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DISSOCIATION IN CIRCULANT GRAPHS AND INTEGER DISTANCE GRAPHS

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Abstract

A dissociation set of a graph G is a set of vertices which induces a subgraph of G with maximum degree at most 1, or equivalently, a set of vertices whose complement in G is a 3-path vertex cover (intersecting every 3-path of G). The maximum cardinality of a dissociation set of G is called the dissociation number of G. We study the dissociation number of a circulant graph (a Cayley graph of the group $\mathbb{Z}_n$) and generalize this concept to the dissociation ratio of an integer distance graph (a Cayley graph of the group $\mathbb{Z}$).

Keywords: dissociation number, dissociation ratio, circulant graph, integer distance graph.

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1. Introduction

Let $G = (V, E)$ be a (finite, simple, undirected) graph with vertex set V and edge set E. Given a positive integer k, a *k-path vertex cover* of G is a set $T \subseteq V$ such that every path in G with k vertices contains some vertex in T. Let $\psi_k(G)$ denote the minimum cardinality of a k-path vertex cover of G. With motivations based on security conerns for communications in wireless sensor networks [3], minimum k-path vertex covers have been widely studied, especially for $k \leq 3$ [4, Section 1].

In fact, one sees that $\psi_1(G) = |V|$ and $\psi_2(G) = |V| - \alpha(G)$, where $\alpha(G)$ is the maximum cardinality of an independent set of G, known as the *independence number* of G. For $k = 3$, we have $T \subseteq V$ is a k-path vertex cover of G if and only if $V - T$ is a *dissociation set* of G, i.e., a set of vertices inducing a subgraph of G with maximum degree at most 1. Thus $\psi_3(G) = |V| - \operatorname{diss}(G)$, where the *dissociation number* $\operatorname{diss}(G)$ of G is the maximum cardinality of a dissociation set of G. The dissociation number was first studied by Yannakakis [19] and Yannakakis and Papadimitriou [14]. The computational complexity of finding a maximum dissociation set has been determined for many families of graphs [1, 3, 5, 13] and it is NP-hard even if G is bipartite or planar [19]. On the other hand, various lower and upper bounds for $\operatorname{diss}(G)$ have been obtained and we summarize some of them below [2, 3].

Theorem 1 [2, 3]**.** *Let G be a graph with n vertices.*

1. *If each vertex v has $d(v) > 0$ neighbors, then* $\operatorname{diss}(G) \geq \frac{4}{3}\sum_{v\in V(G)} \frac{1}{d(v)+1}$.
2. *If G has maximum degree $\Delta := \max_{v\in V} d(v)$, then* $\operatorname{diss}(G) \geq n/\lceil(\Delta+1)/2\rceil$.
3. *If G has m edges, then* $\operatorname{diss}(G) \geq (4n-m)/6$.
4. *If $d(v) = d \geq 2$ for every vertex v in G, then* $\operatorname{diss}(G) \leq nd/(2d-1)$.

Recently, Tu, Zhang and Du [17] and Tu, Zhang and Shi [18] investigated the maximum number of maximum dissociation sets in trees.

In this paper, we focus on circulant graphs and integer distance graphs. Let $\mathbb{Z}_n$ be the finite cyclic group of integers modulo n. Let S be a subset of $\{1, 2, \ldots, \lfloor n/2\rfloor\}$. A *circulant graph* $C(n, S)$ is a finite Cayley graph whose vertex set is $\mathbb{Z}_n$ and whose edge set is $\{ij : i, j \in \mathbb{Z}_n,\ |i-j| \in S\}$. We exclude 0 from S to make sure $C(n, S)$ has no loop. If $\gcd(S \cup \{n\}) = d$, then

$$\operatorname{diss}(C(n, S)) = d \cdot \operatorname{diss}(C(n/d, S/d)) \tag{1}$$

since $C(n, S)$ is a disjoint union of d copies of $C(n/d, S/d)$, where $S/d := \{s/d : s \in S\}$. Thus we may assume $\gcd(S \cup \{n\}) = 1$ if necessary.

Circulant graphs have many nice properties, such as high symmetry, fault-tolerance, and good routing capabilities. They are important topological structures for interconnection networks and widely used in telecommunication networks, VLSI design, and distributed computation (see, e.g., Monakhova [11]). When $|S| = 1$ (respectively, 2) the circulant graph $C(n, S)$ is called a *single-loop network* (respectively, *double-loop network*).

We study the dissociation number of the single-loop and double-loop networks in Section 2. We show that

$$\operatorname{diss}(C(n, \{s\})) = \begin{cases} n & \text{if } s = n/2, \\ \lfloor 2n/3\rfloor & \text{if } 1 \leq s < n/2 \text{ and } \gcd(n, s) = 1. \end{cases}$$

One can handle the case $\gcd(n, s) \neq 1$ by (1). For the double-loop network $\operatorname{diss}(C(n, \{s, t\}))$, where $1 \leq s < t \leq n/2$, we show that

$$\lceil n/3\rceil \leq \operatorname{diss}(C(n, \{s, t\})) \leq \lfloor n/2\rfloor$$

and give some sufficient conditions for $\mathrm{diss}(C(n, \{s, t\})) = \lfloor n/2 \rfloor$. In particular, the dissociation number of $C(n, \{1, s\})$, where $1 < s \leq n/2$, reaches the upper bound $\lfloor n/2 \rfloor$ unless $2 \mid s$ and the least nonnegative residue r of n modulo $2s$ satisfies $r > 0$, $2 \mid r$ and $(4 \mid r \Rightarrow 4 \mid s)$; in that case we have $\mathrm{diss}(C(n, \{1, s\})) = n/2 - 1$.

