Discussiones Mathematicae
Graph Theory xx (xxxx) 1–9
https://doi.org/10.7151/dmgt.2562

POISON GAME FOR SEMIKERNELS OF ARBITRARY DIGRAPHS

Michał Walicki

Department of Informatics
University of Bergen, Norway
e-mail: Michal.Walicki@uib.no

Abstract

A modified version of Duchet and Meyniel's Poison Game is presented, which can be played on arbitrary digraphs without sinks. Player A has a winning strategy if and only if the graph has a semikernel; the winning condition for this player is that the game is infinite. Even for the uncountable graphs, it suffices to consider games with up to ω steps.

Keywords: infinite digraphs, kernels, semikernels.

2020 Mathematics Subject Classification: 05C57, 05C63, 05C99.

A *kernel* of a graph (which here always means a digraph) is an independent subset K of vertices with an incoming edge from every vertex $v \notin K$. Kernels have been studied since their introduction in [13], as this notion captures various important concepts in graph theory and its different applications. For instance, kernels played an important role in the search for a proof of the strong perfect graph theorem, presented eventually in [5] (since a graph is perfect if and only if every orientation of it, for which each complete subgraph is acyclic, has a kernel). They provide tools for analysing some games, where their existence implies a winning strategy, e.g., [4, 13]. In classical logic, kernels can be used to capture logical consistency and, with graph cycles modelling self-reference, support study of paradoxes, e.g., [6, 14].

A *semikernel*, generalising the notion of a kernel, is a nonempty independent subset L of vertices with an incoming edge from every out-neighbour of every vertex $v \in L$. A graph is *kernel perfect, KP*, if each of its nonempty induced subgraphs has a kernel. As shown in [10], a graph is KP if each of its nonempty induced subgraphs has a semikernel. By this fact, semikernels play often a central role in proving the existence of a kernel, by showing kernel perfectness.

A finite graph without odd cycles has a kernel. This (consequence of) Richardson's theorem [11] has been extended in many ways, e.g., [1–4, 7, 9]. The theorem holds also for infinite graphs with no infinite out-branching or no *rays* (infinite simple outgoing paths), but these are very restrictive conditions. A concise characterization of kernel perfectness for infinite graphs in [12] provides, unfortunately, no structural conditions on the graphs. Such conditions from [15], sufficient for graphs with finitely many ends, are only expected but not known to suffice for all graphs. Providing sufficient structural conditions for the existence of kernels in arbitrary infinite graphs remains an open problem.

This note contributes a tool for verifying the (non)existence of a semikernel in a graph, rather than any such general conditions. In particular, it shows that a certain form of cardinality restriction is possible when considering infinite graphs. It introduces a two-player, sequential, perfect information game, which modifies Poison Game from [8], extending its applicability from finite to arbitrary graphs. In both variants of the game, Player A has a winning strategy if and only if the graph has a semikernel. The central feature of the new game is that a winning strategy for Player B, on an arbitrary graph, allows him to win in finite time. Thus, even on an uncountable graph, the existence of a semikernel is witnessed by a strategy allowing A to survive merely ω steps of each play.

Sinks (vertices with no out-neighbours) are semikernels, so we skip the game's trivial generalisation to graphs with sinks and consider only those without.

For a brief comparison, we recall the original game from [8].

Poison Game I. Player A starts by choosing a vertex, and then B and A choose alternately vertices from the out-neighbours of the opponent's last choice. B poisons the visited vertices, but can re-visit them. On entering a poisoned vertex, Player A dies and B wins the play. Player A wins by surviving (in particular, when B has no move).

By Theorem 1 from [8], a graph with no loops, no rays and no infinite out-branching has a semikernel if and only if A has a winning strategy. We extend this result to the following version of the game played on arbitrary sink-free graphs.

