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# SOME RESULTS ON THE GLOBAL TRIPLE ROMAN DOMINATION IN GRAPHS

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**Abstract**

A triple Roman dominating function (TRDF) on a graph $G$ with vertex set $V$ is a function $f : V \to \{0, 1, 2, 3, 4\}$ such that for any vertex $v \in V$ with $f(v) < 3$, $\sum_{x \in N(v) \cup \{v\}} f(x) \geq |\{x \in N(v) : f(x) \geq 1\}| + 3$, where $N(v)$ is the open neighborhood of $v$. The weight of a TRDF $f$ is the value $\sum_{v \in V} f(v)$. A global triple Roman dominating function (GTRDF) on $G$ is a TRDF on both $G$ and its complement. The minimum weight of a GTRDF on $G$ is called the global triple Roman domination number $\gamma_{g[3R]}(G)$ of $G$. We first show that for any tree $T$ on $n \geq 5$ vertices, $\gamma_{g[3R]}(T) \leq 7n/4$ and characterize all extremal trees. We also show that for any graph $G$ on $n$ vertices, $\gamma_{g[3R]}(G) \neq 3n - 3$, and further characterize all graphs $G$ with $\gamma_{g[3R]}(G) = 3n - k$ for each $k \in \{4, 5, 6, 7\}$, which improves the results given by Nahani Pour *et al.* (2022).

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# 1. Introduction

In this paper, $G$ is a simple graph with vertex set $V(G)$ and edge set $E(G)$. For $v \in V(G)$, the *open neighborhood* of $v$ is the set $N(v) = N_G(v) = \{u \in V(G) : uv \in E(G)\}$ and its *closed neighborhood* is the set $N[v] = N_G[v] = N(v) \cup \{v\}$. We denote the *degree* of a vertex $v$ in $G$ by $d(v) = d_G(v) = |N(v)|$. The minimum degree among all vertices of $G$ is denoted by $\delta$. For $u, v \in V(G)$, the length of a shortest $(u,v)$-path in $G$ is the *distance* $d(u,v)$ between $u$ and $v$. The *diameter* $\mathrm{diam}(G)$ of $G$ is the maximum distance among all pairs of vertices. A shortest path whose length equals $\mathrm{diam}(G)$ is called a *diametral path* of $G$.

We write $G[S]$ for the subgraph induced by a subset $S$ of $V(G)$. A *clique* of a graph $G$ is a complete subgraph of $G$. The maximum order of a clique of $G$ is called the *clique number* of $G$ and denoted by $\omega(G)$. A vertex of degree one is called a *leaf* and a vertex adjacent to (exactly) one leaf is called a (*weak*) *support vertex*. For $r, s \geq 1$, a *double star* $S(r,s)$ is a tree with exactly two adjacent vertices that are not leaves, one of which is adjacent to $r$ leaves and the other is adjacent to $s$ leaves. As usual, the path, cycle and complete graph with $n$ vertices are denoted by $P_n$, $C_n$ and $K_n$, respectively. We denote by $K_{r,s}$ the complete bipartite graph having partite sets of cardinality $r$ and $s$.

We denote by $G - e$ the graph obtained from $G$ by deleting one edge $e$. The *complement* of a graph $G$ is the graph $\overline{G}$, where $V(\overline{G}) = V(G)$ and $uv \in E(\overline{G})$ if and only if $uv \notin E(G)$. The *union* $H_1 \cup H_2$ of two graphs $H_1$ and $H_2$ is the graph with vertex set $V(H_1) \cup V(H_2)$ and edge set $E(H_1) \cup E(H_2)$. We denote by $kG$ a disjoint union of $k$ copies of a graph $G$. The *weight* of a real-valued function $h : V(G) \to \mathbb{R}$ is the value $\omega(h) = \sum_{x \in V(G)} h(x)$.

A function $f : V(G) \to \{0,1,2\}$ is a *Roman dominating function* (RDF) on a graph $G$ if any vertex assigned 0 under $f$ is adjacent to at least one vertex assigned 2. The minimum weight of an RDF on $G$ is called the *Roman domination number* of $G$. The literature on Roman domination and its variations have been surveyed and detailed in two book chapters and three surveys [4–8].

Beeler *et al.* [3] introduced a stronger version of Roman domination, namely, double Roman domination. A function $f : V(G) \to \{0,1,2,3\}$ is a *double Roman dominating function* (DRDF) on a graph $G$ if any vertex assigned 0 under $f$ is adjacent to at least one vertex assigned 3 or two vertices assigned 2, and any vertex assigned 1 under $f$ is adjacent to at least one vertex assigned at least 2. The minimum weight of a DRDF on $G$ is called the *double Roman domination number* of $G$. The double Roman domination, with its many variations, is now well studied [2, 9, 11–13, 15, 17–19].

Recently, Abdollahzadeh Ahangar *et al.* [1] proposed a generalization of the Roman domination and double Roman domination, namely, $[k]$-Roman domination. Let $k$ be a positive integer. A $[k]$-*Roman dominating function* ($[k]$-RDF) on a graph $G$ is a function $f : V(G) \to \{0, 1, \ldots, k + 1\}$ such that for any vertex $v \in V(G)$ with $f(v) < k$, $\sum_{x \in AN(v) \cup \{v\}} f(x) \geq |AN(v)| + k$, where $AN(v) = \{x \in N(v) : f(x) \geq 1\}$ is the *active neighbourhood* of a vertex $v$ in $G$. The $[k]$-*Roman domination number* of a graph $G$ equals the minimum weight of a $[k]$-RDF on $G$. It is worth pointing out that [1]-Roman domination is the Roman domination and [2]-Roman domination is the double Roman domination.

In [1], [3]-Roman domination was also called triple Roman domination, which can be stated in the following equivalent but more explicit form. A *triple Roman dominating function* (TRDF) on a graph $G$ is a function $f : V(G) \to \{0, 1, 2, 3, 4\}$ such that

(1) any vertex assigned 0 under $f$ must be adjacent to one vertex assigned 4, or two vertices assigned 3, or one vertex assigned 2 and one vertex assigned 3, or three vertices assigned 2;

(2) any vertex assigned 1 under $f$ must be adjacent to one vertex assigned at least 3, or two vertices assigned 2;

(3) Any vertex assigned 2 under $f$ must be adjacent to one vertex assigned at least 2.

The minimum weight of a TRDF on a graph $G$ is called the *triple Roman domination number* of $G$, denoted by $\gamma_{[3R]}(G)$. Triple Roman domination has been studied by several authors (see for instance [10, 16]).

A *global triple Roman dominating function* (GTRDF) on a graph $G$ is a TRDF on both $G$ and its complement $\overline{G}$. The *global triple Roman domination number* $\gamma_{g[3R]}(G)$ of $G$ is the minimum weight of a GTRDF on $G$. A GTRDF on $G$ with weight $\gamma_{g[3R]}(G)$ is called a $\gamma_{g[3R]}(G)$-*function*. For a sake of simplicity, any $\gamma_{g[3R]}(G)$-function $f$ will be represented by the ordered partition $(V_0^f, V_1^f, \ldots, V_4^f)$ of $V(G)$ induced by $f$, where $V_i^f = \{v \in V(G) : f(v) = i\}$ for $i \in \{0, 1, \ldots, 4\}$. Nahani Pour *et al.* [14] introduced the global triple Roman domination and derived some results as follows.

**Proposition A** [14]**.** *For any connected graph $G$ on $n \geq 3$ vertices, $\gamma_{g[3R]}(G) \leq 3n$ with equality if and only if $G = K_n$.*

**Proposition B** [14]**.** *There are no connected graphs $G$ on $n \geq 3$ vertices such that $\gamma_{g[3R]}(G) = 3n - 1$.*

**Proposition C** [14]**.** *For any connected graph $G$ on $n \geq 3$ vertices, $\gamma_{g[3R]}(G) = 3n - 2$ if and only if $G = K_n - e$.*

Our purpose in this paper is to continue the study of the global triple Roman domination in graphs. We give an upper bound on this domination parameter for trees and characterize the extremal trees attaining this bound. Moreover, we improve the results presented in [14] and prove that for any graph $G$ on $n$ vertices, $\gamma_{g[3R]}(G) \neq 3n-3$ and characterize all graphs $G$ with $\gamma_{g[3R]}(G) \in \{3n-4, 3n-5, 3n-6, 3n-7\}$.

## 2. An Upper Bound for Trees

In this section we present an upper bound on the global triple Roman domination number of trees and characterize all extremal trees. For this purpose, we first give some needed results.

