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BOUNDS ON THE k-CONVERSION NUMBER

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Abstract

We consider a graphical model of the spread of influence through social networks, where the goal is to find a set of vertices in the network, such that if this initial set is "influenced", then after the application of a certain propagation process eventually every vertex in the graph will also be influenced. In particular, we seek a minimum set of vertices to be initially influenced and follow an iterative process, where for a fixed integer threshold $k \geq 0$, a vertex outside the influenced set becomes influenced if at least k of its neighbors are influenced. We determine bounds on the minimum number of vertices required in such a set for every integer $k \geq 0$ and focus our study on the case for $k = 2$.

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1. Introduction

Graph theoretical models of the spread of influence through social networks abound and are prominent areas of research. In the classic *Target Set Selection* (also called *Conversion Set*), influence spread throughout a network is modeled by a graph G, where each vertex v in G is assigned a threshold $t(v)$. The goal is to find a set of vertices, called a *target set* (also called a *conversion set* or a *seed set*), that iteratively spreads influence throughout the whole graph by converting (influencing) a vertex v if $t(v)$ of its neighbors are influenced. For examples, see the papers by Kempe *et al.* [15, 16], Fazli *et al.* [9], Roberts [23], and Chen [3], and the PhD thesis of Dreyer [6].

In social network applications of this model, the goal is to find a relatively small number of vertices in a network such that if the vertices in this set adopt a given product, then ultimately every vertex in the graph will also adopt the product. This influence threshold model can be applied to many other things, such as the likelihood that someone will vote for something if sufficiently many of their neighbors vote for it. It could also indicate the likelihood that someone will get a virus, if sufficiently many of their neighbors are contagious, or that someone will believe misinformation if sufficiently many of their neighbors do.

Here we consider the *irreversible k-threshold conversion process model* suggested by Dreyer and Roberts [7]. For this variation of the target set selection problem, every vertex is assigned the same fixed threshold k, for some integer $k \geq 0$, and once a vertex is influenced it is remains influenced. The task is to find a conversion set of vertices that influences the whole graph after the following iterative process. Begin with a set S_0 of vertices, which are said to be influenced. For a fixed integer k and for each step $j \geq 1$, S_j is obtained from S_{j-1} by adding the vertices that have at least k neighbors in S_{j-1}. Such vertices are said to be *influenced* or *converted* by S_{j-1}. A set $S_0 \subseteq V$ is an *irreversible k-conversion set* or just a *k-conversion set* of a graph $G = (V, E)$ if $S_t = V$ for some $t \geq 0$. The *k-conversion number* $c_k(G)$ is the minimum cardinality of any k-conversion set of G and a k-conversion set of cardinality $c_k(G)$ is called a *c_k-set* of G.

Most of the research on the irreversible k-threshold conversion process model has focused on the computational complexity of determining, or even approximating, the minimum number of vertices in a k-conversion set. Dreyer and Roberts [7] showed that it is NP-hard to compute c_k for any constant $k \geq 3$. Furthermore, Chen [3] showed that this problem is NP-hard when $k = 2$, even for bounded bipartite graphs. Therefore, it is interesting to establish bounds on the k-conversion number as well as exact values for specific graphs, even for $k = 2$. Some research along these lines has been done on particular graphs, for example, paths, cycles, and complete multipartite graphs [7]; block-cactus graphs and chordal graphs [5]; hexagonal grids [4]; Cartesian products and grid graphs [14];

and regular graphs [18]. In this paper, we determine lower and upper bounds on the k-conversion number, focusing mainly on $k = 2$.

We close this section by giving some additional but standard notation and definitions used in the following. Let G be a graph with vertex set $V = V(G)$ and edge set $E = E(G)$. Two vertices v and w are *neighbors* in G if they are adjacent; that is, if $vw \in E$. The *open neighborhood* $N(v)$ of a vertex v in G is the set of neighbors of v, and the *degree* of v is $|N(v)|$. For a set S of vertices in a graph G, let $G[S]$ denote the subgraph of G induced by S.

A set S of vertices in a graph G is an *independent set* if no two vertices in S are adjacent, and the *independence number* $\alpha(G)$ is the maximum cardinality of an independent set of G. A set S of vertices in a graph G is a *dominating set* if every vertex in $V \setminus S$ has a neighbor in S. The *domination number* $\gamma(G)$ equals the minimum cardinality of a dominating set of G. For a positive integer k, a subset S of vertices in a graph G is a *k-dominating set* if every vertex not in S is adjacent to at least k vertices of S. The *k-domination number* $\gamma_k(G)$ is the minimum cardinality of a k-dominating set of G. Note that every graph G has a k-dominating set, since $V(G)$ is such a set. Also the 1-domination number $\gamma_1(G)$ is the usual *domination number* $\gamma(G)$. The concept of k-domination was introduced by Fink and Jacobson [10, 11]. For more details on k-domination see the book chapter by Hansberg and Volkmann [13].

We denote by P_n the path on n vertices, by C_n the cycle on n vertices, and by K_n the complete graph on n vertices. The complete bipartite graph with partite sets of cardinality r and s is denoted by $K_{r,s}$. The graph $K_{1,s}$ is called a *star*. A vertex of degree one is called a *leaf*, and the neighbor of a leaf is called a *support* vertex. A support vertex is said to be *strong* if it has at least two leaf neighbors. A *double star* $S(r, s)$, for $r, s \geq 1$, is a tree with exactly two (adjacent) vertices that are not leaves, with one of the vertices having r leaf neighbors and the other s leaf neighbors.

2. k-Conversion Number

The first lower bound we mention was observed in [6].

