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CONTRACTIBLE SUBGRAPHS OF QUASI 5-CONNECTED GRAPHS

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Abstract

Let G be a quasi 5-connected graph on at least 14 vertices. If there is a vertex $x \in V_4(G)$ such that $G[N_G(x)] \cong K_{1,3}$ or $G[N_G(x)] \cong C_4$, then G can be contracted to a smaller quasi 5-connected graph H such that $0 < |V(G)| - |V(H)| < 4$.

Keywords: quasi 5-connected, contraction, minor.

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1. Introduction

In this paper, we only consider finite simple undirected graphs, with undefined terms and notations following [1]. For a graph G, let $V(G)$ and $E(G)$ denote the set of vertices of G and the set of edges of G, respectively. We denote the set of end vertices of an edge e by $V(e)$. For $x \in V(G)$, let $N_G(x) = \{y \in V(G) : xy \in E(G)\}$. We define the *degree* of $x \in V(G)$ by $d_G(x)$, namely $d_G(x) = |N_G(x)|$. Let $V_k(G)$ denote the set of vertices of degree k in G. Let $\delta(G)$ denote the minimum degree of G. For $S \subseteq V(G)$, we define $N_G(S) = \cup_{x \in S} N_G(x) - S$. Furthermore, let $G[S]$ denote the subgraph induced by S, and let $G - S$ denote the graph obtained from G by deleting the vertices of S together with the edges incident with them. Let $K_{1,n}$ denote the complete bipartite graph with partite sets of cardinality 1 and n. Let C_n denote a cycle of order n.

An edge $e = xy$ of G is said to be *contracted* if it is deleted and its ends are identified, the resulting graph is denoted by G/e. And the new vertex in G/e

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is denoted by $\overline{xy}$. Note that, in the contraction, we replace each resulting pair of double edges by a single edge. A subgraph H of G is said to be contracted by identifying each component to a single vertex, removing each of the resulting loops and, finally, replacing each of the resulting duplicate edges by a single edge. The resulting graph is denoted by G/H. Let k be an integer such that $k \geq 2$ and let G be a k-connected graph with $|V(G)| \geq k+2$. An edge e of G is said to be *k-contractible* if the contraction of the edge results in a k-connected graph. A k-connected graph without a k-contractible edge is said to be a *contraction critical k-connected graph*. A subgraph H of G is said to be k-contractible if G/H is still k-connected.

A *cut* of a connected graph G is a subset $V'(G)$ of $V(G)$ such that $G - V'(G)$ is disconnected. A *k-cut* is a cut of k elements. Suppose T is a k-cut of G. We say that T is a *nontrivial k-cut*, if the components of $G-T$ can be partitioned into subgraphs G_1 and G_2 such that $|V(G_1)| \geq 2$ and $|V(G_2)| \geq 2$. A $(k-1)$-connected graph is *quasi k-connected* if it has no nontrivial $(k-1)$-cuts. Clearly, every k-connected graph is quasi k-connected.

Let G be a quasi k-connected graph. An edge e of G is said to be *quasi k-contractible* if G/e is still quasi k-connected. If G does not have a quasi k-contractible edge, then G is said to be a *contraction critical quasi k-connected graph*. A subgraph H of G is said to be quasi k-contractible if its contraction G/H results again in a quasi k-connected graph.

Tutte's [8] famous wheel theorem implies that every 3-connected graph on more than four vertices contains an edge whose contraction yields a new 3-connected graph. One can give an inductive proof of Kuratowski's theorem by the wheel theorem [6]. Results on the distribution of 3-contractible edges led also to coloring theorems on planar graphs [3, 5]. So the existence and the distribution of k-contractible subgraphs is an attractive research area within graph connectivity theory.

Thomassen [7] showed that for $k \geq 4$, there are infinitely many contraction critical k-connected k-regular graphs. However, every 4-connected graph on at least seven vertices can be reduced to a smaller 4-connected graph by contracting at most two edges. So, naturally, Kriesell posted the following conjecture [2].

Conjecture 1 [2]**.** *There exist positive integers b and h such that every k-connected graph on more than b vertices can be reduced to a smaller k-connected graph by contracting less than h edges for every $k \geq 1$.*

Clearly, Conjecture 1 is true for $k = 1$ and $k = 2$. For $k = 3$ and $k = 4$, the smallest appropriate values for b, h would be 4, 2 and 6, 3, respectively. But for $k \geq 6$, such a statement fails since toroidal triangulations of large face width is a counterexample [2]. In [4], Kriesell proved that every quasi 6-connected graph on at least 13 vertices can be reduced to a smaller 5-connected graph by contracting less than five edges subsequently. The conjecture is still open for $k = 5$.

We focus on quasi 5-connected graphs and obtain the following results.

Theorem 2. *Let G be a quasi 5-connected graph on at least 13 vertices. If there is a vertex $x \in V_4(G)$ such that $G[N_G(x)] \cong K_{1,3}$, then G can be reduced to a smaller quasi 5-connected graph by contracting less than three edges subsequently.*

Theorem 3. *Let G be a quasi 5-connected graph on at least 14 vertices. If there is a vertex $x \in V_4(G)$ such that $G[N_G(x)] \cong C_4$, then G can be reduced to a smaller quasi 5-connected graph by contracting less than four edges subsequently.*

By Combining Theorems 2 and 3, we have the following corollary.

Corollary 4. *Let G be a quasi 5-connected graph on at least 14 vertices. If there is a vertex $x \in V_4(G)$ such that $G[N_G(x)] \cong K_{1,3}$ or $G[N_G(x)] \cong C_4$, then G can be contracted to a smaller quasi 5-connected graph H such that $0 < |V(G)| - |V(H)| < 4$.*

2. Preliminaries

In this section, we introduce more definitions and several preliminary lemmas.

Let G be a non-complete connected graph and let $\kappa(G)$ denote the vertex connectivity of G. By $\mathcal{T}(G) := \{T \subseteq V(G) : T \text{ is a cut of } G, |T| = \kappa(G)\}$, we denote the set of *smallest cuts* of G. For $T \in \mathcal{T}(G)$, the union of the vertex sets of at least one but not of all components of $G-T$ is called a *T-fragment* of G or, briefly, a *fragment*. Let F be a T-fragment, and let $\overline{F} = V(G) - (F \cup T)$. Clearly, $\overline{F} \neq \emptyset$, and $\overline{F}$ is also a T-fragment such that $N_G(F) = T = N_G(\overline{F})$.

Let G be a quasi k-connected graph and let $E_0 = \{e \in E(G) : G/e \text{ is } (k-1)\text{-connected, but not quasi } k\text{-connected}\}$. For $xy \in E_0$, G/xy has a nontrivial $(k-1)$-cut T' by the definition of quasi k-connected. Furthermore, $\overline{xy} \in T'$, for otherwise, T' is also a nontrivial $(k-1)$-cut of G, contradicts the fact that G is quasi k-connected. This implies that $T = (T' - \overline{xy}) \cup \{x, y\}$ is a k-cut of G. Moreover, $G-T$ can be partitioned into subgraphs F and $\overline{F}$ such that $|V(F)| \geq 2$ and $|V(\overline{F})| \geq 2$. Each of these subgraphs is called a *quasi T-fragment* of G or, briefly, a *quasi fragment*. For an edge e of G, a quasi fragment F of G is said to be a *quasi fragment with respect to e* if $V(e) \subseteq N_G(F)$. For a set of edges $E' \subseteq E(G)$, we say that F is a *quasi fragment with respect to E'* if F is a quasi fragment with respect to some $e \in E'$. A quasi fragment with respect to e or E' with least cardinality is called a *quasi atom* with respect to e and E', respectively.