**Observation 1.** *Let $T$ be the tree obtained from a disjoint union of two trees $T_1$ and $T_2$ by joining exactly one vertex of $T_1$ to exactly one vertex of $T_2$. Then $\gamma_{g[3R]}(T) \leq \gamma_{g[3R]}(T_1) + \gamma_{g[3R]}(T_2)$.*

***Proof.*** For each $i \in \{1, 2\}$, let $g_i$ be a $\gamma_{g[3R]}(T_i)$-function. One can check that the function $h$ defined by $h(x) = g_1(x)$ for each $x \in V(T_1)$ and $h(x) = g_2(x)$ for each $x \in V(T_2)$, is a GTRDF on $T$ and so $\gamma_{g[3R]}(T) \leq \omega(h) = \omega(g_1) + \omega(g_2) = \gamma_{g[3R]}(T_1) + \gamma_{g[3R]}(T_2)$, as desired. ∎

If at least one of $T_1$ and $T_2$ is a path on three or four vertices, then the upper bound of Observation 1 can be improved slightly.

**Observation 2.** *Let $T$ be the tree obtained from a tree $T'$ on at least two vertices by adding a path $P_3$ and joining exactly one vertex of $T'$ to exactly one leaf of $P_3$. Then $\gamma_{g[3R]}(T) \leq \gamma_{g[3R]}(T') + 4$.*

***Proof.*** Let $u$ be a vertex of $T'$, $P_3 = u_1u_2u_3$ and let $uu_1 \in E(T)$. Now let $g$ be a $\gamma_{g[3R]}(T')$-function and let $V_i = \{x \in V(T')\backslash\{u\} : g(x) = i\}$ for each $i \in \{0, 1, 2, 3, 4\}$. If $|V_2| \geq 3$, or if $|V_2| \geq 1$ and $|V_3| \geq 1$, or if $|V_3| \geq 2$, or if $|V_4| \geq 1$, then the function $h$ defined by $h(u_1) = h(u_3) = 0$, $h(u_2) = 4$ and $h(x) = g(x)$ for each $x \in V(T')$, is a GTRDF on $T$ and so $\gamma_{g[3R]}(T) \leq \omega(h) = \omega(g) + 4 = \gamma_{g[3R]}(T') + 4$, as desired. Hence we may assume that $V_4 = \emptyset$ and one of the following holds.

(1) $|V_2| \in \{0, 1, 2\}$ and $|V_3| = 0$.

(2) $|V_2| = 0$ and $|V_3| = 1$.

First, suppose that (1) holds. Note that $|V(T')\backslash\{u\}| \geq 1$ and $V_3 = V_4 = \emptyset$. Thus if $|V_2| \in \{0, 1\}$, then it is a contradiction to our assumption that $g$ is a $\gamma_{g[3R]}(T')$-function. This forces $|V_2| = 2$ and hence we conclude from the definition of $\gamma_{g[3R]}(T')$-function that $g(u) \geq 3$. Then the function $f$ defined by $f(u_1) = 0$, $f(u_2) = f(u_3) = 2$ and $f(x) = g(x)$ for each $x \in V(T')$, is a GTRDF on $T$, implying that $\gamma_{g[3R]}(T) \leq \omega(f) = \omega(g) + 4 = \gamma_{g[3R]}(T') + 4$, as desired.

Second, suppose that (2) holds. It is clear that $V(T')\setminus\{u\} = V_0 \cup V_1 \cup V_3$. Moreover, since $|V_3| = 1$, it follows from the definition of $\gamma_{g[3R]}(T')$-function that $g(u) \geq 3$ and hence the function $f$ defined earlier is a GTRDF on $T$, implying that $\gamma_{g[3R]}(T) \leq \omega(f) = \omega(g) + 4 = \gamma_{g[3R]}(T') + 4$, as desired. ■

**Observation 3.** *Let $T$ be the tree obtained from a tree $T'$ by adding a path $P_4$ and joining exactly one vertex of $T'$ to exactly one vertex of $P_4$. Then $\gamma_{g[3R]}(T) \leq \gamma_{g[3R]}(T') + 7$.*

***Proof.*** Let $u$ be a vertex of $T'$ and let $P_4 = u_1u_2u_3u_4$. Without loss of generality, assume that $uu_i \in E(T)$ for some $i \in \{1, 2\}$. Let $g$ be a $\gamma_{g[3R]}(T')$-function and for each $i \in \{2, 3, 4\}$, let $V_i = \{x \in V(T') : g(x) = i\}$. If $|V_4| = |V_3| = 0$ and $|V_2| \leq 1$, then it is a contradiction to our assumption that $g$ is a $\gamma_{g[3R]}(T')$-function. Hence we may assume that $|V_4| \geq 1$, or $|V_3| \geq 1$, or $|V_2| \geq 2$. Then it is easy to check that the function $h$ defined by $h(u_1) = h(u_2) = 2$, $h(u_3) = 0$, $h(u_4) = 3$ and $h(x) = g(x)$ for each $x \in V(T')$, is a GTRDF on $T$ and so $\gamma_{g[3R]}(T) \leq \omega(h) = \omega(g) + 7 = \gamma_{g[3R]}(T') + 7$, as desired. ■

For the global triple Roman domination number of paths, Nahani Pour *et al.* [14] showed the following result.

**Proposition 4** [14]. *For $n \geq 6$,*

$$\gamma_{g[3R]}(P_n) = \begin{cases} 4\lfloor n/3 \rfloor, & \textit{if } n \equiv 0 \pmod 3, \\ 4\lfloor n/3 \rfloor + 3, & \textit{if } n \equiv 1 \pmod 3, \\ 4\lfloor n/3 \rfloor + 4, & \textit{if } n \equiv 2 \pmod 3. \end{cases}$$

**Theorem 5.** *For any tree $T$ on $n \geq 5$ vertices, $\gamma_{g[3R]}(T) \leq 7n/4$.*

***Proof.*** We proceed by induction on $n$. Let $P = u_0u_1 \cdots u_{\mathrm{diam}(T)}$ be a diametral path of $T$. If $\mathrm{diam}(T) = i \in \{2, 3\}$, then the function $h$ defined by $h(u_{i-2}) = h(u_{i-1}) = 4$ and $h(x) = 0$ otherwise, is a GTRDF on $T$ and hence $\gamma_{g[3R]}(T) = 8 < 7n/4$. Hence we may assume that $\mathrm{diam}(T) \geq 4$. If $n = 5$, then $T = P_5$ and hence the function $g$ defined by $g(u_1) = g(u_4) = 4$ and $g(u_0) = g(u_2) = g(u_3) = 0$, is a GTRDF on $P_5$, implying that $\gamma_{g[3R]}(P_5) \leq 8 < 7n/4$. Assume, then, that $n \geq 6$ and that for any tree $T'$ with $5 \leq |V(T')| < n$, $\gamma_{g[3R]}(T') \leq 7|V(T')|/4$. Let $T$ be a tree on $n \geq 6$ vertices with $\mathrm{diam}(T) \geq 4$.

If $T = P$, that is, if $T$ is a path on $n$ vertices, then by Proposition 4, we have $\gamma_{g[3R]}(T) < 7n/4$. So in the following we may assume that $T \neq P$. This forces that there must exist some vertex of $P = u_0u_1 \cdots u_{\mathrm{diam}(T)}$ with degree at least three. Let $u_t$ be the first vertex of $P$ with degree at least three. Without loss of generality, we choose the path $P$ satisfying that $t$ is as small as possible. According to the values of $t$, we now consider the following four cases.

