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ANTIDIRECTED SPANNING TRAILS OF DIGRAPHS WITH α_2-STABLE NUMBER 3

Hong Yang[1], Juan Liu[2] and Jixiang Meng[1]

[1] *College of Mathematics and System Sciences*
Xinjiang University, Urumqi, Xinjiang 830046, P.R. China

[2] *College of Big Data Statistics, Guizhou University of*
Finance and Economics, Guiyang, Guizhou, 550000, P.R. China

e-mail: honger9506@126.com
liujuan1999@126.com
mjx@xju.edu.cn

Abstract

Let D be a digraph with vertex set $V(D)$ and arc set $A(D)$. An antidirected spanning trail of D is a spanning trail in which consecutive arcs have opposite directions and each arc of D occurs at most once. Let $\alpha_2(D) = \max\{|X| : X \subseteq V(D)$ and $D[X]$ has no 2-cycle$\}$ be the $\alpha_2(D)$-stable number. When $\alpha_2(D) = 2$, we have demonstrated that every weakly connected digraph D with $\alpha_2(D) = 2$ has an antidirected Hamiltonian path, and have provided a necessary and sufficient condition for strongly connected digraph D with $\alpha_2(D) = 2$ to have an antidirected Hamiltonian cycle. In this paper, we first determine two families $\mathcal{D}_1$ and $\mathcal{D}_2$ of well-characterized strongly connected digraphs with α_2-stable number 3 such that, for any strongly connected digraph $D \in \mathcal{D}_1 \cup \mathcal{D}_2$, D does not have an antidirected spanning trail. And, we further prove that every strongly connected digraph D with $\alpha_2(D) = 3$ has an antidirected spanning trail if and only if $D \notin \mathcal{D}_1 \cup \mathcal{D}_2$.

Keywords: α_2-stable set, antidirected spanning trail, complete graph, digraph.

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1. Introduction

For standard terminology and notation in graph theory and digraph theory, not specifically defined in this paper, the reader is referred to [2] for graphs and [1] for digraphs. All graphs $G = (V(G), E(G))$ with vertex set $V(G)$ and edge set

$E(G)$, and all digraphs $D = (V(D), A(D))$ with vertex set $V(D)$ and arc set $A(D)$ considered in this paper are finite, loopless and without parallel edges or arcs. We use (x, y) to denote an arc oriented from a vertex x to a vertex y of D and use xy to denote an edge in G incident with both vertices x and y. Sometimes, we also use $x \to y$ to denote an arc oriented from a vertex x to a vertex y of D. A digraph D with n vertices is *complete digraph* if, for any two distinct vertices $x, y \in V(D)$, we have $(x, y), (y, x) \in A(D)$, denoted by K_n^*.

We shall adopt the following notational convenience. For vertex subsets $X, Y \subseteq V(D)$, define

$$(X, Y)_D = \{(x, y) \in A(D) : x \in X, y \in Y\} \text{ and } [X, Y]_D = (X, Y)_D \cup (Y, X)_D.$$

If $X = \{x\}$ (respectively, $Y = \{y\}$), then we often use $(x, Y)_D$ (respectively, $(X, y)_D$) for $(X, Y)_D$. Hence $(x, y)_D = (\{x\}, \{y\})_D$. For a vertex $x \in V(D)$, let $N_D^+(x) = \{y \in V(D) : (x, y) \in A(D)\}$ and $N_D^-(x) = \{y \in V(D) : (y, x) \in A(D)\}$. Thus, $d_D^+(x) = |N_D^+(x)|$ and $d_D^-(x) = |N_D^-(x)|$ are the *out-degree* and the *in-degree* of vertex x in D, respectively. For vertex subset $X \subseteq V(D)$, $D - X$ denotes the subdigraph of D whose vertex set is $V(D) \setminus X$ and whose arc set consists of all arcs of D which have both end-vertices in $V(D) \setminus X$. $D[X]$ denotes the vertex induced subdigraph of D induced by X whose vertex set is X and whose arc set consists of all arcs of D which have both ending vertices in X. The corresponding notation for graphs are similarly defined. In particular, for a graph G and vertex subsets $X, Y \subseteq V(G)$, $[X, Y]_G = \{xy \in E(G) : x \in X, y \in Y\}$.

Let $D = (V(D), A(D))$ be a digraph. An alternating sequence of vertices and arcs in D, beginning and ending with vertices, is called a *walk*. If all arcs in a walk are distinct, then the walk is called a *trail*, and if, in addition, the vertices are also distinct, then the trail is a *path*. A path in which the beginning and end-vertices are the same will be said to be *cycle*. A cycle that contains k arcs is called a *k-cycle*. A trail (respectively, path) from a vertex x to a vertex y is often called an *(x, y)-trail* (respectively, *(x, y)-path*). And we say that x and y are the *ending vertices* of this trail (respectively, path). If a digraph D contains a path which includes all the vertices of D, then the path is called a *Hamiltonian path* of D. A cycle in D is a *Hamiltonian cycle* of D if the cycle contains all vertices of D. D is *Hamiltonian* if D has a Hamiltonian cycle. A digraph D is *strongly connected* if, for any two distinct vertices x and y of D, there exist an (x, y)-walk and a (y, x)-walk in D. D is *weakly connected* if underlying graph $G(D)$ is connected, where the *underlying graph* $G(D)$ of D is the graph obtained from D by erasing all orientations on the arcs of D.

A digraph D is *k-strongly connected* if $|V(D)| \geq k + 1$ and there exists no vertex subset X_1 of D with less than k vertices such that $D - X_1$ is not strongly connected, and a graph G is *k-connected* if $|V(G)| \geq k + 1$ and there exists no vertex subset X_2 of G with less than k vertices such that $G - X_2$ is not connected.

The largest integer k such that digraph D is k-strongly connected is the *vertex connectivity* of D (denoted by $\kappa(D)$), and the largest integer k such that graph G is k-connected is the *vertex connectivity* of G (denoted by $\kappa(G)$). Let D be a digraph and $\mathcal{D}$ be a digraph family. Digraph D is *$\mathcal{D}$-free* if D does not have a vertex induced subgraph isomorphic to a member in $\mathcal{D}$.