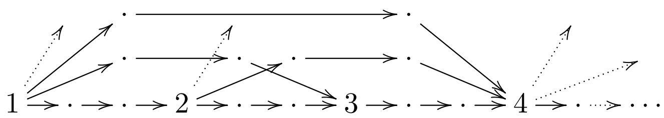
Poison Game II. Player A starts by choosing a vertex, and then B and A choose alternately vertices: A from the out-neighbours of the last vertex chosen by B, while B from the out-neighbours of all vertices chosen so far by A. Player A poisons all (in- and out-)neighbours of the chosen vertices. Player A loses visiting a poisoned vertex and wins by surviving. (B can visit poisoned vertices unharmed.) In a transfinite play, B starts after each limit ordinal.

The limitations of Game I motivate the changes in Game II, as shown by the examples below.

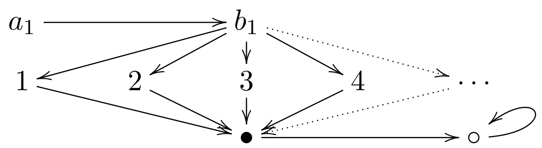
1. Let Y be the graph $(\omega, <)$, with the natural numbers as vertices and edges from each number to all greater ones. It has no semikernel, but A wins Game

I on it. Retaining the poisoning rule, we could allow B to choose from the out-neighbours of all choices of A. Player A still wins such a game, choosing always a vertex past all chosen by B.

2. Similarly, it does not suffice to change merely the poisoning rule to the one in Game II, but keep B choosing only from the out-neighbourhood of the last choice of A. The graph below is obtained from Y by subdividing each edge $i < j$ twice (the numbered vertices are from the original Y). It has no semikernel, but A wins such a game choosing any out-neighbour of the last choice of B. (On this graph, A wins also Game I.)



3. Player A cannot choose freely from the out-neighbours of all vertices chosen by B, since then A could avoid providing witness to some choices of B. The graph below has no semikernel but, with such a liberal rule, A wins from a_1 choosing forever $1, 2, 3, \ldots$, also after B chose $\bullet$.



The following notation is used. For a graph G, we let $\mathbf{V}$ denote its vertex set, $\mathbf{E}$ its edge relation, and $\mathbf{E}^-$ the converse of $\mathbf{E}$. For $x \in \mathbf{V}$, by $\mathbf{E}(x)$ we denote the set $\{y \in \mathbf{V} : \mathbf{E}(x, y)\}$ of out-neighbours of x, by $\mathbf{E}^-(x)$ its in-neighbours $\{y \in \mathbf{V} : \mathbf{E}(y, x)\}$, and we set $\mathbf{E}^\pm(x) = \mathbf{E}(x) \cup \mathbf{E}^-(x)$. For $X \subseteq \mathbf{V}$, we let $\mathbf{E}(X) = \bigcup_{x \in X} \mathbf{E}(x)$ and similarly for $\mathbf{E}^-(X)$ and $\mathbf{E}^\pm(X)$. A kernel is a subset K of $\mathbf{V}$ such that $K = \mathbf{V} \setminus \mathbf{E}^-(K)$, while a semikernel is a nonempty subset L of $\mathbf{V}$ such that $\mathbf{E}(L) \subseteq \mathbf{E}^-(L) \subseteq \mathbf{V} \setminus L$.

We consider also transfinite games, but only to show that they are not needed. For an ordinal κ, a κ-game is a function $\kappa \to \mathbf{V}$, indicated by a sequence $a_0 b_1 a_1 b_2 a_2 \cdots b_i a_i \cdots$, with $i \leq \kappa$ ($i < \kappa$ for limit κ), of pairs of vertices obeying the rules of Poison Game II. (Dropping b_0 in the numbering reflects A starting with a_0 and then responding to b_i with $a_i \in \mathbf{E}(b_i)$.) For $i \leq \kappa$, we let A_i denote the set of vertices visited by A up to step i; similarly for B_i. Player A wins a κ-game if A_κ is *independent* (no edge joins any two vertices in it). Otherwise, Player B wins.

Player B reaches a winning position choosing a b_j such that $\mathbf{E}(b_j) \subseteq \mathbf{E}^\pm(A_i)$, for some i with $1 \leq i < j$. Player A must then choose an $a_j \in \mathbf{E}^\pm(a_k)$, for some $a_k \in A_i$, destroying independence of the set A_j. (Although B has won, the play can continue.) Each vertex visited in a game on a graph G is reachable from a_0 by a G-path of the alternating moves of A and B. This path is a finite subsequence of the game.