*Case* 1. $t = 1$. Let $T'$ be the connected component of $T - u_1u_2$ that contains $u_2$. Since $\mathrm{diam}(T) \geq 4$, we have $|V(T')| \geq 3$. If $|V(T')| = 3$, then $\mathrm{diam}(T) = 4$ and so the function $h$ defined by $h(u_1) = h(u_4) = 4$ and $h(x) = 0$ otherwise, is a GTRDF on $T$, implying that

$$\gamma_{g[3R]}(T) \leq \omega(h) = 8 < 7n/4.$$

Now let $|V(T')| = 4$. Moreover, since $\mathrm{diam}(T) \geq 4$, it is clear that $\mathrm{diam}(T) \in \{4, 5\}$. If $\mathrm{diam}(T) = 4$, then either $T' = K_{1,3}$ or $T' = uu_2u_3u_4$ is a path on four vertices (where $u$ is the unique leaf adjacent to $u_2$ in $T$), and hence the function $h$ defined by $h(u_1) = h(u_2) = h(u_3) = 4$ and $h(x) = 0$ otherwise, is a GTRDF on $T$; and if $\mathrm{diam}(T) = 5$, then $T' = u_2u_3u_4u_5$ is a path on four vertices and hence the function $h$ defined by $h(u_1) = h(u_2) = h(u_5) = 4$ and $h(x) = 0$ otherwise, is a GTRDF on $T$. In either case, we have

$$\gamma_{g[3R]}(T) \leq \omega(h) = 12 < 7n/4.$$

Suppose next that $|V(T')| \geq 5$. Let $g = (V_0^g, V_1^g, \ldots, V_4^g)$ be a $\gamma_{g[3R]}(T')$-function. By considering the case when $V_0^g \neq \emptyset$, (respectively, $V_0^g = \emptyset$ and $V_1^g \neq \emptyset$, $V_0^g = V_1^g = \emptyset$) it follows from the definition of $\gamma_{g[3R]}(T')$-function that one of the following holds. (1) $|V_4^g| \geq 1$; (2) $|V_3^g| \geq 1$ and $|V_3^g| + |V_2^g| \geq 2$; (3) $|V_2^g| \geq 3$. Thus one can check that the function $h$ defined by $h(u_1) = 4$, $h(x) = 0$ for each $x \in N(u_1)\backslash\{u_2\}$ and $h(x) = g(x)$ for each $x \in V(T')$, is a GTRDF on $T$ and so by the induction hypothesis, we have

$$\gamma_{g[3R]}(T) \leq \omega(h) = \omega(g) + 4 \leq 7|V(T')|/4 + 4 \leq 7(n-3)/4 + 4 < 7n/4.$$

In the following, we may assume that $t \geq 2$. Before going further, we let $T_1$ (respectively, $T_2$) be the connected component of $T - u_2u_3$ that contains $u_2$ (respectively, $u_3$). Since $\mathrm{diam}(T) \geq 4$, we have $|V(T_2)| \geq 2$. Moreover, it follows from the choice of the diametral path $P$ that $d(u_{\mathrm{diam}(T)-1}) = 2$.

*Case* 2. $t = 2$ and $|V(T_1)| = 4$. In this case, $u_2$ is a weak support vertex of $T$ and $T_1$ is a path on four vertices. Note that $|V(T_2)| \geq 2$. If $|V(T_2)| = 2$, then $\mathrm{diam}(T) = 4$ and $T_2 = u_3u_4$ is a path on two vertices, implying that the function $h_1$ defined by $h_1(u_0) = h_1(u_4) = 3$, $h_1(u_1) = h_1(u_3) = 0$ and $h_1(x) = 2$ otherwise, is a GTRDF on $T$ and hence

$$\gamma_{g[3R]}(T) \leq \omega(h_1) = 10 < 7n/4.$$

If $|V(T_2)| = 3$, then since $d(u_{\mathrm{diam}(T)-1}) = 2$, we have that $\mathrm{diam}(T) = 5$ and $T_2 = u_3u_4u_5$ is a path on three vertices, implying that the function $h_2$ defined by $h_2(u_1) = h_2(u_2) = h_2(u_5) = 4$ and $h_2(x) = 0$ otherwise, is a GTRDF on $T$ and hence

$$\gamma_{g[3R]}(T) \leq \omega(h_2) = 12 < 7n/4.$$

If $|V(T_2)| \geq 5$, then by Observation 3 and the induction hypothesis, we have

$$\gamma_{g[3R]}(T) \leq \gamma_{g[3R]}(T_2) + 7 \leq 7|V(T_2)|/4 + 7 = 7(n-4)/4 + 7 = 7n/4.$$

Now we consider the last case that $|V(T_2)| = 4$. Since $d(u_{\mathrm{diam}(T)-1}) = 2$, it is clear that either $\mathrm{diam}(T) = 6$ and $T_2 = u_3u_4u_5u_6$ is a path on four vertices or $\mathrm{diam}(T) = 5$ and $T_2 = uu_3u_4u_5$ is a path on four vertices (where $u$ is the unique leaf adjacent to $u_3$ in $T$). If the former holds, then the function $h_2$ defined earlier is a GTRDF on $T$, implying that

$$\gamma_{g[3R]}(T) \leq \omega(h_2) = 12 < 7n/4.$$

If the latter holds, then the function $h_3$ defined by $h_3(u_0) = h_3(u_5) = 3$, $h_3(u_2) = h_3(u_3) = 4$ and $h_3(x) = 0$ otherwise, is a GTRDF on $T$, implying that

$$\gamma_{g[3R]}(T) \leq \omega(h_3) = 14 = 7n/4.$$

*Case* 3. $t = 2$ and $|V(T_1)| \geq 5$. Since $|V(T_1)| \geq 5$, we conclude from the induction hypothesis that

$$\gamma_{g[3R]}(T_1) \leq 7|V(T_1)|/4. \tag{1}$$

Note that $|V(T_2)| \geq 2$. According to the values of $|V(T_2)|$, we distinguish the following two subcases.

*Subcase* 3.1. $|V(T_2)| \in \{2, 3\}$. If $|V(T_2)| = 3$, then since $d(u_{\mathrm{diam}(T)-1}) = 2$, we have that $\mathrm{diam}(T) = 5$ and $T_2 = u_3u_4u_5$ is a path on three vertices and hence by Observation 2 and (1),

$$\gamma_{g[3R]}(T) \leq \gamma_{g[3R]}(T_1) + 4 \leq 7|V(T_1)|/4 + 4 = 7(n-3)/4 + 4 < 7n/4.$$

Now we assume that $|V(T_2)| = 2$. This forces that $\mathrm{diam}(T) = 4$ and $T_2 = u_3u_4$ is a path on two vertices. If every vertex of $N(u_2)\backslash\{u_1, u_3\}$ is a leaf, then the function $h$ defined by $h(u_1) = h(u_2) = 4$, $h(u_4) = 3$ and $h(x) = 0$ otherwise, is a GTRDF on $T$ and hence

$$\gamma_{g[3R]}(T) \leq \omega(h) = 11 < 7n/4.$$

So in the following we may assume that $N(u_2)\backslash\{u_1, u_3\}$ has a support vertex. Moreover, since $t = 2$, it follows from the choice of the diametral path $P$ that every support vertex in $N(u_2)\backslash\{u_1, u_3\}$ is a weak support vertex. Thus if $u_2$ is a support vertex, then the function $h$ defined by $h(u_2) = 4$, $h(x) = 0$ for each $x \in N(u_2)$ and $h(x) = 3$ otherwise, is a GTRDF on $T$ and so

$$\gamma_{g[3R]}(T) \leq \omega(h) \leq 3(n-2)/2 + 4 < 7n/4;$$

and if $u_2$ is not a support vertex, then every vertex in $N(u_2)$ is a weak support vertex and so the function $h$ defined by $h(x) = 0$ for each $x \in N(u_2)$ and $h(x) = 3$ otherwise, is a GTRDF on $T$, implying that

$$\gamma_{g[3R]}(T) \leq \omega(h) = 3(n-1)/2 + 3 < 7n/4.$$

*Subcase* 3.2. $|V(T_2)| \geq 4$. If $|V(T_2)| = 4$, then since $d(u_{\mathrm{diam}(T)-1}) = 2$, we have that either $\mathrm{diam}(T) = 6$ and $T_2 = u_3u_4u_5u_6$ is a path on four vertices or $\mathrm{diam}(T) = 5$ and $T_2 = uu_3u_4u_5$ is a path on four vertices (where $u$ is the unique leaf adjacent to $u_3$ in $T$). In either case, it follows from Observation 3 and (1) that

$$\gamma_{g[3R]}(T) \leq \gamma_{g[3R]}(T_1) + 7 \leq 7|V(T_1)|/4 + 7 = 7(n-4)/4 + 7 = 7n/4.$$

If $|V(T_2)| \geq 5$, then by the induction hypothesis, $\gamma_{g[3R]}(T_2) \leq 7|V(T_2)|/4$ and hence by Observation 1 and (1),

$$\gamma_{g[3R]}(T) \leq \gamma_{g[3R]}(T_1) + \gamma_{g[3R]}(T_2) \leq 7|V(T_1)|/4 + 7|V(T_2)|/4 = 7n/4.$$

*Case* 4. $t \geq 3$. In this case, we conclude from the choice of the diametral path $P$ that $\mathrm{diam}(T) \geq 6$ and so $|V(T_2)| \geq 5$. Note that $T_1 = u_0u_1u_2$ is a path on three vertices. Then by Observation 2 and the induction hypothesis, we have

$$\gamma_{g[3R]}(T) \leq \gamma_{g[3R]}(T_2) + 4 \leq 7|V(T_2)|/4 + 4 = 7(n-3)/4 + 4 < 7n/4.$$

This completes the proof. ∎

Next, we shall characterize the trees attaining the upper bound of Theorem 5. In order to state the characterization, let $\mathcal{T}$ be the family of trees obtained from the disjoint union of $l \geq 2$ paths $P_4^i = u_1^iu_2^iu_3^iu_4^i$ $(1 \leq i \leq l)$ by adding $l-1$ edges incident with $u_2^i$'s such that the resulting graph is a tree (for $l = 5$, a tree in the family $\mathcal{T}$ is shown in Figure 1).