Observation 1 [6]**.** *For any graph G of order n and integer $k \geq 0$, $c_k(G) \geq k$ if $n \geq k$ and $c_k(G) = n$ otherwise.*

We give a straightforward constructive characterization, which follows directly from the definition of a k-conversion set of the graphs having $c_k(G) = k$. Let $\mathcal{G}_k$ for $k \geq 1$ be the family of graphs G of order $n \geq k$ that can be recursively constructed as follows. Begin with any graph of order k with vertex set labeled $\{v_1, v_2, \ldots, v_k\}$, and for each i where $k + 1 \leq i \leq n$, add a vertex v_i and add edges so that v_i is adjacent to at least k vertices in $\{v_1, v_2, \ldots, v_{i-1}\}$. Recall that for $k \geq 1$, a k-tree is a graph that can be built starting from a complete graph K_{k+1} and then iteratively adding a vertex joined to a complete subgraph of order k. Thus, k-trees are examples of graphs in $\mathcal{G}_k$. We note that complete bipartite graphs $K_{2,s}$ and maximal outerplanar graphs, which are a well-known subclass of 2-trees, are in $\mathcal{G}_2$.

Proposition 1. *A graph G has $c_k(G) = k$ if and only if $G \in \mathcal{G}_k$.*

Proof. Let $G \in \mathcal{G}_k$. By Observation 1, $c_k(G) \geq k$. By construction, $S = \{v_1, v_2, \ldots, v_k\}$ is a k-conversion set, so $c_k(G) \leq |S| = k$. Hence, $c_k(G) = k$.

Let G be a graph of order n with $c_k(G) = k$. Let $S = \{v_1, v_2, \ldots, v_k\}$ be a c_k-set of G and let $\pi = (v_1, v_2, \ldots, v_k, v_{k+1}, \ldots, v_n)$ be a sequence that lists first the vertices in S, in any order, followed by the remaining vertices in the order they are iteratively converted (influenced). Note that this sequence is not necessarily unique. Let S_j be the set containing the first j vertices of π. Since $c_k(G) \geq k$ and $|S| = k$, if $n = k$, then $G \in \mathcal{G}_k$. Assume that $n > k$. Then v_i, for each $i \in \{k+1, k+2, \ldots, n\}$, is influenced by S_{i-1}. That is, v_i has at least k neighbors in $\{v_1, v_2, \ldots, v_{i-1}\}$. Thus, $G \in \mathcal{G}_k$. ∎

Obviously, for every integer $k \geq 1$, any k-dominating set of a graph G is a k-conversion set of G. Also, if $L_k(G)$ is the set of vertices of degree less than k in G, then $L_k(G) \subseteq S$ for any k-conversion set S, which leads to the following observation.

Observation 2. *For any graph G and any integer $k \geq 1$, $|L_k(G)| \leq c_k(G) \leq \gamma_k(G)$.*

Note that for $k \geq 2$, both bounds of Observation 2 are sharp for stars $K_{1,k}$ and double stars $S(k, k)$.

Trivially, the 0-conversion number of any graph equals 0, since the empty set is a 0-conversion set for any graph. Moreover, the 1-conversion number of a graph G is simply the number of components in G. Thus, $c_1(G) = 1$ for any connected graph G. If v is a vertex of degree at least k in G, then $V \setminus \{v\}$ is a k-conversion set of G. Thus, if G is a graph with order n and maximum degree $\Delta(G) = \Delta \geq k$, then $c_\Delta(G) < n$. Since every $(k+1)$-conversion set is also a k-conversion set, we make the following observation.

Observation 3. *For an integer $k \geq 0$, let G be a graph of order n with* $\mathrm{com}(G)$ *components and $\Delta(G) \geq k$. Then*

$$c_0(G) = 0 < c_1(G) = \mathrm{com}(G) \leq c_2(G) \leq \cdots \leq c_\Delta(G) < n.$$

The following result gives the exact value of $c_\Delta(G)$. Let X_Δ denote the set of vertices of graph G with maximum degree $\Delta(G) = \Delta$.

Proposition 2. *For every graph G of order n, $c_\Delta(G) = n - \alpha(G[X_\Delta])$.*

Proof. Let S be a c_Δ-set of G. Note that S contains all vertices of degree less than Δ, and thus every vertex in $V \setminus S$ belongs to X_Δ. We also note that the only way for a vertex in $V \setminus S$ to be Δ-influenced is to have all its neighbors in S, that is $V \setminus S$ is an independent set. Hence, $\alpha(G[X_\Delta]) \geq |V \setminus S| = n - c_\Delta(G)$. Moreover, if A is a maximum independent set of $G[X_\Delta]$, then $V \setminus A$ is a Δ-conversion set of G, and thus, $c_\Delta(G) \leq n - |A| = n - \alpha(G[X_\Delta])$. Therefore, $c_\Delta(G) = n - \alpha(G[X_\Delta])$. ∎

Since for regular graphs G, $X_\Delta = V$, Proposition 2 leads to the following result, first observed in [6].

Corollary 3 [6]**.** *For every regular graph G of order n, $c_\Delta(G) = n - \alpha(G)$.*

Next we provide a lower bound on the k-conversion number $c_k(G)$ for every integer $k \geq 2$ in terms of the order and size of the graph G.

Proposition 4. *For every integer $k \geq 2$ and every graph G with $n \geq k$ vertices and m edges, $c_k(G) \geq n - \frac{m}{k}$.*

Proof. Let S be a c_k-set of G and let $\pi = (v_1, v_2, \ldots, v_n)$ be a sequence that lists first the vertices in S, in any order, followed by the remaining vertices in the order they are iteratively influenced. Note that this sequence is not necessarily unique. Let S_j be the set containing the first j vertices of π. Clearly, if $|S| = n$, then we are finished. Hence, we assume that $|S| < n$, and thus, $V \setminus S \neq \emptyset$. By definition, $S_k \subseteq S$, implying that $|S| \geq k$. Hence, for every $j \in \{|S|+1, \ldots, n\}$, vertex v_j is influenced by S_{j-1}, that is, v_j has at least k neighbors in S_{j-1}. Consequently, $m \geq k(n - |S|)$, which yields the desired lower bound $c_k(G) = |S| \geq n - \frac{m}{k}$. ∎

Restricted to trees and unicyclic graphs, the following corollaries to Proposition 4 hold for $k = 2$.