Now let S be a finite set of positive integers. The *integer distance graph* $\mathbb{Z}(S)$ is the Cayley graph with vertex set $\mathbb{Z}$ and edge set $\{ij : i, j \in \mathbb{Z},\ |i - j| \in S\}$. It can be viewed as the limit of the circulant graph $C(n, S)$ as $n \to \infty$. One can extend many parameters from a finite graph to an integer distance graph. For example, the independence ratio of an integer distance graph is related to the chromatic number and has been extensively studied (see, e.g., Carraher, Galvin, Hartke, Radcliffe and Stolee [6] and Liu [10]). The domination ratio of an integer distance graph was studied in our earlier work [8, 9] and an equivalent problem in terms of linear coverings of $\mathbb{Z}$ was investigated by Schmidt and Tuller [15, 16], Frankl, Kupavskii and Sagdeev [7], and others.

In Section 3 we study the dissociation ratio $\mathrm{diss}(\mathbb{Z}(S))$ of the integer distance graph $\mathbb{Z}(S)$, which is defined as the supremum of the (*upper*) *density*

$$\delta(D) := \limsup_{n \to \infty} \frac{|D \cap [-n, n]|}{2n + 1}$$

over all dissociation sets D of $\mathbb{Z}(S)$. If $\gcd S = d$ then

$$\mathrm{diss}(\mathbb{Z}(S)) = \mathrm{diss}(\mathbb{Z}(S/d)) \tag{2}$$

as $\mathbb{Z}(S)$ is the disjoint union of d copies of $\mathbb{Z}(S/d)$. Thus we may assume $\gcd S = 1$ if necessary. Our results on the dissociation ratio of integer distance graphs are summarized below.

Theorem 2. *Let S be a set of positive integers with $|S| = d < \infty$.*

1. *If $d = 1$, then* $\mathrm{diss}(\mathbb{Z}(S)) = 2/3$.
2. *If $d = 2$, then* $\mathrm{diss}(\mathbb{Z}(S)) = 1/2$.
3. *If $d = 3$, and $||T|| \neq ||T'||$ for all distinct $T, T' \subseteq S$, where $||T||$ denotes the sum of the elements of T, then* $\mathrm{diss}(\mathbb{Z}(S)) \leq 1/2$*; the equality holds if in addition, all elements of S are odd.*
4. *We have* $\mathrm{diss}(\mathbb{Z}(S)) \leq 2d/(4d - 1)$.

Furthermore, we show that the dissociation ratio of every integer distance graph must be achieved by a periodic dissociation set. We also relate the dissociation ratio of an integer distance graph to the dissociation number of a corresponding circulant graph.

In Section 4 we conclude the paper with some questions for future study.

2. Single-Loop and Double-Loop Networks

In this section we study dissociation in single-loop and double-loop networks.

For the single-loop network $C(n,\{s\})$, where $1 \le s \le n/2$, we may assume $\gcd(n,s) = 1$ if necessary, thanks to (1). Applying the bounds in Theorem 1 gives

$$\begin{cases} n/2 \le \operatorname{diss}(C(n,\{s\})) \le 2n/3 & \text{if } 1 \le s < n/2, \\ 2n/3 \le \operatorname{diss}(C(n,\{s\})) \le n & \text{if } s = n/2. \end{cases}$$

We determine the precise value of $\operatorname{diss}(C(n,\{s\}))$ below.

Proposition 3. (i) *If $s = n/2$, then* $\operatorname{diss}(C(n,\{s\}) = n$.

(ii) *If $1 \le s < n/2$ and* $\gcd(n,s) = 1$, *then* $\operatorname{diss}(C(n,\{s\}) = \lfloor 2n/3 \rfloor$.

Proof. (i) If $s = n/2$, then $C(n,\{s\})$ is a disjoint union of $n/2$ copies of K_2, and thus we have $\operatorname{diss}(C(n,\{s\})) = n$.

(ii) Suppose $1 \le s < n/2$ and $\gcd(n,s) = 1$. We have an isomorphism from the graph $C(n,\{1\})$ to the graph $C(n,\{s\})$ by sending each $i \in \mathbb{Z}_n$ to si. Thus we may assume that $s = 1$, which implies $n \ge 3$.

We have $\operatorname{diss}(C(n,\{1\}) \le 2n/3$ since a dissociation set of $C(n,\{1\})$ contains at most two of every three consecutive integers (modulo n). On the other hand, we have a dissociation set

$$D = \begin{cases} \{3i, 3i+1 : i = 0,1,\ldots,k-1\} & \text{if } n = 3k \text{ or } n = 3k+1, \\ \{3i, 3i+1 : i = 0,1,\ldots,k-1\} \cup \{3k\} & \text{if } n = 3k+2. \end{cases}$$

This implies $\operatorname{diss}(C(n,\{1\}) = \lfloor 2n/3 \rfloor$. ■

Remark 4. We can use Proposition 3 together with equation (1) to calculate $\operatorname{diss}(C(n,\{s\}))$ when $\gcd(n,s) \ne 1$, and the result does not always reach $\lfloor 2n/3 \rfloor$. For example, we have

$$\operatorname{diss}(C(8,\{2\})) = 2 \cdot \operatorname{diss}(C(4,\{1\})) = 2 \cdot 2 < \lfloor 2 \cdot 8/3 \rfloor.$$

Now consider a double loop network $C(n,S)$ with $S = \{s,t\}$ for distinct integers s and t satisfying $1 \le s, t \le n/2$. Applying Theorem 1 gives

$$\begin{cases} n/3 \le \operatorname{diss}(C(n,\{s,t\})) \le 4n/7 & \text{if } 1 \le s < t < n/2, \\ n/2 \le \operatorname{diss}(C(n,\{s,t\})) \le 3n/5 & \text{if } 1 \le s < t = n/2. \end{cases} \tag{3}$$

We provide a better result below.

