

By (F1), (F3) and the above tables, we can observe that $s(H) \leq n(H) - \left\lceil \frac{n(H)}{3} \right\rceil$ $(\leq n(H) - \left\lceil \frac{n(H)}{m} \right\rceil)$, so $B(H)$ has a perfect matching. And we can know that $f(H) \geq \left\lceil \frac{n(H)}{3} \right\rceil$ and $\left\lceil \frac{n(H)}{3} \right\rceil \leq 5$, combining Remark, the perfect matching of $B(H)$ induces a descending equitable 6 cluster m-partition of H or H'. Thereby, we get an equitable 6 cluster m-partition of G, this leads to a contradiction. ■

Lemma 8. *If x is a bad 3-vertex, where $T(x) = (0, 2, 1)$, and y is the 3^+-neighbor of x, then*

(i) $d(y) = 3$ *with* $t(y) \leq 1$, *or*

(ii) $d(y) = 4$ *with* $t(y) \leq 1$ *or* $T(y) = (0, 2, 2), (0, 3, 1), (1, 1, 2)$, *or*

(iii) $d(y) = 5$ *with* $t(y) \leq 3$ *or* $T(y) = (0, 4, 1), (1, 2, 2)$, *or*

(iv) $d(y) \geq 6$.

Proof. Let x be a bad 3-vertex, and y is the 3^+-neighbor of x. Suppose to the contrary that $d(y) = 3$ with $t(y) \geq 2$, $d(y) = 4$ with $t(y) \geq 2$ and $T(y) \neq (0, 2, 2), (0, 3, 1), (1, 1, 2)$, and $d(y) = 5$ with $t(y) \geq 4$ and $T(y) \neq (0, 4, 1), (1, 2, 2)$. By Lemma 7(1)–(3), if $d(y) = 3$, then $2 \leq t(y) \leq 3$ and $T(y) \neq (1, 0, 2)$, if $d(y) = 4$, then $2 \leq t(y) \leq 4$ and $T(y) \neq (2, 0, 2), (0, 2, 2), (0, 3, 1)$, $(1, 1, 2)$ and if $d(y) = 5$, then $4 \leq t(y) \leq 6$ and $T(y) \neq (3, 0, 2), (2, 1, 2), (0, 4, 1)$, $(1, 2, 2)$. Let H be the graph induced by x, y and the 2-vertices in their incident chains. By Lemma 6, y is only incident with 1-chains and 2-chains. By Remark, $\delta(G-H) \geq 2$ or $\delta(G - H') \geq 2$. Then we can get the following table.

$n(H)$	6	7	8	9	10
$\left\lceil \frac{n(H)}{3} \right\rceil$	2	3	3	3	4
$n(H) - \left\lceil \frac{n(H)}{3} \right\rceil$	4	4	5	6	6
$s(H)$	4	4	5	6	6
$f(H)$	2	3	3	3	4

Table 6. The 3^+-neighbors of bad 3-vertex.

Similarly to the proof of Lemma 7, we can obtain an equitable 6 cluster m-partition of G, this leads to a contradiction. ■

Lemma 9. *Every 3-vertex y with $t(y) = 0$ is adjacent to at most two bad 3-vertices.*

Proof. Suppose to the contrary that there is a 3-vertex y with $t(y) = 0$ that is adjacent to three bad 3-vertices x_1, x_2 and x_3. Let H be the graph induced by x_1, x_2, x_3, y and the 2-vertices in their incident chains. By Remark, $\delta(G - H) \geq 2$ or $\delta(G - H') \geq 2$. By the minimality of G, the graph $G - H$ or $G - H'$ admits an ascending equitable 6 cluster m-partition. $s(H) = d(x_1) + d(x_2) + d(x_3) - 3 = 6$, $n(H) = t(x_1) + t(x_2) + t(x_3) + 4 = 10$. By calculation, we have $s(H) \leq n(H) - \left\lceil \frac{n(H)}{3} \right\rceil \leq n(H) - \left\lceil \frac{n(H)}{m} \right\rceil$, so $B(H)$ has a perfect matching. And we know that $f(H) \geq \left\lceil \frac{n(H)}{3} \right\rceil$ and $\left\lceil \frac{n(H)}{3} \right\rceil \leq 5$, thus the perfect matching induces an equitable 6 cluster m-partition of H or H'. Thereby, we get an equitable 6 cluster m-partition of G, this leads to a contradiction. ■

Lemma 10. *Let x be a B-3-vertex, where $d(x) = 3$, $t(x) = 0$, and x is adjacent to two bad 3-vertex x_1 and x_2. If y is the 3^+-neighbor of x other than x_1 and x_2, then*

(i) *$d(y) = 4$ with $T(x_3) = (0, 0, 4)$ or $T(y) = (0, 2, 2)$, or*

(ii) *$d(y) \geq 5$.*

Proof. Suppose to the contrary that $d(y) = 3$, $d(y) = 4$ with $T(y) \neq (0, 0, 4)$ and $T(y) \neq (0, 2, 2)$. By Lemma 7(1)(2) and Lemma 9, if $d(y) = 3$, then $0 \leq t(y) \leq 3$ and $T(y) \neq (1, 0, 2), (0, 2, 1)$, if $d(y) = 4$, then $t(y) \leq 4$ and $T(y) \neq (0, 0, 4), (0, 2, 2), (2, 0, 2)$. Let H be the graph induced by x, x_1, x_2, y and the 2-vertices on their incident chains. By Lemma 6, y is only incident with 1-chains and 2-chains. By Remark, $\delta(G - H) \geq 2$ or $\delta(G - H') \geq 2$. Then we can get the following table.

$n(H)$	8	9	10	11	12
$\left\lceil \frac{n(H)}{3} \right\rceil$	3	3	4	4	4
$n(H) - \left\lceil \frac{n(H)}{3} \right\rceil$	5	6	6	7	8
$s(H)$	5	6	6	6,7	7
$f(H)$	3	3	4	5,4	5

Table 7. The other neighbors of B-3-vertex x.

Similarly to the proof of Lemma 7, we can obtain an equitable 6 cluster m-partition of G, this leads to a contradiction. ■

Lemma 11. *Let y be an A-3-vertex, where $d(y) = 3$, $t(y) = 1$, and x is a bad 3-vertex that is adjacent to y. If z is a 3^+-vertex that is adjacent to y other than x, then*

(i) $d(z) = 3$ *with* $t(z) = 1$, *or*

(ii) $d(z) = 4$ *with* $t(z) \leq 2$ *and* $T(z) \neq (1, 0, 3)$, *or*

(iii) $d(z) \geq 5$.