Corollary 5. *If G is a tree of order n, then $c_2(T) \geq \frac{n+1}{2}$.*

Corollary 6. *If G is a unicyclic graph G of order n, then $c_2(G) \geq \frac{n}{2}$.*

3. GRAPHS HAVING 2-CONVERSION NUMBER AT LEAST THEIR ORDER MINUS 2

Since the vertex set of any graph is a k-conversion set, for any graph G of order n, $c_k(G) \leq n$. Trivially, $c_2(K_1) = 1$ and $c_2(K_2) = 2$. As previously mentioned, if G has a vertex v of degree at least 2, then $V \setminus \{v\}$ is a 2-conversion set of G and so $c_2(G) \leq |V \setminus \{v\}| = n - 1$. It follows that K_1 and K_2 are the only connected graphs G of order n such that $c_2(G) = n$.

We characterize the connected graphs G of order n for which $c_2(G) = n - 1$ and also those for which $c_2(G) = n - 2$.

Theorem 7. *Let G be a connected graph of order $n \geq 3$. Then $c_2(G) \leq n - 1$, with equality if and only if G is the complete graph K_3, the path P_4, or a star $K_{1,t}$, for $t \geq 2$.*

Proof. Since G is a connected graph of order $n \geq 3$, the upper bound of $n - 1$ follows from our previous comment.

It is straightforward to see that $c_2(K_3) = 2$, $c_2(P_4) = 3$, and $c_2(K_{1,t}) = t$.

For the converse, assume that $c_2(G) = n - 1$. If $n = 3$, then $G = K_3$ or $G = P_3 = K_{1,2}$, and thus the result holds. Hence, assume that $n \geq 4$.

Let S be a c_2-set of G, and let $V \setminus S = \{x\}$. Let $N(x) = \{v_1, v_2, \ldots, v_p\}$. Note that $N(x) \subseteq S$ and $p \geq 2$.

If any vertex $y \in S$ has at least two neighbors in S, then $S \setminus \{y\} = V \setminus \{x, y\}$ is a 2-conversion set with cardinality less than $n - 1$, a contradiction. (We note that if $p = 2$ and $y \in \{v_1, v_2\}$, then for the set $S \setminus \{y\}$, vertex y would be converted first followed by vertex x.) Thus, every vertex in S has at most one neighbor in S. Since G is connected, we deduce that every vertex in $S \setminus N(x)$ is adjacent to exactly one vertex in $N(x)$ and to no other vertex in S.

Assume first that $p \geq 3$. If some $v_i \in N(x)$ has a neighbor in S, then $S \setminus \{v_i\}$ is a 2-conversion set of G of cardinality $n - 2 < c_2(G)$, a contradiction. Thus, no $v_i \in N(x)$ has a neighbor in S. Since G is connected, it follows that every vertex in S is adjacent to x. Therefore, G is a star of order at least four centered at x and the result holds.

Next assume that $p = 2$, that is, $N(x) = \{v_1, v_2\}$. Since $n \geq 4$, $S \setminus N(x) \neq \emptyset$. Since G is connected, every vertex $y \in S \setminus N(x)$ is adjacent to exactly one of v_1 and v_2. Since every vertex in S has at most one neighbor in S and $S \setminus N(x) \neq \emptyset$, it folllows that v_1 is not adjacent to v_2 and $1 \leq |S \setminus N(x)| \leq 2$. If $|S \setminus N(x)| = 2$, then $G = P_5$. But $c_2(P_5) = 3 < 4 = n - 1$, a contradiction. If $|S \setminus N(x)| = 1$, then $G = P_4$ and the result holds. ■

Let $K_4 - e$ denote the complete graph K_4 with an edge removed.

Theorem 8. *Let G be a connected graph of order n. Then $c_2(G) = n - 2$ if and only if $G \in \{C_4, K_4, K_4 - e, C_5, P_6\}$ or G is a double star $S_{r,s}$ with $r + s \geq 3$ or G is isomorphic to any graph H_i in Figure* 1.

Proof. Let G be a connected graph of order n such that $c_2(G) = n - 2$. Our previous comments and Theorem 7 imply that $n \geq 4$, and G is not the path P_4 nor a star $K_{1,t}$.

Let S be a c_2-set of G and let $V \setminus S = \{x, y\}$. Note that each of x and y has at least one neighbor in S, and at least one of x and y has at least two neighbors

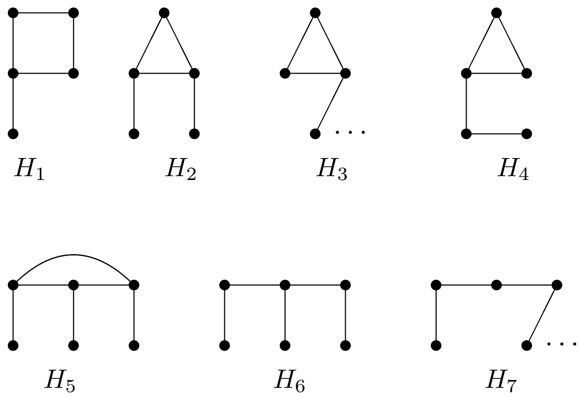

Figure 1. Connected graphs G with order n and $c_2(G) = n-2$, where in H_3 and H_7, the notation "..." following a leaf vertex indicates that its support vertex is adjacent to one or more leaves.

in S. As seen in the proof of Theorem 7, every vertex $w \in S$ has at most one neighbor in S, else $S \setminus \{w\}$ is a 2-conversion set with cardinality less than $c_2(G)$. Let $X = N(x) \cap S = \{x_1, x_2, \ldots, x_r\}$ and $Y = N(y) \cap S = \{y_1, y_2, \ldots, y_s\}$. Without loss of generality, assume that $r \geq s$, and hence $r \geq 2$. Let us examine the following two cases.