For digraphs, we may consider the following definition. Given a digraph $D = (V(D), A(D))$ and a vertex subset $X \subseteq V(D)$, X is an *$\alpha_2(D)$-stable set* of D if $D[X]$ has no 2-cycles. Define $\alpha_2(D) = \max\{|X| : X$ is an $\alpha_2(D)$-stable set of $D\}$ to be the *$\alpha_2(D)$-stable number*. Readers interested in problems and results on this subject can refer to the well structured survey of Jackson and Ordaz [10]. Let $\mathcal{M}_2$ and $\mathcal{M}_3$ denote the digraph families as depicted in Figure 1. Chakroun and Sotteau [5] proved that every 2-strongly connected digraph with $\alpha_2(D) \leq 2$ is Hamiltonian except if D is isomorphic to a digraph of the family $\mathcal{M}_2$, and every 3-strongly connected digraph with $\alpha_2(D) \leq 3$ is Hamiltonian except if D is isomorphic to a digraph of the family $\mathcal{M}_3$. It is easy to check that every digraph in families $\mathcal{M}_2$ and $\mathcal{M}_3$ has a Hamiltonian path. Hence, it is routine to deduce that if $\alpha_2(D) \leq \kappa(D)$ and $\alpha_2(D) \leq 3$, then D has a Hamiltonian path.

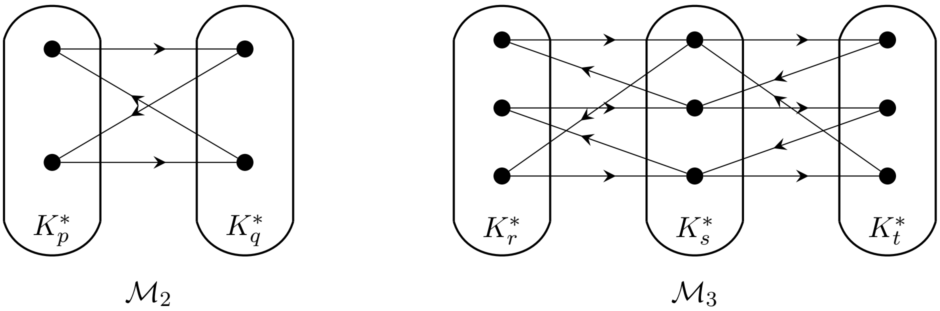

Figure 1. Digraph families $\mathcal{M}_2$ and $\mathcal{M}_3$ with $p, q \geq 2$, $r, s, t \geq 3$ and $p + q \geq 5$.

The notion of an antidirected Hamiltonian path was introduced by Grünbaum in [8] and defined as follows: a path in a digraph $D = (V(D), A(D))$ is an *antidirected path*, provided that every two adjacent arcs of the path have opposing orientations. An *antidirected Hamiltonian path* (respectively, *cycle*) in D is an antidirected path (respectively, cycle) containing all vertices of D. Let $x_1 \to x_2 \leftarrow \cdots x_k$ be an antidirected Hamiltonian path of D, x_1 is called a *starting vertex* and x_k is called a *starting vertex* (respectively, *terminal vertex*) of this path if k is odd (respectively, even).

There have been lots of investigations on antidirected Hamiltonian path and cycle problems. The first result about antidirected Hamiltonian path was given by Grünbaum [8], Grünbaum indicated concerning the existence of antidirected Hamiltonian paths in tournaments and proved that except for T_3^c, T_5^c and T_7^c (see Figure 2), every tournament contains an antidirected Hamiltonian path. In 1972, Rosenfeld [13] improved proof of Grünbaum's theorem concerning the existence of antidirected Hamiltonian paths in tournaments. And he further proved that for every tournament T with $n \geq 12$ vertices, there is an antidirected Hamiltonian path starting at any vertex. In 1974, Rosenfeld [14] showed that every tournament T with $n = 2k \geq 28$ vertices has an antidirected Hamiltonian cycle, as conjectured by Grünbaum [8]. In 1983, Petrović [11] proved that any even tournament with at least 16 vertices has an antidirected Hamiltonian cycle, which was the best result supporting this conjecture by far. In [16], we showed that every weakly connected digraph with $\alpha_2(D) = 2$ has an antidirected Hamiltonian path. We also gave a necessary and sufficient condition for a strongly connected digraph D with $\alpha_2(D) = 2$ to have an antidirected Hamiltonian cycle. Additional researches on antidirected Hamiltonian problem can be found in [3, 4, 6, 9, 11, 12, 15], among others.

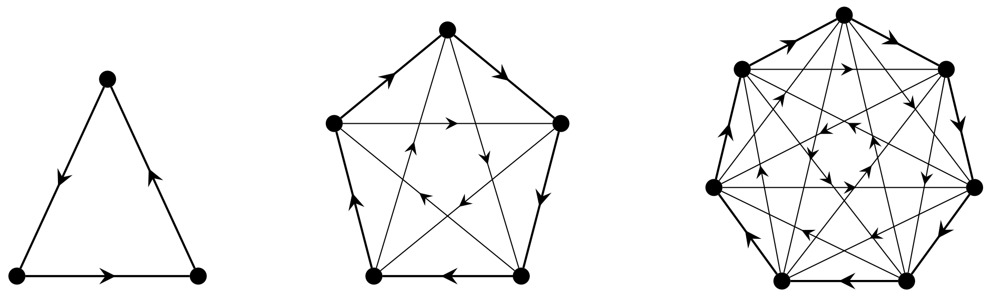

Figure 2. Three distinct tournaments T_3^c, T_5^c and T_7^c.

An *antidirected trail* in a digraph is a trail that alternates between forward and backward arcs. If both the beginning vertex and end-vertex of an antidirected trail are the same, then it is called an *antidirected closed trail.* A trail that starts and ends with forward arcs and alternates between forward and backward arcs is referred to as a *forward antidirected trail.* Similarly, a *forward-backward antidirected trail* is a walk that begins with a forward arc and ends with a backward arc, with no repeated arcs, and where the arcs alternate between forward and backward directions. A *backward-forward antidirected trail* is defined analogously. If an antidirected trail contains all vertices of a digraph, then the antidirected trail is called *antidirected spanning* (*ADS*) *trail* of the digraph. An ADS trail with beginning vertex and ending vertex being same, is called *ADS closed trail* of digraph. If $x \rightarrow x_1 \leftarrow \cdots y$ (respectively, $x \leftarrow x_1 \rightarrow \cdots y$) is an ADS trail in D, then x is also called a *starting vertex* (respectively, *terminal vertex*). Both x and y are called *ending vertices* of this antidirected trail.

The purpose of this paper is to seek a sufficient and necessary condition for digraph D with $\alpha_2(D) = 3$ which has an ADS trail. The rest of the paper is arranged as follows. In section 2, we summarize and develop some of the preliminaries that will be needed in the proofs. In section 3, we construct well-characterized strongly connected digraph families $\mathcal{D}_1$ and $\mathcal{D}_2$, which are then applied to show that every strongly connected digraph D with $\alpha_2(D) = 3$ has an ADS trail if and only if $D \notin \mathcal{D}_1 \cup \mathcal{D}_2$.