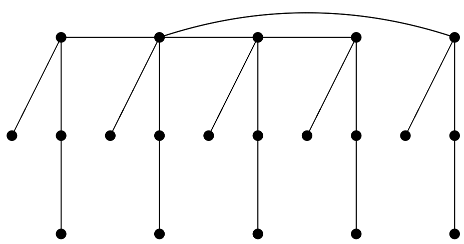

Figure 1. A tree in the family $\mathcal{T}$.

**Proposition 6.** *For any tree $T \in \mathcal{T}$ on $n$ vertices, $\gamma_{g[3R]}(T) = 7n/4$.*

***Proof.*** Let $T \in \mathcal{T}$ be the tree obtained as described above and let $g$ be a $\gamma_{g[3R]}(T)$-function. By the definition of $\gamma_{g[3R]}(T)$-function, it is easy to check that $g(u_1^i) + g(u_2^i) \geq 3$ and $g(u_3^i) + g(u_4^i) \geq 3$ for each $i \in \{1, 2, \ldots, l\}$. Moreover, it is not possible that there exists some $i \in \{1, 2, \ldots, l\}$ such that $g(u_1^i) + g(u_2^i) = g(u_3^i) + g(u_4^i) = 3$ (for otherwise, $g(u_1^i) = g(u_4^i) = 3$ and $g(u_2^i) = g(u_3^i) = 0$, a contradiction to our assumption that $g$ is a $\gamma_{g[3R]}(T)$-function). Thus we have that for each $i \in \{1, 2, \ldots, l\}$, $g(u_1^i) + g(u_2^i) \geq 4$ or $g(u_3^i) + g(u_4^i) \geq 4$, implying that $g(u_1^i) + g(u_2^i) + g(u_3^i) + g(u_4^i) \geq 7$ and so $\gamma_{g[3R]}(T) = \omega(g) \geq 7n/4$. On the other hand, the function $h$ defined by $h(u_1^i) = h(u_3^i) = 0$, $h(u_2^i) = 4$ and $h(u_4^i) = 3$ for each $i \in \{1, 2, \ldots, l\}$, is a GTRDF on $T$ and so $\gamma_{g[3R]}(T) \leq \omega(h) = 7n/4$. As a result, we obtain $\gamma_{g[3R]}(T) = 7n/4$. ∎

**Theorem 7.** *For any tree $T$ on $n \geq 5$ vertices, $\gamma_{g[3R]}(T) = 7n/4$ if and only if $T \in \mathcal{T}$.*

***Proof.*** By Proposition 6, the sufficiency is trivial. To show the necessity, we demand to the proof of Theorem 5. Let $\gamma_{g[3R]}(T) = 7n/4$. Clearly $n \equiv 0 \pmod 4$. The proof is by induction on $n$. Now let $n = 8$. From the proof of Theorem 5, there is only one case, namely, a subcase of Case 2, where it is possible to achieve equality. Using the terminology from this proof, we have $t = 2$, $|V(T_1)| = |V(T_2)| = 4$, $\operatorname{diam}(T) = 5$ and $T_2 = uu_3u_4u_5$ is a path on four vertices (where $u$ is the unique leaf adjacent to $u_3$ in $T$). This forces $T \in \mathcal{T}$. Assume, then, that $n > 8$ and that for any tree $T'$ with $5 \leq |V(T')| < n$, if $\gamma_{g[3R]}(T') = 7|V(T')|/4$, then $T' \in \mathcal{T}$. Let $T$ be a tree on $n > 8$ vertices. From the proof of Theorem 5, there are only two cases, namely, a subcase of Case 2 and Subcase 3.2 of Case 3, where they are possible to achieve equality.

First, suppose that the tree $T$ satisfies the conditions of the subcase of Case 2. Using the terminology from the proof of Theorem 5, we have that $T$ is the tree satisfying that $t = 2$, $|V(T_1)| = 4$ and $|V(T_2)| \geq 5$. Note that that $u_2$ is a weak support vertex of degree 3 in $T$ and $T_1$ is a path on four vertices. Then by Observation 3 and Theorem 5, $7n/4 = \gamma_{g[3R]}(T) \leq \gamma_{g[3R]}(T_2) + 7 \leq 7|V(T_2)|/4 + 7 = 7(n-4)/4 + 7 = 7n/4$ and so we must have equality throughout this inequality chain. In particular, $\gamma_{g[3R]}(T_2) = 7|V(T_2)|/4$ and hence by the induction hypothesis, $T_2 \in \mathcal{T}$. Thus if $d_{T_2}(u_3) \geq 3$, then since $u_2$ is a weak support vertex of degree 3 in $T$ and $T_1$ is a path on four vertices, it is easy to check that $T \in \mathcal{T}$. So in the following we may assume that $d_{T_2}(u_3) \in \{1, 2\}$. Now let $T_1'$ (respectively, $T_2'$) be the connected component of $T - u_3u_4$ that contains $u_3$ (respectively, $u_4$). Since $T_2 \in \mathcal{T}$, we have $|V(T_2)| \geq 8$. If $d_{T_2}(u_3) = 1$, then $|V(T_1')| = |V(T_1)| + 1 = 5$ and $|V(T_2')| = |V(T_2)| - 1 \geq 7$ and if $d_{T_2}(u_3) = 2$, then $|V(T_1')| = |V(T_1)| + 2 = 6$ and $|V(T_2')| = |V(T_2)| - 2 \geq 6$. In either case, $T_i' \notin \mathcal{T}$

for $i \in \{1, 2\}$. Moreover, by Observation 1 and Theorem 5,

$$7n/4 = \gamma_{g[3R]}(T) \leq \gamma_{g[3R]}(T_1') + \gamma_{g[3R]}(T_2') \leq 7|V(T_1')|/4 + 7|V(T_2')|/4 = 7n/4$$

and so we must have equality throughout this inequality chain. In particular, $\gamma_{g[3R]}(T_i') = 7|V(T_i')|/4$ for each $i \in \{1, 2\}$ and hence by the induction hypothesis, we have $T_i' \in \mathcal{T}$, which is a contradiction to the fact that $T_i' \notin \mathcal{T}$.

Second, suppose that the tree $T$ satisfies the conditions of Subcase 3.2 of Case 3. Using the terminology from the proof of Theorem 5, we have that $T$ is the tree satisfying that $t = 2$, $|V(T_1)| \geq 5$ and $|V(T_2)| \geq 4$. By the choice of the diametral path $P$, we have $\mathrm{diam}(T_1) \in \{3, 4\}$ and hence $T_1 \notin \mathcal{T}$. Then by Theorem 5 and the induction hypothesis,

$$\gamma_{g[3R]}(T_1) < 7|V(T_1)|/4. \tag{2}$$

Recall that $d(u_{\mathrm{diam}(T)-1}) = 2$. Thus if $|V(T_2)| = 4$, then either $\mathrm{diam}(T) = 6$ and $T_2 = u_3u_4u_5u_6$ is a path on four vertices or $\mathrm{diam}(T) = 5$ and $T_2 = uu_3u_4u_5$ is a path on four vertices (where $u$ is the unique leaf adjacent to $u_3$ in $T$). In either case, it follows from Observation 3 and (2) that

$$\gamma_{g[3R]}(T) \leq \gamma_{g[3R]}(T_1) + 7 < 7|V(T_1)|/4 + 7 = 7(n-4)/4 + 7 = 7n/4,$$

a contradiction. If $|V(T_2)| \geq 5$, then by Theorem 5, $\gamma_{g[3R]}(T_2) \leq 7|V(T_2)|/4$ and hence by Observation 1 and (2),

$$\gamma_{g[3R]}(T) \leq \gamma_{g[3R]}(T_1) + \gamma_{g[3R]}(T_2) < 7|V(T_1)|/4 + 7|V(T_2)|/4 = 7n/4,$$

a contradiction. This completes the proof. ■

## 3. Graphs with Large Global Triple Roman Domination Number

In this section, we shall improve the results of Propositions A, B and C by characterizing graphs with large global triple Roman domination number.