Case 1. $r \geq 3$. If some $x_i \in X$ has a neighbor in S, then $S \setminus \{x_i\}$ is a 2-conversion set of G of cardinality $n-3 < c_2(G)$, a contradiction. Hence, no vertex $x_i \in X$ has a neighbor in S. If $s \geq 3$, then the same holds for y_i, that is, no $y_i \in Y$ has a neighbor in S.

Assume first that $s \geq 3$. Hence, no vertex in $X \cup Y$ has a neighbor in S. The connectivity of G implies that $S \setminus (X \cup Y) = \emptyset$ and that $xy \in E(G)$. If x and y have a common neighbor in S, say x_i, then $S \setminus \{x_i\}$ is a 2-conversion set of G of cardinality $n-3$, a contradiction. Hence, $X \cap Y = \emptyset$. Therefore, G is a double star $S_{r,s}$ with $r \geq s \geq 3$.

Assume next that $s = 2$. We consider $|X \cap Y|$. If $|X \cap Y| = 2$, then relabeling the vertices if necessary, let $y_1 = x_1$ and $y_2 = x_2$. Since no vertex in X has a neighbor in S, neither y_1 nor y_2 has a neighbor in S. Since G is connected, it follows that $S \setminus X = \emptyset$. If $xy \in E(G)$, then $S \setminus \{x_1\}$ is a 2-conversion set of G of cardinality $n-3$, a contradiction. Furthermore, if $r \geq 4$, then $(S \setminus \{x_1, x_2\}) \cup \{y\}$ is a 2-conversion set of G of cardinality $n-3$, a contradiction. Hence, $r = 3$, and therefore, G is isomorphic to H_1.

Now, assume that $|X \cap Y| = 1$, that is, x_1 is the only common neighbor of x and y in S. Hence, $x_1 = y_1$ and so y_1 has no neighbor in S. If $xy \in E(G)$, then $S \setminus \{x_1\}$ is a 2-conversion set of G, a contradiction. Thus, we may assume that x and y are not adjacent. Since G is connected, every vertex in $S \setminus (X \cup Y)$ is adjacent to y_2. As previously established, every vertex in S, in particular y_2, has at most one neighbor in S, so there is at most one vertex in $S \setminus (X \cup Y)$. If $|S \setminus (X \cup Y)| = 1$, then $(S \setminus \{y_1, y_2\}) \cup \{y\}$ is a 2-conversion set of cardinality $n - 3$, a contradiction. If $S \setminus (X \cup Y) = \emptyset$, then G is isomorphic to H_7.

Thus, we can assume that $X \cap Y = \emptyset$. The connectedness of G implies that $xy \in E(G)$. In this case, it can be seen that neither y_1 nor y_2 has a neighbor in S, and therefore the connectedness of G implies that $S \setminus (X \cup Y) = \emptyset$. Hence, G is a double star $S_{2,r}$ with $r \geq 3$.

Finally, let $s = 1$. Since y_1 is the only neighbor of y in S, vertex y is influenced after vertex x is influenced by S, implying that $xy \in E(G)$. If x and y have a common neighbor, that is, if $y_1 = x_1$, then G is the graph H_3. Assume that x and y have no common neighbor. If y_1 has a neighbor in S, then $(S \setminus \{y_1\}) \cup \{y\}$ is a c_2-set of G corresponding to a situation previously considered when $s = 2$. Hence, we assume that y_1 has no neighbor in S, and it follows that G is a double star $S_{1,r}$ with $r \geq 3$.

Case 2. $r = 2$. Then $s \in \{1, 2\}$. Assume first that $s = 2$. Suppose that x and y have two common neighbors, that is, $y_1 = x_1$ and $y_2 = x_2$. If $n = 4$, then $G \in \{K_4, C_4, K_4 - e\}$ and the result holds. Hence, assume that $n \geq 5$, implying that $S \setminus X \neq \emptyset$. Since G is connected and every vertex in S has at most one neighbor in S, there exists a vertex $z \in S \setminus X$ such that, without loss of generality, z is adjacent to x_1. If $xy \in E(G)$, then $(S \setminus \{x_1, x_2\}) \cup \{x\}$ is a 2-conversion set of G of cardinality $n - 3$, a contradiction. Hence, x is not adjacent to y. Similarly, x_2 has no neighbor in S (otherwise, $(S \setminus \{x_1, x_2\}) \cup \{x\}$ is a 2-conversion set of G). Thus, G is isomorphic to H_1.

Next we assume that x and y have exactly one common neighbor, say $x_1 = y_1$, in S. If $xy \in E(G)$, then x_2 has no neighbor in S, else $S \setminus \{x_2\}$ would be a 2-conversion set of G. Likewise y_2 has no neighbor in S. Thus, depending on whether x_1 has a neighbor in S, two graphs are possible, namely, H_2 and H_5, and the result holds.

Hence, assume that $xy \notin E(G)$. If $S \setminus (X \cup Y) = \emptyset$, then $n = 5$. Since every vertex in S has at most one neighbor in S, there is at most one edge in $G[S]$, implying that $G \in \{P_5, C_5, H_4\}$. Note that P_5 is isomorphic to the graph H_7 where each support is adjacent to exactly one leaf, and so, the result holds.

Hence, assume that $S \setminus (X \cup Y) \neq \emptyset$, and so, $n \geq 6$. Since G is connected and every vertex in S has at most one neighbor in S, it follows that $|S \setminus (X \cup Y)| \leq 3$. If both x_2 and y_2 have a neighbor in $S \setminus (X \cup Y)$, then $(S \setminus \{x_1, x_2, y_2\}) \cup \{x, y\}$ is a 2-conversion set of G with cardinality $n - 3$, a contradiction. Hence, at most one of x_2 and y_2 has a neighbor in $S - (X \cup Y)$. If both x_1 and x_2 have a neighbor in $S \setminus (X \cup Y)$, then $(S - \{x_1, x_2\}) \cup \{x\}$ is a 2-conversion set of G, again a contradiction. Hence, at most one of x_1 and x_2 has a neighbor in $S \setminus (X \cup Y)$.