Proposition 5. *If $1 \le s < t \le n/2$, then $\lceil n/3 \rceil \le \operatorname{diss}(C(n,\{s,t\})) \le \lfloor n/2 \rfloor$, and the upper bound is reached when $2 \nmid s$ and $t = n/2$.*

Proof. By the definition, every dissociation set D of $C(n, \{s, t\})$ must satisfy

$$\begin{cases} |D \cap \{i, i+t\}| = 2 \Rightarrow |D \cap \{i+s, i+s+t\}| = 0, \\ |D \cap \{i+s, i+s+t\}| = 2 \Rightarrow |D \cap \{i, i+t\}| = 0, \end{cases} \tag{4}$$

which implies $|D \cap \{i, i+s, i+t, i+s+t\}| \le 2$ for all $i \in \mathbb{Z}_n$. Thus

$$4|D| \le \sum_{i \in \mathbb{Z}_n} |D \cap \{i, i+s, i+t, i+s+t\}| \le 2n, \tag{5}$$

where the first inequality holds because the sum counts every element of $j \in D$ at least four times (for $i \in \{j, j-s, j-t, j-s-t\}$). It follows that $\mathrm{diss}(C(n, \{s, t\}) \le \lfloor n/2 \rfloor$. Combining this with the lower bounds in (3) gives the desired result when $1 \le s < t < n/2$.

Now suppose $1 \le s < t = n/2$ and $2 \nmid s$. It is not hard to explicitly construct a maximum dissociation set for $C(n, \{s, t\}))$. For instance, let $D = \{2i : i = 0, 1, \ldots, n/2 - 1\}$. Every vertex $2i \in D$ is adjacent to at most one vertex $2i + t \equiv 2i - t \pmod{n}$ in D and its other two neighbors $2i \pm s$ are not in D since $2 \nmid (2i \pm s)$. Thus D is a dissociation set with $|D| = n/2 = \mathrm{diss}(C(n, \{s, t\}))$. ■

We next give another sufficient condition for $\mathrm{diss}(C(n, \{s, t\})) = n/2$.

Theorem 6. *If* $2k \mid n$, $1 \le s, t \le n/2$, $s \equiv k \pmod{2k}$, $t \equiv r \pmod{2k}$ *with* $k/2 \le r \le 3k/2$ *and* $s \ne t$, *then* $\mathrm{diss}(C(n, \{s, t\})) = n/2$.

Proof. By Proposition 5, it suffices to exhibit a dissociation set D of $C(n, \{s, t\}))$ with $|D| = n/2$. Let

$$D = \{2ki + 1, 2ki + 2, \ldots, 2ki + k : i = 0, 1, \ldots, (n/2k) - 1\}.$$

Suppose, for the sake of contradiction, that some vertex $2ki+j \in D$ $(1 \le j \le k)$ is adjacent to two other vertices in D. We have $2ki+j\pm s \notin D$ since $2ki+j\pm s \equiv j+k \pmod{2k}$ and $k+1 \le j+k \le 2k$. Thus $2ki + j \pm t \in D$, i.e.,

$$2ki + j + t = 2ka + b \quad \text{and} \quad 2ki + j - t = 2kc + d$$

for some integers a, b, c, d with $1 \le b, d \le k$. It follows that

$$4ki + 2j = 2k(a+c) + b + d \quad \text{and} \quad 2t = 2k(a-c) + b - d.$$

If $a + c$ is odd then the first equation reduces to $2j \equiv 2k + b + d \pmod{4k}$, which is absurd since $1 \le j, b, d \le k$. Thus $a + c$ is even, and so is $a - c$. Then

$$t = k(a-c) + (b-d)/2 \equiv (b-d)/2 \pmod{2k}, \quad 1 - k \le b - d \le k - 1.$$

This contradicts the hypothesis that $t \equiv r \pmod{2k}$ with $k/2 \le r \le 3k/2$. ■

Example 7. If s and t are distinct odd numbers and n is even then $C(n, \{s, t\})$ has a dissociation set $\{2i : i = 0, 1, \ldots, \frac{n}{2} - 1\}$. If $4 \mid n$, $s \equiv 2 \pmod 4$ and $2 \nmid t$ then $C(n, \{s, t\})$ has a dissociation set $\{4i, 4i + 1 : i = 0, 1, \ldots, \frac{n}{4} - 1\}$. In either case we have $\mathrm{diss}(C(n, \{s, t\})) = n/2$.

Now we consider the double-loop network $C(n, \{1, s\})$, where $2 \le s \le n/2$.

Lemma 8. *Let D be a dissociation set of $C(n, \{1, s\})$, where $2 \le s < n/2$. If s is even, then there exists some $j \in \mathbb{Z}_n$ such that $|D \cap \{j, j + 1\}| = 0$.*

Proof. Working toward a contradiction, we may assume $|D \cap \{i, i + 1\}| = 1$ for all $i \in \mathbb{Z}_n$ by (4). Then $D = \{0, 2, 4, \ldots, n - 2\}$ or $D = \{1, 3, 5, \ldots, n - 1\}$ (in particular, n is even). In the former case we have a vertex $s \in D$ with two neighbors $0, 2s \in D$ and in the latter case we have a vertex $s + 1 \in D$ with two neighbors $1, 2s + 1 \in D$. Thus D cannot be a dissociation set. ∎

Theorem 9. (i) *If s is odd and $2 \le s \le n/2$, then* $\mathrm{diss}(C(n, \{1, s\})) = \lfloor n/2 \rfloor$.

