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# CONTRACTIBLE SUBGRAPHS OF QUASI 5-CONNECTED GRAPHS

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### Abstract

Let G be a quasi 5-connected graph on at least 14 vertices. If there is a vertex  $x \in V_4(G)$  such that  $G[N_G(x)] \cong K_{1,3}$  or  $G[N_G(x)] \cong C_4$ , then G can be contracted to a smaller quasi 5-connected graph H such that 0 < |V(G)| - |V(H)| < 4.

Keywords: quasi 5-connected, contraction, minor.

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# 1. INTRODUCTION

In this paper, we only consider finite simple undirected graphs, with undefined terms and notations following [1]. For a graph G, let V(G) and E(G) denote the set of vertices of G and the set of edges of G, respectively. We denote the set of end vertices of an edge e by V(e). For  $x \in V(G)$ , let  $N_G(x) = \{y \in V(G) : xy \in E(G)\}$ . We define the *degree* of  $x \in V(G)$  by  $d_G(x)$ , namely  $d_G(x) = |N_G(x)|$ . Let  $V_k(G)$  denote the set of vertices of degree k in G. Let  $\delta(G)$  denote the minimum degree of G. For  $S \subseteq V(G)$ , we define  $N_G(S) = \bigcup_{x \in S} N_G(x) - S$ . Furthermore, let G[S] denote the subgraph induced by S, and let G - S denote the graph obtained from G by deleting the vertices of S together with the edges incident with them. Let  $K_{1,n}$  denote the complete bipartite graph with partite sets of cardinality 1 and n. Let  $C_n$  denote a cycle of order n.

An edge e = xy of G is said to be *contracted* if it is deleted and its ends are identified, the resulting graph is denoted by G/e. And the new vertex in G/e

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is denoted by  $\overline{xy}$ . Note that, in the contraction, we replace each resulting pair of double edges by a single edge. A subgraph H of G is said to be contracted by identifying each component to a single vertex, removing each of the resulting loops and, finally, replacing each of the resulting duplicate edges by a single edge. The resulting graph is denoted by G/H. Let k be an integer such that  $k \ge 2$  and let G be a k-connected graph with  $|V(G)| \ge k + 2$ . An edge e of G is said to be k-contractible if the contraction of the edge results in a k-connected graph. A kconnected graph without a k-contractible edge is said to be a contraction critical k-connected graph. A subgraph H of G is said to be k-contractible if G/H is still k-connected.

A cut of a connected graph G is a subset V'(G) of V(G) such that G - V'(G)is disconnected. A *k*-cut is a cut of *k* elements. Suppose T is a *k*-cut of G. We say that T is a nontrivial *k*-cut, if the components of G - T can be partitioned into subgraphs  $G_1$  and  $G_2$  such that  $|V(G_1)| \ge 2$  and  $|V(G_2)| \ge 2$ . A (k-1)connected graph is quasi *k*-connected if it has no nontrivial (k-1)-cuts. Clearly, every *k*-connected graph is quasi *k*-connected.

Let G be a quasi k-connected graph. An edge e of G is said to be quasi k-contractible if G/e is still quasi k-connected. If G does not have a quasi k-contractible edge, then G is said to be a contraction critical quasi k-connected graph. A subgraph H of G is said to be quasi k-contractible if its contraction G/H results again in a quasi k-connected graph.

Tutte's [8] famous wheel theorem implies that every 3-connected graph on more than four vertices contains an edge whose contraction yields a new 3connected graph. One can give an inductive proof of Kuratowski's theorem by the wheel theorem [6]. Results on the distribution of 3-contractible edges led also to coloring theorems on planar graphs [3, 5]. So the existence and the distribution of k-contractible subgraphs is an attractive research area within graph connectivity theory.

Thomassen [7] showed that for  $k \ge 4$ , there are infinitely many contraction critical k-connected k-regular graphs. However, every 4-connected graph on at least seven vertices can be reduced to a smaller 4-connected graph by contracting at most two edges. So, naturally, Kriesell posted the following conjecture [2].

**Conjecture 1** [2]. There exist positive integers b and h such that every kconnected graph on more than b vertices can be reduced to a smaller k-connected graph by contracting less than h edges for every  $k \ge 1$ .

Clearly, Conjecture 1 is true for k = 1 and k = 2. For k = 3 and k = 4, the smallest appropriate values for b, h would be 4, 2 and 6, 3, respectively. But for  $k \ge 6$ , such a statement fails since toroidal triangulations of large face width is a counterexample [2]. In [4], Kriesell proved that every quasi 6-connected graph on at least 13 vertices can be reduced to a smaller 5-connected graph by contracting less than five edges subsequently. The conjecture is still open for k = 5.

We focus on quasi 5-connected graphs and obtain the following results.

**Theorem 2.** Let G be a quasi 5-connected graph on at least 13 vertices. If there is a vertex  $x \in V_4(G)$  such that  $G[N_G(x)] \cong K_{1,3}$ , then G can be reduced to a smaller quasi 5-connected graph by contracting less than three edges subsequently.

**Theorem 3.** Let G be a quasi 5-connected graph on at least 14 vertices. If there is a vertex  $x \in V_4(G)$  such that  $G[N_G(x)] \cong C_4$ , then G can be reduced to a smaller quasi 5-connected graph by contracting less than four edges subsequently.

By Combining Theorems 2 and 3, we have the following corollary.

**Corollary 4.** Let G be a quasi 5-connected graph on at least 14 vertices. If there is a vertex  $x \in V_4(G)$  such that  $G[N_G(x)] \cong K_{1,3}$  or  $G[N_G(x)] \cong C_4$ , then G can be contracted to a smaller quasi 5-connected graph H such that 0 < |V(G)| - |V(H)| < 4.

### 2. Preliminaries

In this section, we introduce more definitions and several preliminary lemmas.

Let G be a non-complete connected graph and let  $\kappa(G)$  denote the vertex connectivity of G. By  $\mathcal{T}(G) := \{T \subseteq V(G) : T \text{ is a cut of } G, |T| = \kappa(G)\}$ , we denote the set of *smallest cuts* of G. For  $T \in \mathcal{T}(G)$ , the union of the vertex sets of at least one but not of all components of G - T is called a *T*-fragment of G or, briefly, a fragment. Let F be a T-fragment, and let  $\overline{F} = V(G) - (F \cup T)$ . Clearly,  $\overline{F} \neq \emptyset$ , and  $\overline{F}$  is also a T-fragment such that  $N_G(F) = T = N_G(\overline{F})$ .

Let G be a quasi k-connected graph and let  $E_0 = \{e \in E(G) : G/e \text{ is } (k-1) \text{-} \text{connected}, \text{ but not quasi } k\text{-connected} \}$ . For  $xy \in E_0$ , G/xy has a nontrivial (k-1)-cut T' by the definition of quasi k-connected. Furthermore,  $\overline{xy} \in T'$ , for otherwise, T' is also a nontrivial (k-1)-cut of G, contradicts the fact that G is quasi k-connected. This implies that  $T = (T' - \overline{xy}) \cup \{x, y\}$  is a k-cut of G. Moreover, G-T can be partitioned into subgraphs F and  $\overline{F}$  such that  $|V(F)| \geq 2$  and  $|V(\overline{F})| \geq 2$ . Each of these subgraphs is called a quasi T-fragment of G or, briefly, a quasi fragment. For an edge e of G, a quasi fragment F of G is said to be a quasi fragment with respect to e if  $V(e) \subseteq N_G(F)$ . For a set of edges  $E' \subseteq E(G)$ , we say that F is a quasi fragment with respect to E' if F is a quasi fragment with respect to e or E' with least cardinality is called a quasi atom with respect to e and E', respectively.