Proof. Suppose to the contrary that $d(z) = 3$ with $t(z) = 0$ or $t(z) \geq 2$, $d(z) = 4$ with $t(z) \geq 3$ or $T(z) = (1, 0, 3)$. By Lemma 7(1)(2), if $d(z) = 3$, then $t(z) = 0$ or $2 \leq t(z) \leq 3$ and $T(z) \neq (1, 0, 2)$; if $d(z) = 4$, then $3 \leq t(z) \leq 4$ and $T(z) \neq (2, 0, 2)$, or $T(z) = (1, 0, 3)$. Let H be the graph induced by x, y, z and the 2-vertices in their incident chains. By Lemma 6, z is only incident with 1-chains and 2-chains. By Remark, $\delta(G - H) \geq 2$ or $\delta(G - H') \geq 2$. Then we can get the following table.

$n(H)$	6	8	9	10
$\lceil \frac{n(H)}{3} \rceil$	2	3	3	4
$n(H) - \lceil \frac{n(H)}{3} \rceil$	4	5	6	6
$s(H)$	4	5	5,6	6
$f(H)$	2	3	4,3	4

Table 8. The other neighbors of A-3-vertex y.

Similarly to the proof of Lemma 7, we can obtain an equitable 6 cluster m-partition of G, this leads to a contradiction. ∎

Lemma 12. *Let x be a C-3-vertex, where $d(x) = 3$, $t(x) = 1$, and x is adjacent to an A-3-vertex y. If z is the 3^+-neighbor of x other than y, then*

(i) $d(z) = 4$ *with* $T(z) = (0, 0, 4)$ *or* $T(z) = (0, 2, 2)$, *or*

(ii) $d(z) \geq 5$.

Proof. Suppose to the contrary that $d(z) = 3$, $d(z) = 4$ with $T(z) \neq (0, 0, 4)$ and $T(z) \neq (0, 2, 2)$. By Lemma 7(1)(2), if $d(z) = 3$, then $t(z) \leq 3$ and $T(z) \neq (1, 0, 2)$, if $d(z) = 4$, then $t(z) \leq 4$ and $T(z) \neq (0, 0, 4), (0, 2, 2)$ and $(2, 0, 2)$. Let H be the graph induced by x, y, z and the 2-vertices on their incident chains. By Lemma 6, z is only incident with 1-chains and 2-chains. By Remark, $\delta(G - H) \geq 2$ or $\delta(G - H') \geq 2$. We can get the following table.

$n(H)$	8	9	10	11	12
$\left\lceil \frac{n(H)}{3} \right\rceil$	3	3	4	4	4
$n(H) - \left\lceil \frac{n(H)}{3} \right\rceil$	5	6	6	7	8
$s(H)$	5	6	6	6,7	7
$f(H)$	3	3	4	5,4	5

Table 9. The other neighbors of C-3-vertex x.

Similarly to the proof of Lemma 7, we can obtain an equitable 6 cluster m-partition of G, this leads to a contradiction. ∎

Lemma 13. *Let x be a terrible 3-vertex with $T(x) = (0, 3, 0)$, and let y be a vertex that is loosely 1-adjacent to x, then*

(i) *$d(y) = 4$ with $T(y) = (0, 1, 3)$ or $T(y) = (0, 3, 1)$, or*

(ii) *$d(y) \geq 5$.*

Proof. Suppose to the contrary that $d(y) = 3$, $d(y) = 4$ with $T(y) \neq (0, 1, 3)$ and $T(y) \neq (0, 3, 1)$. By Lemma 7 (1)(2), if $d(y) = 3$, then $t(y) \leq 3$ and $T(y) \neq (1, 0, 2)$, if $d(y) = 4$, then $t(y) \leq 4$ and $T(y) \neq (0, 1, 3)$, $(0, 3, 1)$, $(2, 0, 2)$. Let H be the graph induced by x, y and the 2-vertices on their incident chains. By Lemma 6, y is only incident with 1-chains and 2-chains. By Remark, $\delta(G - H) \geq 2$ or $\delta(G - H') \geq 2$. Then we can get the following table.

$n(H)$	5	6	7	8
$\left\lceil \frac{n(H)}{3} \right\rceil$	2	2	3	3
$n(H) - \left\lceil \frac{n(H)}{3} \right\rceil$	3	4	4	5
$s(H)$	3	4	4	5
$f(H)$	2	2	3	3

Table 10. The vertices that are loosely 1-adjacent to terrible 3-vertex x with $T(x) = (0, 3, 0)$.

Similarly to the proof of Lemma 7, we can obtain an equitable 6 cluster m-partition of G, this leads to a contradiction. ∎

Lemma 14. *Let x be a terrible 3-vertex with $T(x) = (1, 1, 1)$, and let y be the 3^+-vertex that is loosely 1-adjacent to x, then*

(i) *$d(y) = 4$ with $T(y) = (0, 1, 3)$ or $T(y) = (0, 3, 1)$, or*

(ii) *$d(y) \geq 5$.*

Proof. Suppose to the contrary that $d(y) = 3$, $d(y) = 4$ with $T(y) \neq (0,1,3)$ and $T(y) \neq (0,3,1)$. By Lemma 7(1)(2), if $d(y) = 3$, then $t(y) \leq 3$ and $T(y) \neq (1,0,2)$; if $d(y) = 4$, then $t(y) \leq 4$ and $T(y) \neq (0,1,3)$, $(0,3,1)$, $(2,0,2)$. Let H be the graph induced by x, y and the 2-vertices on their incident chains. By Lemma 6, y is only incident with 1-chains and 2-chains. By Remark, $\delta(G-H) \geq 2$ or $\delta(G-H') \geq 2$. Then we can get the following table.

$n(H)$	5	6	7	8
$\lceil \frac{n(H)}{3} \rceil$	2	2	3	3
$n(H) - \lceil \frac{n(H)}{3} \rceil$	3	4	4	5
$s(H)$	3	4	4	5
$f(H)$	2	2	3	3

Table 11. The vertices that are loosely 1-adjacent to terrible 3-vertex x with $T(x) = (1,1,1)$.

Similarly to the proof of Lemma 7, we can obtain an equitable 6 cluster m-partition of G, this leads to a contradiction. ■

Lemma 15. *Let x be a terrible 3-vertex with $T(x) = (1,1,1)$, and let y be the 3^+-vertex that is adjacent to x, then*

(i) *$d(y) = 4$ with $T(y) = (0,0,4)$ or $T(y) = (0,2,2)$, or*

(ii) *$d(y) \geq 5$.*

Proof. Suppose to the contrary that $d(y) = 3$, $d(y) = 4$ with $T(y) \neq (0,0,4)$ and $T(y) \neq (0,2,2)$. By Lemma 7 (1)(2), if $d(y) = 3$, then $t(y) \leq 3$ and $T(y) \neq (1,0,2)$, if $d(y) = 4$, then $t(y) \leq 4$ and $T(y) \neq (0,0,4), (0,2,2)$ and $(2,0,2)$. Let H be the graph induced by x, y and the 2-vertices on their incident chains. By Lemma 6, y is only incident with 1-chains and 2-chains. By Remark, $\delta(G-H) \geq 2$ or $\delta(G-H') \geq 2$. Then we get the following table.