2. Preliminaries

First, let us review the following results that will be applied later, and then introduce additional findings that will support our arguments.

Theorem 1 [7]**.** *Let G be a graph with $n \geq 3$ vertices. If $\alpha(G) \leq \kappa(G)$, then G is Hamiltonian.*

Theorem 2 [7]**.** *Let G be a graph with n vertices and $n \neq 2$. If $\alpha(G) \leq \kappa(G) - 1$, then for any two vertices $x, y \in V(G)$, G contains a Hamiltonian path such that x and y are ending vertices.*

The proof of the following lemma is very similar to that of Lemma 1 in [16], and hence we omit it.

Lemma 3. *Let D be a complete digraph with n vertices. Then each of the following holds.*

(i) *D contains a forward* ADS *trail with x as the starting vertex and y as the terminal vertex for any two distinct vertices $x, y \in V(D)$ if and only if $n \geq 2$.*

(ii) *D contains a forward-backward* ADS *trail with x and y as starting vertices and contains a backward-forward* ADS *trail with x and y as terminal vertices for any two distinct vertices $x, y \in V(D)$ if and only if $n \geq 3$.*

(iii) *D contains an* ADS *closed trail starting at x and contains an* ADS *closed trail terminating at x for any vertex $x \in V(D)$ if and only if $n \neq 2$.*

(iv) *D contains a forward* ADS *trail with starting at x and terminating at x if and only if $n \geq 3$.*

In the following, a special graph from the digraph D will be introduced, which plays a key role in our arguments.

Definition 4. Given a digraph $D = (V(D), A(D))$, the graph G_D of D is constructed as follows. The vertex set of G_D is $V(D)$ and two distinct vertices x and y of G_D are adjacent if and only if $(x, y), (y, x) \in A(D)$.

A digraph D can be covered by k vertex-disjoint complete digraphs if $V(D)$ can be partitioned into k vertex-disjoint vertex subsets $X_1, X_2, \ldots, X_k$ such that

$D[X_i]$ is a complete digraph for any $i \in \{1, 2, \ldots, k\}$. A graph G that can be covered by k vertex-disjoint complete graphs is defined analogously. In [16], the following result was presented.

Lemma 5 [16]**.** *Let G be a connected graph with $\kappa(G) < 2$ and $\alpha(G) = 2$. Then G can be covered by two vertex-disjoint complete graphs.*

3. Antidirected Spanning Trails in Digraphs with α_2-Stable Number 3

In this section, a sufficient and necessary condition will be provided for strongly connected digraph with $\alpha_2(D) = 3$ to have an ADS trail. To complete our proof, the following notation will be defined.

Let D be a digraph and G_D be a graph defined as in Definition 4. We say that $\mathcal{T} = (T, T', T'')$ is a *spanning* 3*-tuple of mixed trails*, if T is a trail of G_D with x and x' as ending vertices, T'' is a trail of G_D with y and y' as end-vertices and T' is an antidirected trail of D with x' and y' as end-vertices such that $V(T) \cap V(T') = \{x'\}$, $V(T') \cap V(T'') = \{y'\}$, $V(T) \cap V(T'') = \emptyset$ for $|V(T')| \geq 2$ or $V(T) \cap V(T'') = V(T')$ for $|V(T')| = 1$, and $V(T) \cup V(T') \cup V(T'') = V(D)$.

By the definition of G_D, for any edge $zz' \in E(G_D)$, we have that $(z, z'), (z', z) \in A(D)$. Thus, for any trail $z_1 z_2 \cdots z_k$ of G_D, D has an antidirected trail with z_1 as the starting vertex which contains all vertices in $\{z_1, z_2, \ldots, z_k\}$, and has an antidirected trail with z_1 as the terminal vertex which contains all vertices in $\{z_1, z_2, \ldots, z_k\}$. Therefore, we conclude the following.

(1) If $\mathcal{T}$ is a spanning 3-tuple of mixed trails, then D contains an ADS trail.

Especially, if $|V(T')| = 1$, then $T \cup T''$ is a spanning trail of G_D. Thus, by (1), we also conclude that as follows.

(2) If G_D contains a spanning trail, then D contains an ADS trail.

We now consider two special digraph families of digraphs, referred to as $\mathcal{D}_1$ and $\mathcal{D}_2$. Let $\mathcal{D}_1 = \{D_i : 1 \leq i \leq 16\} \cup \{C_3\}$ denote the collection of all digraphs depicted as in Figure 3.

Definition 6. Let H_1, H_2 and H_3 be three vertex-disjoint complete digraphs with $V(H_2) = \{h_2, h_2'\}$. We construct a digraph family $\mathcal{D}_2$ such that every digraph $D \in \mathcal{D}_2$ satisfies $V(D) = V(H_1) \cup V(H_2) \cup V(H_3)$, $(V(H_1), V(H_2))_D = (V(H_1), h_2)_D$, $(V(H_2), V(H_3))_D = (h_2, V(H_3))_D$, $(V(H_3), V(H_2))_D = (V(H_3), h_2')_D$, $(V(H_2), V(H_1))_D = (h_2', V(H_1))_D$ and $[V(H_1), V(H_3)]_D = \emptyset$ (see Figure 4).