Lemma 5. *Let G be a quasi 5-connected graph. If $xy \in E(G)$ and $\delta(G/xy) \geq 4$, then G/xy is 4-connected.*

Proof. Suppose that G/xy is not 4-connected, then there exists a 3-cut T' of G/xy. Since $\delta(G/xy) \geq 4$, we see that each component of $G/xy - T'$ has at least two vertices. Furthermore, $\overline{xy} \in T'$, for otherwise, T' is also a 3-cut of G, a contradiction. Hence, $T = (T' - \{\overline{xy}\}) \cup \{x, y\}$ is a 4-cut of G. And each component of $G - T$ has at least two vertices. It follows that T is a nontrivial 4-cut of G, which contradicts the quasi 5-connectivity of G. ■

Lemma 6. *Let G be a quasi 5-connected graph and let $x \in V_4(G)$ such that $N_G(x) = \{y, z, u, v\}$. If $\{yz, uv\} \subseteq E(G)$ and $\delta(G/\triangle xyz) \geq 4$, then $G/\triangle xyz$ is 4-connected.*

Proof. If $G/\triangle xyz$ is not 4-connected, then $G/\triangle xyz$ has a 3-cut T'. Since $\delta(G/\triangle xyz) \geq 4$, each component of $G/\triangle xyz - T'$ has at least two vertices. Furthermore, the vertex resulting from the contraction of triangle xyz belongs to T'. Hence, G has a 5-cut T such that $\{x, y, z\} \subseteq T$ and $G - T$ can be partitioned into two parts, say F and $\overline{F}$, where each part has at least two vertices. Since $uv \in E(G)$, we see that $N_G(x) \cap F = \emptyset$ or $N_G(x) \cap \overline{F} = \emptyset$, which implies that $T - \{x\}$ is a nontrivial 4-cut of G, a contradiction. ■

Lemma 7. *Let G be a quasi 5-connected graph on at least 8 vertices. If there is a vertex $x \in V(G)$ such that $N_G(x) = \{x_1, x_2, x_3, x_4\}$ and $G[\{x_1, x_2, x_3\}] \cong K_3$, then xx_4 is a quasi 5-contractible edge.*

Proof. For $i = 1, 2, 3$, we have $d_G(x_i) \geq 5$, for otherwise, $N_G(\{x, x_i\})$ is a nontrivial 4-cut of G since $|V(G)| \geq 8$, which contradicts that G is quasi 5-connected. This implies $\delta(G/xx_4) \geq 4$. Thus Lemma 5 assures us that G/xx_4 is 4-connected. Suppose that G/xx_4 is not quasi 5-connected, then we see that G/xx_4 has a nontrivial 4-cut T'. Furthermore, $\overline{xx_4} \in T'$, for otherwise, T' is also a nontrivial 4-cut of G, a contradiction. Thus, G has a 5-cut T such that $\{x, x_4\} \subseteq T$. And $G - T$ can be partitioned into two parts, say F and $\overline{F}$, where each part has at least two vertices. However, since $G[\{x_1, x_2, x_3\}] \cong K_3$, we observe that $N_G(x) \cap F = \emptyset$ or $N_G(x) \cap \overline{F} = \emptyset$. It follows that $T - \{x\}$ is a nontrivial 4-cut of G, a contradiction. ■

3. Proof of Theorem 2

In this section, we give a proof of Theorem 2. We first introduce a useful lemma.

Lemma 8. *Let G be a quasi 5-connected graph on at least 13 vertices. If Figure 1 is a subgraph of G, then the graph G' obtained from G by contracting $A := \{a, a_0\}$, $B := \{p, p_0\}$ to vertices A', B', respectively, is still quasi 5-connected.*

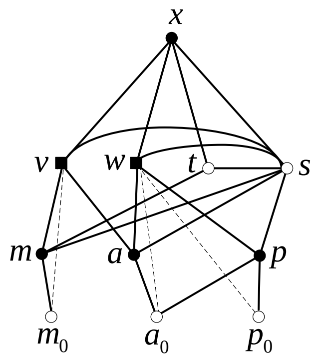


Figure 1. The graph in Lemma 8. Solid edges indicate edges that must exist, and dashed edges indicate edges that may exist. A black circular vertex indicates that the vertex has reached its maximum degree. A black square vertex indicates that its neighbor set is a subset of all vertices adjacent to it by solid and dashed edges. Specifically, the vertex adjacent to it by a solid edge must be a neighbor of it, while the vertex adjacent to it by a dashed edge is not necessarily so.

Proof. We first show that G' is 4-connected. Note that $\{v, w, s, B'\} \subseteq N_{G'}(A')$, then $d_{G'}(A') \geq 4$. If $d_{G'}(B') < 4$, then we see that $N_G(p_0) = \{p, w, s, a_0\}$, and hence $\{t, s, m_0, a_0\}$ is a nontrivial 4-cut of G since $|V(G)| \geq 13$, which contradicts that G is quasi 5-connected. Thus, $d_{G'}(B') \geq 4$. Moreover, it is easy to find that every vertex in G', except A' and B', has degree at least 4. It follows $\delta(G') = 4$. Suppose that G' is not 4-connected, then G' has a 3-cut T' such that each component of $G' - T'$ has at least 2 vertices and $T' \cap \{A', B'\} \neq \emptyset$. If $|T' \cap \{A', B'\}| = 1$, then we find that G has a nontrivial 4-cut, a contradiction. Thus, $\{A', B'\} \subseteq T'$. It follows that G has a 5-cut T containing $\{a, a_0, p, p_0\}$. Since $N_G(p) = \{a_0, p_0, w, s\}$ and $ws \in E(G)$, we see that $T - \{p\}$ is a nontrivial 4-cut of G, a contradiction. Thus, G' is 4-connected.

Suppose, to the contrary, that G' is not quasi 5-connected, then G' has a nontrivial 4-cut T' such that $|T' \cap \{A', B'\}| \geq 1$. Let F' be a T'-fragment of G' and let $\overline{F'} = G' - (T' \cup F')$. Let F, T, $\overline{F}$ be the sets in G corresponding to F', T', $\overline{F'}$ in G'. That is, in each of these sets, we replace the vertices A', B' by the vertices in the sets A, B, respectively.

If $\{A', B'\} \subseteq T'$, then one of v and w is in F, and the other is in $\overline{F}$. Otherwise, $N_G(\{a, p\}) \cap F = \emptyset$ or $N_G(\{a, p\}) \cap \overline{F} = \emptyset$, which implies that $T - \{a, p\}$ is a nontrivial 4-cut of G, a contradiction. Without loss of generality, we may assume that $v \in F$ and $w \in \overline{F}$. Since $\{sv, sw, xv, xw\} \subseteq E(G)$, we have $T = \{a, a_0, p, p_0, s, x\}$. Hence we obtain $N_G(w) \subseteq T$. Since $mv \in E(G)$, $m \in F$. Then the fact that $mt \in E(G)$ assures us that $t \in F$. Hence, we see that $N_G(\{a, p, x\}) \cap \overline{F} = \{w\}$ and, thus, $\{a_0, p_0, s\}$ forms a 3-cut of G since $|\overline{F}| \geq 2$, a contradiction.