$n(H)$	5	6	7	8	9
$\lceil \frac{n(H)}{3} \rceil$	2	2	3	3	3
$n(H) - \lceil \frac{n(H)}{3} \rceil$	3	4	4	5	6
$s(H)$	3	4	4	4,5	5
$f(H)$	2	2	3	4,3	4

Table 12. The neighbors of terrible 3-vertex x with $T(x) = (1,1,1)$.

Similarly to the proof of Lemma 7, we can obtain an equitable 6 cluster m-partition of G, this leads to a contradiction. ■

Lemma 16. *Let x be a terrible 3-vertex with $T(x) = (1,1,1)$ or $T(x) = (0,3,0)$ and let y be a 4-vertex with $T(y) = (0,3,1)$ that is loosely 1-adjacent to x. If z is the 3^+-neighbor of y, then*

(i) *$d(z) = 4$ with $T(z) = (0,0,4)$ or $T(y) = (0,2,2)$, or*

(ii) *$d(z) \geq 5$.*

Proof. Suppose to the contrary that $d(z) = 3$, $d(z) = 4$ with $T(z) \neq (0,0,4)$ and $T(z) \neq (0,2,2)$. By Lemma 7(1)(2), if $d(z) = 3$, then $t(z) \leq 3$ and $T(z) \neq (1,0,2)$, if $d(z) = 4$, then $t(z) \leq 4$ and $T(z) \neq (0,0,4), (0,2,2)$ and $(2,0,2)$. Let H be the graph induced by x, y, z and the 2-vertices on their incident chains. By Lemma 6, z is only incident with 1-chains and 2-chains. By Remark, $\delta(G-H) \geq 2$ or $\delta(G-H') \geq 2$. Then we get the following table.

$n(H)$	8	9	10	11	12
$\left\lceil \frac{n(H)}{3} \right\rceil$	3	3	4	4	4
$n(H) - \left\lceil \frac{n(H)}{3} \right\rceil$	5	6	6	7	8
$s(H)$	5	6	6	6,7	7
$f(H)$	3	3	4	5,4	5

Table 13. The neighbors of some 4-vertex.

Similarly to the proof of Lemma 7, combined with the Remark, we can obtain an equitable 6 cluster m-partition of G, this leads to a contradiction. ∎

Lemma 17. *Every 4-vertex y with $T(y) = (0,3,1)$ is*

(1) *loosely 1-adjacent to at most one terrible 3-vertex x;*

(2) *not loosely 1-adjacent to a terrible 3-vertex and adjacent to a bad 3-vertex, a A-3-vertex, a B-3-vertex or a C-3-vertex at the same time.*

Proof. (1) Suppose to the contrary that there exists a 4-vertex y with $T(y) = (0,3,1)$ that is loosely 1-adjacent to at least two terrible 3-vertices x_1 and x_2. Let H be the graph induced by y, x_1, x_2 and the 2-vertices on their incident chains. By Remark, $\delta(G-H) \geq 2$ or $\delta(G-H') \geq 2$. By the minimality of G, the graph $G-H$ or $G-H'$ admits an ascending equitable 6 cluster m-partition. $n(H) = t(y) + t(x_1) + t(x_2) + 1 = 10$, $s(H) = d(y) + d(x_1) + d(x_2) - 4 = 6$. By calculation, we obtain $s(H) \leq n(H) - \left\lceil \frac{n(H)}{3} \right\rceil$. Combining with Remark, $B(H)$ or $B(H')$ has a perfect matching. And we know that $f(H) \geq \left\lceil \frac{n(H)}{3} \right\rceil$ and $\left\lceil \frac{n(H)}{3} \right\rceil \leq 5$. Thus, the perfect matching induces an descending equitable 6 cluster m-partition of H or H'. Thereby, we get an equitable 6 cluster m-partition of G. This leads to a contradiction.

(2) Suppose to the contrary that there exists a 4-vertex y with $T(y) = (0, 3, 1)$ that is loosely 1-adjacent to a terrible 3-vertex x_1 and a bad 3-vertex, an A-3-vertex, a B-3-vertex or a C-3-vertex x_2 at the same time. Let H be the graph induced by y, x_1, x_2 and the 2-vertices on their incident chains. By Remark, $\delta(G - H) \geq 2$ or $\delta(G - H') \geq 2$. We get the following table.

$n(H)$	10	12	14
$\lceil \frac{n(H)}{3} \rceil$	4	4	5
$n(H) - \lceil \frac{n(H)}{3} \rceil$	6	8	9
$s(H)$	6	7	8
$f(H)$	4	5	6

Table 14. The possible 4-vertices with three 1-chains.

Similarly to the proof of Lemma 7, we can obtain an equitable 6 cluster m-partition of G. This leads to a contradiction. ■

Lemma 18. *Every* 4*-vertex* y *with* $T(y) = (0, 1, 3)$ *is*

(1) *adjacent to at most two bad* 3*-vertices;*

(2) *adjacent to at most one terrible* 3*-vertex;*

(3) *adjacent to at most one* A*-*3*-vertex;*

(4) *not adjacent to* B*-*3*-vertex;*

(5) *not adjacent to* C*-*3*-vertex;*

(6) *not adjacent to a bad* 3*-vertex and* (*loosely* 1*-*)*adjacent to a terrible* 3*-vertex at the same time;*

(7) *not adjacent and loosely* 1*-adjacent to terrible* 3*-vertex at the same time;*

(8) *not adjacent to two bad* 3*-vertices and adjacent to an* A*-*3*-vertex at the same time;*

(9) *not adjacent to an* A*-*3*-vertex and* (*loosely* 1*-*)*adjacent to a terrible* 3*-vertex at the same time.*

Proof. In the nine cases of the lemma, we suppose to the contrary as follows. (1) there exists a 4-vertex y with $T(y) = (0, 1, 3)$ that is adjacent to three bad 3-vertices x_1, x_2 and x_3; (2) y is adjacent to at least two terrible 3-vertices x_1 and x_2; (3) y is adjacent to at least two A-3-vertices x_1 and x_2; (4) y is adjacent to at least one B-3-vertex x; (5) y is adjacent to at least one C-3-vertex x; (6) y is adjacent to a bad 3-vertex x_1 and (loosely 1-) adjacent to a terrible 3-vertex x_2 at the same time; (7) y is adjacent to a terrible 3-vertex x_1 and loosely 1-adjacent to a terrible 3-vertex x_2 at the same time; (8) y is adjacent to two bad 3-vertices x_1 x_2 and adjacent to an A-3-vertex x_3 at the same time; (9) y is adjacent to an A-3-vertex x_1 and (loosely 1-) adjacent to a terrible 3-vertex x_2 at the same time. Let H be the graph induced by the vertices involved and the 2-vertices on their incident chains. By Remark, $\delta(G-H) \geq 2$ or $\delta(G-H') \geq 2$. We get the following table.