(ii) *Suppose s is even and write $n = 2sk + r \ge 2s$ for some $k \ge 0$ and some $r \in \{0, 1, \ldots, 2s - 1\}$. Then*

$$\mathrm{diss}(C(n, \{1, s\})) = \begin{cases} \lfloor n/2 \rfloor & \text{if } r = 0,\ 2 \nmid r \text{ or } (4 \nmid s \text{ and } 4 \mid r), \\ \frac{n}{2} - 1 & \text{if } r > 0,\ 2 \mid r \text{ and } (4 \mid r \Rightarrow 4 \mid s). \end{cases}$$

Proof. (i) Suppose that s is odd and $2 \le s < n/2$ (the case $s = n/2$ is solved by Proposition 5). It is routine to check that $C(n, \{1, s\})$ has a dissociation set $D = \{0, 2, 4, \ldots, 2(\lfloor n/2 \rfloor - 1)\}$ whose cardinality is $\lfloor n/2 \rfloor$. Combining this with Proposition 5, we have $\mathrm{diss}(C(n, \{1, s\})) = \lfloor n/2 \rfloor$.

(ii) Suppose that s is even and $2 \le s \le n/2$. We can write $n = 2sk + r$ for some $k \ge 0$ and some $r \in \{0, 1, \ldots, 2s-1\}$. It is routine to check that $C(n, \{1, s\})$ has a dissociation set

$$\begin{aligned} &D(n, s) \\ &:= \{i+1, i+3, \ldots, i+s-1, i+s, i+s+2, \ldots, i+2s-2 : i = 0, 2s, 4s, \ldots, 2(k-1)s\} \\ &\cup \{2ks + 1, 2ks + 3, \ldots, 2ks + 2\lfloor (r + 1)/4 \rfloor - 1, n - 2\lfloor (r - 1)/4 \rfloor, \ldots, n - 4, n - 2\} \end{aligned}$$

whose cardinality is

$$|D(n, s)| = \begin{cases} ks & \text{if } r = 0, \\ ks + \lfloor (r + 1)/4 \rfloor + \lfloor (r - 1)/4 \rfloor & \text{if } r > 0. \end{cases}$$

If $r = 0$ or $2 \nmid r$ then $|D(n, s)| = \lfloor n/2 \rfloor$ and thus $\mathrm{diss}(C(n, \{1, s\})) = \lfloor n/2 \rfloor$ by Proposition 5.

If $4 \mid r$ and $4 \nmid s$ then we also have $\mathrm{diss}(C(n, \{1, s\})) = \lfloor n/2 \rfloor$ by Theorem 6 (the case $k = 2$ as discussed in Example 7).

Assume now that $r > 0$, $2 \mid r$ and $(4 \mid r \Rightarrow 4 \mid s)$. Then $|D(n,s)| = \frac{n}{2} - 1$. It suffices to show that every dissociation set D of $C(n, \{1, s\})$ must satisfy $|D| \leq \frac{n}{2} - 1$. We may assume $|D \cap \{i, i+1, i+s, i+s+1\}| = 2$ for all $i \in \mathbb{Z}_n$ by (5). By Lemma 8, $|D \cap \{j, j+1\}| = 0$ for some $j \in \mathbb{Z}_n$. Let $j = 0$, without loss of generality. Then

$$|D \cap \{s, s+1\}| = 2,\ |D \cap \{2s, 2s+1\} = 0,\ |D \cap \{3s, 3s+1\}| = 2, \ldots$$

It follows that

$$D = \{i+2, i+3 : i = 0, 4, 8, \ldots, n/4 - 1\}$$

and in particular, $4 \mid n$ (which implies $4 \mid r$) and $4 \nmid s$ (otherwise $3 \in D$ is adjacent to $2, s+2 \in D$). But this contradicts to the hypothesis on r and s. ■

Example 10. We can use a diagram to represent $D(n,s)$, where $\square$ and $\times$ correspond to integers inside and outside $D(n,s)$, respectively, with vertical bars dividing integers into blocks. Let $s = 4$ and $n = 8k + r$, where $r \in \{0, 1, \ldots, 7\}$. If $r \in \{0, 1, 2\}$ we have the following diagrams for $D(n,2) = \{i+1, i+3, i+4, i+6 : i = 0, 8, \ldots, k-1\}$.

$$\cdots|\times\square\times\square\square\times\square\times|\times\square\times\square\square\times\square\times|\times\square\times\square\square\times\square\times|\times\square\times\square\square\times\square\times|\cdots$$

$$\cdots|\times\square\times\square\square\times\square\times|\times\square\times\square\square\times\square\times|\times|\times\square\times\square\square\times\square\times|\times\square\times\square\square\times\square\times|\cdots$$

$$\cdots|\times\square\times\square\square\times\square\times|\times\square\times\square\square\times\square\times|\times\times|\times\square\times\square\square\times\square\times|\times\square\times\square\square\times\square\times|\cdots.$$

If $r \in \{3, 4\}$ then we have $D(n,2) = \{i+1, i+3, i+4, i+6 : i = 0, 8, \ldots, k-1\} \cup \{8k+1\}$:

$$\cdots|\times\square\times\square\square\times\square\times|\times\square\times\square\square\times\square\times|\times\square\times|\times\square\times\square\square\times\square\times|\times\square\times\square\square\times\square\times|\cdots$$

$$\cdots|\times\square\times\square\square\times\square\times|\times\square\times\square\square\times\square\times|\times\square\times\times|\times\square\times\square\square\times\square\times|\times\square\times\square\square\times\square\times|\cdots$$