Consequently, $|T' \cap \{A', B'\}| = 1$. Suppose that $A' \notin T'$ and $B' \in T'$. We may assume that $A' \in F'$ without loss of generality. Then $\{w, s\} \subseteq F \cup T$ because $\{aw, as\} \subseteq E(G)$, and hence $N_G(p) \cap \overline{F} = \emptyset$. This implies that $T - \{p\}$ is a nontrivial 4-cut of G, a contradiction. Thus, $A' \in T'$ and $B' \notin T'$. We may assume that $B' \in F'$ without loss of generality. Then $|F| \geq 3$. If $N_G(a) \cap F = \emptyset$ or $N_G(a) \cap \overline{F} = \emptyset$, then $T - \{a\}$ is a nontrivial 4-cut of G, a contradiction. So $N_G(a) \cap F \neq \emptyset$ and $N_G(a) \cap \overline{F} \neq \emptyset$. Then we see that $s \in T$ since $N_G(a) = \{a_0, v, w, s\}$ and $\{vs, ws\} \subseteq E(G)$. Since $p \in F$ and $wp \in E(G)$, we have $w \in F$ and $v \in \overline{F}$. Notice that $\{xw, xv\} \subseteq E(G)$, then $x \in T$. If $N_G(\{a, x\}) \cap F = \{w\}$, then we find that $(T - \{a, x\}) \cup \{w\}$ is a nontrivial 4-cut of G since $|F| \geq 3$, a contradiction. Hence, $t \in F$, and thus $N_G(\{a, x\}) \cap \overline{F} = \{v\}$. This implies that $|\overline{F}| = 2$, for otherwise, $(T - \{a, x\}) \cup \{v\}$ forms a nontrivial 4-cut of G. Let $\overline{F} = \{v, z\}$. Then $d_G(z) = 4$ and $\{v, s, a_0\} \subseteq N_G(z)$. Clearly, $z \neq m$ since m is not adjacent to a_0. If $vm_0 \notin E(G)$, then we obtain a contradiction, since there is no such a vertex z in G. If $vm_0 \in E(G)$, then $z = m_0$, and thus $N_G(m_0) = \{v, s, a_0, m\}$. It follows that $\{t, s, a_0, p_0\}$ forms a nontrivial 4-cut of G by the fact that $|V(G)| \geq 13$, a contradiction. ■

Now we are prepared to prove Theorem 2.

***Proof of Theorem* 2.** If G has a quasi 5-contractible edge, Theorem 2 holds immediately. Thus we assume that G is a contraction critical quasi 5-connected graph. Let $N_G(x) = \{x_1, x_2, x_3, x_4\}$. Without loss of generality, we suppose that $\{x_1x_4, x_2x_4, x_3x_4\} \subseteq E(G)$. If $d_G(x_4) = 4$, then $\{x_1, x_2, x_3\}$ forms a 3-cut of G. So, $d_G(x_4) \geq 5$. Hence, for $i = 1, 2, 3$, $\delta(G/xx_i) \geq 4$, and hence Lemma 5 assures us that G/xx_i is 4-connected.

Let $E' = \{xx_1, xx_2, xx_3\}$ and let F_1 be a quasi atom with respect to E'. Let $T_1 = N_G(F_1)$ and let $\overline{F_1} = V(G) - (F_1 \cup T_1)$. Then $|\overline{F_1}| \geq |F_1| \geq 2$. Without loss of generality, we assume that F_1 is a quasi fragment with respect to xx_1. Since $x_2x_4 \in E(G)$ and $x_3x_4 \in E(G)$, we see that $x_4 \in T_1$ and we may assume that $x_2 \in F_1$, $x_3 \in \overline{F_1}$. Let F_2 be a quasi fragment with respect to xx_2 and let $T_2 = N_G(F_1)$, $\overline{F_2} = V(G) - (F_2 \cup T_2)$. Then $x_4 \in T_2$ and we may assume that $x_1 \in F_2$, $x_3 \in \overline{F_2}$. Now, we find that $\{x, x_4\} \subseteq T_1 \cap T_2$, $x_1 \in T_1 \cap F_2$, $x_2 \in F_1 \cap T_2$ and $x_3 \in \overline{F_1} \cap \overline{F_2}$. Let $X_1 = (T_1 \cap F_2) \cup (T_1 \cap T_2) \cup (F_1 \cap T_2)$, $X_2 = (T_1 \cap F_2) \cup (T_1 \cap T_2) \cup (\overline{F_1} \cap T_2)$, $X_3 = (\overline{F_1} \cap T_2) \cup (T_1 \cap T_2) \cup (T_1 \cap \overline{F_2})$, $X_4 = (F_1 \cap T_2) \cup (T_1 \cap T_2) \cup (T_1 \cap \overline{F_2})$.

Claim 1. $F_1 \cap T_2 = \{x_2\}$.

Proof. Assume, to the contrary, that $|F_1 \cap T_2| \geq 2$. Then $|(T_1 \cap T_2) \cup (\overline{F_1} \cap T_2)| \leq 3$. Since $x_3 \in \overline{F_1} \cap \overline{F_2}$, we see that $|X_3| \geq 4$. Hence, $|T_1 \cap \overline{F_2}| \geq 1$, and thus, $|T_1 \cap F_2| \leq 2$. If $|T_1 \cap F_2| = 1$, then $|X_2| \leq 4$. Since $N_G(x) \cap (\overline{F_1} \cap F_2) = \emptyset$, we observe that $\overline{F_1} \cap F_2 = \emptyset$. This implies that $|F_2| < |F_1|$, a contradiction.

Therefore, $|T_1 \cap F_2| = 2$. It follows that $|T_1 \cap T_2| = |F_1 \cap T_2| = 2$ and $|T_1 \cap \overline{F_2}| = |\overline{F_1} \cap T_2| = 1$. Now, $|X_2| = 5$ and $|X_3| = 4$. Since G has no nontrivial 4-cuts, we find that $|\overline{F_1} \cap F_2| \leq 1$ and $\overline{F_1} \cap \overline{F_2} = \{x_3\}$. This implies that $|\overline{F_1}| \leq 3$. Hence, $|F_1| \leq |\overline{F_1}| \leq 3$, which implies that $|V(G)| \leq 11$, a contradiction. □

By Claim 1, we have $F_1 \cap F_2 \neq \emptyset$. Otherwise, we see that $N_G(x_1) \cap F_1 = \emptyset$ since $x_1x_2 \notin E(G)$, and then $T_1 - \{x_1\}$ is a nontrivial 4-cut of G, a contradiction. Since $N_G(x) \cap (F_1 \cap F_2) = \emptyset$, $|X_1| \geq 5$. Hence, $|F_1 \cap T_2| \geq |T_1 \cap \overline{F_2}|$. Then we have $|T_1 \cap \overline{F_2}| \leq 1$ by Claim 1.

Claim 2. $|T_1 \cap \overline{F_2}| = 1$ *and* $|F_1 \cap F_2| = 1$.