Similarly, at most one of x_1 and y_2 has a neighbor in $S \setminus (X \cup Y)$. It follows that there is exactly one vertex z in $S \setminus (X \cup Y)$. Without loss of generality, z is adjacent to either x_1 or x_2. Assume that z is adjacent to x_1. If $x_2y_2 \in E(G)$, then $(S \setminus \{x_1, x_2\}) \cup \{x\}$ is a 2-conversion set of G, a contradiction. If $x_2y_2 \notin E(G)$, G is isomorphic to H_6. Hence, assume that z is adjacent to x_2. If $x_1y_2 \in E(G)$, then $(S \setminus \{x_1, x_2, y_2\}) \cup \{x, y\}$ is a 2-conversion set of G with cardinality $n-3$, a contradiction. Thus, $x_1y_2 \notin E(G)$, and so, $G = P_6$.

Finally, assume that x and y have no common neighbor. If $x_1y_1 \in E(G)$, then $(S \setminus \{y_1\}) \cup \{y\}$ is a c_2-set of G such that $s = r = 2$, where x and y_1 share a common neighbor, and such a situation has just been considered. Hence, we assume that no edge joins x_i to y_j for any $i, j \in \{1, 2\}$. Since no vertex of S has two neighbors in S, the connectedness of G implies that $xy \in E(G)$. If x_1 has a neighbor in S, then $S \setminus \{x_1\}$ is a 2-conversion set of G, a contradiction. Likewise, none of x_2, y_1, and y_2 has a neighbor in S, implying that $S = \{x_1, x_2, y_1, y_2\}$ and G is the double star $S_{2,2}$.

We can now consider the last situation when $s = 1$. Since S is a 2-conversion set, $xy \in E(G)$ in order to influence y. Assume first that $x_1 = y_1$. Since no vertex in S has two neighbors in S, it follows from the connectedness of G that $n \leq 6$. If $n = 4$, then $G \in \{K_4 - e, H_3\}$. If $n = 5$, then $G \in \{H_2, H_4\}$. If $n = 6$, then each of x_1 and x_2 has a neighbor in $S \setminus (X \cup Y)$ and $(S \setminus \{x_1, x_2\}) \cup \{x\}$ is a 2-conversion set of G, a contradiction.

Assume now that x and y have no common neighbor in S. If y_1 has a neighbor in S, then $S' = (S \setminus \{y_1\}) \cup \{y\}$ is a c_2-set of G such that $r = 3$ and $s = 2$, and such a case has been considered. Hence, y_1 has no neighbor in S. If $n = 5$, then G is the double star $S_{1,2}$ or $G = H_4$. Thus, assume that $n \geq 6$, and so, $S \setminus (X \cup Y) \neq \emptyset$. If both x_1 and x_2 have a neighbor in S, then $(S \setminus \{x_1, x_2\}) \cup \{x\}$ is a 2-conversion set of G, a contradiction. It follows that, without loss of generality, there is exactly one vertex z in $S \setminus (X \cup Y)$ and $x_1z \in E(G)$. Hence, $G = H_6$.

For the converse, by Theorem 7, $c_2(G) \leq n-2$. It is straightforward to check the graphs for equality. ∎

4. Upper Bounds on the 2-Conversion Number

For a path P_n, it is shown in [6] that $c_2(P_n) = n - \alpha(P_{n-2}) = n - \lceil (n-2)/2 \rceil = \lceil (n+1)/2 \rceil$. The first upper bound is in terms of the order and the length of a longest path of a graph.

Proposition 9. *If G is a connected graph of order $n \geq 2$ and $\ell(G)$ is the length of a longest path in G, then*

$$c_2(G) \leq n - \left\lfloor \frac{\ell(G)}{2} \right\rfloor .$$

Proof. Since the result is valid for connected graphs G with $\ell(G) \in \{1,2\}$, we assume that $\ell(G) \geq 3$. Let $P = v_0v_1 \cdots v_\ell$ be a longest path of G, where $\ell = \ell(G)$, and let S be a c_2-set of P. Then $|S| = \lceil(\ell+2)/2\rceil$ and $S \cup (V(G) \setminus V(P))$ is a 2-conversion set of G. Hence, $c_2(G) \leq |S \cup (V(G) \setminus V(P))| = \lceil(\ell+2)/2\rceil + (n - \ell - 1) = n - \lfloor \frac{\ell}{2} \rfloor$. ■

The bound of Proposition 9 is sharp for paths P_n.

The next upper bound is in terms of order and minimum degree.

Proposition 10. *For any graph G of order n and minimum degree $\delta(G)$,*

$$c_2(G) \leq n + 1 - \delta(G).$$

Proof. If $\delta(G) \in \{0,1\}$, then the inequality holds. Hence, assume that $\delta(G) \geq 2$. Let X be any set of vertices of cardinality $\delta(G) - 1$, and consider $S = V \setminus X$. Now every vertex in X has at least two neighbors in S, and so S is a 2-conversion set of G. Thus, $c_2(G) \leq |S| = |V \setminus X| = n - (\delta(G) - 1) = n + 1 - \delta(G)$. ■

The bound of Proposition 10 is sharp for complete graphs K_n.

Note that for any graph G and a dominating set D of G, D along with a 1-conversion set of $G[V \setminus D]$ is a 2-conversion set of G. Thus, to build a 2-conversion set from any γ-set D of G, we can select one vertex from each component of $G[V \setminus D]$. Hence, we have the following upper bound.