If $r \in \{5, 6\}$ then $D(n,2) = \{i+1, i+3, i+4, i+6 : i = 0, 8, \ldots, k-1\} \cup \{8k+1, n-2\}$:

$$\cdots|\times\square\times\square\square\times\square\times|\times\square\times\square\square\times\square\times|\times\square\times\square\times|\times\square\times\square\square\times\square\times|\times\square\times\square\square\times\square\times|\cdots$$

$$\cdots|\times\square\times\square\square\times\square\times|\times\square\times\square\square\times\square\times|\times\square\times\times\square\times|\times\square\times\square\square\times\square\times|\times\square\times\square\square\times\square\times|\cdots$$

If $r = 7$ then $D(n,2) = \{i+1, i+3, i+4, i+6 : i = 0, 8, \ldots, k-1\} \cup \{8k+1, 8k+3, n-2\}$:

$$\cdots|\times\square\times\square\square\times\square\times|\times\square\times\square\square\times\square\times|\times\square\times\square\times\square\times|\times\square\times\square\square\times\square\times|\times\square\times\square\square\times\square\times|\cdots$$

Corollary 11. *The following results hold.*

(i) *For $n \geq 4$ we have*

$$\operatorname{diss}(C(n,\{1,2\}) = \begin{cases} 2k & \text{if } n \in \{4k, 4k+1, 4k+2\}, \\ 2k+1 & \text{if } n = 4k+3. \end{cases}$$

(ii) *For $n \geq 8$ we have*

$$\operatorname{diss}(C(n,\{1,4\}) = \begin{cases} 4k & \text{if } n \in \{8k, 8k+1, 8k+2\}, \\ 4k+1 & \text{if } n \in \{8k+3, 8k+4\}, \\ 4k+2 & \text{if } n \in \{8k+5, 8k+6\}, \\ 4k+3 & \text{if } n = 8k+7. \end{cases}$$

(iii) *For $n \geq 12$ we have*

$$\operatorname{diss}(C(n,\{1,6\}) = \begin{cases} 6k & \text{if } n \in \{12k, 12k+1, 12k+2\}, \\ 6k+1 & \text{if } n = 12k+3, \\ 6k+2 & \text{if } n \in \{12k+4, 12k+5, 12k+6\}, \\ 6k+3 & \text{if } n = 12k+7, \\ 6k+4 & \text{if } n \in \{12k+8, 12k+9, 12k+10\}, \\ 6k+5 & \text{if } n = 12k+11. \end{cases}$$

3. Dissociation Ratio of Integer Distance Graphs

Recall that an *integer distance graph* $\mathbb{Z}(S)$ is the Cayley graph with vertex set $\mathbb{Z}$ and edge set $\{ij : i, j \in \mathbb{Z},\ |i-j| \in S\}$, where S is a finite set of positive integers. In this section we study the *dissociation ratio* $\operatorname{diss}(\mathbb{Z}(S))$, which is defined as the supremum of the (*upper*) *density*

$$\delta(D) := \limsup_{n\to\infty} \frac{|D \cap [-n, n]|}{2n+1}$$

over all dissociation sets D of $\mathbb{Z}(S)$. We start with the case $|S| = 1$.

Proposition 12. *For any positive integer s we have* $\operatorname{diss}(\mathbb{Z}(\{s\})) = 2/3$.

Proof. We may assume $s = 1$ since $\operatorname{diss}(\mathbb{Z}(\{s\})) = \operatorname{diss}(\mathbb{Z}(\{1\}))$ by (2). Every dissociation set D of $\mathbb{Z}(\{1\})$ must satisfy $|D \cap \{i, i+1, i+2\}| \leq 2$ for all $i \in \mathbb{Z}$ by its definition. Thus

$$3|D \cap [-n, n]| \leq \sum_{i=-n-2}^{n} |D \cap \{i, i+1, i+2\}| \leq 2(2n+3).$$

It follows that

$$\delta(D) = \limsup_{n\to\infty} \frac{|D \cap [-n,n]|}{2n+1} \le \limsup_{n\to\infty} \frac{2(2n+3)}{3(2n+1)} = \frac{2}{3}.$$

On the other hand, $\{3i, 3i+1 : i \in \mathbb{Z}\}$ is a dissociation set of $\mathbb{Z}(\{1\})$ with density 2/3. Hence $\text{diss}(\mathbb{Z}(\{1\})) = 2/3$. ■

Next, we determine $\text{diss}(\mathbb{Z}(S))$ for $|S| = 2$. Recall that the *hypercube* Q_d has vertex set consisting of binary strings of length d and edge set consisting of unordered pairs of binary strings differing in exactly one position. Given $T \subseteq \mathbb{Z}$, let $||T||$ denote the sum of all elements of T.

Lemma 13. *Let S be a finite set of d positive integers such that $||T|| \ne ||T'||$ for all distinct subsets T, T' of S. Then* $\text{diss}(\mathbb{Z}(S)) \le \text{diss}(Q_d)/2^d$.

Proof. By the hypothesis on S, for any $i \in \mathbb{Z}$ there are exactly 2^d distinct elements in the set

$$S_i := \{i + ||T|| : T \subseteq S\}.$$

This set induces a subgraph of $\mathbb{Z}(S)$, which contains a copy of the hypercube Q_d as a spanning subgraph. Thus any dissociation set D of $\mathbb{Z}(S)$ can intersect S_i at most $\text{diss}(Q_d)$ times. Then

$$2^d|D \cap [-n,n]| \le \sum_{i=-n-||S||}^{n} |D \cap S_i| \le \text{diss}(Q_d)(2n+1+||S||)$$

which implies

$$\delta(D) = \limsup_{n\to\infty} \frac{|D \cap [-n,n]|}{2n+1} \le \limsup_{n\to\infty} \frac{\text{diss}(Q_d)(2n+1+||S||)}{2^d(2n+1)} = \frac{\text{diss}(Q_d)}{2^d}.$$

The result follows. ■

One can check that $||T|| \ne ||T'||$ for all distinct subsets T, T' of S whenever $|S| = 2$ but it may not hold when $|S| \ge 3$ (e.g., $S = \{1, 2, 3\}$). Using Lemma 13 we establish more upper bounds.