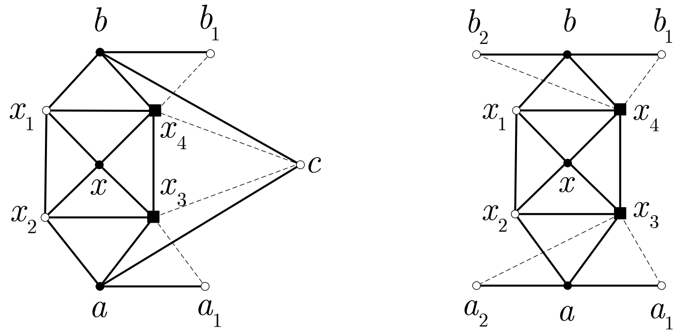
Proof. Suppose $T_1 \cap \overline{F_2} = \emptyset$. Then $|X_1| = 6$ and $|X_3| = 4$. This implies $\overline{F_1} \cap \overline{F_2} = \{x_3\}$. Then $|\overline{F_2}| < |F_1|$, a contradiction. So $|T_1 \cap \overline{F_2}| = 1$. Then we see that $|(T_1 \cap F_2) \cup (T_1 \cap T_2)| = 4$, and hence $|X_1| = 5$. This implies $|F_1 \cap F_2| = 1$ since $N_G(x) \cap (F_1 \cap F_2) = \emptyset$ and G has no nontrivial 4-cuts. □

Let $F_1 \cap F_2 = \{a\}$. Then we see that $d_G(a) = 4$ and $\{x_1, x_2, x_4\} \subseteq N_G(a)$. Let $N_G(a) = \{x_1, x_2, x_4, a_0\}$. Note that $a_0 \in (T_1 \cap F_2) \cup (T_1 \cap T_2)$. We next show that G/aa_0 is 4-connected. Suppose $\delta(G/aa_0) < 4$. It follows that there exist a vertex $z \in N_G(a) \cap N_G(a_0)$ such that $d_G(z) = 4$. Clearly, $z \neq x_4$ since $d_G(x_4) \geq 5$. If $z = x_1$, then we see that $N_G(x_1) \cap \overline{F_1} = \emptyset$, which implies that $T_1 - \{x_1\}$ is a nontrivial 4-cut of G, a contradiction. If $z = x_2$, then $N_G(x_2) \cap \overline{F_2} = \emptyset$, which implies that $T_2 - \{x_2\}$ is a nontrivial 4-cut of G, a contradiction. Thus, there is no such a vertex z in G. Then $\delta(G/aa_0) \geq 4$, and then Lemma 5 assures us that G/aa_0 is 4-connected.

Let F_3 and F_4 be the quasi fragments with respect to xx_3 and aa_0, respectively. For $i = 3, 4$, let $T_i = N_G(F_i)$, $\overline{F_i} = V(G) - (F_i \cup T_i)$. Since $\{x_1x_4, x_2x_4\} \subseteq E(G)$, we see that $x_4 \in T_3$ and we may assume that $x_1 \in F_3$, $x_2 \in \overline{F_3}$ without loss of generality. Then $a \in T_3$ since $\{ax_1, ax_2\} \subseteq E(G)$. Similarly, $\{x_4, x\} \subseteq T_4$ and we may assume that $x_1 \in F_4$, $x_2 \in \overline{F_4}$. If $a_0 \in T_3$, then $N_G(\{x, a\}) \cap F_3 = \{x_1\}$ and $N_G(\{x, a\}) \cap \overline{F_3} = \{x_2\}$. Since $|V(G)| \geq 13$, we have $|F_3| \geq 4$ or $|\overline{F_3}| \geq 4$. It follows that $(T_3 - \{x, a\}) \cup \{x_1\}$ or $(T_3 - \{x, a\}) \cup \{x_2\}$ is a nontrivial 4-cut of G, a contradiction. Thus $a_0 \notin T_3$. Similarly, $x_3 \notin T_4$. Now, we see that $x_1 \in F_3 \cap F_4$, $x_2 \in \overline{F_3} \cap \overline{F_4}$ and $\{x, x_4, a\} \subseteq T_3 \cap T_4$. Since $x_3 \notin T_4$ and $a_0 \notin T_3$, $x_3 \in (T_3 \cap F_4) \cup (T_3 \cap \overline{F_4})$ and $a_0 \in (T_4 \cap F_3) \cup (T_4 \cap \overline{F_3})$. Let $X_5 = (T_3 \cap F_4) \cup (T_3 \cap T_4) \cup (F_3 \cap T_4)$, $X_6 = (T_3 \cap F_4) \cup (T_3 \cap T_4) \cup (\overline{F_3} \cap T_4)$, $X_7 = (\overline{F_3} \cap T_4) \cup (T_3 \cap T_4) \cup (T_3 \cap \overline{F_4})$, $X_8 = (F_3 \cap T_4) \cup (T_3 \cap T_4) \cup (T_3 \cap \overline{F_4})$.

Claim 3. $|F_3 \cap F_4| \leq 2$ *and* $|\overline{F_3} \cap \overline{F_4}| \leq 2$.

Proof. We only show that $|F_3 \cap F_4| \leq 2$. Suppose $|F_3 \cap F_4| \geq 3$. Let $W = (F_3 \cap F_4) - \{x_1\}$. Thus $|W| \geq 2$. Since G is quasi 5-connected, we see that $|N_G(W)| \geq 5$. Note that $N_G(\{x, a\}) \cap (F_3 \cap F_4) = \{x_1\}$. It follows $N_G(\{x, a\}) \cap W = \emptyset$, and hence, $|X_5| \geq 6$. Thus $|X_7| \leq 4$. This implies that $\overline{F_3} \cap \overline{F_4} = \{x_2\}$ and $\overline{F_3} \cap T_4 = \emptyset$



(a) (b)

Figure 2. The explanations for Figures 2(a) and 2(b) are identical to the explanation for the graph in Figure 1.

or $T_3 \cap \overline{F_4} = \emptyset$. Without loss of generality, we may assume $\overline{F_3} \cap T_4 = \emptyset$. Then $|X_6| \leq 5$, Since $N_G(\{x,a\}) \cap (\overline{F_3} \cap F_4) = \emptyset$, we find $\overline{F_3} \cap F_4 = \emptyset$. Then $|\overline{F_3}| = 1$, a contradiction. □

Claim 4. *Either* $F_3 \cap \overline{F_4} = \emptyset$ *or* $\overline{F_3} \cap F_4 = \emptyset$.

Proof. Note that, $N_G(\{x,a\}) \cap (F_3 \cap \overline{F_4}) = \emptyset$ and $N_G(\{x,a\}) \cap (\overline{F_3} \cap F_4) = \emptyset$. This implies that $|X_8| \geq 6$ if $F_3 \cap \overline{F_4} \neq \emptyset$ and $|X_6| \geq 6$ if $\overline{F_3} \cap F_4 \neq \emptyset$. By the fact that $|X_6| + |X_8| = 10$, we have $F_3 \cap \overline{F_4} = \emptyset$ or $\overline{F_3} \cap F_4 = \emptyset$. □

If $F_3 \cap \overline{F_4} = \emptyset$, then $|\overline{F_3} \cap F_4| \geq 2$. Otherwise, we see that $|V(G)| \leq 12$ by Claim 3, which contradicts the fact that $|V(G)| \geq 13$. This implies that $|X_6| \geq 7$ since $N_G(\{x,a\}) \cap (\overline{F_3} \cap F_4) = \emptyset$. Thus, $|T_3 \cap T_4| = 3$, $|T_3 \cap F_4| = |\overline{F_3} \cap T_4| = 2$ and $F_3 \cap T_4 = T_3 \cap \overline{F_4} = \emptyset$. It follows that $x_3 \in T_3 \cap F_4$ and $a_0 \in \overline{F_3} \cap T_4$. Furthermore, $|F_3| = |F_3 \cap F_4| = 2$ and $|\overline{F_4}| = |\overline{F_3} \cap \overline{F_4}| = 2$ by Claim 3. Let $F_3 \cap F_4 = \{x_1, m\}$ and let $T_3 \cap F_4 = \{x_3, m_0\}$. Let $\overline{F_3} \cap \overline{F_4} = \{x_2, p\}$ and let $\overline{F_3} \cap T_4 = \{a_0, p_0\}$. Now, we find that G has a subgraph isomorphic to the graph in Figure 1. Then $G/aa_0/pp_0$ is quasi 5-connected by Lemma 8. If $\overline{F_3} \cap F_4 = \emptyset$, we can also obtain that G has a subgraph isomorphic to the graph in Figure 1 by similar argument. And then Theorem 2 holds by Lemma 8. ■