**Lemma 5.** Let G be a quasi 5-connected graph. If  $xy \in E(G)$  and  $\delta(G/xy) \ge 4$ , then G/xy is 4-connected.

**Proof.** Suppose that G/xy is not 4-connected, then there exists a 3-cut T' of G/xy. Since  $\delta(G/xy) \geq 4$ , we see that each component of G/xy - T' has at least two vertices. Furthermore,  $\overline{xy} \in T'$ , for otherwise, T' is also a 3-cut of G, a contradiction. Hence,  $T = (T' - \{\overline{xy}\}) \cup \{x, y\}$  is a 4-cut of G. And each component of G - T has at least two vertices. It follows that T is a nontrivial 4-cut of G, which contradicts the quasi 5-connectivity of G.

**Lemma 6.** Let G be a quasi 5-connected graph and let  $x \in V_4(G)$  such that  $N_G(x) = \{y, z, u, v\}$ . If  $\{yz, uv\} \subseteq E(G)$  and  $\delta(G/\triangle xyz) \ge 4$ , then  $G/\triangle xyz$  is 4-connected.

**Proof.** If  $G/\triangle xyz$  is not 4-connected, then  $G/\triangle xyz$  has a 3-cut T'. Since  $\delta(G/\triangle xyz) \geq 4$ , each component of  $G/\triangle xyz - T'$  has at least two vertices. Furthermore, the vertex resulting from the contraction of triangle xyz belongs to T'. Hence, G has a 5-cut T such that  $\{x, y, z\} \subseteq T$  and G - T can be partitioned into two parts, say F and  $\overline{F}$ , where each part has at least two vertices. Since  $uv \in E(G)$ , we see that  $N_G(x) \cap F = \emptyset$  or  $N_G(x) \cap \overline{F} = \emptyset$ , which implies that  $T - \{x\}$  is a nontrivial 4-cut of G, a contradiction.

**Lemma 7.** Let G be a quasi 5-connected graph on at least 8 vertices. If there is a vertex  $x \in V(G)$  such that  $N_G(x) = \{x_1, x_2, x_3, x_4\}$  and  $G[\{x_1, x_2, x_3\}] \cong K_3$ , then  $xx_4$  is a quasi 5-contractible edge.

**Proof.** For i = 1, 2, 3, we have  $d_G(x_i) \geq 5$ , for otherwise,  $N_G(\{x, x_i\})$  is a nontrivial 4-cut of G since  $|V(G)| \geq 8$ , which contradicts that G is quasi 5connected. This implies  $\delta(G/xx_4) \geq 4$ . Thus Lemma 5 assures us that  $G/xx_4$ is 4-connected. Suppose that  $G/xx_4$  is not quasi 5-connected, then we see that  $G/xx_4$  has a nontrivial 4-cut T'. Furthermore,  $\overline{xx_4} \in T'$ , for otherwise, T' is also a nontrivial 4-cut of G, a contradiction. Thus, G has a 5-cut T such that  $\{x, x_4\} \subseteq T$ . And G - T can be partitioned into two parts, say F and  $\overline{F}$ , where each part has at least two vertices. However, since  $G[\{x_1, x_2, x_3\}] \cong K_3$ , we observe that  $N_G(x) \cap F = \emptyset$  or  $N_G(x) \cap \overline{F} = \emptyset$ . It follows that  $T - \{x\}$  is a nontrivial 4-cut of G, a contradiction.

#### 3. Proof of Theorem 2

In this section, we give a proof of Theorem 2. We first introduce a useful lemma.

**Lemma 8.** Let G be a quasi 5-connected graph on at least 13 vertices. If Figure 1 is a subgraph of G, then the graph G' obtained from G by contracting  $A := \{a, a_0\}, B := \{p, p_0\}$  to vertices A', B', respectively, is still quasi 5-connected.



Figure 1. The graph in Lemma 8. Solid edges indicate edges that must exist, and dashed edges indicate edges that may exist. A black circular vertex indicates that the vertex has reached its maximum degree. A black square vertex indicates that its neighbor set is a subset of all vertices adjacent to it by solid and dashed edges. Specifically, the vertex adjacent to it by a solid edge must be a neighbor of it, while the vertex adjacent to it by a dashed edge is not necessarily so.

**Proof.** We first show that G' is 4-connected. Note that  $\{v, w, s, B'\} \subseteq N_{G'}(A')$ , then  $d_{G'}(A') \geq 4$ . If  $d_{G'}(B') < 4$ , then we see that  $N_G(p_0) = \{p, w, s, a_0\}$ , and hence  $\{t, s, m_0, a_0\}$  is a nontrivial 4-cut of G since  $|V(G)| \geq 13$ , which contradicts that G is quasi 5-connected. Thus,  $d_{G'}(B') \geq 4$ . Moreover, it is easy to find that every vertex in G', except A' and B', has degree at least 4. It follows  $\delta(G') = 4$ . Suppose that G' is not 4-connected, then G' has a 3-cut T' such that each component of G' - T' has at least 2 vertices and  $T' \cap \{A', B'\} \neq \emptyset$ . If  $|T' \cap \{A', B'\}| = 1$ , then we find that G has a nontrivial 4-cut, a contradiction. Thus,  $\{A', B'\} \subseteq T'$ . It follows that G has a 5-cut T containing  $\{a, a_0, p, p_0\}$ . Since  $N_G(p) = \{a_0, p_0, w, s\}$  and  $ws \in E(G)$ , we see that  $T - \{p\}$  is a nontrivial 4-cut of G, a contradiction. Thus, G' is 4-connected.

Suppose, to the contrary, that G' is not quasi 5-connected, then G' has a nontrivial 4-cut T' such that  $|T' \cap \{A', B'\}| \ge 1$ . Let F' be a T'-fragment of G' and let  $\overline{F'} = G' - (T' \cup F')$ . Let  $F, T, \overline{F}$  be the sets in G corresponding to F',  $T', \overline{F'}$  in G'. That is, in each of these sets, we replace the vertices A', B' by the vertices in the sets A, B, respectively.

If  $\{A', B'\} \subseteq T'$ , then one of v and w is in F, and the other is in  $\overline{F}$ . Otherwise,  $N_G(\{a, p\}) \cap F = \emptyset$  or  $N_G(\{a, p\}) \cap \overline{F} = \emptyset$ , which implies that  $T - \{a, p\}$  is a nontrivial 4-cut of G, a contradiction. Without loss of generality, we may assume that  $v \in F$  and  $w \in \overline{F}$ . Since  $\{sv, sw, xv, xw\} \subseteq E(G)$ , we have  $T = \{a, a_0, p, p_0, s, x\}$ . Hence we obtain  $N_G(w) \subseteq T$ . Since  $mv \in E(G)$ ,  $m \in F$ . Then the fact that  $mt \in E(G)$  assures us that  $t \in F$ . Hence, we see that  $N_G(\{a, p, x\}) \cap \overline{F} = \{w\}$  and, thus,  $\{a_0, p_0, s\}$  forms a 3-cut of G since  $|\overline{F}| \ge 2$ , a contradiction.