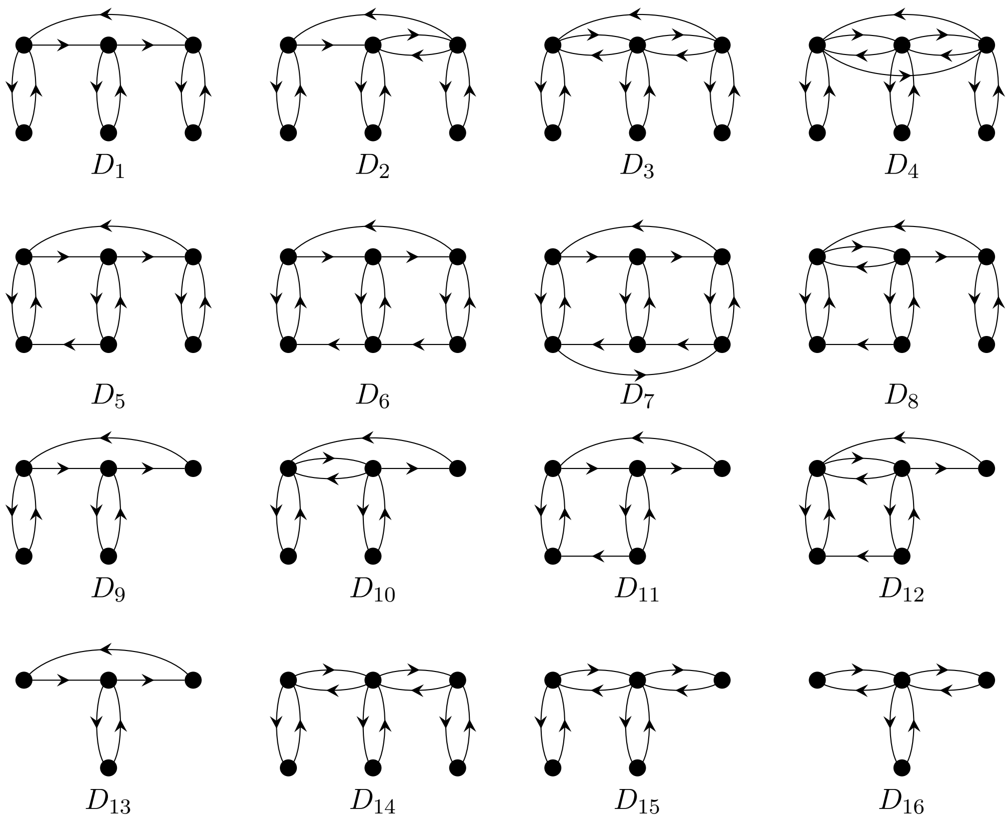

Figure 3. The sixteen digraphs $D_1, D_2, \ldots, D_{16}$.

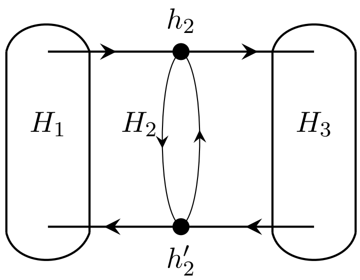

Figure 4. Digraph family $\mathcal{D}_2$.

Let D be a digraph. By the definition of ADS trail, if D contains an ADS trail T, then all arcs in ADS trail T of D are distinct, and there exist at most two distinct vertices $x, y \in V(D)$ such that $\max\{d_D^+(x), d_D^-(x)\} = 1$ and $\max\{d_D^+(y), d_D^-(y)\} = 1$. Thus, every digraph in $\{D_1, D_2, D_3, D_4, D_9, D_{10}, D_{13},$

$D_{14}, D_{15}, D_{16}\}$ does not contain an ADS trail. It is also not difficult to check that every digraph in $(\mathcal{D}_1 - \{D_1, D_2, D_3, D_4, D_9, D_{10}, D_{13}, D_{14}, D_{15}, D_{16}\}) \cup \mathcal{D}_2$ does not contain an ADS trail. Therefore, we conclude the following.

(3) Every digraph $D \in \mathcal{D}_1 \cup \mathcal{D}_2$ does not contain an ADS trail.

Let us state a proposition that is a particular case of Theorem 9 and is useful to make the proof of the theorem more clear.

Proposition 7. *Let D be a strongly connected digraph with $\alpha_2(D) = 3$ and $D \notin \mathcal{D}_1 \cup \mathcal{D}_2$. If D can be covered by three vertex-disjoint complete digraphs H_1, H_2 and H_3, then D contains an* ADS *trail.*

Proof. Let D be a strongly connected digraph with $\alpha_2(D) = 3$ and $D \notin \mathcal{D}_1 \cup \mathcal{D}_2$, and let us consider graph G_D of D defined as in Definition 4. Clearly, $\alpha_2(D) = \alpha(G_D) = 3$. Assume that D can be covered by three vertex-disjoint complete digraphs H_1, H_2 and H_3. Then $G_D[V(H_i)]$ is a complete graph for any i with $i \in \{1, 2, 3\}$, denoted by H'_i. By contradiction, assume that D does not contain an ADS trail. Since D is strongly connected, we may assume that either $(V(H_1), V(H_2))_D \neq \emptyset$, $(V(H_2), V(H_3))_D \neq \emptyset$ and $(V(H_3), V(H_1))_D \neq \emptyset$, or $(V(H_1), V(H_2))_D \neq \emptyset$, $(V(H_2), V(H_3))_D \neq \emptyset$, $(V(H_3), V(H_2))_D \neq \emptyset$, $(V(H_2), V(H_1))_D \neq \emptyset$ and $[V(H_1), V(H_3)]_D = \emptyset$. Next, we will consider the following two cases.

Case 1. $(V(H_1), V(H_2))_D \neq \emptyset$, $(V(H_2), V(H_3))_D \neq \emptyset$ and $(V(H_3), V(H_1))_D \neq \emptyset$.

Then we have the following Claim.

Claim 8. (i) *There exist vertices $h_i \in V(H_i)$ for any $i \in \{1, 2, 3\}$ such that $(V(H_1), V(H_2))_D = \{(h_1, h_2)\}$, $(V(H_2), V(H_3))_D = \{(h_2, h_3)\}$ and $(V(H_3), V(H_1))_D = \{(h_3, h_1)\}$.*

(ii) *$|V(H)_i| \leq 2$ for any i with $i \in \{1, 2, 3\}$ and there exists a complete digraph $H_{i'}$ with $i' \in \{1, 2, 3\}$ such that $|V(H_{i'})| = 2$.*

(iii) *For any vertex $h'_i \in V(H_i) \setminus \{h_i\}$, $(h'_i, h_{i+2}), (h_{i-2}, h'_i) \notin A(D)$ (consider module* 3*).*

Proof. If there exists a complete digraph, says H_2, and two distinct vertices $h_{21}, h_{22} \in V(H_2)$ such that $(V(H_1), h_{21})_D \neq \emptyset$ and $(h_{22}, V(H_3))_D \neq \emptyset$, then let arcs $(h_{11}, h_{21}) \in (V(H_1), h_{21})_D$ and $(h_{22}, h_{31}) \in (h_{22}, V(H_3))_D$. It is clear that H'_1 contains a Hamiltonian path with h_{11} as the ending vertex, denoted by $P^1_{h_{11}}$, and H'_3 contains a Hamiltonian path with h_{31} as the end-vertex, denoted by $P^3_{h_{31}}$. It follows by Lemma 3(i) that H_2 contains a forward ADS trail with h_{22} as the starting vertex and h_{21} as the terminal vertex, denoted by $h_{22} \to \cdots \to h_{21}$.