Theorem 14. *Let S be a finite set of positive integers. If $|S| = 2$ or if $|S| = 3$ and $||T|| \ne ||T'||$ for all distinct subsets T, T' of S, then* $\text{diss}(\mathbb{Z}(S)) \le 1/2$.

Proof. It follows from work of Brešar, Jakovac, Katrenič, Semanišin and Taranenko [2] that

$$2^{d-1} \le \text{diss}(Q_d) \le 2^d \cdot d/(2d-1)$$

where $d = |S|$. For $d = 2, 3$, taking the floor of the above upper bound of $\text{diss}(Q_d)$ and applying Lemma 13 we have the desired result. ■

Proposition 15. *Let S be a finite set of odd positive integers. Then* $\operatorname{diss}(\mathbb{Z}(S)) \geq 1/2$ *and the equality holds if $|S| = 2$ or if $|S| = 3$ and $||T|| \neq ||T'||$ for all distinct $T, T' \subseteq S$.*

Proof. One sees that $D = \{2i : i \in \mathbb{Z}\}$ is a dissociation set of $\mathbb{Z}(S)$ since S has no even element. Thus $\operatorname{diss}(\mathbb{Z}(S)) \geq \delta(D) = 1/2$. If $|S| = 2$ or if $|S| = 3$ and $||T|| \neq ||T'||$ for all distinct $T, T' \subseteq S$ then $\operatorname{diss}(\mathbb{Z}(S)) = 1/2$ by Theorem 14. ∎

Proposition 16. *We have* $\operatorname{diss}(\mathbb{Z}(\{1, s\})) = 1/2$ *for any $s \geq 2$.*

Proof. We may assume $2 \mid s$ by Proposition 15. One sees that $\mathbb{Z}(\{1, s\})$ has a dissociation set $D = \{4i, 4i + 1 : i \in \mathbb{Z}\}$ if $4 \nmid s$ or

$$D = \{2si, 2si + 2, \ldots, 2si + s - 2, 2si + s + 1, 2si + s + 3, \ldots, 2si + 2s - 1 : i \in \mathbb{Z}\}$$

if $4 \mid s$. Thus $\operatorname{diss}(\mathbb{Z}(\{1, s\})) = \delta(D) = 1/2$ by Theorem 14. ∎

We generalize the above proposition to $\mathbb{Z}(\{s, t\})$.

Theorem 17. *If s and t are distinct positive integers, then* $\operatorname{diss}(\mathbb{Z}(\{s, t\})) = 1/2$.

Proof. Since $S = \{s, t\}$ satisfies the hypothesis of Theorem 14, we must have $\operatorname{diss}(\mathbb{Z}, \{s, t\}) \leq 1/2$. Thus it suffices to construct a dissociation set D for $\mathbb{Z}(\{s, t\})$ with density $\delta(D) = 1/2$. We may assume s is even and t is odd with $\gcd(s, t) = 1$ by (2) and Proposition 15.

There must exist a positive integer m such that $2^m \mid s$ and $2^{m+1} \nmid s$. If $t \equiv \pm 1, \pm 3, \ldots, \pm(2^m - 3) \pmod{2^{m+1}}$ then $\mathbb{Z}(\{s, t\})$ has a dissociation set

$$D = \left\{2^{m+1} \cdot i + r : i \in \mathbb{Z},\ r = 0, 2, 4, \ldots, 2^m - 2, 2^m + 1, 2^m + 3, \ldots, 2^{m+1} - 1\right\}.$$

If $t \equiv \pm(2^m - 1) \pmod{2^{m+1}}$ then $\mathbb{Z}(\{s, t\})$ has a dissociation set

$$D = \left\{2^{m+1} \cdot i + r : i \in \mathbb{Z},\ r = 0, 1, \ldots, 2^m - 1\right\}.$$

In either of the above cases, we have $\delta(D) = 1/2$. ∎

Example 18. Assume s is even and t is odd. We give some examples for the construction of a dissociation set with density $1/2$ in the above proof.

If $s \equiv 2 \pmod 4$, then $t \equiv \pm 1 \pmod 4$ and $D = \{4i, 4i + 1 : i \in \mathbb{Z}\}$ is a dissociation set for $\mathbb{Z}(\{s, t\})$ with $\delta(D) = 1/2$. If $4 \mid s$ then we distinguish two cases below.

If $s \equiv 4 \pmod 8$ and $t \equiv \pm 1 \pmod 8$ then $D = \{8i, 8i + 2, 8i + 5, 8i + 7 : i \in \mathbb{Z}\}$ is a dissociation set for $\mathbb{Z}(\{s, t\})$.

If $s \equiv 4 \pmod 8$ and $t \equiv \pm 3 \pmod 8$ then $D = \{8i, 8i + 1, 8i + 2, 8i + 3 : i \in \mathbb{Z}\}$ is a dissociation set for $\mathbb{Z}(\{s, t\})$.

It remains to consider the case $8 \mid s$.