Consequently,  $|T' \cap \{A', B'\}| = 1$ . Suppose that  $A' \notin T'$  and  $B' \in T'$ . We may assume that  $A' \in F'$  without loss of generality. Then  $\{w, s\} \subseteq F \cup T$ because  $\{aw, as\} \subseteq E(G)$ , and hence  $N_G(p) \cap \overline{F} = \emptyset$ . This implies that  $T - \{p\}$ is a nontrivial 4-cut of G, a contradiction. Thus,  $A' \in T'$  and  $B' \notin T'$ . We may assume that  $B' \in F'$  without loss of generality. Then  $|F| \geq 3$ . If  $N_G(a) \cap F = \emptyset$ or  $N_G(a) \cap \overline{F} = \emptyset$ , then  $T - \{a\}$  is a nontrivial 4-cut of G, a contradiction. So  $N_G(a) \cap F \neq \emptyset$  and  $N_G(a) \cap \overline{F} \neq \emptyset$ . Then we see that  $s \in T$  since  $N_G(a) =$  $\{a_0, v, w, s\}$  and  $\{vs, ws\} \subseteq E(G)$ . Since  $p \in F$  and  $wp \in E(G)$ , we have  $w \in F$ and  $v \in \overline{F}$ . Notice that  $\{xw, xv\} \subseteq E(G)$ , then  $x \in T$ . If  $N_G(\{a, x\}) \cap F = \{w\}$ , then we find that  $(T - \{a, x\}) \cup \{w\}$  is a nontrivial 4-cut of G since  $|F| \ge 3$ , a contradiction. Hence,  $t \in F$ , and thus  $N_G(\{a, x\}) \cap \overline{F} = \{v\}$ . This implies that  $|\overline{F}| = 2$ , for otherwise,  $(T - \{a, x\}) \cup \{v\}$  forms a nontrivial 4-cut of G. Let  $\overline{F} = \{v, z\}$ . Then  $d_G(z) = 4$  and  $\{v, s, a_0\} \subseteq N_G(z)$ . Clearly,  $z \neq m$ since m is not adjacent to  $a_0$ . If  $vm_0 \notin E(G)$ , then we obtain a contradiction, since there is no such a vertex z in G. If  $vm_0 \in E(G)$ , then  $z = m_0$ , and thus  $N_G(m_0) = \{v, s, a_0, m\}$ . It follows that  $\{t, s, a_0, p_0\}$  forms a nontrivial 4-cut of G by the fact that  $|V(G)| \ge 13$ , a contradiction. 

Now we are prepared to prove Theorem 2.

**Proof of Theorem 2.** If G has a quasi 5-contractible edge, Theorem 2 holds immediately. Thus we assume that G is a contraction critical quasi 5-connected graph. Let  $N_G(x) = \{x_1, x_2, x_3, x_4\}$ . Without loss of generality, we suppose that  $\{x_1x_4, x_2x_4, x_3x_4\} \subseteq E(G)$ . If  $d_G(x_4) = 4$ , then  $\{x_1, x_2, x_3\}$  forms a 3-cut of G. So,  $d_G(x_4) \ge 5$ . Hence, for  $i = 1, 2, 3, \delta(G/xx_i) \ge 4$ , and hence Lemma 5 assures us that  $G/xx_i$  is 4-connected.

Let  $E' = \{xx_1, xx_2, xx_3\}$  and let  $F_1$  be a quasi atom with respect to E'. Let  $T_1 = N_G(F_1)$  and let  $\overline{F_1} = V(G) - (F_1 \cup T_1)$ . Then  $|\overline{F_1}| \ge |F_1| \ge 2$ . Without loss of generality, we assume that  $F_1$  is a quasi fragment with respect to  $xx_1$ . Since  $x_2x_4 \in E(G)$  and  $x_3x_4 \in E(G)$ , we see that  $x_4 \in T_1$  and we may assume that  $x_2 \in F_1$ ,  $x_3 \in \overline{F_1}$ . Let  $F_2$  be a quasi fragment with respect to  $xx_2$  and let  $T_2 = N_G(F_1)$ ,  $\overline{F_2} = V(G) - (F_2 \cup T_2)$ . Then  $x_4 \in T_2$  and we may assume that  $x_1 \in F_2$ ,  $x_3 \in \overline{F_2}$ . Now, we find that  $\{x, x_4\} \subseteq T_1 \cap T_2$ ,  $x_1 \in T_1 \cap F_2$ ,  $x_2 \in F_1 \cap T_2$  and  $x_3 \in \overline{F_1} \cap \overline{F_2}$ . Let  $X_1 = (T_1 \cap F_2) \cup (T_1 \cap T_2) \cup (F_1 \cap T_2)$ ,  $X_2 = (T_1 \cap F_2) \cup (T_1 \cap T_2) \cup (\overline{F_1} \cap T_2)$ ,  $X_3 = (\overline{F_1} \cap T_2) \cup (T_1 \cap \overline{F_2})$ ,  $X_4 = (F_1 \cap T_2) \cup (T_1 \cap T_2) \cup (T_1 \cap \overline{F_2})$ .

Claim 1.  $F_1 \cap T_2 = \{x_2\}.$ 

**Proof.** Assume, to the contrary, that  $|F_1 \cap T_2| \ge 2$ . Then  $|(T_1 \cap T_2) \cup (\overline{F_1} \cap T_2)| \le 3$ . Since  $x_3 \in \overline{F_1} \cap \overline{F_2}$ , we see that  $|X_3| \ge 4$ . Hence,  $|T_1 \cap \overline{F_2}| \ge 1$ , and thus,  $|T_1 \cap F_2| \le 2$ . If  $|T_1 \cap F_2| = 1$ , then  $|X_2| \le 4$ . Since  $N_G(x) \cap (\overline{F_1} \cap F_2) = \emptyset$ , we observe that  $\overline{F_1} \cap F_2 = \emptyset$ . This implies that  $|F_2| < |F_1|$ , a contradiction.

Therefore,  $|T_1 \cap F_2| = 2$ . It follows that  $|T_1 \cap T_2| = |F_1 \cap T_2| = 2$  and  $|T_1 \cap \overline{F_2}| = |\overline{F_1} \cap T_2| = 1$ . Now,  $|X_2| = 5$  and  $|X_3| = 4$ . Since G has no nontrivial 4-cuts, we find that  $|\overline{F_1} \cap F_2| \le 1$  and  $\overline{F_1} \cap \overline{F_2} = \{x_3\}$ . This implies that  $|\overline{F_1}| \le 3$ . Hence,  $|F_1| \le |\overline{F_1}| \le 3$ , which implies that  $|V(G)| \le 11$ , a contradiction.

By Claim 1, we have  $F_1 \cap F_2 \neq \emptyset$ . Otherwise, we see that  $N_G(x_1) \cap F_1 = \emptyset$ since  $x_1x_2 \notin E(G)$ , and then  $T_1 - \{x_1\}$  is a nontrivial 4-cut of G, a contradiction. Since  $N_G(x) \cap (F_1 \cap F_2) = \emptyset$ ,  $|X_1| \ge 5$ . Hence,  $|F_1 \cap T_2| \ge |T_1 \cap \overline{F_2}|$ . Then we have  $|T_1 \cap \overline{F_2}| \le 1$  by Claim 1.

**Claim 2.**  $|T_1 \cap \overline{F_2}| = 1$  and  $|F_1 \cap F_2| = 1$ .

**Proof.** Suppose  $T_1 \cap \overline{F_2} = \emptyset$ . Then  $|X_1| = 6$  and  $|X_3| = 4$ . This implies  $\overline{F_1} \cap \overline{F_2} = \{x_3\}$ . Then  $|\overline{F_2}| < |F_1|$ , a contradiction. So  $|T_1 \cap \overline{F_2}| = 1$ . Then we see that  $|(T_1 \cap F_2) \cup (T_1 \cap T_2)| = 4$ , and hence  $|X_1| = 5$ . This implies  $|F_1 \cap F_2| = 1$  since  $N_G(x) \cap (F_1 \cap F_2) = \emptyset$  and G has no nontrivial 4-cuts.