**Lemma 8.** *For any graph $G$ on $n$ vertices with* $\mathrm{diam}(G) \geq 4$, $\gamma_{g[3R]}(G) \leq 3n - 7$ *with equality if and only if $G = P_5$.*

***Proof.*** Let $P = u_0u_1 \cdots u_{\mathrm{diam}(G)}$ be a diametral path of $G$. If $\mathrm{diam}(G) \geq 5$, then the function $h_1$ defined by $h_1(u_0) = h_1(u_2) = h_1(u_3) = h_1(u_5) = 0$, $h_1(u_1) = h_1(u_4) = 4$ and $h_1(x) = 3$ otherwise, is a GTRDF on $G$ and hence $\gamma_{g[3R]}(G) \leq \omega(h_1) = 3(n-6) + 8 = 3n - 10$. Next, assume that $\mathrm{diam}(G) = 4$.

If $d(u_4) \geq 2$ (the case $d(u_0) \geq 2$ is similar), then assume, without loss of generality, that $u_5 \in N(u_4) \backslash \{u_3\}$ and therefore the function $h_1$ defined earlier is a GTRDF on $G$, implying that $\gamma_{g[3R]}(G) \leq \omega(h_1) = 3n - 10$. If $d(u_3) \geq 3$ (the case $d(u_1) \geq 3$ is similar), then assume, without loss of generality, that $u_5 \in N(u_3)\backslash\{u_2, u_4\}$ and therefore the function $h_2$ defined by $h_2(u_1) = h_2(u_2) = h_2(u_4) = h_2(u_5) = 0$, $h_2(u_0) = h_2(u_3) = 4$ and $h_2(x) = 3$ otherwise, is a GTRDF on $G$, implying that $\gamma_{g[3R]}(G) \leq \omega(h_2) = 3(n-6) + 8 = 3n - 10$. Hence we may assume that $d(u_0) = d(u_4) = 1$ and $d(u_1) = d(u_3) = 2$. If $d(u_2) \geq 3$, then assume, without loss of generality, that $u_5 \in N(u_2)\backslash\{u_1, u_3\}$ and therefore the function $h_3$ defined by $h_3(u_1) = h_3(u_3) = 0$, $h_3(u_2) = h_3(u_5) = 2$ and $h_3(x) = 3$ otherwise, is a GTRDF on $G$, implying that $\gamma_{g[3R]}(G) \leq \omega(h_3) = 3(n-4)+4 = 3n-8$. Now let $d(u_2) = 2$. Clearly $G = P_5$. Let $h$ be a $\gamma_{g[3R]}(P_5)$-function. By the definition of $\gamma_{g[3R]}(P_5)$-function, one can check that $h(u_0) + h(u_1) \geq 3$ and $h(u_3) + h(u_4) \geq 3$. If at least one of $h(u_0)+h(u_1)$ and $h(u_3)+h(u_4)$, say $h(u_0)+h(u_1)$, equals 3, then it follows from the definition of $\gamma_{g[3R]}(P_5)$-function that $h(u_0) = 3$ and $h(u_1) = 0$, implying that $h(u_2) \geq 2$ and so $\gamma_{g[3R]}(P_5) = \omega(h) \geq 8$. If $h(u_0) + h(u_1) \geq 4$ and $h(u_3) + h(u_4) \geq 4$, then $\gamma_{g[3R]}(P_5) = \omega(h) \geq 8$. In either case, we have $\gamma_{g[3R]}(P_5) \geq 8$. On the other hand, it follows from the proof of Theorem 5 that $\gamma_{g[3R]}(P_5) \leq 8$. Thus $\gamma_{g[3R]}(P_5) = 8 = 3n - 7$. This completes the proof. ∎

**Proposition 9.** *For each $G \in \{P_4, S(1,2), H_1, H_2, H_3\}$ of order $n$,*

$$\gamma_{g[3R]}(G) = \begin{cases} 3n-4, & \text{if } G = P_4, \\ 3n-6, & \text{if } G = H_1, \\ 3n-7, & \text{if } G \in \{S(1,2), H_2, H_3\}, \end{cases}$$

*where $H_1$ and $H_i$ ($i \in \{2,3\}$) are illustrated in Figures 2 and 3, respectively.*

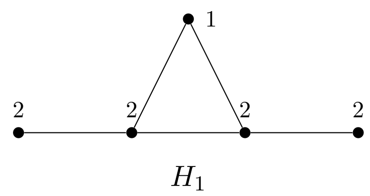


Figure 2. The graph $H_1$ with $\gamma_{g[3R]}(H_1) = 3n - 6$.


Figure 3. The graphs $H_2$ and $H_3$ with $\gamma_{g[3R]}(H_2) = \gamma_{g[3R]}(H_3) = 3n - 7$.

***Proof.*** For each $G \in \{P_4, S(1,2), H_1\}$, let $h = (V_0^h, V_1^h, \ldots, V_4^h)$ be a $\gamma_{g[3R]}(G)$-function. First, suppose that $G = P_4 = u_1u_2u_3u_4$. By the definition of $\gamma_{g[3R]}(G)$-function, we have $h(u_1) + h(u_2) \geq 3$ and $h(u_3) + h(u_4) \geq 3$. If at least one of $h(u_1)+h(u_2)$ and $h(u_3)+h(u_4)$, say $h(u_1)+h(u_2)$, equals 3, then clearly $h(u_1) = 3$ and $h(u_2) = 0$. This forces $h(u_3) \geq 2$ and $h(u_4) = 4$, implying that $\gamma_{g[3R]}(G) = \omega(h) \geq 9$. If $h(u_1)+h(u_2) \geq 4$ and $h(u_3)+h(u_4) \geq 4$, then $\gamma_{g[3R]}(G) = \omega(h) \geq 8$. In either case, we have $\gamma_{g[3R]}(G) \geq 8$. On the other hand, one can check that the function $g$ defined by $g(u_2) = g(u_3) = 4$ and $g(u_1) = g(u_4) = 0$, is a GTRDF on $G$ and hence $\gamma_{g[3R]}(G) \leq 8$. This forces $\gamma_{g[3R]}(G) = 8 = 3n - 4$.

Second, suppose that $G = S(1,2)$. If there exists some vertex $v \in V_0^h$, then by the definition of $\gamma_{g[3R]}(G)$-function, we have $\sum_{x\in N(v)} h(x) \geq 4$ and $\sum_{x\notin N[v]} h(x) \geq 4$, implying that $\gamma_{g[3R]}(G) = \omega(h) = h(v) + \sum_{x\in N(v)} h(x) + \sum_{x\notin N[v]} h(x) \geq 8$. Hence we may assume that $V_0^h = \emptyset$. If $V_1^h = \emptyset$, then $\gamma_{g[3R]}(G) = \omega(h) = 2|V_2^h| + 3|V_3^h| + 4|V_4^h| \geq 2(|V_2^h| + |V_3^h| + |V_4^h|) = 2|V(G)| = 10$. So in the following we may assume that there exists some vertex $v \in V_1^h$. Then by the definition of $\gamma_{g[3R]}(G)$-function, we have $\sum_{x\in N(v)} h(x) \geq 3$ and $\sum_{x\notin N[v]} h(x) \geq 3$. If at least one of $\sum_{x\in N(v)} h(x)$ and $\sum_{x\notin N[v]} h(x)$ equals at least 4, then $\gamma_{g[3R]}(G) = \omega(h) = h(v) + \sum_{x\in N(v)} h(x) + \sum_{x\notin N[v]} h(x) \geq 8$. Now let $\sum_{x\in N(v)} h(x) = \sum_{x\notin N[v]} h(x) = 3$. It follows from the definition of $\gamma_{g[3R]}(G)$-function that there exist two vertices $u \in N(v) \cap V_3^h$ and $w \in (V(G)\backslash N[v]) \cap V_3^h$. Moreover, since $V_0^h = \emptyset$, this forces that for each $x \in V(G)\backslash\{u, v, w\}$, $h(x) \geq 1$. Thus $\gamma_{g[3R]}(G) = \omega(h) \geq 9$. In the above cases, we have $\gamma_{g[3R]}(G) \geq 8$. On the other hand, one can check that the function $g$ defined by $g(x) = 4$ for each support vertex $x$ and $g(x) = 0$ otherwise, is a GTRDF on $G$ and hence $\gamma_{g[3R]}(G) \leq 8$. This forces $\gamma_{g[3R]}(G) = 8 = 3n - 7$.