If $s \equiv 8 \pmod{16}$ and $t \equiv \pm 1, \pm 3, \pm 5 \pmod{16}$ then $\mathbb{Z}(\{s, t\})$ has a dissociation set

$$D = \{16i + r : i \in \mathbb{Z},\ r = 0, 2, 4, 6, 9, 11, 13, 15\}.$$

If $s \equiv 8 \pmod{16}$ and $t \equiv \pm 7 \pmod{16}$ then $\mathbb{Z}(\{s, t\})$ has a dissociation set

$$D = \{16i + r : i \in \mathbb{Z},\ r = 0, 1, \ldots, 7\}.$$

We can repeat the above process to handle the remaining case.

Remark 19. If $4 \mid s$ and $t = 1$ then $\mathbb{Z}(\{1, s\})$ has another dissociation set with density $1/2$:

$$\{2si, 2si + 2, \ldots, 2si + s - 2, 2si + s + 1, 2si + s + 3, \ldots, 2si + 2s - 1 : i \in \mathbb{Z}\}.$$

For each of the integer distance graphs discussed so far, the dissociation ratio is achieved by a periodic dissociation set. We will show that this is not an coincidence. Here a set $U \subseteq \mathbb{Z}$ is *periodic* if there exists a positive integer m such that

$$U \cap [im + 1, im + m] = \{im + j : j \in U \cap [1, m]\}, \quad \forall i \in \mathbb{Z}.$$

The smallest such integer m is the *period* of U. It is easy to calculate the density of a periodic set.

Lemma 20 [8]**.** *If U is a periodic subset of $\mathbb{Z}$ with period m then we have $\delta(U) = |U \cap [1, m]|/m$.*

Suppose that S is a finite set of positive integers. Let

$$a := \max S \cup \{0\}, \quad b := -\min S \cup \{0\}, \quad \text{and} \quad c := a + b.$$

Also let $[m, n]$ denote the set $\{x \in \mathbb{Z} : m \leq x \leq n\}$. A *state* is a subset of $[1, c]$. A state T is *admissible* if there exists a dissociation set D of $\mathbb{Z}(S)$ such that $D \cap [ic + 1, (i + 1)c] = T + ic$ for some $i \in \mathbb{Z}$. A *transition* occurs between two states T and T' if there exists a dissociation set D of $\mathbb{Z}(S)$ such that $D \cap [ic + 1, (i + 1)c] = T + ic$ and $D \cap [(i + 1)c + 1, (i + 2)c] = T' + (i + 1)c$ for some $i \in \mathbb{Z}$. The *state graph* associated with $\mathbb{Z}(S)$ is a graph whose vertices are the admissible states and whose edges are transitions. The *weight* of a state T is $|T|/c$. A *doubly infinite walk* in the state graph is a sequence $(T_i : i \in \mathbb{Z})$ of states such that there is an edge between T_i and T_{i+1} for all $i \in \mathbb{Z}$. The *upper average weight* of this walk is

$$\limsup_{m \to \infty} \sum_{i \in [-m, m]} \frac{|T_i|}{(2m + 1)c}.$$

Proposition 21. *Let S be a finite set of positive integers. Let $a := \max S \cup \{0\}$, $b := -\min S \cup \{0\}$, and $c := a+b$. Then the dissociation ratio of $\mathbb{Z}(S)$ is achieved by some periodic dissociation set with period at most $c2^c$.*

Proof. Any doubly infinite walk $(T_i : i \in \mathbb{Z})$ in the state graph of $\mathbb{Z}(S)$ gives a set

$$D := \bigcup_{i \in \mathbb{Z}} (T_i + ic).$$

We show that D is a dissociation set of $\mathbb{Z}(S)$. Suppose some integer $j \in D$ is adjacent to two other integers $j+s$ and $j+t$ in D, where $s, t \in S$.

Since $|j+s-(j+t)| = |s-t| \le a+b = c$, we must have $\{j, j+s, j+t\} \subseteq [ic+1, (i+2)c]$ for some $i \in \mathbb{Z}$. Since there is an edge between T_i and T_{i+1} in the state graph, there exists a dissociation set D' such that

$$D' \cap [ic+1, (i+1)c] = T_i + ic,$$
$$D' \cap [(i+1)c+1, (i+2)c] = T_{i+1} + (i+1)c.$$

Thus $\{j, j+s, j+t\} \nsubseteq D' \cap [ic+1, (i+2)c] = (T_i + ic) \cup (T_{i+1} + (i+1)c) = D \cap [ic+1, (i+2)c]$. It follows that D is a dissociation set of $\mathbb{Z}(S)$.

Conversely, a dissociation set D of $\mathbb{Z}(S)$ corresponds to a doubly infinite walk $(T_i : i \in \mathbb{Z})$ in the state graph, where $T_i := D \cap [ic+1, (i+1)c]$. The upper average weight of this walk equals

$$\limsup_{m\to\infty} \sum_{i \in [-m,m]} \frac{|T_i|}{(2m+1)c} = \limsup_{m\to\infty} \frac{|D \cap [-mc+1, mc+c]|}{(2m+1)c} = \delta(D),$$

where the last equality follows from a result in previous work [8, Lemma 2.1].

We know that the supremum of the upper average weights of doubly infinite walks is achieved by repeating some simple cycle in the state graph [6, Lemma 3]. Since the length of this cycle is at most 2^c, the dissociation ratio of $\mathbb{Z}(S)$ can be achieved by some periodic dominating set with period at most $c2^c$. ■

Proposition 21 shows that the dissociation ratio of the integer distance graph $\mathbb{Z}(S)$ is achieved by some periodic dominating set of period m. Let $\mathbb{Z}_m := \{1, 2, \ldots, m\}$ be the cyclic group of order m under addition modulo m. Let S_m be the subset of $\mathbb{Z}_m$ consisting of all least positive residues of elements in S modulo m. We conclude this section by relating the dissociation ratio of the integer distance digraph $\mathbb{Z}(S)$ and the dissociation number of the circulant digraph $C(m, S_m)$.