Let  $F_1 \cap F_2 = \{a\}$ . Then we see that  $d_G(a) = 4$  and  $\{x_1, x_2, x_4\} \subseteq N_G(a)$ . Let  $N_G(a) = \{x_1, x_2, x_4, a_0\}$ . Note that  $a_0 \in (T_1 \cap F_2) \cup (T_1 \cap T_2)$ . We next show that  $G/aa_0$  is 4-connected. Suppose  $\delta(G/aa_0) < 4$ . It follows that there exist a vertex  $z \in N_G(a) \cap N_G(a_0)$  such that  $d_G(z) = 4$ . Clearly,  $z \neq x_4$  since  $d_G(x_4) \geq 5$ . If  $z = x_1$ , then we see that  $N_G(x_1) \cap \overline{F_1} = \emptyset$ , which implies that  $T_1 - \{x_1\}$  is a nontrivial 4-cut of G, a contradiction. If  $z = x_2$ , then  $N_G(x_2) \cap \overline{F_2} = \emptyset$ , which implies that  $T_2 - \{x_2\}$  is a nontrivial 4-cut of G, a contradiction. Thus, there is no such a vertex z in G. Then  $\delta(G/aa_0) \geq 4$ , and then Lemma 5 assures us that  $G/aa_0$  is 4-connected.

Let  $F_3$  and  $F_4$  be the quasi fragments with respect to  $xx_3$  and  $aa_0$ , respectively. For i = 3, 4, let  $T_i = N_G(F_i)$ ,  $\overline{F_i} = V(G) - (F_i \cup T_i)$ . Since  $\{x_1x_4, x_2x_4\} \subseteq E(G)$ , we see that  $x_4 \in T_3$  and we may assume that  $x_1 \in F_3$ ,  $x_2 \in \overline{F_3}$  without loss of generality. Then  $a \in T_3$  since  $\{ax_1, ax_2\} \subseteq E(G)$ . Similarly,  $\{x_4, x\} \subseteq T_4$  and we may assume that  $x_1 \in F_4$ ,  $x_2 \in \overline{F_4}$ . If  $a_0 \in T_3$ , then  $N_G(\{x, a\}) \cap F_3 = \{x_1\}$  and  $N_G(\{x, a\}) \cap \overline{F_3} = \{x_2\}$ . Since  $|V(G)| \ge 13$ , we have  $|F_3| \ge 4$  or  $|\overline{F_3}| \ge 4$ . It follows that  $(T_3 - \{x, a\}) \cup \{x_1\}$  or  $(T_3 - \{x, a\}) \cup \{x_2\}$  is a nontrivial 4-cut of G, a contradiction. Thus  $a_0 \notin T_3$ . Similarly,  $x_3 \notin T_4$ . Now, we see that  $x_1 \in F_3 \cap F_4$ ,  $x_2 \in \overline{F_3} \cap \overline{F_4}$  and  $\{x, x_4, a\} \subseteq T_3 \cap T_4$ . Since  $x_3 \notin T_4$  and  $a_0 \notin T_3$ ,  $x_3 \in (T_3 \cap F_4) \cup (T_3 \cap \overline{F_4})$ ,  $A_6 = (T_3 \cap F_4) \cup (T_3 \cap T_4) \cup (\overline{F_3} \cap T_4)$ ,  $X_7 = (\overline{F_3} \cap T_4) \cup (T_3 \cap T_4) \cup (T_3 \cap \overline{F_4})$ ,  $X_8 = (F_3 \cap T_4) \cup (T_3 \cap \overline{F_4})$ .

Claim 3.  $|F_3 \cap F_4| \leq 2$  and  $|\overline{F_3} \cap \overline{F_4}| \leq 2$ .

**Proof.** We only show that  $|F_3 \cap F_4| \leq 2$ . Suppose  $|F_3 \cap F_4| \geq 3$ . Let  $W = (F_3 \cap F_4) - \{x_1\}$ . Thus  $|W| \geq 2$ . Since G is quasi 5-connected, we see that  $|N_G(W)| \geq 5$ . Note that  $N_G(\{x, a\}) \cap (F_3 \cap F_4) = \{x_1\}$ . It follows  $N_G(\{x, a\}) \cap W = \emptyset$ , and hence,  $|X_5| \geq 6$ . Thus  $|X_7| \leq 4$ . This implies that  $\overline{F_3} \cap \overline{F_4} = \{x_2\}$  and  $\overline{F_3} \cap T_4 = \emptyset$ 



Figure 2. The explanations for Figures 2(a) and 2(b) are identical to the explanation for the graph in Figure 1.

or  $T_3 \cap \overline{F_4} = \emptyset$ . Without loss of generality, we may assume  $\overline{F_3} \cap T_4 = \emptyset$ . Then  $|X_6| \leq 5$ , Since  $N_G(\{x, a\}) \cap (\overline{F_3} \cap F_4) = \emptyset$ , we find  $\overline{F_3} \cap F_4 = \emptyset$ . Then  $|\overline{F_3}| = 1$ , a contradiction.

Claim 4. Either  $F_3 \cap \overline{F_4} = \emptyset$  or  $\overline{F_3} \cap F_4 = \emptyset$ .

**Proof.** Note that,  $N_G(\{x, a\}) \cap (F_3 \cap \overline{F_4}) = \emptyset$  and  $N_G(\{x, a\}) \cap (\overline{F_3} \cap F_4) = \emptyset$ . This implies that  $|X_8| \ge 6$  if  $F_3 \cap \overline{F_4} \ne \emptyset$  and  $|X_6| \ge 6$  if  $\overline{F_3} \cap F_4 \ne \emptyset$ . By the fact that  $|X_6| + |X_8| = 10$ , we have  $F_3 \cap \overline{F_4} = \emptyset$  or  $\overline{F_3} \cap F_4 = \emptyset$ .

If  $F_3 \cap \overline{F_4} = \emptyset$ , then  $|\overline{F_3} \cap F_4| \ge 2$ . Otherwise, we see that  $|V(G)| \le 12$  by Claim 3, which contradicts the fact that  $|V(G)| \ge 13$ . This implies that  $|X_6| \ge 7$ since  $N_G(\{x,a\}) \cap (\overline{F_3} \cap F_4) = \emptyset$ . Thus,  $|T_3 \cap T_4| = 3$ ,  $|T_3 \cap F_4| = |\overline{F_3} \cap T_4| = 2$  and  $F_3 \cap T_4 = T_3 \cap \overline{F_4} = \emptyset$ . It follows that  $x_3 \in T_3 \cap F_4$  and  $a_0 \in \overline{F_3} \cap T_4$ . Furthermore,  $|F_3| = |F_3 \cap F_4| = 2$  and  $|\overline{F_4}| = |\overline{F_3} \cap \overline{F_4}| = 2$  by Claim 3. Let  $F_3 \cap F_4 = \{x_1, m\}$ and let  $T_3 \cap F_4 = \{x_3, m_0\}$ . Let  $\overline{F_3} \cap \overline{F_4} = \{x_2, p\}$  and let  $\overline{F_3} \cap T_4 = \{a_0, p_0\}$ . Now, we find that G has a subgraph isomorphic to the graph in Figure 1. Then  $G/aa_0/pp_0$  is quasi 5-connected by Lemma 8. If  $\overline{F_3} \cap F_4 = \emptyset$ , we can also obtain that G has a subgraph isomorphic to the graph in Figure 1 by similar argument. And then Theorem 2 holds by Lemma 8.