$n(H)$	8	9	10	11	12	13
$\lceil \frac{n(H)}{3} \rceil$	3	3	4	4	4	5
$n(H) - \lceil \frac{n(H)}{3} \rceil$	5	6	6	7	8	8
$s(H)$	5	5,6	6	7	8	8
$f(H)$	3	4,3	4	4	4	5

Table 15. The possible cases for a 4-vertex y with one 1-chain.

Similarly to the proof of Lemma 7, we can obtain an equitable 6 cluster m-partition of G. This leads to a contradiction. ■

Lemma 19. *Every* 4*-vertex* y *with* $T(y) = (0, 2, 2)$ *is*

(1) *adjacent to at most one bad* 3*-vertex;*

(2) *adjacent to at most one terrible* 3*-vertex;*

(3) *loosely* 1*-adjacent to at most one terrible* 3*-vertex;*

(4) *adjacent to at most one* B*-3-vertex;*

(5) *adjacent to at most one* A*-3-vertex;*

(6) *adjacent to at most one* C*-3-vertex;*

(7) *not adjacent to a bad* 3*-vertex and adjacent to or loosely* 1*-adjacent to a terrible* 3*-vertex or adjacent to a* B*-3-vertex or adjacent to an* A*-3-vertex or adjacent to a* C*-3-vertex at the same time;*

(8) *not loosely* 1*-adjacent to a terrible* 3*-vertex and adjacent to a terrible* 3*-vertex or a* B*-3-vertex or an* A*-3-vertex or a* C*-3-vertex at the same time;*

(9) *not adjacent to a terrible* 3*-vertex and adjacent to a* B*-3-vertex or an* A*-3-vertex or a* C*-3-vertex at the same time;*

(10) *not adjacent to a* B*-3-vertex and adjacent to an* A*-3-vertex or a* C*-3-vertex at the same time;*

(11) *not adjacent to an* A*-3-vertex and adjacent to a* C*-3-vertex at the same time.*

Proof. In the eleven cases of the lemma, we suppose to the contrary as follows. (1) there exists a 4-vertex y with $T(y) = (0, 2, 2)$ that is adjacent to two bad 3-vertices x_1 and x_2; (2) y is adjacent to two terrible 3-vertices x_1 and x_2; (3) y is loosely 1-adjacent to two terrible 3-vertices x_1 and x_2; (4) y is adjacent to two B-3-vertices x_1 and x_2; (5) y is adjacent to two A-3-vertices x_1 and x_2; (6) y is adjacent to two C-3-vertices x_1 and x_2; (7) y is adjacent to a bad 3-vertex x_1 and adjacent to or loosely 1-adjacent to a terrible 3-vertex or adjacent to a B-3-vertex or adjacent to an A-3-vertex or adjacent to a C-3-vertex x_2 at the same time; (8) y is loosely 1-adjacent to a terrible 3-vertex x_1 and adjacent to a terrible 3-vertex or a B-3-vertex or an A-3-vertex or a C-3-vertex x_2 at the same time; (9) y is adjacent to a terrible 3-vertex x_1 and adjacent to a B-3-vertex or an A-3-vertex or a C-3-vertex x_2 at the same time; (10) y is adjacent to a B-3-vertex x_1 and adjacent to an A-3-vertex or a C-3-vertex x_2 at the same time; (11) y is adjacent to an A-3-vertex x_1 and adjacent to a C-3-vertex x_2 at the same time. Let H be the graph induced by the vertices involved and the 2-vertices on their incident chains. By Remark, $\delta(G-H) \geq 2$ or $\delta(G-H') \geq 2$. We get the following table.

$n(H)$	9	10	11	12	13	14	15	17
$\left\lceil \frac{n(H)}{3} \right\rceil$	3	4	4	4	5	5	5	6
$n(H) - \left\lceil \frac{n(H)}{3} \right\rceil$	6	6	7	8	8	9	10	11
$s(H)$	5,6	6	6,7	7	8	8	9	10
$f(H)$	4,3	4	5,4	5	5	6	6	7

Table 16. The possible cases for a 4-vertex y with two 1-chains.

When $|V(H)| = 17$, although $\left\lceil \frac{n(H)}{3} \right\rceil = 6$, because of $f(H) = 7$, we can always find a way to partition such that H admits an equitable 5 cluster m-partition. Similarly to the proof of Lemma 7, we can obtain an equitable 6 cluster m-partition of G. This leads to a contradiction. ■

Lemma 20. *Every* 4*-vertex* y *with* $T(y) = (1, 1, 2)$ *is adjacent to at most one bad* 3*-vertex.*

Proof. Suppose to the contrary that there exists a 4-vertex y with $T(y) = (1, 1, 2)$ that is adjacent to two bad 3-vertices x_1 and x_2. Let H be the graph induced by x_1, x_2 and the 2-vertices on their incident chains. By Remark, $\delta(G - H) \geq 2$ or $\delta(G - H') \geq 2$. By the minimality of G, the graph $G - H$ or $G - H'$ admits an ascending equitable 6 cluster m-partition. $s(H) = d(y) + d(x_1) + d(x_2) - 4 = 6$, $n(H) = t(y) + t(x_1) + t(x_2) + 3 = 10$. By calculation, we obtain $s(H) \leq n(H) - \left\lceil \frac{n(H)}{3} \right\rceil \leq n(H) - \left\lceil \frac{n(H)}{m} \right\rceil$, so $B(H)$ has a perfect matching. And we can know that $f(H) \geq \left\lceil \frac{n(H)}{3} \right\rceil$ and $\left\lceil \frac{n(H)}{3} \right\rceil \leq 5$, thus the perfect matching induces an equitable 6 cluster m-partition of H or H'. Thereby, we get an equitable 6 cluster m-partition of G. This leads to a contradiction. ■

Lemma 21. *Every* 5*-vertex* y *with* $T(y) = (0, 3, 2)$*,* $T(y) = (0, 4, 1)$*,* $T(y) = (1, 2, 2)$*,* $T(y) = (0, 5, 0)$ *or* $T(y) = (1, 3, 1)$ *is loosely* 1*-adjacent to at most one terrible* 3*-vertex.*

Proof. Suppose to the contrary that there exists a 5-vertex y with $T(y) = (0,3,2)$, $T(y) = (0,4,1)$, $T(y) = (1,2,2)$, $T(y) = (0,5,0)$ or $T(y) = (1,3,1)$ that is loosely 1-adjacent to at least two terrible 3-vertices x_1 and x_2. Let H be the graph induced by y, x_1, x_2 and the 2-vertices on their incident chains. By Remark, $\delta(G-H) \geq 2$ or $\delta(G-H') \geq 2$, and by the minimality of G, the graph $G-H$ or $G-H'$ admits an ascending equitable 6 cluster m-partition. We get the following table.