A *play* is the process of forming a game, with κ-plays yielding κ-games. A winning strategy for a player on a graph G ensures a win for this player in every play on G, no matter the moves of the opponent. Strategies need not be computable and, in general, amount to prescience — A knowing a semikernel, or B how to poison A in a finite time. We establish these two claims.

In Game II, as in I, a winning strategy for A on a graph G is equivalent to G possessing a semikernel. As a minor difference, in Game II such a semikernel contains all moves of A.

Theorem 1. *Player A has a winning strategy on a sink-free graph G if and only if G has a semikernel.*

Proof. If there is a semikernel, then A never gets poisoned by choosing always a vertex from it. For the converse, assume that A has a winning strategy. We design a strategy ensuring that B visits all out-neighbours of all vertices played by A. The reader accepting existence of such a strategy for B can go directly to the last paragraph of this proof.

For a vertex $x \in \mathbf{V}$, let $\vec{\mathbf{E}}(x)$ denote a well-ordering (assumig Axiom of Choice) of the out-neighbourhood $\mathbf{E}(x)$. This ordering is used by B to visit systematically all out-neighbours of each vertex visited by A. The successive moves $a_0a_1\cdots$ of A order respectively the corresponding out-neighbourhoods: $\vec{\mathbf{E}}(a_0) < \vec{\mathbf{E}}(a_1) < \cdots$. The drawing gives a schematic picture of B's strategy.

$$
\begin{array}{lcccccl}
a_0: & b_1 \longrightarrow & b_2 & b_4 & b_7 & b_{11} & \cdots = \vec{\mathbf{E}}(a_0) \\
a_1: & b_3 & b_5 & b_8 & \circ & \circ & \cdots = \vec{\mathbf{E}}(a_1) \\
a_2: & b_6 & b_9 & \circ & \circ & & \cdots = \vec{\mathbf{E}}(a_2) \\
a_3: & b_{10} & \circ & \circ & \circ & & \cdots = \vec{\mathbf{E}}(a_3) \\
\vdots & \vdots & \vdots & & & &
\end{array}
$$

A play starts with $a_0b_1a_1b_2$, where $b_1, b_2 \in \mathbf{E}(a_0)$ are the first two vertices in $\vec{\mathbf{E}}(a_0)$. (If $\mathbf{E}(a_0)$ has only one vertex, then b_2 is the first in $\vec{\mathbf{E}}(a_1)$.) For $j \geq 2$, after $a_j \in \mathbf{E}(b_j)$ there are two cases for the choice of b_{j+1}. Case (1), marked with dotted arrows, occurs when b_j is the very first vertex of $\vec{\mathbf{E}}(a_i)$, for some $i \leq j$. In this case, B finds the first k with $k \leq j$ such that $\vec{\mathbf{E}}(a_k)$ still contains vertices unvisited by B, and chooses as b_{j+1} the first such unvisited vertex from $\vec{\mathbf{E}}(a_k)$. (On the drawing, $k = 0$.) Case (2), marked by solid arrows, occurs when b_j is the n-th vertex of $\vec{\mathbf{E}}(a_i)$, for some $i \leq j$ and $n > 1$. Player B chooses then as b_{j+1} the first unvisited vertex of $\vec{\mathbf{E}}(a_{i+1})$. (If all vertices in $\mathbf{E}(a_{i+1})$ have been visited, then B goes to the first $\mathbf{E}(a_k)$, with $i < k \leq j$, having unvisited vertices. If all such $\mathbf{E}(a_k)$ have been visited in their entirety, then B restarts with $k = 0$ as in case (1).)

In either case, if the search for an unvisited vertex fails, then B has visited all vertices in $\bigcup_1^j \mathbf{E}(a_k)$ and can only make a move b_{j+1} repeating some earlier move. Player A can then repeat earlier answer, and the game can continue thus only repeating earlier moves. The union $\bigcup_1^j \mathbf{E}(a_k)$ of out-neighbourhoods of all vertices played by A is then a finite set, giving a special case of the general situation.