#### 4. Proof of Theorem 3

In this section, we consider the quasi 5-connected graph G that contains a 4degree vertex x such that  $G[N_G(x)] \cong C_4$ . If  $x' \in N_G(x)$  and  $d_G(x') = 4$ , then we see that  $N_G(\{x, x'\})$  is a nontrivial 4-cut of G, which contradicts the quasi 5-connectivity of G. Hence, every neighborhood of x has degree at least 5 (\*).

**Lemma 9.** Let G be a quasi 5-connected graph that contains either Figure 2(a) or Figure 2(b) as a subgraph. If  $d_G(x_3) = 6$  or  $d_G(x_4) = 6$ , then Theorem 3 holds.

**Proof.** If  $d_G(x_3) = 6$ , we see that either  $G[N_G(a)] \cong K_{1,3}$  or  $G[N_G(a)]$  contains a  $K_3$ -subgraph. It follows immediately from Theorem 2 and Lemma 7 that G has a quasi 5-contractible subgraph containing at most two edges. Hence, Theorem 3 holds. If  $d_G(x_4) = 6$ , we can obtain the same result similarly.

**Lemma 10.** Let G be a contraction critical quasi 5-connected graph on at least 14 vertices. If Figure 2(a) is a subgraph of G satisfying  $d_G(x_3) = d_G(x_4) = 5$ , then  $\triangle xx_3x_4$  is a quasi 5-contractible subgraph of G.

**Proof.** Since  $d_G(x_3) = d_G(x_4) = 5$ , either  $a_1 \in N_G(x_3)$  or  $c \in N_G(x_3)$ , and also either  $b_1 \in N_G(x_4)$  or  $c \in N_G(x_4)$ .

Claim 1.  $N_G(x_3) = \{x, x_2, x_4, a, a_1\}$  and  $N_G(x_4) = \{x, x_1, x_3, b, b_1\}.$ 

**Proof.** We only show that  $N_G(x_3) = \{x, x_2, x_4, a, a_1\}$ . Let us assume, to the contrary, that  $N_G(x_3) = \{x, x_2, x_4, a, c\}$ . We next show that  $aa_1$  is quasi 5-contractible, which contradicts that G is a contraction critical quasi 5-connected graph. We first show that  $\delta(G/aa_1) = 4$ . If  $\delta(G/aa_1) < 4$ , then we have that there exist a vertex  $z \in N_G(a) \cap N_G(a_1)$  such that  $d_G(z) = 4$ . Clearly,  $z \neq x_2$  since  $d_G(x_2) \geq 5$ . Similarly,  $z \neq x_3$ . Thus z = c. However, we see that  $N_G(\{a, c\})$  is a nontrivial 4-cut of G, a contradiction. So  $\delta(G/aa_1) = 4$ . Then Lemma 5 assures us that  $G/aa_1$  is 4-connected. Suppose that  $G/aa_1$  is not quasi 5-connected. Let C be a quasi fragment with respect to  $aa_1$  and let  $R = N_G(C)$ ,  $\overline{C} = G - (C \cup R)$ .

Since  $\{x_2x_3, x_3c\} \subseteq E(G), x_3 \in R$  and we may assume that  $x_2 \in C$  and  $c \in \overline{C}$ . Since  $\{xx_2, x_1x_2\} \subseteq E(G), \{x, x_1\} \subseteq C \cup R$ . If  $x_4 \in C \cup R$ , then  $N_G(\{a, x_3\}) \cap \overline{C} = \{c\}$ . It follows that  $|\overline{C}| = 2$ , for otherwise,  $(R - \{a, x_3\}) \cup \{c\}$  forms a nontrivial 4-cut of G. Let  $\overline{C} = \{c, w\}$ . Then we see that  $d_G(w) = 4$  and  $\{c, a_1\} \subseteq N_G(w)$ . Since  $cb \in E(G)$  and  $ba_1 \notin E(G), b \in R$ , and then we have  $w \in N_G(b)$ . This implies that  $w = b_1$ . Since  $c \in N_G(x_4)$  or  $b_1 \in N_G(x_4)$ , we find that  $x_4 \in R$ . It follows that  $N_G(c) \subseteq \{a, b, x_3, b_1, a_1, x_4\}$  and  $N_G(b_1) = \{c, a_1, b, x_4\}$ , which implies that  $\{a_1, x_1, x_2\}$  forms a 3-cut of G since  $|V(G)| \ge 14$ , a contradiction. Therefore,  $x_4 \in \overline{C}$ , and hence  $\{x, x_1\} \subseteq R$ . Then we see that  $R = \{a, a_1, x_3, x, x_1\}$  and  $N_G(\{a, x, x_3\}) \cap C = \{x_2\}$ . Since  $|C| \ge 2$ , we find that  $\{x_1, x_2, a_1\}$  is a 3-cut of G, which is absurd.

In the following, we show that  $\triangle xx_3x_4$  is a quasi 5-contractible subgraph of G. By (\*),  $d_G(x_1) \ge 5$  and  $d_G(x_2) \ge 5$ . Then  $\delta(G/\triangle xx_3x_4) = 4$ , which implies that  $G/\triangle xx_3x_4$  is 4-connected by Lemma 6. Suppose that  $G/\triangle xx_3x_4$ is not quasi 5-connected, then  $G/\triangle xx_3x_4$  has a nontrivial 4-cut T'. Let F' be a T'-fragment of  $G/\triangle xx_3x_4$  and let  $\overline{F'} = G/\triangle xx_3x_4 - (T' \cup F')$ . Let  $F, T, \overline{F}$  be the sets in G corresponding to F', T',  $\overline{F'}$  in  $G/\triangle xx_3x_4$ . So,  $\{x, x_3, x_4\} \subseteq T$ . Without loss of generality, we may assume  $\{x_1, x_2\} \subseteq F \cup T$ .

Claim 2.  $|\{x_1, x_2\} \cap T| = 1.$ 

**Proof.** Suppose  $|\{x_1, x_2\} \cap T| \neq 1$ , then either  $\{x_1, x_2\} \subseteq T$  or  $\{x_1, x_2\} \subseteq F$ . If  $\{x_1, x_2\} \subseteq T$ , then we see that  $N_G(\{x, x_3\}) \cap F = \emptyset$  or  $N_G(\{x, x_3\}) \cap \overline{F} = \emptyset$ , which implies that  $T - \{x, x_3\}$  is a nontrivial 4-cut of G, a contradiction.

If  $\{x_1, x_2\} \subseteq F$ , then  $N_G(x) \cap \overline{F} = \emptyset$ . If  $N_G(x_3) \cap \overline{F} = \emptyset$ , then  $T - \{x, x_3\}$ is a nontrivial 4-cut of G, a contradiction. Thus  $N_G(x_3) \cap \overline{F} \neq \emptyset$ , and then  $a \in T$  and  $a_1 \in \overline{F}$  since  $ax_2 \in E(G)$ . Similarly,  $b \in T$  and  $b_1 \in \overline{F}$ . If  $c \in T$ , then  $\{x, x_3, x_4, a, b\} \cap F = \{x_1, x_2\}$  and  $\{x, x_3, x_4, a, b\} \cap \overline{F} = \{a_1, b_1\}$ , which implies  $|F| = |\overline{F}| = 2$ , for otherwise,  $\{x_1, x_2, c\}$  or  $\{a_1, b_1, c\}$  is a 3-cut of G, a contradiction. It follows |V(G)| = 10, which contradicts  $|V(G)| \ge 14$ . Hence,  $c \notin T$ . If  $c \in F$ , then  $\{x, x_3, x_4, a, b\} \cap F = \{x_1, x_2, c\}$  and  $\{x, x_3, x_4, a, b\} \cap \overline{F} = \{a_1, b_1\}$ . This implies that  $|F| \le 4$  and  $|\overline{F}| = 2$ , for otherwise,  $(T - \{x, x_3, x_4, a, b\}) \cup \{x_1, x_2, c\}$  is a nontrivial 4-cut or  $(T - \{x, x_3, x_4, a, b\}) \cup \{a_1, b_1\}$ is a 3-cut of G, a contradiction. Then we have  $|V(G)| \le 12$ , a contradiction. If  $c \in \overline{F}$ , we can also obtain  $|V(G)| \le 12$  by similar argument.