Third, suppose that $G = H_1$, where $V(G) = \{u_1, u_2, \ldots, u_5\}$ and $E(G) = \{u_1u_2, u_2u_3, u_3u_4, u_2u_5, u_3u_5\}$. By the definition of $\gamma_{g[3R]}(G)$-function, we have $h(u_1) + h(u_2) \geq 3$ and $h(u_3) + h(u_4) \geq 3$. If at least one of $h(u_1) + h(u_2)$ and $h(u_3) + h(u_4)$, say $h(u_1) + h(u_2)$, equals 3, then clearly $h(u_1) = 3$ and $h(u_2) = 0$. This forces at least one of $h(u_3)$ and $h(u_5)$ equals at least 2 and $h(u_4) = 4$, implying that $\gamma_{g[3R]}(G) = \omega(h) \geq 9$. Suppose next that $h(u_1) + h(u_2) \geq 4$ and $h(u_3) + h(u_4) \geq 4$. If at least one of $h(u_1) + h(u_2)$ and $h(u_3) + h(u_4)$ equals at least 5, then $\gamma_{g[3R]}(G) = \omega(h) \geq 9$. Now let $h(u_1) + h(u_2) = h(u_3) + h(u_4) = 4$. If $h(u_5) \geq 1$, then $\gamma_{g[3R]}(G) = \omega(h) \geq 9$. Assume that $h(u_5) = 0$. It follows from the definition of $\gamma_{g[3R]}(G)$-function that $h(u_1)+h(u_4) \geq 4$ and $h(u_2)+h(u_3) \geq 4$. If $h(u_1) + h(u_4) = h(u_2) + h(u_3) = 4$, then since $h(u_5) = 0$, we have that one of $h(u_2)$ and $h(u_3)$ equals 4 and the other equals 0, say $h(u_2) = 4$ and $h(u_3) = 0$. This forces that $h(u_1) = 0$ and $h(u_4) = 4$, a contradiction to our assumption that $h$ is a $\gamma_{g[3R]}(G)$-function. Thus at least one of $h(u_1) + h(u_4)$ and $h(u_2) + h(u_3)$ equals at least 5, and hence $\gamma_{g[3R]}(G) = \omega(h) \geq 9$. In the above cases, we have $\gamma_{g[3R]}(G) \geq 9$. On the other hand, one can check that the function $g$ defined by $g(u_1) = g(u_2) = g(u_3) = g(u_4) = 2$ and $g(u_5) = 1$, is a GTRDF on $G$ and hence $\gamma_{g[3R]}(G) \leq \omega(g) = 9$. This forces $\gamma_{g[3R]}(G) = 9 = 3n - 6$. Using a similar argument we can obtain that if $G \in \{H_2, H_3\}$, then $\gamma_{g[3R]}(G) = 3n - 7$. ■

**Lemma 10.** *For any connected graph $G$ on $n$ vertices with* $\operatorname{diam}(G) = 3$,

$$\gamma_{g[3R]}(G) \leq 3n - 4.$$

*Furthermore, the following hold.*

(a) $\gamma_{g[3R]}(G) = 3n - 4$ *if and only if* $G = P_4$,

(b) $\gamma_{g[3R]}(G) \neq 3n - 5$,

(c) $\gamma_{g[3R]}(G) = 3n - 6$ *if and only if* $G = H_1$,

(d) $\gamma_{g[3R]}(G) = 3n - 7$ *if and only if* $G \in \{S(1,2), H_2, H_3\}$ *or* $\overline{G} \in \{S(1,2), H_2, H_3\}$,

*where $H_1$ and $H_i$ ($i \in \{2,3\}$) are illustrated in Figures* 2 *and* 3, *respectively.*

***Proof.*** If $n = 4$, that is, if $G = P_4$, then by Proposition 9, $\gamma_{g[3R]}(G) = 3n - 4$. Suppose next that $n \geq 5$. Let $P = u_1u_2u_3u_4$ be a diametral path of $G$. Without loss of generality, we may assume that $d(u_1) \leq d(u_4)$. If $d(u_1) \geq 2$ or $d(u_4) \geq 3$, then $d(u_1) + d(u_4) \geq 4$ and the function $h$ defined by $h(u_1) = h(u_4) = 4$, $h(x) = 0$ for each $x \in N(u_1) \cup N(u_4)$ and $h(x) = 3$ otherwise, is a GTRDF on $G$, implying that $\gamma_{g[3R]}(G) \leq \omega(h) = 3(n - d(u_1) - d(u_4) - 2) + 8 \leq 3n - 10$. Hence we may assume that $d(u_1) = 1$ and $d(u_4) \in \{1, 2\}$.

**Claim 1.** *If $d(u_1) = d(u_4) = 1$, then $\gamma_{g[3R]}(G) \leq 3n - 6$ with equality if and only if $G = H_1$. Moreover, $\gamma_{g[3R]}(G) = 3n - 7$ if and only if $G \in \{S(1,2), H_2\}$.*

***Proof.*** If $d(u_3) \geq 5$, then the function $h$ defined by $h(u_1) = h(u_3) = 4$, $h(x) = 0$ for each $x \in N(u_3) \backslash \{u_2\}$ and $h(x) = 3$ otherwise, is a GTRDF on $G$ and so $\gamma_{g[3R]}(G) \leq \omega(h) = 3(n - d(u_3) - 1) + 8 \leq 3n - 10$. Thus we may assume that $d(u_3) \leq 4$. Similarly we may assume that $d(u_2) \leq 4$. This forces $|N(u_2) \cap N(u_3)| \in \{0, 1, 2\}$. Consider the following three cases.

*Case* 1. $|N(u_2) \cap N(u_3)| = 0$. If $n = 5$, then clearly $G = S(1,2)$ and hence by Proposition 9, $\gamma_{g[3R]}(G) = 3n - 7$. Now let $n \geq 6$. Moreover, since $d(u_1) = d(u_4) = 1$, we have $d(u_2) + d(u_3) \geq 5$. Suppose now that $d(u_2) + d(u_3) = 5$. Without loss of generality, we may assume that $d(u_2) = 2$ and $d(u_3) = 3$. Let $N(u_3) \backslash \{u_2, u_4\} = \{v\}$. Since $n \geq 6$, this forces that there must exist some vertex, say $w$, in $N(v) \backslash \{u_3\}$. Note that $w \notin N(u_1) \cup N(u_2) \cup N(u_3) \cup N(u_4)$. This implies that $d(u_1, w) = 4 > \operatorname{diam}(G)$, a contradiction. Thus $d(u_2) + d(u_3) \geq 6$. Recall that $|N(u_2) \cap N(u_3)| = 0$. One can check that the function $h$ defined by $h(u_2) = h(u_3) = 4$, $h(x) = 0$ for each $x \in (N(u_2) \cup N(u_3)) \backslash \{u_2, u_3\}$ and $h(x) = 3$

otherwise, is a GTRDF on $G$ and so $\gamma_{g[3R]}(G) \leq \omega(h) = 3(n-d(u_2)-d(u_3))+8 \leq 3n-10$.

*Case* 2. $|N(u_2)\cap N(u_3)| = 1$. If $n = 5$, then $G = H_1$ and hence by Proposition 9, $\gamma_{g[3R]}(G) = 3n-6$. Now let $n \geq 6$ and let $N(u_2) \cap N(u_3) = \{w\}$. Without loss of generality, assume that $d(u_2) \geq d(u_3)$. Note that $d(u_3) \geq 3$. If $d(u_2) = d(u_3) = 3$, then since $n \geq 6$ and $d(u_1) = d(u_4) = 1$, there must exist some vertex, say $v$, in $V(G)\setminus\{u_1, u_2, u_3, u_4, w\}$ such that $v \in N(w)$ and so the function $h$ defined by $h(u_2) = h(u_3) = 0$, $h(v) = h(w) = 2$ and $h(x) = 3$ for each $x \in V(G)\setminus\{u_2, u_3, v, w\}$, is a GTRDF on $G$, implying that $\gamma_{g[3R]}(G) \leq \omega(h) = 3(n-4)+4 = 3n-8$. So in the following we may assume that $d(u_2) \geq 4$. Moreover, since $d(u_2) \leq 4$ by our earlier assumptions, this forces $d(u_2) = 4$. Further, we note that $d(u_3) \in \{3, 4\}$ since $d(u_3) \leq d(u_2)$.

If $d(u_3) = 4$, then since $N(u_2) \cap N(u_3) = \{w\}$, one can check that the function $h$ defined by $h(u_2) = h(u_3) = 4$, $h(x) = 0$ for each $x \in (N(u_2) \cup N(u_3))\setminus\{u_2, u_3, w\}$ and $h(x) = 3$ otherwise, is a GTRDF on $G$ and so $\gamma_{g[3R]}(G) \leq \omega(h) = 3(n-6)+8 = 3n-10$. Suppose next that $d(u_3) = 3$. Let $N(u_2)\setminus\{u_1, u_3, w\} = \{v\}$. Note that $v \notin N(u_3)$.