$n(H)$	10	11	12
$\left\lceil \frac{n(H)}{3} \right\rceil$	4	4	4
$n(H) - \left\lceil \frac{n(H)}{3} \right\rceil$	6	7	8
$s(H)$	6	6,7	7
$f(H)$	4	5,4	5

Table 17. Some 5-vertices y.

Similarly to the proof of Lemma 7, we can obtain an equitable 6 cluster m-partition of G. This leads to a contradiction. ■

Lemma 22. *Every* 5*-vertex* y *with* $T(y) = (1,4,0)$ *is not loosely* 1*-adjacent to terrible* 3*-vertex.*

Proof. Suppose to the contrary that there exists a 5-vertex y with $T(y) = (1,4,0)$ that is loosely 1-adjacent to a terrible 3-vertex x. Let H be the graph induced by x, y and the 2-vertices on their incident chains. By Remark, $\delta(G-H) \geq 2$ or $\delta(G-H') \geq 2$. By the minimality of G, the graph $G-H$ or $G-H'$ admits an ascending equitable 6 cluster m-partition. $s(H) = d(y) + d(x) - 2 = 6$, $n(H) = t(y) + t(x) + 1 = 10$. By calculation, we obtain $s(H) \leq n(H) - \left\lceil \frac{n(H)}{3} \right\rceil \leq n(H) - \left\lceil \frac{n(H)}{m} \right\rceil$, so $B(H)$ has a perfect matching. And we know that $f(H) \geq \left\lceil \frac{n(H)}{3} \right\rceil$ and $\left\lceil \frac{n(H)}{3} \right\rceil \leq 5$, thus the perfect matching induces an equitable 6 cluster m-partition of H or H'. Thereby, we get an equitable 6 cluster m-partition of G. This leads to a contradiction. ■

Lemma 23. *Every* 5*-vertex* y *with* $T(y) = (2,2,1)$ *is not loosely* 1*-adjacent to terrible* 3*-vertex.*

Proof. Suppose to the contrary that there exists a 5-vertex y with $T(y) = (2,2,1)$ that is loosely 1-adjacent to a terrible 3-vertices x. Let H be the graph induced by x, y and the 2-vertices on their incident chains. By Remark, $\delta(G-H) \geq 2$ or $\delta(G-H') \geq 2$. By the minimality of G, the graph $G-H$ or $G-H'$ admits an ascending equitable 6 cluster m-partition. $s(H) = d(x) + d(y) - 2 = 6$,

$n(H) = t(x) + t(y) + 2 = 10$. By calculation, we obtain $s(H) \leq n(H) - \left\lceil \frac{n(H)}{3} \right\rceil \leq n(H) - \left\lceil \frac{n(H)}{m} \right\rceil$, so $B(H)$ has a perfect matching. And we know that $f(H) \geq \left\lceil \frac{n(H)}{3} \right\rceil$ and $\left\lceil \frac{n(H)}{3} \right\rceil \leq 5$, thus the perfect matching induces an equitable 6 cluster m-partition of H or H'. Thereby, we get an equitable 6 cluster m-partition of G. This leads to a contradiction. ■

Lemma 24. *Every* 6*-vertex* y *with* $T(y) = (0,5,1)$*,* $T(y) = (0,6,0)$ *or* $T(y) = (1,4,1)$ *is loosely* 1*-adjacent to at most one terrible* 3*-vertex.*

Proof. Suppose to the contrary that there exists a 6-vertex y with $T(y) = (0,5,1)$, $T(y) = (0,6,0)$ or $T(y) = (1,4,1)$ that is loosely 1-adjacent to at least two terrible 3-vertices x_1 and x_2. Let H be the graph induced by y, x_1, x_2 and the 2-vertices on their incident chains. By Remark, $\delta(G - H) \geq 2$ or $\delta(G - H') \geq 2$. By the minimality of G, the graph $G - H$ or $G - H'$ admits an ascending equitable 6 cluster m-partition. We get the following table.

$n(H)$	12	13
$\left\lceil \frac{n(H)}{3} \right\rceil$	4	5
$n(H) - \left\lceil \frac{n(H)}{3} \right\rceil$	8	8
$s(H)$	8	8
$f(H)$	4	5

Table 18. Some 6-vertices y.

Similarly to the proof of Lemma 7, we can obtain an equitable 6 cluster m-partition of G. This leads to a contradiction. ■

Lemma 25. *Every* 6*-vertex* y *with* $T(y) = (1,5,0)$ *or* $T(y) = (2,3,1)$ *is not loosely* 1*-adjacent to terrible* 3*-vertex.*

Proof. Suppose to the contrary that there is a 6-vertex y with $T(y) = (1,5,0)$ or $T(y) = (2,3,1)$ that is loosely 1-adjacent to a terrible 3-vertex x. Let H be the graph induced by x, y and the 2-vertices on their incident chains. By Remark, $\delta(G - H) \geq 2$ or $\delta(G - H') \geq 2$. By the minimality of G, the graph $G - H$ or $G - H'$ admits an ascending equitable 6 cluster m-partition. $s(H) = d(x) + d(y) - 2 = 7$, $n(H) = t(x) + t(y) + 1 = 11$. By calculation, we obtain $s(H) \leq n(H) - \left\lceil \frac{n(H)}{3} \right\rceil \leq n(H) - \left\lceil \frac{n(H)}{m} \right\rceil$, so $B(H)$ has a perfect matching. And we know that $f(H) \geq \left\lceil \frac{n(H)}{3} \right\rceil$ and $\left\lceil \frac{n(H)}{3} \right\rceil \leq 5$, thus the perfect matching induces an equitable 6 cluster m-partition of H or H'. Thereby, we get an equitable 6 cluster m-partition of G. This leads to a contradiction. ■

Lemma 26. *Every 7-vertex y with $T(y) = (0,7,0)$, $T(y) = (1,6,0)$ or $T(y) = (2,5,0)$ is loosely 1-adjacent to at most one terrible 3-vertex.*

Proof. Suppose to the contrary that there exists a 7-vertex y with $T(y) = (0,7,0)$, $T(y) = (1,6,0)$ or $T(y) = (2,5,0)$ that is loosely 1-adjacent to at least two terrible 3-vertices x_1 and x_2. Let H be the graph induced by y, x_1, x_2 and the 2-vertices on their incident chains. By Remark, $\delta(G-H) \geq 2$ or $\delta(G-H') \geq 2$. By the minimality of G, the graph $G-H$ or $G-H'$ admits an ascending equitable 6 cluster m-partition. We get the following table.