We may assume that  $x_2 \in F$  and  $x_1 \in T$  without loss of generality. Similar to what is described above, we have  $a \in T$ ,  $a_1 \in \overline{F}$  and  $\{b, b_1\} \cap \overline{F} \neq \emptyset$ . It follows  $\{b, b_1\} \cap F = \emptyset$  since  $bb_1 \in E(G)$ . If  $c \notin F$ , then  $N_G(\{x, x_3, x_4, a\}) \cap$  $F = \{x_2\}$ , and then  $(T - \{x, x_3, x_4, a\}) \cup \{x_2\}$  is a 3-cut of G since  $|F| \ge 2$ , a contradiction. Thus,  $c \in F$ , and thus  $b \in T$  and  $b_1 \in \overline{F}$ . Then we can find that  $|N_G(\{x, x_3, x_4, a, b\}) \cap F| = |N_G(\{x, x_3, x_4, a, b\}) \cap \overline{F}| = 2$ , which implies  $|F| = |\overline{F}| = 2$ . It follows |V(G)| = 10, a contradiction.

**Lemma 11.** Let G be a contraction critical quasi 5-connected graph on at least 14 vertices. If Figure 2(b) is a subgraph of G satisfying  $d_G(x_3) = d_G(x_4) = 5$ , and without loss of generality, we assume  $N_G(x_3) = \{x, x_2, x_4, a, a_1\}$  and  $N_G(x_4) = \{x, x_1, x_3, b, b_1\}$ , then one of the following statements holds.

- (i)  $G/\triangle xx_3x_4$  is quasi 5-connected.
- (ii) The graph obtained from G by contracting  $A := \{b, b_2\}, B := \{b_1, x_4\}, C := \{x, x_1\}$  to vertices A', B', C', respectively, is quasi 5-connected.

**Proof.** Assume neither (i) nor (ii) holds. Similar to Lemma 10, we have that  $G/\triangle xx_3x_4$  is 4-connected. Since  $G/\triangle xx_3x_4$  is not quasi 5-connected, we know that  $G/\triangle xx_3x_4$  has a nontrivial 4-cut  $T'_1$ . Let  $F'_1$  be a  $T'_1$ -fragment of  $G/\triangle xx_3x_4$  and let  $\overline{F'_1} = G/\triangle xx_3x_4 - (T'_1 \cup F'_1)$ . Let  $F_1, T_1, \overline{F_1}$  be the sets in G corresponding to  $F'_1, T'_1, \overline{F'_1}$  in  $G/\triangle xx_3x_4$ . Then we have  $T_1 \supset \{x, x_3, x_4\}$ . Without loss of generality, we may assume  $\{x_1, x_2\} \subseteq F_1 \cup T_1$ . It follows that  $N_G(x) \cap \overline{F_1} = \emptyset$ . If  $N_G(x_3) \cap \overline{F_1} = \emptyset$ , then we see that  $N_G(\{x, x_3\}) \cap \overline{F_1} = \emptyset$ , and hence  $T_1 - \{x, x_3\}$  is

a nontrivial 4-cut of G, a contradiction. Consequently,  $\{a, a_1\} \cap \overline{F_1} \neq \emptyset$ . Similarly,  $\{b, b_1\} \cap \overline{F_1} \neq \emptyset$ . It follows  $\{a, a_1, b, b_1\} \subseteq T_1 \cup \overline{F_1}$  since  $\{aa_1, bb_1\} \subseteq E(G)$ .

**Claim 1.**  $\{x_1, x_2\} \subseteq F_1$ .

**Proof.** Suppose  $\{x_1, x_2\} \notin F_1$ . If  $\{x_1, x_2\} \subseteq T_1$ , then  $N_G(x) \cap F_1 = \emptyset$ . Since  $\{a, a_1\} \subseteq T_1 \cap \overline{F_1}$ , we see that  $N_G(x_3) \cap F_1 = \emptyset$ . This implies that  $T_1 - \{x, x_3\}$  is a nontrivial 4-cut of G, a contradiction. Thus  $|\{x_1, x_2\} \cap T_1| = 1$ . Without loss of generality, we assume that  $x_1 \in T_1$  and  $x_2 \in F_1$ . Then  $a \in T_1$  and  $a_1 \in \overline{F_1}$  since  $ax_2 \in E(G)$ . Now, we observe that  $N_G(\{x, x_3, x_4\}) \cap F_1 = \{x_2\}$ . It follows that  $|F_1| = 2$ , for otherwise,  $(T_1 - \{x, x_3, x_4\}) \cup \{x_2\}$  is a nontrivial 4-cut of G. Let  $|F_1| = \{x_2, z\}$ , then  $d_G(z) = 4$  and  $\{x_1, x_2, a\} \subseteq N_G(z)$ . This implies that  $z = a_2$ . Let  $N_G(a_2) = \{x_1, x_2, a, z'\}$ . Then we see that  $\{a_1, x_1, x_4, z'\}$  is a nontrivial 4-cut of G since  $|V(G)| \ge 14$ , a contradiction.

By Claim 1, we see that  $\{a, b\} \subseteq T_1$  and  $\{a_1, b_1\} \subseteq \overline{F_1}$  because  $\{ax_2, bx_1\} \subseteq E(G)$ .

Claim 2.  $\{a_2, b_2\} \subseteq \overline{F_1}$ .

**Proof.** If  $a_2 \in T_1$ , then  $N_G(\{x, x_3, a\}) \cap F_1 = \{x_2\}$  and  $N_G(\{x, x_3, a\}) \cap \overline{F_1} = \{a_1\}$ . Since  $|V(G)| \ge 14$ ,  $|F_1| \ge 3$  or  $|\overline{F_3}| \ge 3$ , which implies that  $(T_1 - \{x, x_3, a\}) \cup \{x_2\}$  or  $(T_1 - \{x, x_3, a\}) \cup \{a_1\}$  is a nontrivial 4-cut of G, a contradiction. Thus  $a_2 \notin T_1$ . Similarly,  $b_2 \notin T_1$ . In the following, we show that  $a_2 \in \overline{F_1}$ , and the other one can be handled similarly. Suppose  $a_2 \notin \overline{F_1}$ . Hence,  $a_2 \in F_1$  and, thus,  $N_G(\{x, x_3, a\}) \cap \overline{F_1} = \{a_1\}$ . It follows that  $|\overline{F_1}| = \{a_1, b_1\}$ . Let  $T_1 = \{x, x_3, x_4, a, b, u\}$ . Then we see that  $N_G(a_1) = \{x_3, a, b_1, u\}$  and  $N_G(b_1) = \{x_4, b, a_1, u\}$ . Note that  $u \neq b_2$ , for otherwise,  $\{x_1, x_2, a_2, b_2\}$  is a nontrivial 4-cut of G.