Thus, $(P^1_{h_{11}}, h_{11} \to h_{21} \leftarrow \cdots \leftarrow h_{22} \to h_{31}, P^3_{h_{31}})$ is a spanning 3-tuple of mixed trails, and so by (1), D contains an ADS trail, a contradiction. This proves (i).

By (i), assume that there exist vertices $h_i \in V(H_i)$ for any i with $i \in \{1,2,3\}$ such that $(V(H_1), V(H_2))_D = \{(h_1, h_2)\}$, $(V(H_2), V(H_3))_D = \{(h_2, h_3)\}$ and $(V(H_3), V(H_1))_D = \{(h_3, h_1)\}$. It is clear that H'_i contains a Hamiltonian path with h_i as the end-vertex, denoted by $P^i_{h_i}$ for any i with $i \in \{1,2,3\}$.

To prove (ii). If there exists a complete digraph, says H_3, such that $|V(H_3)| \geq 3$, then by Lemma 3 (iv), H_3 contains a forward ADS trail such that starting at h_3 and terminating at h_3, denoted by $h_3 \to \cdots \to h_3$. Thus, $(P^2_{h_2}, h_2 \to h_3 \leftarrow \cdots \leftarrow h_3 \to h_1, P^1_{h_1})$ is a spanning 3-tuple of mixed trails, and so by (1), D contains an ADS trail, a contradiction. Hence for any H_i with $i \in \{1,2,3\}$, $|V(H_i)| \leq 2$. If $|V(H_1)| = |V(H_2)| = |V(H_3)| = 1$, then, as D is C_3-free, we may assume that $(h_1, h_3) \in A(D)$. Thus, $h_3 \leftarrow h_1 \to h_2$ is an ADS trail of D, a contradiction. Hence (ii) holds.

To prove (iii). If there exists a vertex $h'_j \in V(H_j) \setminus \{h_j\}$ for some j with $j \in \{1,2,3\}$ such that $(h'_j, h_{j+2}) \in A(D)$ (consider module 3), then by Lemma 3(i), H_j contains a forward ADS trail with h'_j as the starting vertex and h_j as the terminal vertex, denoted by $h'_j \to \cdots \to h_j$. Thus, $(P^{j+2}_{h_{j+2}}, h_{j+2} \to h_j \leftarrow \cdots \leftarrow h'_j \to h_{j+2} \leftarrow h_{j+1}, P^{j+1}_{h_{j+1}})$ is a spanning 3-tuple of mixed trails, and so by (1), D contains an ADS trail, a contradiction. If there exists a vertex $h'_{j'} \in V(H_{j'}) \setminus \{h_{j'}\}$ for some j' with $j' \in \{1,2,3\}$ such that $(h_{j'-2}, h'_{j'}) \in A(D)$ (consider module 3), then by Lemma 3(i), $H_{j'}$ contains a forward ADS trail with $h_{j'}$ as the starting vertex and $h'_{j'}$ as the terminal vertex, denoted by $h_{j'} \to \cdots \to h'_{j'}$. Thus, $(P^{j'+2}_{h_{j'+2}}, h_{j'+2} \leftarrow h_{j'-2} \to h'_{j'} \leftarrow \cdots \leftarrow h_{j'} \to h_{j'-2}, P^{j'-2}_{h_{j'-2}})$ is a spanning 3-tuple of mixed trails, and so by (1), D contains an ADS trail, a contradiction. This proves (iii) and completes the proof of Claim 8. $\square$

By Claim 8(ii), without loss of generality, we assume that $|V(H_3)| = 2$. Let $V(H_3) = \{h_3, h'_3\}$. Assume first that $|V(H_1)| = 2$ and $|V(H_2)| = 2$. Let $V(H_1) = \{h_1, h'_1\}$ and $V(H_2) = \{h_2, h'_2\}$. If there exist two distinct integers $i_1, i_2 \in \{1,2,3\}$ such that $(h_{i_1}, h_{i_1-1}), (h'_{i_2}, h'_{i_2-1})_D \in A(D)$ (consider module 3), then assume that $(h'_1, h'_3) \in A(D)$. Thus, $h'_2 \leftarrow h_2 \to h_1 \leftarrow h'_1 \to h'_3 \leftarrow h_3$ or $h_1 \leftarrow h'_1 \to h'_3 \leftarrow h_3 \to h_2 \leftarrow h'_2$ is an ADS trail of D, a contradiction. Therefore, for any integer $i \in \{1,2,3\}$ (consider module 3), if $(h_i, h_{i-1}) \in A(D)$, then $(h'_{i+1}, h'_i), (h'_{i+2}, h'_{i+1}) \notin A(D)$, and if $(h'_i, h'_{i-1}) \in A(D)$, then $(h_{i+1}, h_i), (h_{i+2}, h_{i+1}) \notin A(D)$, and so by Claim 8(i) and (iii), $D \in \{D_1, D_2, \ldots, D_8\}$, contrary to $D \notin \mathcal{D}_1$.

Assume now that $|V(H_1)| = 1$. It is clear that H_2 contains an ADS trail with h_2 as the starting vertex, denoted by $h_2 \to \cdots$, and contains an ADS trail with h_2 as the terminal vertex, denoted by $h_2 \leftarrow \cdots$. If $(h_1, h_3) \in A(D)$, then $h'_3 \to h_3 \leftarrow h_1 \to h_2 \leftarrow \cdots$ is an ADS trail of D, a contradiction. If $(h_2, h_1) \in A(D)$, then $h'_3 \leftarrow h_3 \to h_1 \leftarrow h_2 \to \cdots$ is an ADS trail of D, a contradiction. Hence $(h_1, h_3), (h_2, h_1) \notin A(D)$, and so by Claim 8(i) and (iii), the out-degree and in-degree of h_1 are 1 in D. Again by Claim 8(i) and (iii), we have that $(h_2, h_3'), (h_3', h_2) \notin A(D)$ and if $|V(H_2)| = 2$, choose $h_2' \in V(H_2) \setminus \{h_2\}$, then $(h_2', h_3), (h_3, h_2'), (h_2', h_3') \notin A(D)$. As D is $\{D_9, D_{10}, D_{11}, D_{12}, D_{13}\}$-free, we have that $|V(H_2)| = 1$ and $(h_3, h_2) \in A(D)$. Thus, $h_3' \leftarrow h_3 \rightarrow h_2 \leftarrow h_1$ is an ADS trail of D, a contradiction.