$n(H)$	14	15	16
$\lceil \frac{n(H)}{3} \rceil$	5	5	6
$n(H) - \lceil \frac{n(H)}{3} \rceil$	9	10	10
$s(H)$	9	9	9
$f(H)$	5	6	7

Table 19. Some 7-vertices y.

When $|V(H)| = 16$, although $\left\lceil \frac{n(H)}{3} \right\rceil = 6$, because of $f(H) = 7$, we can always find a way to partition such that H admits an equitable 5 cluster m-partition. Similarly to the proof of Lemma 7, we can obtain an equitable 6 cluster m-partition of G. This leads to a contradiction. ■

3. Discharging

Consider the minimal counterexample G to Theorem 3, we know $mad(G) < \frac{8}{3}$. For any $x \in V(G)$, let $\mu(x) = d(x) - \frac{8}{3}$ be the initial charge. We have

$$\sum_{x \in V(G)} \mu(x) = \sum_{x \in V(G)} (d(x) - \frac{8}{3}) < 0.$$

Next, we redistribute the charges among vertices according to the following rules.

(R1) Every 3^+-vertex gives $\frac{1}{3}$ to each 2-vertex on its incident chains.

(R2) Every 3^+-vertex y gives $\frac{1}{3}$ to each bad 3-vertex x which is adjacent to y.

(R3) Every 3^+-vertex except bad 3-vertex gives $\frac{1}{3}$ to each of its A-3-neighbor.

(R4) Every 4^+-vertex gives $\frac{1}{3}$ to each of its B-3-neighbor.

(R5) Every 4^+-vertex gives $\frac{1}{3}$ to each of its C-3-neighbor.

(R6) Every 4^+-vertex y gives $\frac{2}{9}$ to the terrible 3-vertex x with $T(x) = (0, 3, 0)$ which is loosely 1-adjacent to y.

(R7) Every 4^+-vertex y gives $\frac{1}{3}$ to the terrible 3-vertex x with $T(x) = (1, 1, 1)$ which is adjacent to y.

(R8) Every 4^+-vertex y gives $\frac{1}{3}$ to the terrible 3-vertex x with $T(x) = (1, 1, 1)$ which is loosely 1-adjacent to y.

Let $\mu'(x)$ be the final charge of x after applying rules $R1$–$R8$. Then, we prove $\mu'(x) \geq 0$ for all $x \in V(G)$, and next are some cases to be discussed.

Case 1. $d(x) = 2$. If $d(x) = 2$, by Lemma 6, $\Delta(G) \geq 3$, then $\mu'(x) = \left(2 - \frac{8}{3}\right) + \frac{1}{3} \times 2 = 0$ by $R1$.

Case 2. $d(x) = 3$. If $d(x) = 3$, by Lemma 7(1) $t(x) \leq 3$ and $T(x) \neq (1, 0, 2)$, and according to discharging rules 3-vertex only sends charge to 2-vertex, bad 3-vertex and A-3-vertex. Then we will discuss several kinds of situations that may appear.

Case 2.1. $t(x) = 0$. Lemma 9 and Lemma 11 imply that every 3-vertex y with $t(y) = 0$ is adjacent to at most two bad 3-vertices and it is not adjacent to any A-3-vertex. By Lemma 10, the neighbor of B-3-vertex except for the two bad 3-vertices adjacent in the definition of B-3-vertex is 4^+-vertex, then $\mu'(x) \geq \left(3 - \frac{8}{3}\right) - \frac{1}{3} \times 2 + \frac{1}{3} \times 1 = 0$ by $R2$ and $R4$.

Case 2.2. $t(x) = 1$. Lemma 11 implies that every 3-vertex y with $t(y) = 1$ is adjacent to at most one bad 3-vertex and A-3-vertex is not adjacent to the other bad 3-vertices except for the bad 3-vertex in the definition of A-3-vertex. By Lemma 12, the neighbor of C-3-vertex except for the A-3-vertex and 2-vertex adjacent in the definition of C-3-vertex is 4^+-vertex, then $\mu'(x) \geq \left(3 - \frac{8}{3}\right) - \frac{1}{3} \times 1 - \frac{1}{3} \times 1 + \frac{1}{3} \times 1 = 0$ by $R1$, $R2$, $R3$ and $R5$.

Case 2.3. $t(x) = 2$. According to Lemma 8, we know bad 3-vertex is not adjacent to bad 3-vertex, and Lemma 11 implies that A-3-vertex is not adjacent to the other bad 3-vertices except for the bad 3-vertex in the definition of A-3-vertex, in other words, bad 3-vertex x is not adjacent to A-3-vertex that is adjacent to the other bad 3-vertices different from x, so $\mu'(x) \geq \left(3 - \frac{8}{3}\right) - \frac{1}{3} \times 2 + \frac{1}{3} \times 1 = 0$ by $R1$ and $R2$.

Case 2.4. $t(x) = 3$. Lemmas 13, 14 and 15 imply that the neighbor of terrible 3-vertex is 4^+-vertex and the vertex that loosely 1-adjacent to terrible 3-vertex is 4^+-vertex. If $T(x) = (0, 3, 0)$, then $\mu'(x) = (3 - \frac{8}{3}) - \frac{1}{3} \times 3 + \frac{2}{9} \times 3 = 0$ by $R1$ and $R6$. If $T(x) = (1, 1, 1)$, Lemma 8 implies that terrible 3-vertex is not adjacent to bad 3-vertex, and Lemma 11 implies that terrible 3-vertex is not adjacent to A-3-vertex, then $\mu'(x) \geq \left(3 - \frac{8}{3}\right) - \frac{1}{3} \times 3 + \frac{1}{3} \times 1 + \frac{1}{3} \times 1 = 0$ by $R1$ and $R7$ and $R8$.

According to discharging rules, 4^+-vertex may send charge to 2-vertex, bad 3-vertex, B-3-vertex, A-3-vertex, C-3-vertex and terrible 3-vertex.

Case 3. $d(x) = 4$. If $d(x) = 4$, by Lemma 7(2), $t(x) \leq 4$ and $T(x) \neq (2, 0, 2)$, next we will analyze various possible situations.