We next show that either  $aa_2$  or  $bb_2$  is quasi 5-contractible, which contradicts that G is a contraction critical quasi 5-connected graph. Suppose that neither  $aa_2$ nor  $bb_2$  is quasi 5-contractible. Clearly,  $\delta(G/aa_2) = 4$ . Then Lemma 5 assures us that  $G/aa_2$  is 4-connected. Let C be a quasi fragment with respect to  $aa_2$ and let  $R = N_G(C)$ ,  $\overline{C} = G - (R \cup C)$ . Then  $x_3 \in R$  and we may assume that  $x_2 \in C$ ,  $a_1 \in \overline{C}$  without loss of generality. If  $N_G(x_3) \cap C = \{x_2\}$ , then we see that  $x \in R$  since  $xx_2 \in E(G)$ . Furthermore,  $N_G(\{a, x_3\}) = \{x_2\}$ . This implies that |C| = 2. Let  $C = \{x_2, z\}$ . Then  $d_G(z) = 4$  and  $zx \in E(G)$ , which contradicts to (\*). Thus  $\{x, x_4\} \cap C \neq \emptyset$ . Then  $N_G(\{a, x_3\}) \cap \overline{C} = \{a_1\}$ , which implies that  $|\overline{C}| = 2$ . Let  $\overline{C} = \{a_1, w\}$ . Then we see that the vertex w satisfies  $d_G(w) = 4$  and  $\{a_1, a_2\} \subseteq N_G(w)$ . So we must have w = u. By similar argument for  $bb_2$ , we can deduce that  $ub_2 \in E(G)$ . Thus,  $N_G(u) = \{a_1, a_2, b_1, b_2\}$ , and thus,  $\{x_1, x_2, a_2, b_2\}$ is a nontrivial 4-cut of G since  $|V(G)| \geq 14$ , a contradiction. By Claim 2, we see that  $N_G(\{x, x_3, x_4, a, b\}) \cap F_1 = \{x_1, x_2\}$ . Hence,  $F_1 = \{x_1, x_2\}$ , for otherwise,  $(T_1 - \{x, x_3, x_4, a, b\}) \cup \{x_1, x_2\}$  is a 3-cut of G, a contradiction. Let  $T_1 - \{x, x_3, x_4, a, b\} = \{v\}$ . Then we have that  $N_G(x_1) = \{x, x_2, x_4, b, v\}$  and  $N_G(x_2) = \{x, x_1, x_3, a, v\}$ . Let  $G' = G/bb_2/b_1x_4/xx_1$ . Clearly,  $\delta(G') = 4$ .

# Claim 3. G' is 4-connected.

**Proof.** Assume, to the contrary, that G' is not 4-connected, then G' has a 3cut T' such that each component of G' - T' has at least 2 vertices and  $|T' \cap \{A', B', C'\}| \ge 2$ . Let T be the set in G corresponding to T' in G'. We first find that  $T' \cap \{A', B', C'\} \ne \{A', B'\}$  and  $T' \cap \{A', B', C'\} \ne \{A', C'\}$ , for otherwise,  $T - \{b\}$  is a nontrivial 4-cut of G. Therefore, the set  $T' \cap \{A', B', C'\}$  is  $\{B', C'\}$ or  $\{A', B', C'\}$ . In the former,  $T - \{x\}$  forms a nontrivial 4-cut of G. In the latter,  $T - \{b, x_4, x\}$  forms a 3-cut of G. Both of which contradict the fact that G is quasi 5-connected.

Since G' is not quasi 5-connected, there exists a nontrivial 4-cut  $T'_2$  of G' by Claim 3. Furthermore,  $|T'_2 \cap \{A', B', C'\}| \ge 1$ . Let  $F'_2$  be a  $T'_2$ -fragment of G' and let  $\overline{F'_2} = G' - (T'_2 \cup F'_2)$ . Let  $F_2, T_2, \overline{F_2}$  be the sets in G corresponding to  $F'_2, T'_2, \overline{F'_2}$  in G'. Note that the three vertices A', B', C' are adjacent to each other in G'. Hence, we may assume that the vertices in  $\{A', B', C'\} - T'_2$  belong to  $F'_2$  without loss of generality.

Claim 4.  $|T'_2 \cap \{A', B', C'\}| \neq 1$ .

**Proof.** Suppose  $|T'_2 \cap \{A', B', C'\}| = 1$ . Then  $|T_2| = 5$  and  $|F_2| \ge 4$ . If  $T'_2 \cap \{A', B', C'\} = \{A'\}$ , then we see that  $N_G(b) \cap \overline{F_2} = \emptyset$ , which implies that  $T_2 - \{b\}$  is a nontrivial 4-cut of G, a contradiction. If  $T'_2 \cap \{A', B', C'\} = \{B'\}$ , we can get that  $T_2 - \{x_4\}$  is a nontrivial 4-cut of G.

Therefore,  $T'_2 \cap \{A', B', C'\} = \{C'\}$ . If  $N_G(x) \cap \overline{F_2} = \emptyset$ , then  $T_2 - \{x\}$  is a nontrivial 4-cut of G, a contradiction. Hence,  $x_3 \in T_2$  and  $x_2 \in \overline{F_2}$  since  $x_3x_4 \in E(G)$ . If  $N_G(x_3) \cap F_2 = \{x_4\}$ , then  $(T_2 - \{x, x_3\}) \cup \{x_4\}$  is a nontrivial 4-cut of G. Hence,  $a_1 \in F_2$  and  $a \in T_2$  since  $\{ax_2, aa_1\} \subseteq E(G)$ . Note that  $N_G(\{x, x_3\}) \cap \overline{F_2} = \{x_2\}$ , which implies that  $N_G(a) \cap \overline{F_2} \neq \{x_2\}$ . Otherwise,  $(T_2 - \{x, x_3, a\}) \cup \{x_2\}$  is a 3-cut of G, which is absurd. So  $a_2 \in \overline{F_2}$ . It follows  $N_G(\{x, x_3, a\}) \cap F_2 = \{x_4, a_1\}$ . This implies  $(T_2 - \{x, x_3, a\}) \cup \{x_4, a_1\}$  is a nontrivial 4-cut of G, a contradiction.

Claim 5.  $|T'_2 \cap \{A', B', C'\}| \neq 2.$ 

**Proof.** Suppose  $|T'_2 \cap \{A', B', C'\}| = 2$ . Then  $|T_2| = 6$  and  $|F_2| \ge 3$ . If  $T'_2 \cap \{A', B', C'\} = \{A', B'\}$ , then  $x_3 \in F_2 \cap T_2$  since  $xx_3 \in E(G)$ . This implies  $N_G(\{b, x_4\}) \cap \overline{F_2} = \emptyset$ , and hence  $T_2 - \{b, x_4\}$  is a nontrivial 4-cut of G, a contradiction.

If  $T'_2 \cap \{A', B', C'\} = \{A', C'\}$ , then we see that  $N_G(b) \cap \overline{F_2} = \emptyset$ . If  $N_G(x) \cap \overline{F_2} = \emptyset$ , then  $T_2 - \{b, x\}$  is a nontrivial 4-cut of G, a contradiction. So,  $N_G(x) \cap \overline{F_2} \neq \emptyset$ . This implies  $x_2 \in \overline{F_2}$  and  $x_3 \in T_2$  since  $x_3x_4 \in E(G)$ . Since  $vx_2 \in E(G)$ , we see that  $v \in \overline{F_2} \cup T_2$ . It follows  $N_G(\{x, x_1\}) \cap F_2 = \{x_4\}$ . Then  $N_G(x_3) \cap F_2 \neq \{x_4\}$ , for otherwise,  $(T_2 - \{x, x_1, x_3\}) \cup \{x_4\}$  is a nontrivial 4-cut since  $|F_2| \geq 3$ . Hence,  $a_1 \in F_2$  and  $a \in T_2$  since  $ax_2 \in E(G)$ . Now,  $T_2 = \{b, b_2, x, x_1, x_3, a\}$  and  $v \in \overline{F_2}$ . Then we see that  $|\overline{F_2}| = 2$ , for otherwise,  $\{a, v, b_2\}$  is a 3-cut of G. However, we observe that  $d_G(v) < 4$  since  $va \notin E(G)$ , a contradiction.