First, suppose that $n = 6$. If $vw \in E(G)$, then the function $h$ defined by $h(u_1) = h(u_2) = 2$, $h(u_3) = h(w) = 0$ and $h(u_4) = h(v) = 3$, is a GTRDF on $G$ and so $\gamma_{g[3R]}(G) \leq \omega(h) = 10 = 3n-8$. If $vw \notin E(G)$, then clearly $G = H_2$ and hence by Proposition 9, $\gamma_{g[3R]}(G) = 3n-7$.

Second, suppose that $n \geq 7$. Since $d(u_1) = d(u_4) = 1$, $d(u_2) = 4$ and $d(u_3) = 3$, this forces that $xw \in E(G)$ for each $x \in V(G)\setminus\{u_1, u_2, u_3, u_4, v, w\}$ (for otherwise, $\operatorname{diam}(G) \geq d(x, u_4) \geq 4$, a contradiction). It is easy to see that the function $h$ defined by $h(u_2) = h(u_3) = h(w) = 4$ and $h(x) = 0$ otherwise, is a GTRDF on $G$ and so $\gamma_{g[3R]}(G) \leq \omega(h) = 12 \leq 3n-9$.

*Case* 3. $|N(u_2) \cap N(u_3)| = 2$. In this case, one can check that the function $h$ defined by $h(u_1) = h(u_2) = h(u_3) = h(u_4) = 2$, $h(x) = 1$ for each $x \in N(u_2) \cap N(u_3)$ and $h(x) = 3$ otherwise, is a GTRDF on $G$ and so $\gamma_{g[3R]}(G) \leq \omega(h) = 3(n-6)+10 = 3n-8$.

Claim 1 holds. □

**Claim 2.** *If* $d(u_1) = 1$ *and* $d(u_4) = 2$, *then* $\gamma_{g[3R]}(G) \leq 3n-7$ *with equality if and only if* $G = H_3$ *or* $\overline{G} \in \{S(1, 2), H_2, H_3\}$.

***Proof.*** Let $N(u_4)\setminus\{u_3\} = \{v\}$. According to the values of $|N(v) \cap \{u_2, u_3\}|$, we distinguish the following three cases.

*Case* 1. $|N(v)\cap\{u_2, u_3\}| = 0$. In this case, there must exist some vertex, say $w$, in $V(G)\setminus\{u_1, u_2, u_3, u_4, v\}$ adjacent to both $v$ and $u_2$ (for otherwise, $\operatorname{diam}(G) \geq d(u_1, v) \geq 4$, which is a contradiction). Observe that the function $h$ defined by $h(u_1) = h(u_2) = h(u_4) = h(v) = 2$, $h(u_3) = h(w) = 1$ and $h(x) = 3$ otherwise, is a GTRDF on $G$ and so $\gamma_{g[3R]}(G) \leq \omega(h) = 3(n-6)+10 = 3n-8$.

*Case* 2. $|N(v) \cap \{u_2, u_3\}| = 1$. If $n = 5$, then it is not difficult to check that $G = H_3$ or $\overline{G} = H_3$, and hence by Proposition 9, $\gamma_{g[3R]}(G) = \gamma_{g[3R]}(\overline{G}) = 3n - 7$. Now let $n \geq 6$. Recall that $d(u_1) = 1$ and $d(u_4) = 2$. This forces that there must exist some vertex, say $w$, in $V(G)\backslash\{u_1, u_2, u_3, u_4, v\}$ adjacent to at least one of $u_2$, $u_3$ and $v$. Note that $|N(v) \cap \{u_2, u_3\}| = 1$. Thus if $w \in N(u_i)\backslash N(u_{5-i})$ for some $i \in \{2, 3\}$, then the function $h$ defined by $h(u_2) = h(u_3) = 4$, $h(u_1) = h(u_4) = h(w) = h(v) = 0$ and $h(x) = 3$ otherwise, is a GTRDF on $G$ and so $\gamma_{g[3R]}(G) \leq \omega(h) = 3(n - 6) + 8 = 3n - 10$; and if $w \in N(u_2) \cap N(u_3)$, then the function $h$ defined by $h(u_1) = h(u_2) = h(u_3) = h(u_4) = 2$, $h(w) = h(v) = 1$ and $h(x) = 3$ otherwise, is a GTRDF on $G$ and so $\gamma_{g[3R]}(G) \leq \omega(h) = 3(n-6)+10 = 3n - 8$. So in the following we may assume that $w \in N(v)\backslash(N(u_2) \cup N(u_3))$. Recall that $|N(v) \cap \{u_2, u_3\}| = 1$. If $N(v) \cap \{u_2, u_3\} = \{u_2\}$, then the function $h$ defined by $h(u_2) = h(v) = 0$, $h(u_3) = h(u_4) = 2$ and $h(x) = 3$ otherwise, is a GTRDF on $G$ and so $\gamma_{g[3R]}(G) \leq \omega(h) = 3(n - 4) + 4 = 3n - 8$; and if $N(v) \cap \{u_2, u_3\} = \{u_3\}$, then the function $h$ defined by $h(u_1) = h(v) = 4$, $h(u_2) = h(u_3) = h(u_4) = h(w) = 0$ and $h(x) = 3$ otherwise, is a GTRDF on $G$ and so $\gamma_{g[3R]}(G) \leq \omega(h) = 3(n - 6) + 8 = 3n - 10$.

*Case* 3. $|N(v) \cap \{u_2, u_3\}| = 2$. If $n = 5$, then clearly $\overline{G} = S(1, 2)$ and hence by Proposition 9, $\gamma_{g[3R]}(G) = \gamma_{g[3R]}(\overline{G}) = 3n - 7$. Assume next that $n \geq 6$. If $d(u_2) \geq 5$, then the function $h$ defined by $h(u_1) = h(u_2) = 4$, $h(x) = 0$ for each $x \in N(u_2)\backslash\{u_1\}$ and $h(x) = 3$ otherwise, is a GTRDF on $G$ and hence $\gamma_{g[3R]}(G) \leq \omega(h) = 3(n - |N(u_2)| - 1) + 8 \leq 3n - 10$. So in the following we may assume that $d(u_2) \in \{3, 4\}$. Note that $d(u_1) = 1$ and $d(u_4) = 2$. Thus if $n \geq 7$, then it is not difficult to verify that the function $h$ defined by $h(u_4) = 1$ and $h(x) = 2$ for each $x \in V(G)\backslash\{u_4\}$, is a GTRDF on $G$ and so $\gamma_{g[3R]}(G) \leq \omega(h) = 2n - 1 \leq 3n - 8$. Therefore it suffices for us to consider the case when $n = 6$. Let $V(G)\backslash\{u_1, u_2, u_3, u_4, v\} = \{w\}$. This forces $w \in N(u_2) \cup N(u_3) \cup N(v)$ and $w \notin N(u_1) \cup N(u_4)$. Recall that $d(u_2) \in \{3, 4\}$.

First, suppose that $d(u_2) = 3$. It is clear that $w \in (N(u_3) \cup N(v))\backslash N(u_2)$. If $w \in N(u_3)\backslash N(v)$, then define the GTRDF $h$ on $G$ by $h(u_2) = h(v) = 0$, $h(u_3) = h(w) = 2$ and $h(u_1) = h(u_4) = 3$; if $w \in N(v)\backslash N(u_3)$, then define the GTRDF $h$ on $G$ by $h(u_2) = h(u_3) = 0$, $h(v) = h(w) = 2$ and $h(u_1) = h(u_4) = 3$; and if $w \in N(u_3) \cap N(v)$, then define the GTRDF $h$ on $G$ by $h(u_1) = 4$, $h(u_2) = h(u_3) = h(v) = 0$ and $h(u_4) = h(w) = 3$. In the above three cases, we have $\gamma_{g[3R]}(G) \leq \omega(h) = 10 = 3n - 8$.

Second, suppose that $d(u_2) = 4$. This implies that $w \in N(u_2)$. If $w \notin N(u_3) \cup N(v)$, then define the GTRDF $h$ on $G$ by $h(u_1) = h(u_2) = 2$, $h(u_3) = h(v) = 0$ and $h(u_4) = h(w) = 3$; if $w \in N(v)\backslash N(u_3)$, then define the GTRDF $h$ on $G$ by $h(u_1) = h(v) = 2$, $h(u_2) = h(u_3) = 3$ and $h(u_4) = h(w) = 0$; and if $w \in N(u_3)\backslash N(v)$, then define the GTRDF $h$ on $G$ by $h(u_1) = h(u_3) = 2$, $h(u_2) = h(v) = 3$ and $h(u_4) = h(w) = 0$. In the above cases, we have $\gamma_{g[3R]}(G) \leq \omega(h) =$

$10 = 3n - 8$. If $w \in N(u_3) \cap N(v)$, then it is easy to see that $\overline{G} = H_2$ and hence by Proposition 9, $\gamma_{g[3R]}(G) = \gamma_{g[3R]}(\overline{G}) = 3n - 7$. Thus Claim 2 holds. $\square$

This completes the proof. $\blacksquare$

In the following, we shall consider the graphs with diameter two. Let $\mathcal{H}$ be the family of all graphs on five vertices with minimum degree one and maximum degree four.