Case 3.1. $t(x) = 0$. $\mu'(x) \geq \left(4 - \frac{8}{3}\right) - \frac{1}{3} \times 4 = 0$ by $R2$–$R5$ and $R7$.

Case 3.2. $t(x) = 1$. According to Lemma 18, we have $\mu'(x) \geq \left(4 - \frac{8}{3}\right) - \frac{1}{3} \times 1 - \frac{1}{3} \times 2 = \frac{1}{3}$ by $R1$–$R8$.

Case 3.3. $t(x) = 2$. If $T(x) = (0, 2, 2)$, then according to Lemma 19, we have $\mu'(x) \geq \left(4 - \frac{8}{3}\right) - \frac{1}{3} \times 2 - \frac{1}{3} \times 1 = \frac{1}{3}$ by $R1$–$R8$. If $T(x) = (1, 0, 3)$, then according to Lemmas 8, 15, 11, 10 and 12, we obtain that every 4-vertex x with $T(x) = (1, 0, 3)$ is not adjacent to bad 3-vertex, terrible 3-vertex, A-3-vertex, B-3-vertex or C-3-vertex, then $\mu'(x) \geq \left(4 - \frac{8}{3}\right) - \frac{1}{3} \times 2 = \frac{2}{3}$ by $R1$.

Case 3.4. $t(x) = 3$. If $T(x) = (0, 3, 1)$, then according to Lemmas 16 and 17, we obtain $\mu'(x) \geq \left(4 - \frac{8}{3}\right) - \frac{1}{3} \times 3 - \frac{1}{3} \times 1 = 0$ by $R1$–$R8$. If $T(x) = (1, 1, 2)$, then according to Lemmas 15, 11, 10, 12, 13 and 14, we obtain that every 4-vertex x with $T(x) = (1, 1, 2)$ is not adjacent to a terrible 3-vertex, A-3-vertex, B-3-vertex or C-3-vertex, and it is not loosely 1-adjacent to a terrible 3-vertex, combining with Lemma 20, we get $\mu'(x) \geq \left(4 - \frac{8}{3}\right) - \frac{1}{3} \times 3 - \frac{1}{3} \times 1 = 0$ by $R1$ and $R2$.

Case 3.5 $t(x) = 4$. If $T(x) = (0, 4, 0)$, then according to Lemma 13 and Lemma 14, we know that every 4-vertex x with $T(x) = (0, 4, 0)$ is not loosely 1-adjacent to terrible 3-vertex, then $\mu'(x) \geq \left(4 - \frac{8}{3}\right) - \frac{1}{3} \times 4 = 0$ by $R1$. If $T(x) = (1, 2, 1)$, then according to Lemmas 8, 15, 11, 10, 12, 13 and 14, we know that every 4-vertex x with $T(x) = (1, 2, 1)$ is not adjacent to bad 3-vertex, terrible 3-vertex, A-3-vertex, B-3-vertex or C-3-vertex, and it is not loosely 1-adjacent to terrible 3-vertex, then $\mu'(x) \geq \left(4 - \frac{8}{3}\right) - \frac{1}{3} \times 4 = 0$ by $R1$.

Case 4. $d(x) = 5$. If $d(x) = 5$, by Lemma 7(3), $t(x) \leq 6$ and $T(x) \neq (3, 0, 2)$, $T(x) \neq (2, 1, 2)$.

Case 4.1. $t(x) = 0$. $\mu'(x) \geq \left(5 - \frac{8}{3}\right) - \frac{1}{3} \times 5 = \frac{2}{3}$ by $R1$–$R8$.

Case 4.2. $t(x) = 1$. $\mu'(x) \geq \left(5 - \frac{8}{3}\right) - \frac{1}{3} \times 1 - \frac{1}{3} \times 5 = \frac{1}{3}$ by $R1$–$R8$.

Case 4.3. $t(x) = 2$. If $T(x) = (0, 2, 3)$, then $\mu'(x) \geq \left(5 - \frac{8}{3}\right) - \frac{1}{3} \times 2 - \frac{1}{3} \times 5 = 0$ by $R1$–$R8$. If $T(x) = (1, 0, 4)$, then $\mu'(x) \geq \left(5 - \frac{8}{3}\right) - \frac{1}{3} \times 2 - \frac{1}{3} \times 4 = \frac{1}{3}$ by $R1$–$R5$ and $R7$.

Case 4.4. $t(x) = 3$. If $T(x) = (0, 3, 2)$, according to Lemma 21, then $\mu'(x) \geq \left(5 - \frac{8}{3}\right) - \frac{1}{3} \times 3 - \frac{1}{3} \times 1 - \frac{1}{3} \times 2 = \frac{1}{3}$ by $R1$–$R8$. If $T(x) = (1, 1, 3)$, then $\mu'(x) \geq \left(5 - \frac{8}{3}\right) - \frac{1}{3} \times 3 - \frac{1}{3} \times 4 = 0$ by $R1$–$R8$.

Case 4.5. $t(x) = 4$. If $T(x) = (0, 4, 1)$, according to Lemma 21, then $\mu'(x) \geq \left(5 - \frac{8}{3}\right) - \frac{1}{3} \times 4 - \frac{1}{3} \times 1 - \frac{1}{3} \times 1 = \frac{1}{3}$ by $R1$–$R8$. If $T(x) = (1, 2, 2)$, according to Lemma 21, then $\mu'(x) \geq \left(5 - \frac{8}{3}\right) - \frac{1}{3} \times 4 - \frac{1}{3} \times 1 - \frac{1}{3} \times 2 = 0$ by $R1$–$R8$. If $T(x) = (2, 0, 3)$, then $\mu'(x) \geq \left(5 - \frac{8}{3}\right) - \frac{1}{3} \times 4 - \frac{1}{3} \times 3 = 0$ by $R1$–$R5$ and $R7$.

Case 4.6. $t(x) = 5$. If $T(x) = (0, 5, 0)$, according to Lemma 21, then $\mu'(x) \geq \left(5 - \frac{8}{3}\right) - \frac{1}{3} \times 5 - \frac{1}{3} \times 1 = \frac{1}{3}$ by $R1$, $R6$ and $R8$. If $T(x) = (1, 3, 1)$, according to Lemma 21, then $\mu'(x) \geq \left(5 - \frac{8}{3}\right) - \frac{1}{3} \times 5 - \frac{1}{3} \times 1 - \frac{1}{3} \times 1 = 0$ by $R1$–$R8$.

Case 4.7. $t(x) = 6$. If $T(x) = (1, 4, 0)$, according to Lemma 22, then $\mu'(x) \geq \left(5 - \frac{8}{3}\right) - \frac{1}{3} \times 6 = \frac{1}{3}$ by $R1$. If $T(x) = (2, 2, 1)$, according to Lemma 23, then $\mu'(x) \geq \left(5 - \frac{8}{3}\right) - \frac{1}{3} \times 6 - \frac{1}{3} \times 1 = 0$ by $R1$–$R5$ and $R7$.