In general, for a countable graph, this strategy of B exhausts in ω steps the out-neighbourhood of each vertex played by A. For an uncountable graph, the play continues in this way transfinitely. At step $\lambda + 1$, for each limit ordinal λ, Player B chooses as in case (1) the least vertex unvisited by B in $\vec{\mathbf{E}}(a_i)$, for the least i with $i < \lambda$ and $\mathbf{E}(a_i)$ still having such vertices. The play continues until B exhausts out-neighbourhoods of all choices of A.

For the (least) ordinal κ at which B has visited all out-neighbours of all vertices visited by A, we have $\mathbf{E}(A_\kappa) \subseteq B_\kappa \subseteq \mathbf{E}^-(A_\kappa)$. The second inclusion holds since A provided an out-neighbour for each vertex in B_κ. Since A survived, no vertex in A_κ is poisoned, $A_\kappa \cap \mathbf{E}^\pm(A_\kappa) = \emptyset$, which means that A_κ is independent. Thus A_κ is a semikernel. ∎

Consequently, the game is determined; on each sink-free graph, exactly one of the players has a winning strategy. Player A wins moving always in a semikernel, if one exists. Otherwise B, searching exhaustively through the out-neighbours of all vertices visited by A, forces eventually A to choose a vertex with an edge joining it to some vertex chosen earlier by A.

If B follows this brute-force strategy on a countable graph, then the game ends in no more than ω steps. If the graph has no semikernel, then B wins after finitely many steps.

Fact 2. *If B has a winning strategy on a sink-free countable graph, then B can win every play after finitely many steps.*

On uncountable graphs, this brute-force strategy of B may require uncountable plays. To show that such plays are not needed, we consider κ-plays with ordinals κ possibly smaller than the cardinality of $\mathbf{V}$. Such games are also determined. If B has a winning strategy, then A does not have it. The less obvious opposite implication holds because A wins by not losing.

Fact 3. *For every graph and ordinal κ, if B does not have a winning strategy for κ-plays, then A has it.*

Proof. If B has no winning strategy for κ-plays, then A can start with some a_0 after which B still does not have it. Any move b_1 of B can be then answered by A with some a_1, after which B still does not have a winning strategy. This ensures survival of A for ω steps without giving B any winning strategy also when making the first move after the limit. The same holds for all greater successor and limit ordinals until κ. ∎

One more fact is used.

Fact 4. *For each graph and limit ordinal λ, if A has a strategy σ for winning κ-plays for all $\kappa < \lambda$, then σ is also a winning strategy for A in λ-plays.*

Proof. Every λ-play g_λ is the limit of its prefixes, κ-plays g_κ, for $\kappa < \lambda$. If B wins g_λ, in which A follows σ, then A_λ contains a pair $\{a_m, a_n\}$ with $a_n \in \mathbf{E}^{\pm}(a_m)$. Since $A_\lambda = \bigcup_{\kappa<\lambda} A_\kappa$, such a pair is contained in A_κ for some $\kappa < \lambda$. Hence B wins already g_κ, but A following σ wins all κ-plays. This contradiction establishes the fact. ∎

The three facts give the second main claim.

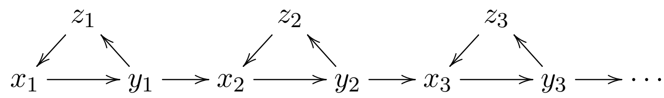
Theorem 5. *For each sink-free graph, B has a winning strategy if and only if B has one for winning in finitely many steps.*

Proof. The if direction is obvious, so we show the other implication. For countable graphs the claim is Fact 2, so consider an arbitrary uncountable graph G. If B has a winning strategy for all plays on G, then B can use it to win every ω-play, which happens in finitely many steps. However, a winning strategy on G might possibly require B to play uncountably many moves. So suppose that B has a winning strategy for the uncountable plays on G, but does not have it for the countable ones. By Fact 3, A has then a winning strategy for the countable plays which, by Fact 4, gives also a winning strategy for A in ω_1-plays, for the first uncountable ordinal ω_1. This contradicts B having a winning strategy for all uncountable plays. Hence, if B has a winning strategy for the uncountable plays on G, then B has one also for the countable plays, in particular, for ω-plays. Thus B can ensure that no play on G continues past ω, winning each in finitely many steps. ∎