4. Proof of Theorem 3

In this section, we consider the quasi 5-connected graph G that contains a 4-degree vertex x such that $G[N_G(x)] \cong C_4$. If $x' \in N_G(x)$ and $d_G(x') = 4$, then we see that $N_G(\{x,x'\})$ is a nontrivial 4-cut of G, which contradicts the quasi 5-connectivity of G. Hence, every neighborhood of x has degree at least 5 $(*)$.

Lemma 9. *Let G be a quasi 5-connected graph that contains either Figure 2(a) or Figure 2(b) as a subgraph. If $d_G(x_3) = 6$ or $d_G(x_4) = 6$, then Theorem 3 holds.*

Proof. If $d_G(x_3) = 6$, we see that either $G[N_G(a)] \cong K_{1,3}$ or $G[N_G(a)]$ contains a K_3-subgraph. It follows immediately from Theorem 2 and Lemma 7 that G has a quasi 5-contractible subgraph containing at most two edges. Hence, Theorem 3 holds. If $d_G(x_4) = 6$, we can obtain the same result similarly. ■

Lemma 10. *Let G be a contraction critical quasi 5-connected graph on at least 14 vertices. If Figure 2(a) is a subgraph of G satisfying $d_G(x_3) = d_G(x_4) = 5$, then $\triangle xx_3x_4$ is a quasi 5-contractible subgraph of G.*

Proof. Since $d_G(x_3) = d_G(x_4) = 5$, either $a_1 \in N_G(x_3)$ or $c \in N_G(x_3)$, and also either $b_1 \in N_G(x_4)$ or $c \in N_G(x_4)$.

Claim 1. $N_G(x_3) = \{x, x_2, x_4, a, a_1\}$ *and* $N_G(x_4) = \{x, x_1, x_3, b, b_1\}$.

Proof. We only show that $N_G(x_3) = \{x, x_2, x_4, a, a_1\}$. Let us assume, to the contrary, that $N_G(x_3) = \{x, x_2, x_4, a, c\}$. We next show that aa_1 is quasi 5-contractible, which contradicts that G is a contraction critical quasi 5-connected graph. We first show that $\delta(G/aa_1) = 4$. If $\delta(G/aa_1) < 4$, then we have that there exist a vertex $z \in N_G(a) \cap N_G(a_1)$ such that $d_G(z) = 4$. Clearly, $z \neq x_2$ since $d_G(x_2) \geq 5$. Similarly, $z \neq x_3$. Thus $z = c$. However, we see that $N_G(\{a, c\})$ is a nontrivial 4-cut of G, a contradiction. So $\delta(G/aa_1) = 4$. Then Lemma 5 assures us that G/aa_1 is 4-connected. Suppose that G/aa_1 is not quasi 5-connected. Let C be a quasi fragment with respect to aa_1 and let $R = N_G(C)$, $\overline{C} = G - (C \cup R)$.

Since $\{x_2x_3, x_3c\} \subseteq E(G)$, $x_3 \in R$ and we may assume that $x_2 \in C$ and $c \in \overline{C}$. Since $\{xx_2, x_1x_2\} \subseteq E(G)$, $\{x, x_1\} \subseteq C \cup R$. If $x_4 \in C \cup R$, then $N_G(\{a, x_3\}) \cap \overline{C} = \{c\}$. It follows that $|\overline{C}| = 2$, for otherwise, $(R - \{a, x_3\}) \cup \{c\}$ forms a nontrivial 4-cut of G. Let $\overline{C} = \{c, w\}$. Then we see that $d_G(w) = 4$ and $\{c, a_1\} \subseteq N_G(w)$. Since $cb \in E(G)$ and $ba_1 \notin E(G)$, $b \in R$, and then we have $w \in N_G(b)$. This implies that $w = b_1$. Since $c \in N_G(x_4)$ or $b_1 \in N_G(x_4)$, we find that $x_4 \in R$. It follows that $N_G(c) \subseteq \{a, b, x_3, b_1, a_1, x_4\}$ and $N_G(b_1) = \{c, a_1, b, x_4\}$, which implies that $\{a_1, x_1, x_2\}$ forms a 3-cut of G since $|V(G)| \geq 14$, a contradiction. Therefore, $x_4 \in \overline{C}$, and hence $\{x, x_1\} \subseteq R$. Then we see that $R = \{a, a_1, x_3, x, x_1\}$ and $N_G(\{a, x, x_3\}) \cap C = \{x_2\}$. Since $|C| \geq 2$, we find that $\{x_1, x_2, a_1\}$ is a 3-cut of G, which is absurd. □

In the following, we show that $\triangle xx_3x_4$ is a quasi 5-contractible subgraph of G. By $(*)$, $d_G(x_1) \geq 5$ and $d_G(x_2) \geq 5$. Then $\delta(G/\triangle xx_3x_4) = 4$, which implies that $G/\triangle xx_3x_4$ is 4-connected by Lemma 6. Suppose that $G/\triangle xx_3x_4$ is not quasi 5-connected, then $G/\triangle xx_3x_4$ has a nontrivial 4-cut T'. Let F' be a T'-fragment of $G/\triangle xx_3x_4$ and let $\overline{F'} = G/\triangle xx_3x_4 - (T' \cup F')$. Let F, T, $\overline{F}$

be the sets in G corresponding to F', T', $\overline{F'}$ in $G/\triangle xx_3x_4$. So, $\{x, x_3, x_4\} \subseteq T$. Without loss of generality, we may assume $\{x_1, x_2\} \subseteq F \cup T$.

Claim 2. $|\{x_1, x_2\} \cap T| = 1$.

Proof. Suppose $|\{x_1, x_2\} \cap T| \neq 1$, then either $\{x_1, x_2\} \subseteq T$ or $\{x_1, x_2\} \subseteq F$. If $\{x_1, x_2\} \subseteq T$, then we see that $N_G(\{x, x_3\}) \cap F = \emptyset$ or $N_G(\{x, x_3\}) \cap \overline{F} = \emptyset$, which implies that $T - \{x, x_3\}$ is a nontrivial 4-cut of G, a contradiction.