Case 5. $d(x) = 6$. If $d(x) = 6$, by Lemma 7(4), $t(x) \leq 7$ and $T(x) \neq (3, 1, 2)$.

Case 5.1. $t(x) \leq 4$. $\mu'(x) \geq \left(6 - \frac{8}{3}\right) - \frac{1}{3} \times 4 - \frac{1}{3} \times 6 = 0$ by $R1$–$R8$.

Case 5.2. $t(x) = 5$. If $T(x) = (0, 5, 1)$, according to Lemma 24, then $\mu'(x) \geq \left(6 - \frac{8}{3}\right) - \frac{1}{3} \times 5 - \frac{1}{3} \times 1 - \frac{1}{3} \times 1 = \frac{3}{3} = 1$ by $R1$–$R8$. If $T(x) = (1, 3, 2)$, then $\mu'(x) \geq \left(6 - \frac{8}{3}\right) - \frac{1}{3} \times 5 - \frac{1}{3} \times 5 = 0$ by $R1$–$R8$. If $T(x) = (2, 1, 3)$, then $\mu'(x) \geq \left(6 - \frac{8}{3}\right) - \frac{1}{3} \times 5 - \frac{1}{3} \times 4 = \frac{1}{3}$ by $R1$–$R8$.

Case 5.3. $t(x) = 6$. If $T(x) = (0, 6, 0)$, according to Lemma 24, then $\mu'(x) \geq \left(6 - \frac{8}{3}\right) - \frac{1}{3} \times 6 - \frac{1}{3} \times 1 = \frac{3}{3} = 1$ by $R1$, $R6$ and $R8$. If $T(x) = (1, 4, 1)$, according to Lemma 24, then $\mu'(x) \geq \left(6 - \frac{8}{3}\right) - \frac{1}{3} \times 6 - \frac{1}{3} \times 1 - \frac{1}{3} \times 1 = \frac{2}{3}$ by $R1$–$R8$. If $T(x) = (2, 2, 2)$, then $\mu'(x) \geq \left(6 - \frac{8}{3}\right) - \frac{1}{3} \times 6 - \frac{1}{3} \times 4 = 0$ by $R1$–$R8$. If $T(x) = (3, 0, 3)$, then $\mu'(x) \geq \left(6 - \frac{8}{3}\right) - \frac{1}{3} \times 6 - \frac{1}{3} \times 3 = \frac{1}{3}$ by $R1$–$R5$ and $R7$.

Case 5.4. $t(x) = 7$. If $T(x) = (1, 5, 0)$, according to Lemma 25, then $\mu'(x) \geq \left(6 - \frac{8}{3}\right) - \frac{1}{3} \times 7 = \frac{3}{3} = 1$ by $R1$. If $T(x) = (2, 3, 1)$, according to Lemma 25, then $\mu'(x) \geq \left(6 - \frac{8}{3}\right) - \frac{1}{3} \times 7 - \frac{1}{3} \times 1 = \frac{2}{3}$ by $R1$–$R5$ and $R7$.

Case 6. $d(x) = 7$. If $d(x) = 7$, by Lemma 7(5), $t(x) \leq 9$ and $T(x) \neq (3, 2, 2)$, $T(x) \neq (4, 1, 2)$.

Case 6.1. $t(x) \leq 6$. $\mu'(x) \geq \left(7 - \frac{8}{3}\right) - \frac{1}{3} \times 6 - \frac{1}{3} \times 7 = 0$ by $R1$–$R8$.

Case 6.2. $t(x) = 7$. If $T(x) = (0, 7, 0)$, according to Lemma 26, then $\mu'(x) \geq \left(7 - \frac{8}{3}\right) - \frac{1}{3} \times 7 - \frac{1}{3} \times 1 = \frac{5}{3}$ by $R1$, $R6$ and $R8$. If $T(x) = (1, 5, 1)$, then $\mu'(x) \geq (7 - \frac{8}{3}) - \frac{1}{3} \times 7 - \frac{1}{3} \times 6 = 0$ by $R1$–$R8$. If $T(x) = (2, 3, 2)$, then $\mu'(x) \geq \left(7 - \frac{8}{3}\right) - \frac{1}{3} \times 7 - \frac{1}{3} \times 5 = \frac{1}{3}$ by $R1$–$R8$. If $T(x) = (3, 1, 3)$, then $\mu'(x) \geq \left(7 - \frac{8}{3}\right) - \frac{1}{3} \times 7 - \frac{1}{3} \times 4 = \frac{2}{3}$ by $R1$–$R8$.

Case 6.3. $t(x) = 8$. If $T(x) = (1, 6, 0)$, according to Lemma 26, then $\mu'(x) \geq \left(7 - \frac{8}{3}\right) - \frac{1}{3} \times 8 - \frac{1}{3} \times 1 = \frac{4}{3}$ by $R1$–$R8$. If $T(x) = (2, 4, 1)$, then $\mu'(x) \geq \left(7 - \frac{8}{3}\right) - \frac{1}{3} \times 8 - \frac{1}{3} \times 5 = 0$ by $R1$–$R8$. If $T(x) = (4, 0, 3)$, then $\mu'(x) \geq \left(7 - \frac{8}{3}\right) - \frac{1}{3} \times 8 - \frac{1}{3} \times 3 = \frac{2}{3}$ by $R1$–$R5$ and $R7$.

Case 6.4. $t(x) = 9$. If $T(x) = (2, 5, 0)$, according to Lemma 26, then $\mu'(x) \geq \left(7 - \frac{8}{3}\right) - \frac{1}{3} \times 9 - \frac{1}{3} \times 1 = \frac{3}{3} = 1$ by $R1$, $R6$ and $R8$. If $T(x) = (3, 3, 1)$, then $\mu'(x) \geq \left(7 - \frac{8}{3}\right) - \frac{1}{3} \times 9 - \frac{1}{3} \times 4 = 0$ by $R1$–$R8$.

Case 7. $d(x) \geq 8$. If $d(x) \geq 8$, then $\mu'(x) \geq (d(x) - \frac{8}{3}) - \frac{1}{3} \times d(x) - \frac{1}{3} \times d(x) = \frac{1}{3} \times d(x) - \frac{8}{3} \geq 0$ by $R1$–$R8$.

We have proved that $\mu'(x) \geq 0$ for all $x \in V(G)$, then $\sum_{x \in V(G)} \mu'(x) \geq 0$. This contradicts the fact that $\sum_{x \in V(G)} \mu(x) < 0$. This completes the proof.

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