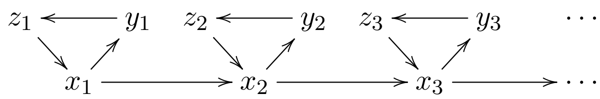
In short, a sink-free graph G of an arbitrary cardinality has no semikernel if and only if A, starting from any vertex, gets poisoned by optimally playing B in finitely many steps. Dually, G has a semikernel if and only if A has a strategy for surviving ω-plays.

Some examples

(A) The available results ensure typically not only the existence of a kernel but kernel perfectness. (E.g., the existence of kernels in finite graphs without odd cycles gives their kernel perfectness, since the absence of odd cycles is inherited by the induced subgraphs.) No such results give kernel existence in non-KP graphs, e.g., in the following one, on which A wins trivially, choosing always next x_i.



(B) The following two examples from [8] show that Poison Game I is insufficient on graphs with rays or infinite out-branching. A wins that game on the following graph starting with any x_i and continuing always from x_k to x_{k+1}.



The graph, however, has no semikernel. Player A looses Game II on it arriving at some x_i from which B poisons A playing y_i.

Likewise, A survives Poison Game I on graph G in Figure 1, having no semikernel.

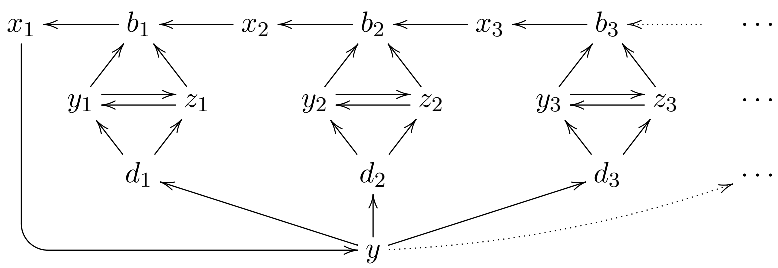


Figure 1. Graph G.

Player A starts with any d_i and plays $y_i \leftrightarrows z_i$ until B moves to b_i. After that, A reaches x_1 and then B, forced to play y, enables A to choose a fresh d_k. This strategy does not work in Game II, since A choosing d_i poisons y_i, z_i and y. After B plays y_i or z_i, A must choose b_i. Then, B reaches x_1 after which A must play y poisoned by the initial choice of d_i.

One verifies easily that A loses Poison Game II on G no matter the starting position.

(C) A winning strategy for A on a given graph can be used also when arbitrary new out-going edges or paths are added from vertices that only B can choose when A follows this strategy. Player A simply moves as before from such vertices with new out-going edges. For instance, A still wins following the strategy from (A) on the graph there extended with edges (y_j, z_i), for all $i < j$, which add infinitely many odd cycles to the graph.

Now, A wins trivially on a ray $0 \to 1 \to 2 \to \cdots$, starting from any vertex, e.g., from 0. The same strategy works then on the following graph, modifying the ray by adding edges from each odd vertex to all greater odd vertices.

$$\begin{tikzcd}
0 \arrow[r] & 1 \arrow[r] \arrow[rr, bend left] \arrow[rrrr, bend left] \arrow[rrrrrr, bend left] & 2 \arrow[r] & 3 \arrow[r] \arrow[rr, bend left] \arrow[rrrr, bend left] & 4 \arrow[r] & 5 \arrow[r] \arrow[rr, bend left, dotted] & 6 \arrow[r] & \cdots
\end{tikzcd}$$

A *variant* of this graph is obtained by removing some even vertices. If only finitely many even vertices are removed, then A wins by starting from any even vertex after all missing ones. But if infinitely many even vertices are removed, then A loses every game: with B reaching easily an odd vertex just before some missing even one, A must move to an odd vertex, after which B's move to the next odd vertex just before a missing even one leads to poisoning A in the next move. Thus, a variant of this graph has a semikernel if and only if at most finitely many even vertices are missing.