If $\{x_1, x_2\} \subseteq F$, then $N_G(x) \cap \overline{F} = \emptyset$. If $N_G(x_3) \cap \overline{F} = \emptyset$, then $T - \{x, x_3\}$ is a nontrivial 4-cut of G, a contradiction. Thus $N_G(x_3) \cap \overline{F} \neq \emptyset$, and then $a \in T$ and $a_1 \in \overline{F}$ since $ax_2 \in E(G)$. Similarly, $b \in T$ and $b_1 \in \overline{F}$. If $c \in T$, then $\{x, x_3, x_4, a, b\} \cap F = \{x_1, x_2\}$ and $\{x, x_3, x_4, a, b\} \cap \overline{F} = \{a_1, b_1\}$, which implies $|F| = |\overline{F}| = 2$, for otherwise, $\{x_1, x_2, c\}$ or $\{a_1, b_1, c\}$ is a 3-cut of G, a contradiction. It follows $|V(G)| = 10$, which contradicts $|V(G)| \geq 14$. Hence, $c \notin T$. If $c \in F$, then $\{x, x_3, x_4, a, b\} \cap F = \{x_1, x_2, c\}$ and $\{x, x_3, x_4, a, b\} \cap \overline{F} = \{a_1, b_1\}$. This implies that $|F| \leq 4$ and $|\overline{F}| = 2$, for otherwise, $(T - \{x, x_3, x_4, a, b\}) \cup \{x_1, x_2, c\}$ is a nontrivial 4-cut or $(T - \{x, x_3, x_4, a, b\}) \cup \{a_1, b_1\}$ is a 3-cut of G, a contradiction. Then we have $|V(G)| \leq 12$, a contradiction. If $c \in \overline{F}$, we can also obtain $|V(G)| \leq 12$ by similar argument. □

We may assume that $x_2 \in F$ and $x_1 \in T$ without loss of generality. Similar to what is described above, we have $a \in T$, $a_1 \in \overline{F}$ and $\{b, b_1\} \cap \overline{F} \neq \emptyset$. It follows $\{b, b_1\} \cap F = \emptyset$ since $bb_1 \in E(G)$. If $c \notin F$, then $N_G(\{x, x_3, x_4, a\}) \cap F = \{x_2\}$, and then $(T - \{x, x_3, x_4, a\}) \cup \{x_2\}$ is a 3-cut of G since $|F| \geq 2$, a contradiction. Thus, $c \in F$, and thus $b \in T$ and $b_1 \in \overline{F}$. Then we can find that $|N_G(\{x, x_3, x_4, a, b\}) \cap F| = |N_G(\{x, x_3, x_4, a, b\}) \cap \overline{F}| = 2$, which implies $|F| = |\overline{F}| = 2$. It follows $|V(G)| = 10$, a contradiction. ■

Lemma 11. *Let G be a contraction critical quasi 5-connected graph on at least 14 vertices. If Figure 2(b) is a subgraph of G satisfying $d_G(x_3) = d_G(x_4) = 5$, and without loss of generality, we assume $N_G(x_3) = \{x, x_2, x_4, a, a_1\}$ and $N_G(x_4) = \{x, x_1, x_3, b, b_1\}$, then one of the following statements holds.*

(i) *$G/\triangle xx_3x_4$ is quasi 5-connected.*

(ii) *The graph obtained from G by contracting $A := \{b, b_2\}$, $B := \{b_1, x_4\}$, $C := \{x, x_1\}$ to vertices A', B', C', respectively, is quasi 5-connected.*

Proof. Assume neither (i) nor (ii) holds. Similar to Lemma 10, we have that $G/\triangle xx_3x_4$ is 4-connected. Since $G/\triangle xx_3x_4$ is not quasi 5-connected, we know that $G/\triangle xx_3x_4$ has a nontrivial 4-cut T'_1. Let F'_1 be a T'_1-fragment of $G/\triangle xx_3x_4$ and let $\overline{F'_1} = G/\triangle xx_3x_4 - (T'_1 \cup F'_1)$. Let F_1, T_1, $\overline{F_1}$ be the sets in G corresponding to F'_1, T'_1, $\overline{F'_1}$ in $G/\triangle xx_3x_4$. Then we have $T_1 \supset \{x, x_3, x_4\}$. Without loss of generality, we may assume $\{x_1, x_2\} \subseteq F_1 \cup T_1$. It follows that $N_G(x) \cap \overline{F_1} = \emptyset$. If $N_G(x_3) \cap \overline{F_1} = \emptyset$, then we see that $N_G(\{x, x_3\}) \cap \overline{F_1} = \emptyset$, and hence $T_1 - \{x, x_3\}$ is a nontrivial 4-cut of G, a contradiction. Consequently, $\{a, a_1\} \cap \overline{F_1} \neq \emptyset$. Similarly, $\{b, b_1\} \cap \overline{F_1} \neq \emptyset$. It follows $\{a, a_1, b, b_1\} \subseteq T_1 \cup \overline{F_1}$ since $\{aa_1, bb_1\} \subseteq E(G)$.

Claim 1. $\{x_1, x_2\} \subseteq F_1$.

Proof. Suppose $\{x_1, x_2\} \nsubseteq F_1$. If $\{x_1, x_2\} \subseteq T_1$, then $N_G(x) \cap F_1 = \emptyset$. Since $\{a, a_1\} \subseteq T_1 \cap \overline{F_1}$, we see that $N_G(x_3) \cap F_1 = \emptyset$. This implies that $T_1 - \{x, x_3\}$ is a nontrivial 4-cut of G, a contradiction. Thus $|\{x_1, x_2\} \cap T_1| = 1$. Without loss of generality, we assume that $x_1 \in T_1$ and $x_2 \in F_1$. Then $a \in T_1$ and $a_1 \in \overline{F_1}$ since $ax_2 \in E(G)$. Now, we observe that $N_G(\{x, x_3, x_4\}) \cap F_1 = \{x_2\}$. It follows that $|F_1| = 2$, for otherwise, $(T_1 - \{x, x_3, x_4\}) \cup \{x_2\}$ is a nontrivial 4-cut of G. Let $|F_1| = \{x_2, z\}$, then $d_G(z) = 4$ and $\{x_1, x_2, a\} \subseteq N_G(z)$. This implies that $z = a_2$. Let $N_G(a_2) = \{x_1, x_2, a, z'\}$. Then we see that $\{a_1, x_1, x_4, z'\}$ is a nontrivial 4-cut of G since $|V(G)| \geq 14$, a contradiction. □

By Claim 1, we see that $\{a, b\} \subseteq T_1$ and $\{a_1, b_1\} \subseteq \overline{F_1}$ because $\{ax_2, bx_1\} \subseteq E(G)$.

Claim 2. $\{a_2, b_2\} \subseteq \overline{F_1}$.

Proof. If $a_2 \in T_1$, then $N_G(\{x, x_3, a\}) \cap F_1 = \{x_2\}$ and $N_G(\{x, x_3, a\}) \cap \overline{F_1} = \{a_1\}$. Since $|V(G)| \geq 14$, $|F_1| \geq 3$ or $|\overline{F_3}| \geq 3$, which implies that $(T_1 - \{x, x_3, a\}) \cup \{x_2\}$ or $(T_1 - \{x, x_3, a\}) \cup \{a_1\}$ is a nontrivial 4-cut of G, a contradiction. Thus $a_2 \notin T_1$. Similarly, $b_2 \notin T_1$. In the following, we show that $a_2 \in \overline{F_1}$, and the other one can be handled similarly. Suppose $a_2 \notin \overline{F_1}$. Hence, $a_2 \in F_1$ and, thus, $N_G(\{x, x_3, a\}) \cap \overline{F_1} = \{a_1\}$. It follows that $|\overline{F_1}| = \{a_1, b_1\}$. Let $T_1 = \{x, x_3, x_4, a, b, u\}$. Then we see that $N_G(a_1) = \{x_3, a, b_1, u\}$ and $N_G(b_1) = \{x_4, b, a_1, u\}$. Note that $u \neq b_2$, for otherwise, $\{x_1, x_2, a_2, b_2\}$ is a nontrivial 4-cut of G.