Proposition 11. *Let D be a γ-set of G, where* $\mathrm{com}(G[V \setminus D])$ *is the number of components of $G[V \setminus D]$. Then $c_2(G) \leq \gamma(G) + \mathrm{com}(G[V \setminus D])$.*

Corollary 12. *If G has a γ-set D such that $G[V \setminus D]$ is connected, then $c_2(G) \leq \gamma(G) + 1$.*

The *corona* $G = H \circ K_1$ is the graph obtained from a graph H by adding for each vertex $v \in V(H)$ a new vertex v' and the edge vv'. Thus, $G = H \circ K_1$ has order $n = 2|V(H)|$ and has $|V(H)|$ leaves. Moreover, the set of leaves of G is a γ-set of G, and so, by Proposition 11, $c_2(G) \leq |V(H)| + \mathrm{com}(H)$. Since every c_2-set of G contains all the leaves of G and each leaf is adjacent to exactly one vertex in $V(H)$, to complete a 2-conversion set of G, we must select a 1-conversion set of H. Hence, $c_2(G) \geq |V(H)| + \mathrm{com}(H)$, and equality follows.

Corollary 13. *If G is the corona of a connected graph H, then $c_2(G) = |V(H)| + 1 = \gamma(G) + 1$.*

Next we use Corollary 5 to derive a lower bound on $c_2(T)$ for trees T in terms of their domination number. We recall a classic result due to Ore [19] that for every graph G of order n with no isolated vertices, $\gamma(G) \leq \frac{n}{2}$.

Proposition 14. *For every nontrivial tree T, $c_2(T) \geq \gamma(T) + 1$.*

Proof. By Corollary 5, $c_2(T) \geq \frac{n+1}{2} \geq \gamma(T) + \frac{1}{2}$, and since $c_2(T)$ is an integer, the desired bound follows. ■

Note that the difference $c_2(T) - (\gamma(T) + 1)$ can be arbitrarily large for trees as may be seen by considering a star of large order.

Our next aim is to characterize the trees T such that $c_2(T) = \gamma(T) + 1$. Let X be the set of vertices of a tree T that are neither leaf nor support vertices of T. Let ℓ and s be the number of leaves and support vertices, respectively, of T.

Let $\mathcal{T}$ be the family of trees T such that one of the following holds.

(1) $s = \ell$ and $X = \emptyset$, that is, T is the corona $T = T' \circ K_1$ of some tree T'.

(2) $s = \ell - 1$ and $X = \emptyset$.

(3) $s = \ell$ and $|X| = 1$.

(4) $s = \ell$ and $G[X]$ is a path P_3 whose center vertex has degree two in T.

We note that the trees of $\mathcal{T}$ are precisely the trees of order n having $\gamma(T) = \left\lfloor \frac{n}{2} \right\rfloor$, as shown in [1, 12, 20, 21]. Now we are ready to characterize the nontrivial trees T such that $c_2(T) = \gamma(T) + 1$.

Theorem 15. *A tree T has $c_2(T) = \gamma(T) + 1$ if and only if $T \in \mathcal{T}$.*

Proof. Assume that T is a tree of order n in $\mathcal{T}$. If (1) holds, then T is a corona of a tree T', and by Corollary 13, $c_2(T) = \gamma(T) + 1$. Assume that (2)-(4) holds. By Proposition 14, $c_2(T) \geq \gamma(T) + 1$. To show equality, we show that $c_2(T) \leq \gamma(T) + 1$. We note that the set of support vertices is a γ-set of T and $\gamma(T) = s$ if (2) or (3) holds, while one more vertex is needed if (4) holds and $\gamma(T) = s + 1$. The set of leaves forms a 2-conversion set if (2) holds and so $c_2(T) = \ell = s + 1 = \gamma(T) + 1$. If (3) holds, the set of leaves along with the vertex in X is a 2-conversion set, implying that $c_2(T) \leq \ell + 1 = s + 1 = \gamma(T) + 1$. Finally, if (4) holds, then the set of leaves along with two vertices from X forms a 2-conversion set of T, and so, $c_2(T) \leq \ell + 2 = s + 2 = \gamma(T) + 1$.

Conversely, assume that T is a tree T of order $n \geq 2$ such that $c_2(T) = \gamma(T) + 1$. To show that $T \in \mathcal{T}$, by our previous comments, it suffices to show that $\gamma(T) = \left\lfloor \frac{n}{2} \right\rfloor$. By Ore's result, $\gamma(T) \leq \left\lfloor \frac{n}{2} \right\rfloor$. By Corollary 5, $c_2(T) \geq \frac{n+1}{2}$. If n is even, then since $c_2(T)$ is an integer, $c_2(T) \geq \frac{n+2}{2}$. Thus, $c_2(T) = \gamma(T) + 1 \geq \frac{n+2}{2} = \frac{n}{2} + 1$, implying that $\gamma(T) \geq \frac{n}{2}$, and so, $\gamma(T) = \frac{n}{2}$.

If n is odd, then $\gamma(T) \leq \left\lfloor \frac{n}{2} \right\rfloor = \frac{n-1}{2}$ and $c_2(T) \geq \frac{n+1}{2}$, implying that $\frac{n+1}{2} \leq c_2(T) = \gamma(T) + 1 \leq \frac{n-1}{2} + 1 = \frac{n+1}{2}$. Thus, we have equality thoughout, and so, $\gamma(T) = \frac{n-1}{2} = \left\lfloor \frac{n}{2} \right\rfloor$. ■

We turn our attention to graphs having minimum degree at least 2.

Theorem 16. *If G is a connected graph with domination number $\gamma(G) = \gamma$ and minimum degree $\delta(G) = \delta \geq 2$, then $c_2(G) \leq \frac{n+(\delta-1)\gamma}{\delta}$.*

Proof. Let G be a graph with domination number $\gamma(G) = \gamma$ and minimum degree $\delta(G) = \delta$. Let D be a minimum dominating set of G. To build a 2-conversion set S of G, we partition the vertices of $V \setminus D$ into the sets X, Y, and Z as follows.