(D) Let graph G have real numbers as vertices and edges from each vertex to all greater by at most 1, that is, $\mathbf{E} = \{(x, y)\colon x < y \le x + 1\}$. To A starting at 0, let B answer by 0.2. Now A must choose some $y \in \mathbf{E}(0.2)$ outside the interval $(0, 1]$, poisoned by the first move. Any such move $y \in (1, 1.2]$ poisons all vertices in $[1, 2]$ (and more), so B wins now by playing anywhere between the first move of A and of B. For instance, B playing 0.1 forces A to choose from the poisoned $\mathbf{E}(0.1) = (0.1, 1.1]$. This strategy for B works no matter where A starts, so G has no semikernel.

Acknowledgment

Valentin Goranko deserves thanks for some suggestions, and L. Meltzers Høyskolefond for partial financial support.

References

[1] C. Berge and P. Duchet, *Perfect graphs and kernels*, Bull. Inst. Math. Acad. Sin. **16** (1988) 263–274.

[2] C. Berge and P. Duchet, *Recent problems and results about kernels in directed graphs*, Discrete Math. **86** (1990) 27–31.
https://doi.org/10.1016/0012-365X(90)90346-J

[3] M. Blidia, *A parity digraph has a kernel*, Combinatorica **6(1)** (1986) 23–27.
https://doi.org/10.1007/BF02579405

[4] E. Boros and V. Gurvich, *Perfect graphs, kernels, and cores of cooperative games*, Discrete Math. **306** (2006) 2336–2354.
https://doi.org/10.1016/j.disc.2005.12.031

[5] M. Chudnovsky, N. Robertson, P. Seymour, and R. Thomas, *The strong perfect graph theorem*, Ann. of Math. (2) **164** (2006) 51–229.
https://doi.org/10.4007/annals.2006.164.51

[6] R.T. Cook, *Patterns of paradox*, J. Symb. Log. **69** (2004) 767–774. https://doi.org/10.2178/jsl/1096901765

[7] P. Duchet and H. Meyniel, *Une généralisation du théorème de Richardson sur l'existence de noyaux dans les graphes orientés*, Discrete Math. **43** (1983) 21–27. https://doi.org/10.1016/0012-365X(83)90017-1

[8] P. Duchet and H. Meyniel, *Kernels in directed graphs: a poison game*, Discrete Math. **115** (1993) 273–276. https://doi.org/10.1016/0012-365X(93)90496-G

[9] H. Galeana-Sánchez and V. Neumann-Lara, *On kernels and semikernels of digraphs*, Discrete Math. **48** (1984) 67–76. https://doi.org/10.1016/0012-365X(84)90131-6

[10] V. Neumann-Lara, Seminúcleos de una Digráfica, Technical Report (Anales del Instituto de Matemáticas II, Universidad Nacional Autónoma México, 1971).

[11] M. Richardson, *Solutions of irreflexive relations*, Ann. of Math. (2) **58** (1953) 573–590. https://doi.org/10.2307/1969755

[12] R. Sánchez-López, *Infinite kernel perfect digraphs*, AKCE Int. J. Graphs Comb. **14** (2017) 165–171. https://doi.org/10.1016/j.akcej.2017.02.005

[13] J. von Neumann and O. Morgenstern, Theory of Games and Economic Behavior (Princeton University Press, 1944).

[14] M. Walicki, *Resolving infinitary paradoxes*, J. Symb. Log. **82** (2017) 709–723. https://doi.org/10.1017/jsl.2016.18

[15] M. Walicki, *Kernels of digraphs with finitely many ends*, Discrete Math. **342** (2019) 473–486. https://doi.org/10.1016/j.disc.2018.10.026

Received 28 June 2024
Revised 10 September 2024
Accepted 10 September 2024
Available online 25 September 2024