We next show that either aa_2 or bb_2 is quasi 5-contractible, which contradicts that G is a contraction critical quasi 5-connected graph. Suppose that neither aa_2 nor bb_2 is quasi 5-contractible. Clearly, $\delta(G/aa_2) = 4$. Then Lemma 5 assures us that G/aa_2 is 4-connected. Let C be a quasi fragment with respect to aa_2 and let $R = N_G(C)$, $\overline{C} = G - (R \cup C)$. Then $x_3 \in R$ and we may assume that $x_2 \in C$, $a_1 \in \overline{C}$ without loss of generality. If $N_G(x_3) \cap C = \{x_2\}$, then we see that $x \in R$ since $xx_2 \in E(G)$. Furthermore, $N_G(\{a, x_3\}) = \{x_2\}$. This implies that $|C| = 2$. Let $C = \{x_2, z\}$. Then $d_G(z) = 4$ and $zx \in E(G)$, which contradicts to $(*)$. Thus $\{x, x_4\} \cap C \neq \emptyset$. Then $N_G(\{a, x_3\}) \cap \overline{C} = \{a_1\}$, which implies that $|\overline{C}| = 2$. Let $\overline{C} = \{a_1, w\}$. Then we see that the vertex w satisfies $d_G(w) = 4$ and $\{a_1, a_2\} \subseteq N_G(w)$. So we must have $w = u$. By similar argument for bb_2, we can deduce that $ub_2 \in E(G)$. Thus, $N_G(u) = \{a_1, a_2, b_1, b_2\}$, and thus, $\{x_1, x_2, a_2, b_2\}$ is a nontrivial 4-cut of G since $|V(G)| \geq 14$, a contradiction. □

By Claim 2, we see that $N_G(\{x, x_3, x_4, a, b\}) \cap F_1 = \{x_1, x_2\}$. Hence, $F_1 = \{x_1, x_2\}$, for otherwise, $(T_1 - \{x, x_3, x_4, a, b\}) \cup \{x_1, x_2\}$ is a 3-cut of G, a contradiction. Let $T_1 - \{x, x_3, x_4, a, b\} = \{v\}$. Then we have that $N_G(x_1) = \{x, x_2, x_4, b, v\}$ and $N_G(x_2) = \{x, x_1, x_3, a, v\}$. Let $G' = G/bb_2/b_1x_4/xx_1$. Clearly, $\delta(G') = 4$.

Claim 3. *G' is 4-connected.*

Proof. Assume, to the contrary, that G' is not 4-connected, then G' has a 3-cut T' such that each component of $G' - T'$ has at least 2 vertices and $|T' \cap \{A', B', C'\}| \geq 2$. Let T be the set in G corresponding to T' in G'. We first find that $T' \cap \{A', B', C'\} \neq \{A', B'\}$ and $T' \cap \{A', B', C'\} \neq \{A', C'\}$, for otherwise, $T - \{b\}$ is a nontrivial 4-cut of G. Therefore, the set $T' \cap \{A', B', C'\}$ is $\{B', C'\}$ or $\{A', B', C'\}$. In the former, $T - \{x\}$ forms a nontrivial 4-cut of G. In the latter, $T - \{b, x_4, x\}$ forms a 3-cut of G. Both of which contradict the fact that G is quasi 5-connected. □

Since G' is not quasi 5-connected, there exists a nontrivial 4-cut T_2' of G' by Claim 3. Furthermore, $|T_2' \cap \{A', B', C'\}| \geq 1$. Let F_2' be a T_2'-fragment of G' and let $\overline{F_2'} = G' - (T_2' \cup F_2')$. Let F_2, T_2, $\overline{F_2}$ be the sets in G corresponding to F_2', T_2', $\overline{F_2'}$ in G'. Note that the three vertices A', B', C' are adjacent to each other in G'. Hence, we may assume that the vertices in $\{A', B', C'\} - T_2'$ belong to F_2' without loss of generality.

Claim 4. $|T_2' \cap \{A', B', C'\}| \neq 1$.

Proof. Suppose $|T_2' \cap \{A', B', C'\}| = 1$. Then $|T_2| = 5$ and $|F_2| \geq 4$. If $T_2' \cap \{A', B', C'\} = \{A'\}$, then we see that $N_G(b) \cap \overline{F_2} = \emptyset$, which implies that $T_2 - \{b\}$ is a nontrivial 4-cut of G, a contradiction. If $T_2' \cap \{A', B', C'\} = \{B'\}$, we can get that $T_2 - \{x_4\}$ is a nontrivial 4-cut of G.

Therefore, $T_2' \cap \{A', B', C'\} = \{C'\}$. If $N_G(x) \cap \overline{F_2} = \emptyset$, then $T_2 - \{x\}$ is a nontrivial 4-cut of G, a contradiction. Hence, $x_3 \in T_2$ and $x_2 \in \overline{F_2}$ since $x_3x_4 \in E(G)$. If $N_G(x_3) \cap F_2 = \{x_4\}$, then $(T_2 - \{x, x_3\}) \cup \{x_4\}$ is a nontrivial 4-cut of G. Hence, $a_1 \in F_2$ and $a \in T_2$ since $\{ax_2, aa_1\} \subseteq E(G)$. Note that $N_G(\{x, x_3\}) \cap \overline{F_2} = \{x_2\}$, which implies that $N_G(a) \cap \overline{F_2} \neq \{x_2\}$. Otherwise, $(T_2 - \{x, x_3, a\}) \cup \{x_2\}$ is a 3-cut of G, which is absurd. So $a_2 \in \overline{F_2}$. It follows $N_G(\{x, x_3, a\}) \cap F_2 = \{x_4, a_1\}$. This implies $(T_2 - \{x, x_3, a\}) \cup \{x_4, a_1\}$ is a nontrivial 4-cut of G, a contradiction. □

Claim 5. $|T_2' \cap \{A', B', C'\}| \neq 2$.

Proof. Suppose $|T_2' \cap \{A', B', C'\}| = 2$. Then $|T_2| = 6$ and $|F_2| \geq 3$. If $T_2' \cap \{A', B', C'\} = \{A', B'\}$, then $x_3 \in F_2 \cap T_2$ since $xx_3 \in E(G)$. This implies $N_G(\{b, x_4\}) \cap \overline{F_2} = \emptyset$, and hence $T_2 - \{b, x_4\}$ is a nontrivial 4-cut of G, a contradiction.

If $T_2' \cap \{A', B', C'\} = \{A', C'\}$, then we see that $N_G(b) \cap \overline{F_2} = \emptyset$. If $N_G(x) \cap \overline{F_2} = \emptyset$, then $T_2 - \{b, x\}$ is a nontrivial 4-cut of G, a contradiction. So, $N_G(x) \cap \overline{F_2} \neq \emptyset$. This implies $x_2 \in \overline{F_2}$ and $x_3 \in T_2$ since $x_3x_4 \in E(G)$. Since $vx_2 \in E(G)$, we see that $v \in \overline{F_2} \cup T_2$. It follows $N_G(\{x, x_1\}) \cap F_2 = \{x_4\}$. Then $N_G(x_3) \cap F_2 \neq \{x_4\}$, for otherwise, $(T_2 - \{x, x_1, x_3\}) \cup \{x_4\}$ is a nontrivial 4-cut since $|F_2| \geq 3$. Hence, $a_1 \in F_2$ and $a \in T_2$ since $ax_2 \in E(G)$. Now, $T_2 = \{b, b_2, x, x_1, x_3, a\}$ and $v \in \overline{F_2}$. Then we see that $|\overline{F_2}| = 2$, for otherwise, $\{a, v, b_2\}$ is a 3-cut of G. However, we observe that $d_G(v) < 4$ since $va \notin E(G)$, a contradiction.