Let X be the set of vertices in $V \setminus D$ that have at least two neighbors in D and let Y be the set of vertices in $(V \setminus D) \setminus X$ such that X is a 1-conversion set of $X \cup Y$ in $G[V \setminus D]$. In other words, for every vertex in $y \in Y$, there exists a path containing only vertices of $V \setminus D$ connecting y to a vertex in X. Finally, let $Z = (V \setminus D) \setminus (X \cup Y)$. Note that if $z \in Z$, then z has exactly one neighbor in D and no neighbor in $X \cup Y$. Form a set Z' by selecting one vertex from each component in $G[Z]$. Let $S = D \cup Z'$.

To see that S is a 2-conversion set, note that the vertices of $D \cup Z'$ are in S, the vertices of X are influenced by the set S, and the vertices of Y are influenced by S since they each have a exactly one neighbor in D and are influenced iteratively by the 1-conversion set X in $G[X \cup Y]$. Finally, each vertex in $Z \setminus Z'$ is influenced by $S = D \cup Z'$ since they each have exactly one neighbor in D and Z' is a 1-conversion set of $G[Z]$.

Hence, $c_2(G) \leq |S| = |D| + |Z'| = \gamma(G) + \mathrm{com}(G[Z])$, where $\mathrm{com}(G[Z])$ is the number of components of $G[Z]$. Since $\delta(G) = \delta \geq 2$ and each vertex in Z has exactly one neighbor in D, it follows that $\delta(G[Z]) \geq \delta - 1$. Hence, $G[Z]$ has at most $|Z|/\delta$ components. Thus, $c_2(G) \leq \gamma(G) + \frac{|Z|}{\delta} \leq \gamma(G) + \frac{n-\gamma}{\delta} = \frac{n+(\delta-1)\gamma}{\delta}$. ■

We note that in 2008 Favaron *et al.* [8] proved that $\gamma_{k+1}(G) \leq \frac{n+\gamma_k(G)}{2}$ for graphs having $\delta(G) \geq k+1$. Since $c_2(G) \leq \gamma_2(G)$, their result gives our bound in Theorem 16 for the case when $\delta(G) = 2$. However, for $\delta(G) \geq 3$, the bound of Theorem 16 is an improvement over the known bound.

Moreover, it is also worth noting that in 1985, Cockayne, Gamble and Shepherd [2] showed that $\gamma_2(G) \leq \frac{2}{3}n$ for every graph G of order n and minimum degree 2, while in 1993 Stracke and Volkmann [24] showed that $\gamma_2(G) \leq \frac{1}{2}n$ for every graph G with minimum degree at least three. Since $c_2(G) \leq \gamma_2(G)$ for every graph G, one can observe that the bound in Theorem 16 is better than that on $\gamma_2(G)$ when $\delta(G) = 2$ for all graphs G with $\gamma(G) \leq \frac{1}{3}n$, and when $\delta(G) = 3$ for all graphs G with $\gamma(G) \leq \frac{1}{4}n$.

5. Open Problems and Questions

1. Determine additional lower and upper bounds on the k-conversion number $c_k(G)$ for $k \geq 2$.

2. When is $c_2(G) \leq \gamma(G)$? For example, any two vertices at distance at most two apart in a triangular grid of any order (see Figure 5 for a 4 by 10 triangular

grid) forms a c_2-set of G, so $c_2(G) = 2$ and the difference $\gamma(G) - c_2(G)$ can be arbitrarily large.

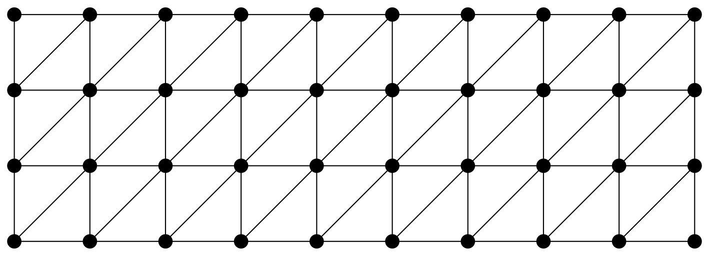

Figure 2. Triangular grid.

3. Determine Nordhaus-Gaddum type results for k-conversion numbers, that is, determine bounds on $c_k(G) + c_k(\overline{G})$ and on $c_k(G) \times c_k(\overline{G})$.

4. Determine $c_2(S(G))$, where $S(G)$ denotes the graph obtained from a graph G by subdividing every edge of G.

5. The *domatic number* $\mathrm{dom}(G)$ of a graph G is the maximum order p of a partition $\pi = \{V_1, V_2, \ldots, V_p\}$ of $V(G)$ into dominating sets. One can define the *c_2-domatic number* $c_2\mathrm{dom}(G)$ to equal the maximum order p of a partition $\pi = \{V_1, V_2, \ldots, V_p\}$ into 2-conversion sets. Investigate $c_2\mathrm{dom}(G)$.

6. In the introduction we defined the k-threshold conversion process as follows. Begin with a set S_0 of vertices, which are said to be influenced. For a fixed integer k and for each step $j \geq 1$, S_j is obtained from S_{j-1} by adding the vertices that have at least k neighbors in S_{j-1}. A set $S_0 \subseteq V$ is a *k-conversion set* of a graph $G = (V, E)$ if $S_t = V$ for some $t \geq 0$. Consider the number of steps t (also called the number of *rounds* in the literature) until $S_t = V$. Among all c_k-sets of a graph G define $tc_k(G)$ and $TC_k(G)$ to equal minimum and maximum number of steps/rounds t until $S_t = V$. Study these minimum and maximum k-conversion times for a graph G. The value $tc_k(G)$ has been studied using different notation in the literature, but as far as we know $TC_k(G)$ has not.