Finally, assume that $|V(H_1)| = 2$ and $|V(H_2)| = 1$. Let $V(H_1) = \{h_1, h_1'\}$. If $(h_2, h_1) \in A(D)$, then $h_3' \rightarrow h_3 \leftarrow h_2 \rightarrow h_1 \leftarrow h_1'$ is an ADS trail of D; if $(h_3, h_2) \in A(D)$, then $h_3' \leftarrow h_3 \rightarrow h_2 \leftarrow h_1 \rightarrow h_1'$ is an ADS trail of D. In both cases, a contradiction occurs. Hence $(h_2, h_1), (h_3, h_2) \notin A(D)$ and by Claim 8(i) and (iii), the out-degree and in-degree of h_2 both are 1 in D. Again by Claim 8(i) and (iii), we have $(h_1, h_3'), (h_3', h_1), (h_3, h_1'), (h_3', h_1'), (h_1', h_3) \notin A(D)$, and so $D \in \{D_9, D_{10}, D_{11}, D_{12}\}$, contrary to $D \notin \mathcal{D}_1$.

Case 2. $(V(H_1), V(H_2))_D \neq \emptyset$, $(V(H_2), V(H_3))_D \neq \emptyset$, $(V(H_3), V(H_2))_D \neq \emptyset$, $(V(H_2), V(H_1))_D \neq \emptyset$ and $[V(H_1), V(H_3)]_D = \emptyset$.

If there exist two distinct vertices $h_{21}, h_{22} \in V(H_2)$ such that $(V(H_1), h_{21})_D \neq \emptyset$ and $(h_{22}, V(H_3))_D \neq \emptyset$, or $(V(H_3), h_{21})_D \neq \emptyset$ and $(h_{22}, V(H_1))_D \neq \emptyset$, then we, without loss of generality, assume that $(V(H_1), h_{21})_D \neq \emptyset$, $(h_{22}, V(H_3))_D \neq \emptyset$ and $(h_{11}, h_{21}), (h_{22}, h_{31}) \in A(D)$ with $h_{11} \in V(H_1)$ and $h_{31} \in V(H_3)$. It is clear that H_1' contains a Hamiltonian path with h_{11} as the end-vertex, denoted by $P^1_{h_{11}}$, and H_3' contains a Hamiltonian path with h_{31} as the end-vertex, denoted by $P^3_{h_{31}}$. It follows by Lemma 3(i) that H_2 contains a forward ADS trail with h_{22} as the starting vertex and h_{21} as the terminal vertex, denoted by $h_{22} \rightarrow \cdots \rightarrow h_{21}$. Thus, $(P^1_{h_{11}}, h_{11} \rightarrow h_{21} \leftarrow \cdots \leftarrow h_{22} \rightarrow h_{31}, P^3_{h_{31}})$ is a spanning 3-tuple of mixed trails, and so by (1), D contains an ADS trail, a contradiction.

Hence assume that there exist two vertices $h_{23}, h_{24} \in V(H_2)$ such that $(V(H_1), V(H_2))_D = (V(H_1), h_{23})_D$, $(V(H_2), V(H_3))_D = (h_{23}, V(H_3))_D$, $(V(H_3), V(H_2))_D = (V(H_3), h_{24})_D$ and $(V(H_2), V(H_1))_D = (h_{24}, V(H_1))_D$. Let $(h_{12}, h_{23}) \in (V(H_1), h_{23})_D$ and $(h_{32}, h_{24}) \in (V(H_3), h_{24})_D$ with $h_{12} \in V(H_1)$ and $h_{32} \in V(H_3)$. It is clear that H_1' contains a Hamiltonian path with h_{12} as the end-vertex, denoted by $P^1_{h_{12}}$, and H_3' contains a Hamiltonian path with h_{32} as the end-vertex, denoted by $P^3_{h_{32}}$. If $|V(H_2)| \neq 2$, then by Lemma 3(ii) or (iii), H_2 contains a backward-forward ADS trail with h_{23} and h_{24} as terminal vertices, denoted by $h_{23} \leftarrow \cdots \rightarrow h_{24}$. Thus, $(P^1_{h_{12}}, h_{12} \rightarrow h_{23} \leftarrow \cdots \rightarrow h_{24} \leftarrow h_{32}, P^3_{h_{32}})$ is a spanning 3-tuple of mixed trails, and so by (1), D contains an ADS trail, a contradiction.

Hence assume that $|V(H_2)| = 2$. As $D \notin \mathcal{D}_2$, we have $h_{23} = h_{24}$. Let $h_{25} \in V(H_2) \setminus \{h_{23}\}$. Since D is $\{D_{14}, D_{15}, D_{16}\}$-free, we have that there exists a complete digraph, says H_1, such that either $|V(H_1)| \geq 3$ or, $|V(H_1)| = 2$ and for any vertex $h_1 \in V(H_1)$, $[h_1, V(H_2)]_D \neq \emptyset$. If $|V(H_1)| \geq 3$, then let $(h_{23}, h_{13}) \in (h_{23}, V(H_1))_D$. By Lemma 3(i) or (iv), H_1 contains a forward ADS trail with h_{12} as the starting vertex and h_{13} as the terminal vertex, denoted by $h_{12} \to \cdots \to h_{13}$. Thus, $(h_{25}h_{23}, h_{23} \to h_{13} \leftarrow \cdots \leftarrow h_{12} \to h_{23} \leftarrow h_{32}, P^3_{h_{32}})$ is a spanning 3-tuple of mixed trails, and so by (1), D contains an ADS trail, a contradiction. If $|V(H_1)| = 2$ and for any vertex $h_1 \in V(H_1)$, $[h_1, V(H_2)]_D \neq \emptyset$, then assume that $V(H_1) = \{h_1', h_1''\}$ such that $(h_1', h_{23}), (h_{23}, h_1'') \in A(D)$ as $(V(H_1), V(H_2))_D = (V(H_1), h_{23})_D$ and $(V(H_2), V(H_1))_D = (h_{23}, V(H_1))_D$. Thus, $(h_{25}h_{23}, h_{23} \to h_1'' \leftarrow h_1' \to h_{23} \leftarrow h_{32}, P^3_{h_{32}})$ is a spanning 3-tuple of mixed trails, and so by (1), D contains an ADS trail, a contradiction. This proves the theorem. ∎

Theorem 9. *Let D be a strongly connected digraph with $\alpha_2(D) = 3$. Then D contains an* ADS *trail if and only if $D \notin \mathcal{D}_1 \cup \mathcal{D}_2$.*

Proof. If $D \in \mathcal{D}_1 \cup \mathcal{D}_2$, then by (3), D does not contain an ADS trail. Hence assume that $D \notin \mathcal{D}_1 \cup \mathcal{D}_2$, we want to prove that D contains an ADS trail. Let D be a strongly connected digraph with $\alpha_2(D) = 3$ and let us consider graph G_D of D defined as in Definition 4. Clearly, $\alpha_2(D) = \alpha(G_D) = 3$. As $\alpha_2(D) = 3$, we have $|V(D)| = |V(G_D)| \geq 3$. If $\kappa(G_D) \geq 3$, then by Theorem 1, G_D has a Hamiltonian cycle, and hence by (2), D contains an ADS trail. Assume now that $\kappa(G_D) \leq 2$. Let S be a minimum vertex cut of G_D. Then $|S| \leq 2$ and $G_D - S$ has at least two connected components and one of the them, says K, is a complete graph since $\alpha(G_D) = 3$.