Therefore, $T_2' \cap \{A', B', C'\} = \{B', C'\}$. If $\{x_2, x_3\} \subseteq F_2 \cup T_2$, then $N_G(\{x_4, x\}) \cap \overline{F_2} = \emptyset$, which implies that $T_2 - \{x_4, x\}$ is a nontrivial 4-cut of G, a contradiction. Then $\{x_2, x_3\} \cap \overline{F_2} \neq \emptyset$, and then $\{x_2, x_3\} \subseteq T_2 \cup \overline{F_2}$ since $x_2x_3 \in E(G)$. If $N_G(x_1) \cap F_2 = \{b\}$, then $N_G(\{x, x_1, x_4\}) \cap F_2 = \{b\}$, and then, $(T_2 - \{x, x_1, x_4\}) \cup \{b\}$ is a nontrivial 4-cut of G. Thus, $v \in F_2$, and thus, $x_2 \in T_2$ and $x_3 \in \overline{F_2}$. If $a \notin \overline{F_2}$, then $N_G(\{x, x_1, x_2, x_4\}) \cap \overline{F_2} = \{x_3\}$. It follows that $(T_1 - \{x, x_1, x_2, x_4\}) \cup \{x_3\}$ is a 3-cut of G, a contradiction. Hence $a \in \overline{F_2}$. This implies $\{a_1, a_2\} \subseteq T_2 \cup \overline{F_2}$. Note that $\{b_1, x, x_1, x_2, x_4\} \subseteq T_2$ and $|T_2| = 6$, we have $\{a_1, a_2\} \cap \overline{F_2} \neq \emptyset$. It follows $|\overline{F_3}| \geq 3$. Then we see that $(T_2 - \{x, x_1, x_4\}) \cup \{x_3\}$ is a nontrivial 4-cut of G, a contradiction. □

By Claims 4 and 5, we have $\{A', B', C'\} \subseteq T_2'$. Then $|T_2| = 7$. Moreover, we see that $N_G(\{b, x_4, x\}) \cap F_2 = \emptyset$ or $N_G(\{b, x_4, x\}) \cap \overline{F_2} = \emptyset$. This implies that $T_2 - \{b, x_4, x\}$ is a nontrivial 4-cut of G, a contradiction. ■

Now we are prepared to prove Theorem 3.

***Proof of Theorem* 3.** If G has a quasi 5-contractible edge, Theorem 3 holds immediately. Thus we assume that G is a contraction critical quasi 5-connected graph. Let $N_G(x) = \{x_1, x_2, x_3, x_4\}$. Without loss of generality, we suppose $\{x_1x_2, x_2x_3, x_3x_4, x_4x_1\} \subseteq E(G)$. For $i = 1, 2, 3, 4$, $d_G(x_i) \geq 5$ by $(*)$. Thus Lemma 5 assures us that G/xx_i is 4-connected.

For $i = 1, 2$, let F_i be quasi fragments with respect to xx_i. Let $T_i = N_G(F_i)$ and $\overline{F_i} = V(G) - (F_i \cup T_i)$. Then $x_3 \in T_1$ and $x_4 \in T_2$. Without loss of generality, we may assume $x_2 \in F_1$, $x_4 \in \overline{F_1}$ and $x_1 \in F_2$, $x_3 \in \overline{F_2}$. Thus, $x \in T_1 \cap T_2$, $x_1 \in T_1 \cap F_2$, $x_2 \in F_1 \cap T_2$, $x_3 \in T_1 \cap \overline{F_2}$ and $x_4 \in \overline{F_1} \cap T_2$. Let $X_1 = (T_1 \cap F_2) \cup (T_1 \cap T_2) \cup (F_1 \cap T_2)$, $X_2 = (T_1 \cap F_2) \cup (T_1 \cap T_2) \cup (\overline{F_1} \cap T_2)$, $X_3 = (\overline{F_1} \cap T_2) \cup (T_1 \cap T_2) \cup (T_1 \cap \overline{F_2})$ and $X_4 = (F_1 \cap T_2) \cup (T_1 \cap T_2) \cup (T_1 \cap \overline{F_2})$.

Claim 1. *There exists* $i \in \{1, 2, 3, 4\}$ *such that* $|X_i| \leq 4$.

Proof. To the contrary, we assume that for all $i = 1, 2, 3, 4$, $|X_i| \geq 5$. The fact $|X_1| + |X_3| = |X_2| + |X_4| = 10$ shows that $|X_1| = |X_2| = |X_3| = |X_4| = 5$. Since $N_G(x) \cap (F_1 \cap F_2) = \emptyset$, we see that $|F_1 \cap F_2| \leq 1$, for otherwise, $X_1 - \{x\}$ is a nontrivial 4-cut of G, a contradiction. Similarly, we have that $|\overline{F_1} \cap F_2| \leq 1$,

$|\overline{F_1} \cap \overline{F_2}| \leq 1$ and $|F_1 \cap \overline{F_2}| \leq 1$. It follows $|V(G)| \leq 13$, which contradicts the fact that $|V(G)| \geq 14$. □

Without loss of generality, we may assume $|X_3| \leq 4$. Since $N_G(x) \cap (\overline{F_1} \cap \overline{F_2}) = \emptyset$, $\overline{F_1} \cap \overline{F_2} = \emptyset$.

Claim 2. $|\overline{F_1} \cap T_2| = |T_1 \cap \overline{F_2}| = 1$.

Proof. We only show that $|\overline{F_1} \cap T_2| = 1$, and the other one can be handled similarly. Suppose $|\overline{F_1} \cap T_2| \geq 2$. Then $|\overline{F_1} \cap T_2| = 2$ and $|T_1 \cap T_2| = |T_1 \cap \overline{F_2}| = 1$. Hence, $|F_1 \cap T_2| = 2$, which implies $|X_4| = 4$. Since $N_G(x) \cap (F_1 \cap \overline{F_2}) = \emptyset$, we see that $F_1 \cap \overline{F_2} = \emptyset$. It follows $|\overline{F_2}| = 1$, a contradiction. □

Claim 3. $|F_1 \cap \overline{F_2}| = |\overline{F_1} \cap F_2| = 1$.

Proof. We only show that $|F_1 \cap \overline{F_2}| = 1$. By Claim 2, we see that $|X_4| = 5$, which implies that $|F_1 \cap \overline{F_2}| \leq 1$ since $N_G(x) \cap (F_1 \cap \overline{F_2}) = \emptyset$. If $F_1 \cap \overline{F_2} = \emptyset$, then we find $|\overline{F_2}| = 1$, a contradiction. So $|F_1 \cap \overline{F_2}| = 1$. □

Let $F_1 \cap \overline{F_2} = \{a\}$ and let $\overline{F_1} \cap F_2 = \{b\}$. Note that $|T_1 \cap T_2| \leq 2$ by Claim 1. If $|T_1 \cap T_2| = 2$, then we see that G has the subgraph shown in Figure 2(a). If $|T_1 \cap T_2| = 1$, then G has the subgraph shown in Figure 2(b). Then Theorem 3 holds by Lemmas 9, 10 and 11. ■

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