REFERENCES

[1] X. Baogen, E.J. Cockayne, T.W. Haynes, S.T. Hedetniemi and Z. Shangchao, *Extremal graphs for inequalities involving domination parameters*, Discrete Math. **216** (2000) 1–10.
https://doi.org/10.1016/S0012-365X(99)00251-4

[2] E.J. Cockayne, B. Gamble and B. Shepherd, *An upper bound for the k-domination number of a graph*, J. Graph Theory **9** (1985) 533–534.
https://doi.org/10.1002/jgt.3190090414

[3] N. Chen, *On the approximability of influence in social networks*, SIAM J. Discrete Math. **23** (2009) 1400–1415.
https://doi.org/10.1137/08073617X

[4] C.-Y. Chiang, L.-H. Huang and H.-G. Yeh, *Target set selection problem for honeycomb networks*, SIAM J. Discrete Math. **27** (2013) 310–328.
https://doi.org/10.1137/120868864

[5] C.-Y. Chiang, L.-H. Huang, B.-J. Li, J. Wu and H.-G. Yeh, *Some results on the target set selection problem*, J. Comb. Optim. **25** (2013) 702–715.
https://doi.org/10.1007/s10878-012-9518-3

[6] P.A. Dreyer, Applications and Variations of Domination in Graphs, PhD Thesis (Rutgers University, Piscataway, NJ, 2000).

[7] P.A. Dreyer Jr. and F.S. Roberts, *Irreversible k-threshold processes: Graph-theoretical threshold models of the spread of disease and of opinion*, Discrete Appl. Math. **157** (2009) 1615–1627.
https://doi.org/10.1016/j.dam.2008.09.012

[8] O. Favaron, A. Hansberg and L. Volkmann, *On k-domination and minimum degree in graphs*, J. Graph Theory **57** (2008) 33–40.
https://doi.org/10.1002/jgt.20279

[9] M.A. Fazli, M. Ghodsi, J. Habibi, P. Jalaly, V. Mirrokni and S. Sadeghian, *On non-progressive spread of influence through social networks*, Theoret. Comput. Sci. **550** (2014) 36–50.
https://doi.org/10.1016/j.tcs.2014.07.009

[10] J.F. Fink and M.S. Jacobson, *n-domination in graphs*, in: Graph Theory with Applications to Algorithms and Computer Science, Y. Alavi and A.J. Schwenk (Ed(s)) (Wiley, New York, 1985) 282–300.

[11] J.F. Fink and M.S. Jacobson, *On n-domination, n-dependence and forbidden subgraphs*, in: Graph Theory with Applications to Algorithms and Computer Science, Y. Alavi and A.J. Schwenk (Ed(s)) (Wiley, New York, 1985) 301–311.

[12] J.F. Fink, M.S. Jacobson, L.F. Kinch and J. Roberts, *On graphs having domination number half their order*, Period. Math. Hungar. **16** (1985) 287–293.
https://doi.org/10.1007/BF01848079

[13] A. Hansberg and L. Volkmann, *Multiple domination*, in: Topics in Domination in Graphs, T.W. Haynes, S.T. Hedetniemi and M.A. Henning (Ed(s)), Dev. Math. **64** (Springer, Cham, 2020) 151–203.
https://doi.org/10.1007/978-3-030-51117-3_6

[14] H. Jiang and H. Wan, *Target set selection in Cartesian product graphs*, Ars Combin. **147** (2019) 361–373.

[15] D. Kempe, J. Kleinberg and É. Tardos, *Maximizing the spread of influence through a social network*, in: Proc. 9th ACM SIGKDD Int. Conf. (Washington, D.C., 2003) 137–146.
https://doi.org/10.1145/956750.956769

[16] D. Kempe, J. Kleinberg and É. Tardos, *Influential nodes in a diffusion model for social networks*, in: Proc. 32nd Int. Colloq. Automata, Languages, and Programming (ICALP 2005), L. Caires, G.F. Italiano, L. Mateiro, C. Palamidessi and M. Yung (Ed(s)), Lecture Notes in Comput. Sci. **3580** (Springer, Berlin, Heidelberg, 2005) 1127–1138.
https://doi.org/10.1007/11523468_91

[17] W. McCuaig and B. Shepherd, *Domination in graphs with minimum degree two*, J. Graph Theory **13** (1989) 749–762.
https://doi.org/10.1002/jgt.3190130610

[18] C.M. Mynhardt and J.L. Wodlinger, *The k-conversion number of regular graphs*, AKCE Int. J. Graphs Comb. **17** (2020) 955–965.
https://doi.org/10.1016/j.akcej.2019.12.016

[19] O. Ore, Theory of Graphs, Amer. Math. Soc. Colloq. Publ. **38** (Providence, RI, 1962).
https://doi.org/10.1090/coll/038

[20] C. Payan and N.H. Xuong, *Domination-balanced graphs*, J. Graph Theory **6** (1982) 23–32.
https://doi.org/10.1002/jgt.3190060104

[21] B. Randerath and L. Volkmann, *Characterization of graphs with equal domination and covering number*, Discrete Math. **191** (1998) 159–169.
https://doi.org/10.1016/S0012-365X(98)00103-4

[22] B.A. Reed, *Paths, stars and the number three*, Combin. Probab. Comput. **5** (1996) 277–295.
https://doi.org/10.1017/S0963548300002042

[23] F.S. Roberts, *Graph-theoretical problems arising from defending against bioterrorism and controlling the spread of fires*, in: Proc. DIMACS/DIMATIA/Renyi Combinatorial Challenges Conf. (Piscataway, NJ, 2006).

[24] C. Stracke and L. Volkmann, *A new domination conception*, J. Graph Theory **17** (1993) 315–323.
https://doi.org/10.1002/jgt.3190170306

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