Therefore,  $T'_2 \cap \{A', B', C'\} = \{B', C'\}$ . If  $\{x_2, x_3\} \subseteq F_2 \cup T_2$ , then  $N_G(\{x_4, x\}) \cap \overline{F_2} = \emptyset$ , which implies that  $T_2 - \{x_4, x\}$  is a nontrivial 4-cut of G, a contradiction. Then  $\{x_2, x_3\} \cap \overline{F_2} \neq \emptyset$ , and then  $\{x_2, x_3\} \subseteq T_2 \cup \overline{F_2}$  since  $x_2 x_3 \in E(G)$ . If  $N_G(x_1) \cap F_2 = \{b\}$ , then  $N_G(\{x, x_1, x_4\}) \cap F_2 = \{b\}$ , and then,  $(T_2 - \{x, x_1, x_4\}) \cup \{b\}$  is a nontrivial 4-cut of G. Thus,  $v \in F_2$ , and thus,  $x_2 \in T_2$  and  $x_3 \in \overline{F_2}$ . If  $a \notin \overline{F_2}$ , then  $N_G(\{x, x_1, x_2, x_4\}) \cap \overline{F_2} = \{x_3\}$ . It follows that  $(T_1 - \{x, x_1, x_2, x_4\}) \cup \{x_3\}$  is a 3-cut of G, a contradiction. Hence  $a \in \overline{F_2}$ . This implies  $\{a_1, a_2\} \subseteq T_2 \cup \overline{F_2}$ . Note that  $\{b_1, x, x_1, x_2, x_4\} \subseteq T_2$  and  $|T_2| = 6$ , we have  $\{a_1, a_2\} \cap \overline{F_2} \neq \emptyset$ . It follows  $|\overline{F_3}| \ge 3$ . Then we see that  $(T_2 - \{x, x_1, x_4\}) \cup \{x_3\}$  is a nontrivial 4-cut of G, a contradiction.

By Claims 4 and 5, we have  $\{A', B', C'\} \subseteq T'_2$ . Then  $|T_2| = 7$ . Moreover, we see that  $N_G(\{b, x_4, x\}) \cap F_2 = \emptyset$  or  $N_G(\{b, x_4, x\}) \cap \overline{F_2} = \emptyset$ . This implies that  $T_2 - \{b, x_4, x\}$  is a nontrivial 4-cut of G, a contradiction.

Now we are prepared to prove Theorem 3.

**Proof of Theorem 3.** If G has a quasi 5-contractible edge, Theorem 3 holds immediately. Thus we assume that G is a contraction critical quasi 5-connected graph. Let  $N_G(x) = \{x_1, x_2, x_3, x_4\}$ . Without loss of generality, we suppose  $\{x_1x_2, x_2x_3, x_3x_4, x_4x_1\} \subseteq E(G)$ . For  $i = 1, 2, 3, 4, d_G(x_i) \geq 5$  by (\*). Thus Lemma 5 assures us that  $G/xx_i$  is 4-connected.

For i = 1, 2, let  $F_i$  be quasi fragments with respect to  $xx_i$ . Let  $T_i = N_G(F_i)$ and  $\overline{F_i} = V(G) - (F_i \cup T_i)$ . Then  $x_3 \in T_1$  and  $x_4 \in T_2$ . Without loss of generality, we may assume  $x_2 \in F_1$ ,  $x_4 \in \overline{F_1}$  and  $x_1 \in F_2$ ,  $x_3 \in \overline{F_2}$ . Thus,  $x \in T_1 \cap T_2$ ,  $x_1 \in T_1 \cap F_2$ ,  $x_2 \in F_1 \cap T_2$ ,  $x_3 \in T_1 \cap \overline{F_2}$  and  $x_4 \in \overline{F_1} \cap T_2$ . Let  $X_1 = (T_1 \cap F_2) \cup (T_1 \cap T_2) \cup (F_1 \cap T_2)$ ,  $X_2 = (T_1 \cap F_2) \cup (T_1 \cap T_2) \cup (\overline{F_1} \cap T_2)$ ,  $X_3 = (\overline{F_1} \cap T_2) \cup (T_1 \cap T_2) \cup (T_1 \cap \overline{F_2})$  and  $X_4 = (F_1 \cap T_2) \cup (T_1 \cap \overline{F_2})$ .

Claim 1. There exists  $i \in \{1, 2, 3, 4\}$  such that  $|X_i| \leq 4$ .

**Proof.** To the contrary, we assume that for all i = 1, 2, 3, 4,  $|X_i| \ge 5$ . The fact  $|X_1| + |X_3| = |X_2| + |X_4| = 10$  shows that  $|X_1| = |X_2| = |X_3| = |X_4| = 5$ . Since  $N_G(x) \cap (F_1 \cap F_2) = \emptyset$ , we see that  $|F_1 \cap F_2| \le 1$ , for otherwise,  $X_1 - \{x\}$  is a nontrivial 4-cut of G, a contradiction. Similarly, we have that  $|\overline{F_1} \cap F_2| \le 1$ ,

 $|\overline{F_1} \cap \overline{F_2}| \leq 1$  and  $|F_1 \cap \overline{F_2}| \leq 1$ . It follows  $|V(G)| \leq 13$ , which contradicts the fact that  $|V(G)| \geq 14$ .

Without loss of generality, we may assume  $|X_3| \leq 4$ . Since  $N_G(x) \cap (\overline{F_1} \cap \overline{F_2}) = \emptyset$ ,  $\overline{F_1} \cap \overline{F_2} = \emptyset$ .

Claim 2.  $|\overline{F_1} \cap T_2| = |T_1 \cap \overline{F_2}| = 1.$ 

**Proof.** We only show that  $|\overline{F_1} \cap T_2| = 1$ , and the other one can be handled similarly. Suppose  $|\overline{F_1} \cap T_2| \ge 2$ . Then  $|\overline{F_1} \cap T_2| = 2$  and  $|T_1 \cap T_2| = |T_1 \cap \overline{F_2}| = 1$ . Hence,  $|F_1 \cap T_2| = 2$ , which implies  $|X_4| = 4$ . Since  $N_G(x) \cap (F_1 \cap \overline{F_2}) = \emptyset$ , we see that  $F_1 \cap \overline{F_2} = \emptyset$ . It follows  $|\overline{F_2}| = 1$ , a contradiction.

Claim 3.  $|F_1 \cap \overline{F_2}| = |\overline{F_1} \cap F_2| = 1.$ 

**Proof.** We only show that  $|F_1 \cap \overline{F_2}| = 1$ . By Claim 2, we see that  $|X_4| = 5$ , which implies that  $|F_1 \cap \overline{F_2}| \le 1$  since  $N_G(x) \cap (F_1 \cap \overline{F_2}) = \emptyset$ . If  $F_1 \cap \overline{F_2} = \emptyset$ , then we find  $|\overline{F_2}| = 1$ , a contradiction. So  $|F_1 \cap \overline{F_2}| = 1$ .

Let  $F_1 \cap \overline{F_2} = \{a\}$  and let  $\overline{F_1} \cap F_2 = \{b\}$ . Note that  $|T_1 \cap T_2| \leq 2$  by Claim 1. If  $|T_1 \cap T_2| = 2$ , then we see that G has the subgraph shown in Figure 2(a). If  $|T_1 \cap T_2| = 1$ , then G has the subgraph shown in Figure 2(b). Then Theorem 3 holds by Lemmas 9, 10 and 11.

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