Let $H = G_D - S - V(K)$. We have $\alpha(H) \leq 2$. Otherwise, if $\alpha(H) \geq 3$, then let $\{h_1, h_2, h_3\}$ be an independent set of H and let $k_1 \in V(K)$ be an arbitrary vertex. Since K and H are connected components of $G_D - S$, we have that $h_1k_1, h_2k_1, h_3k_1 \notin E(G_D)$. Thus $\{h_1, h_2, h_3, k_1\}$ is an independent set of G_D, contrary to $\alpha(G_D) = 3$.

If $\alpha(H) = 1$, then H is complete graph, and so H contains a Hamiltonian cycle or G is an edge; if $\alpha(H) = 2$ and $\kappa(H) \geq 2$, then $|V(H)| \geq 3$ and by Theorem 1, H contains a Hamiltonian cycle. In both cases above, either H contains a Hamiltonian cycle or G is an edge. If H contains a Hamiltonian cycle, then let $C{=}h_1h_2 \cdots h_{n_2}h_1$ be a Hamiltonian cycle of H and let $C_{h_1h_{n_2}} = h_1h_2 \cdots h_{n_2}$; if H is an edge, then let $h_1h_2 \in E(H)$.

First, we investigate the case in which $|S| = 0$. Then $[V(K), V(H)]_D \neq \emptyset$ as D is strongly connected. Without loss of generality, assume that $(V(K), V(H))_D \neq \emptyset$ and $(k, h_1) \in (V(K), V(H))_D$ as the case when $(V(H), V(K))_D \neq \emptyset$ can be justified by a similar argument. It is clear that K contains a Hamiltonian path with k as the end-vertex, denoted by P_k. Thus, $(P_k, k \to h_1, C_{h_1h_{n_2}})$ or $(P_k, k \to h_1, h_1h_2)$ can serve as a spanning 3-tuple of mixed trails. It follows by (1) that D contains an ADS trail.

We now investigate the case in which $|S| = 1$. Let $S = \{s\}$. Since S is a minimum vertex cut of G_D, we have $[V(K), s]_{G_D} \neq \emptyset$ and $[V(H), s]_{G_D} \neq \emptyset$, says

$ks, h_1s \in E(G_D)$. It is clear that K contains a Hamiltonian path with k as the end-vertex, denoted by P_k. Thus, $P_k \cup ksh_1 \cup C_{h_1h_{n_2}}$ or $P_k \cup ksh_1 \cup h_1h_2$ can serve as a Hamiltonian path of G_D. By (2), D contains an ADS trail.

Finally, we investigate the case in which $|S| = 2$. Then $\kappa(G_D) = 2$. Let $S = \{s_1, s_2\}$. As S is a minimum vertex cut of G_D, we have that there exist edges $h_\ell s_1, h_{\ell'}s_2, k_1s_1, k_2s_2 \in E(G_D)$ with $h_\ell, h_{\ell'} \in V(H)$ and $k_1, k_2 \in V(K)$ such that $h_\ell \neq h_{\ell'}$ if $|V(H)| \geq 2$ and $k_1 \neq k_2$ if $|V(K)| \geq 2$. It follows by Theorem 2 that K contains a Hamiltonian path $P_{k_1k_2}$ with k_1 and k_2 as end-vertices. Thus, $s_1k_1 \cup P_{k_1k_2} \cup k_2s_2h_{\ell'}h_{\ell'+1} \cdots h_{n_2}h_1 \cdots h_{\ell'-1}$ or $s_1k_1 \cup P_{k_1k_2} \cup k_2s_2h_{\ell'}h_\ell$ is a Hamiltonian path of G_D, and so by (2), D contains an ADS trail.

So let us now assume that $\alpha(H) = 2$ and $\kappa(H) \leq 1$. Then by Lemma 5, H can be covered by two vertex-disjoint complete graphs, denoted by H_1 and H_2. If $|S| = 0$, then G_D can be covered by three vertex-disjoint complete graphs K, H_1 and H_2, and so D can be covered by three vertex-disjoint complete digraphs. It follows by Proposition 7 that D contains an ADS trail. In the remaining proof, we assume that $1 \leq |S| \leq 2$ and we will now divide the proof into several cases, depending on the connectivity of H.

Case 1. $\kappa(H) = 1$. Then $[V(H_1), V(H_2)]_{G_D} \neq \emptyset$. As $\alpha(H) = 2$, we have $|V(H)| \geq 3$.

Subcase 1.1. $|S| = 1$. Let $S = \{s\}$. As S is a minimum vertex cut of G_D, we have $[s, V(H)]_{G_D} \neq \emptyset$ and $[s, V(K)]_{G_D} \neq \emptyset$. If $G_D[K \cup S]$ is a complete graph, then G_D can be covered by three vertex-disjoint complete graphs, and so D can be covered by three vertex-disjoint complete digraphs. By Proposition 7, D contains an ADS trail. Hence assume that $G_D[K \cup S]$ is not a complete graph. Let $sk_1 \in [s, V(K)]_{G_D}$. It is clear that K contains a Hamiltonian path with k_1 as the end-vertex, denoted by P_{k_1}.

If there exist H_i with $i \in \{1, 2\}$ and three distinct vertices $h_{i1}, h_{i2} \in V(H_i)$ and $h_{(3-i)1} \in V(H_{3-i})$ such that $h_{i1}h_{(3-i)1} \in [V(H_1), V(H_2)]_{G_D}$ and $h_{i2}s \in [V(H_i), s]_{G_D}$, then H_i contains a Hamiltonian path $P^i_{h_{i1}h_{i2}}$ with h_{i1} and h_{i2} as end-vertices, and H_{3-i} contains a Hamiltonian path with $h_{(3-i)1}$ as the end-vertex, denoted by $P^{3-i}_{h_{(3-i)1}}$. Thus, $P^{3-i}_{h_{(3-i)1}} \cup h_{(3-i)1}h_{i1} \cup P^i_{h_{i1}h_{i2}} \cup h_{i2}sk_1 \cup P_{k_1}$ is a Hamiltonian path of G_D, and so by (2), D contains an ADS trail.