Proposition 22. *Assume S is a finite set of positive integers. Let D be a dissociation set of $\mathbb{Z}(S)$ with period m such that $\operatorname{diss}(\mathbb{Z}(S)) = \delta(D) = |D \cap [1,m]|/m$. Then $D \cap [1,m]$ is a maximum dissociation set of $C(m, S_m)$ and $\operatorname{diss}(C(m, S_m)) = |D \cap [1,m]| = m \cdot \operatorname{diss}(\mathbb{Z}(S))$.*

Proof. We first show that $D \cap [1, m]$ is a dissociation set of $C(m, S_m)$. Suppose, for the sake of contradiction, that some $i \in D \cap [1, m]$ has two neighbors in $D \cap [1, m]$, which can be written as $i + s$ and $i + t$ for some distinct $s, t \in S_m$. Then there must exist distinct $s', t' \in S$ such that $s \equiv s' \pmod{m}$ and $t \equiv t' \pmod{m}$. Thus $i + s', i + t' \in D$ since $i + s, i + t \in D \cap [1, m]$ and D has period m, and i is adjacent to both $i + s'$ and $i + t'$ in $\mathbb{Z}(S)$. This gives a contradiction to the hypothesis that D is a dissociation set of $\mathbb{Z}(S)$.

Now let E be a maximum dissociation set of $C(m, S_m)$. We show that $\overline{E} := \cup_{k\in\mathbb{Z}}(E + km)$ is a dissociation set of $\mathbb{Z}(S)$. Suppose a vertex $i \in \overline{E}$ has two neighbors in $\overline{E}$, which can be written as $i+s$ and $i+t$ for some distinct $s, t \in S$. Let i_0, s_0, t_0 be the least positive residue of i, s, t modulo m. Then $i_0, i_0+s_0, i_0+t_0 \in E$ and i_0 is adjacent to $i_0 + s_0$ and $i_0 + t_0$ in $C(m, S_m)$ since $s_0, t_0 \in S_m$. This contradicts the hypothesis that E is a dissociation set of $C(m, S_m)$.

Combining the above two paragraphs we have

$$\mathrm{diss}(\mathbb{Z}(S)) \geq \delta(\overline{E}) = |E|/m = \mathrm{diss}(C(m, S_m))/m \geq |D \cap [1, m]|/m = \mathrm{diss}(\mathbb{Z}(S)),$$

where the two inequalities must both be equalities. The result follows. ∎

We can apply Proposition 22 to the periodic dissociation sets constructed in this section and recover the following results, which are special cases of the results in Section 2:

1. $\mathrm{diss}(C(2, \{1\})) = \mathrm{diss}(C(3, \{1\})) = 2$,
2. $\mathrm{diss}(C(4, \{1, 2\})) = 2$,
3. $\mathrm{diss}(C(2s, \{1, s\})) = s$, and
4. $\mathrm{diss}(C(2^{m+1}, \{2^m, t\})) = 2^m$ if $1 \leq t < 2^m$ is odd.

We can also use Proposition 22 to establish a general upper bound for $\mathrm{diss}(\mathbb{Z}(S))$.

Theorem 23. *Let S be a set of positive integers with $|S| = d < \infty$. Then*

$$\mathrm{diss}(\mathbb{Z}(S)) \leq 2d/(4d - 1).$$

Proof. By Proposition 21, the graph $\mathbb{Z}(S)$ has a dissociation set D with period m such that $\mathrm{diss}(\mathbb{Z}(S)) = \delta(D)$. We may assume that $m > \max S - \min S$, since we can replace the period of D with any of its positive multiples. This guarantees that $|S_m| = |S| = d$. It follows from Proposition 22 and Theorem 1 that

$$\mathrm{diss}(\mathbb{Z}(S)) = |D \cap [1, m]|/m = \mathrm{diss}(C(m, S_m))/m \leq 2d/(4d - 1).$$ ∎

Remark 24. The upper bound Theorem 23 is achieved when $|S| = 1$ by Proposition 12 but not as good as Theorem 14 when $|S| = 2, 3$. The proof of Theorem 14 can be used to show that $\mathrm{diss}(\mathbb{Z}(S)) \leq 9/16$ if $|S| = 4$ and $\mathrm{diss}(\mathbb{Z}(S)) \leq 17/32$ if $|S| = 5, 6, 7, 8$, but these upper bounds are worse than Theorem 23.

4. Conclusion

In this paper we focus on the dissociation number of some single-loop and double-loop networks and extend it to the dissociation ratio of the integer distance graphs. In particular, the dissociation ratio of any integer distance graph $\mathbb{Z}(S)$ with $|S| = 2$ equals $1/2$ by Theorem 17. This suggests that the dissociation number of a double-loop network should be equal or very close to $n/2$ at least when n is large. We have some partial results in Section 2 and one can try to complete this in the future.

It would also be nice to find the dissociation number of $C(n, S)$ and the dissociation ratio of $\mathbb{Z}(S)$ when $|S| \geq 3$; we only have some upper bounds for the latter in Section 3. Our proof in one case involves the well-known hypercube and its dissociation number is yet to be determined.

For any integer distance graph, the domination ratio turns out to be equivalent to the minimal covering density, which was studied by Newman [12], Schmidt and Tuller [15, 16], Frankl, Kupavskii, and Sagdeev [7], and others. It would be interesting to see whether the techniques used to study covering density could be adapted for dissociation.

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