Hence assume that there exist two distinct vertices $h_1 \in V(H_1)$ and $h_2 \in V(H_2)$ such that $[V(H_1), V(H_2)]_{G_D} = [h_1, V(H_2)]_{G_D}$, $h_1h_2 \in [h_1, V(H_2)]_{G_D}$ and if $|[h_1, V(H_2)]_{G_D}| = 1$, then $[s, V(H)]_{G_D} = [s, \{h_1, h_2\}]_{G_D}$; if $|[h_1, V(H_2)]_{G_D}| \geq 2$, then $[s, V(H)]_{G_D} = [s, h_1]_{G_D}$. Since $[s, V(H)]_{G_D} \neq \emptyset$, we may assume that $h_1s \in [s, V(H)]_{G_D}$. It is clear that H_2 contains a Hamiltonian path with h_2 as the end-vertex, denoted by $P^2_{h_2}$. If $|V(H_1)| = 1$, then $P^2_{h_2} \cup h_2h_1sk_1 \cup P_{k_1}$ is a Hamiltonian path of G_D, and so by (2), D contains an ADS trail. If $|V(H_1)| \geq 2$, then, as $\alpha(G_D) = 3$ and $G_D[K \cup S]$ is not a complete graph, we have that

$|[h_1, V(H_2)]_{G_D}| = 1$, $sh_2 \in [s, V(H)]_{G_D}$ and $|V(H_2)| = 1$. It is clear that H_1 contains a Hamiltonian path with h_1 as the end-vertex, denoted by $P^1_{h_1}$. Thus, $P^1_{h_1} \cup h_1h_2sk_1 \cup P_{k_1}$ is a Hamiltonian path of G_D, and so by (2), D contains an ADS trail.

Subcase 1.2. $|S| = 2$. Then $\kappa(G_D) = 2$. Let $S = \{s_1, s_2\}$. As S is a minimum vertex cut of G_D, we have that there exist edges $h_1s_1, h_2s_2, k_1s_1, k_2s_2 \in E(G_D)$ with $h_1, h_2 \in V(H)$, $k_1, k_2 \in V(K)$ and $h_1 \neq h_2$ such that $k_1 \neq k_2$ if $|V(K)| \geq 2$. Assume that $h_1 \in V(H_1)$. If $h_2 \in V(H_2)$, then it is clear that H_1 contains a Hamiltonian path with h_1 as the end-vertex, denoted by $P^1_{h_1}$, H_2 contains a Hamiltonian path with h_2 as the end-vertex, denoted by $P^2_{h_2}$, and K contains a Hamiltonian path $P_{k_1k_2}$ with k_1 and k_2 as end-vertices. Thus, $P^2_{h_2} \cup h_2s_2k_2 \cup P_{k_2k_1} \cup k_1s_1h_1 \cup P^1_{h_1}$ is a Hamiltonian path of G_D, and so by (2), D contains an ADS trail.

Hence assume that $h_2 \in V(H_1)$. As $[V(H_1), V(H_2)]_{G_D} \neq \emptyset$, we may assume that there exists edge $h_{11}h_{21} \in [V(H_1), V(H_2)]_{G_D}$ with $h_{11} \neq h_1$. It is clear that H_1 contains a Hamiltonian $h_{11}h_1$-path $P^1_{h_{11}h_1}$, K contains a Hamiltonian path $P_{k_1k_2}$ with k_1 and k_2 as end-vertices, and H_2 contains a Hamiltonian path with h_{21} as the end-vertex, denoted by $P^2_{h_{21}}$. Thus, $P^2_{h_{21}} \cup h_{21}h_{11} \cup P^1_{h_{11}h_1} \cup h_1s_1k_1 \cup P_{k_1k_2} \cup k_2s_2$ is a Hamiltonian path of G_D, and so by (2), D contains an ADS trail.

Case 2. $\kappa(H) = 0$. Then $[V(H_1), V(H_2)]_{G_D} = \emptyset$ and $|V(H)| \geq 2$. If $|S| = 1$, then, as $\alpha(G_D) = 3$, we have that $G_D[V(K) \cup S]$ is a complete graph or $G_D[V(H_i) \cup S]$ is a complete graph for some i with $i \in \{1, 2\}$. Thus, G_D can be covered by three vertex-disjoint complete graphs, and so D can be covered by three vertex-disjoint complete digraphs. By Proposition 7, D contains an ADS trail.

Hence assume that $|S| = 2$. Then $\kappa(G_D) = 2$. Let $S = \{s_1, s_2\}$. As S is a minimum vertex cut of G_D, we have that there exist edges $s_1h_1, s_2h'_1, s_1h_2, s_2h'_2, s_1k, s_2k' \in E(G_D)$ with $h_1, h'_1 \in V(H_1), h_2, h'_2 \in V(H_2)$ and $k, k' \in V(K)$ such that $h_1 \neq h'_1$ if $|V(H_1)| \geq 2$, $h_2 \neq h'_2$ if $|V(H_2)| \geq 2$ and $k \neq k'$ if $|V(K)| \geq 2$. Since H_1, H_2 and K are complete graphs, we have that H_i contains a Hamiltonian path $P^i_{h_ih'_i}$ with h_i and h'_i as ending vertices for any i with $i \in \{1, 2\}$ and K contains a Hamiltonian path $P_{kk'}$ with k and k' as end-vertices. Thus, $P^1_{h'_1h_1} \cup h_1s_1h_2 \cup P^2_{h_2h'_2} \cup h'_2s_2k' \cup P_{k'k}$ is a spanning trail of G_D, and so by (2), D contains an ADS trail. This completes the proof of the theorem. ∎

Using the definitions of $\mathcal{D}_1$ and $\mathcal{D}_2$, along with Theorem 9, the following corollaries can be immediately obtained.

Corollary 10. *Let D be a 2-strongly connected digraph with $\alpha_2(D) = 3$. Then D contains an* ADS *trail if and only if $D \ncong D_7$.*

Corollary 11. *Every 3-strongly connected digraph D with $\alpha_2(D) = 3$ contains an* ADS *trail.*

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