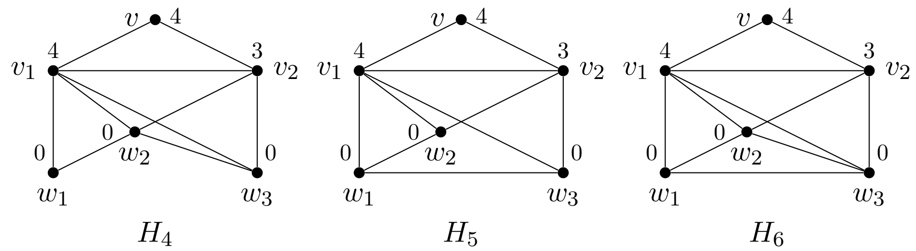


Figure 4. The graphs $H_4$ to $H_6$ with $\gamma_{g[3R]}(H_i) = 3n - 7$.

**Proposition 11.** *For each* $G \in \mathcal{H} \cup \{H_4, H_5, H_6\}$ *of order* $n$, $\gamma_{g[3R]}(G) = 3n - 7$, *where* $H_4, H_5$ *and* $H_6$ *are illustrated in Figure* 4.

***Proof.*** For each $G \in \mathcal{H} \cup \{H_4\}$, let $h = (V_0^h, V_1^h, \ldots, V_4^h)$ be a $\gamma_{g[3R]}(G)$-function. First, suppose that $G \in \mathcal{H}$. If there exists some vertex $w \in V_0^h$, then by the definition of $\gamma_{g[3R]}(G)$-function, we have $\sum_{x \in N(w)} h(x) \geq 4$ and $\sum_{x \notin N[w]} h(x) \geq 4$, implying that $\gamma_{g[3R]}(G) = \omega(h) = h(w) + \sum_{x \in N(w)} h(x) + \sum_{x \notin N[w]} h(x) \geq 8$. Next, assume that $V_0^h = \emptyset$. Let $u$ and $v$ be two vertices of $G$ with $d(u) = 4$ and $d(v) = 1$. Then $u \in V_3^h \cup V_4^h$. If $u \in V_4^h$, then since $V_0^h = \emptyset$, we have $\gamma_{g[3R]}(G) = \omega(h) \geq 8$. Now let $u \in V_3^h$. If $v \in V_2^h \cup V_3^h \cup V_4^h$, then since $V_0^h = \emptyset$, we have $\gamma_{g[3R]}(G) = \omega(h) \geq 8$. If $v \in V_1^h$, then since $h$ is also a TRDF on $\overline{G}$, we have that $v$ must be adjacent to at least one vertex of $V_3^h \cup V_4^h$ or two vertices of $V_2^h$ in $\overline{G}$. implying that $\gamma_{g[3R]}(G) = \omega(h) \geq 9$. In the above cases, we have $\gamma_{g[3R]}(G) \geq 8$. On the other hand, one can check that the function $g$ defined by $g(u) = g(v) = 4$ and $g(x) = 0$ otherwise, is a GTRDF on $G$ and hence $\gamma_{g[3R]}(G) \leq 8$. This forces $\gamma_{g[3R]}(G) = 8 = 3n - 7$.

Second, suppose that $G = H_4$. Since $h$ is a $\gamma_{g[3R]}(G)$-function, we have that $h$ is a TRDF on $\overline{G}$. We next consider the complement $\overline{G}$. In the complement $\overline{G}$, one can check that $h(w_2) + h(v) \geq 3$, $h(w_1) + h(v_2) \geq 3$ and $h(v_1) \geq 3$. If $h(w_3) \geq 2$, then $\gamma_{g[3R]}(G) = \omega(h) \geq 11$.

Now assume that $h(w_3) = 0$. If at least two of $h(w_2) + h(v)$, $h(w_1) + h(v_2)$ and $h(v_1)$ equal at least 4, then $\gamma_{g[3R]}(G) = \omega(h) \geq 11$. Suppose next that at least two of $h(w_2) + h(v)$, $h(w_1) + h(v_2)$ and $h(v_1)$ equal 3. Without loss of generality, we may assume that $f(w_2)+f(v) = 3$. By the definition of $\gamma_{g[3R]}(G)$-function, we have $h(w_2) = 3$ and $h(v) = 0$. Moreover, since $h(w_3) = 0$, we have $h(w_1) = 4$ and so $h(v_1) = 3$. If $h(v_2) \geq 1$, then $\gamma_{g[3R]}(G) = \omega(h) \geq 11$. If $h(v_2) = 0$, then since $v$ is adjacent to exactly two vertices $v_1$ and $v_2$ in $G$ with $h(v) = 0$ and $h(v_1) = 3$, it is a contradiction to our assumption that $h$ is a $\gamma_{g[3R]}(G)$-function.

Next assume that $h(w_3) = 1$. If $h(w_2) + h(v) = h(w_1) + h(v_2) = 3$, then by the definition of $\gamma_{g[3R]}(G)$-function, we have $h(w_2) = h(v_2) = 3$ and $h(v) = h(w_1) = 0$, a contradiction to our assumption that $h(w_3) = 1$. Thus at least one of $h(w_2)+h(v)$ and $h(w_1)+h(v_2)$ equals at least 4, and hence $\gamma_{g[3R]}(G) = \omega(h) \geq 11$.

In each case, we have $\gamma_{g[3R]}(G) \geq 11$. On the other hand, one can check that the function $g$ defined by $g(v) = g(v_1) = 4$, $g(v_2) = 3$ and $g(x) = 0$ otherwise, is a GTRDF on $G$ and hence $\gamma_{g[3R]}(G) \leq 11$. This forces $\gamma_{g[3R]}(G) = 11 = 3n - 7$. Using a similar argument we can obtain that if $G \in \{H_5, H_6\}$, then $\gamma_{g[3R]}(G) = 3n - 7$. ■

**Lemma 12.** *Let $G$ be a connected graph on $n \geq 5$ vertices with diameter two. If $\delta \leq n - 4$, then*

$$\gamma_{g[3R]}(G) \leq 3n - 7$$

*with equality if and only if $G \in \mathcal{H} \cup \{H_4, H_5, H_6\}$, where $H_4, H_5$ and $H_6$ are illustrated in Figure* 4.

***Proof.*** Let $v$ be a vertex of $G$ having minimum degree $\delta$. Since $G$ has diameter two, we have that every vertex in $V(G)\backslash N[v]$ is adjacent to some vertex in $N(v)$. If $\delta \leq n - 5$, then the function $h$ defined by $h(x) = 3$ for each $x \in N[v]$ and $h(x) = 1$ for each $x \in V(G)\backslash N[v]$, is a GTRDF on $G$ and so $\gamma_{g[3R]}(G) \leq \omega(h) = 3(\delta+1) + (n-\delta-1) = n + 2\delta + 2 \leq 3n - 8$. Thus it suffices for us to restrict our attention to the graphs with $\delta = n - 4$.

If $n = 5$, then $\delta = n - 4 = 1$ and since $G$ has diameter two, we have $G \in \mathcal{H}$ and so by Proposition 11, $\gamma_{g[3R]}(G) = 3n - 7$. Hence we may assume that $n \geq 6$. Let $V(G)\backslash N[v] = \{w_1, w_2, w_3\}$. If $|N(w_i) \cap N(v)| \geq 2$ for each $i \in \{1, 2, 3\}$, then the function $h$ defined by $h(v) = 4$, $h(w_1) = h(w_2) = h(w_3) = 0$ and $h(x) = 3$ for each $x \in N(v)$, is a GTRDF on $G$ and so $\gamma_{g[3R]}(G) \leq \omega(h) = 3\delta + 4 = 3n - 8$. Moreover, note that $|N(w_i) \cap N(v)| \geq 1$ for each $i \in \{1, 2, 3\}$ since $G$ has diameter two. So in the following we may assume that there exists some vertex, say $w_1$, in $\{w_1, w_2, w_3\}$ such that $|N(w_1) \cap N(v)| = 1$. This forces $n \leq 7$ (for otherwise, $d(w_1) \geq \delta = n - 4 \geq 4$, implying that $|N(w_1) \cap N(v)| \geq 2$, a contradiction). Moreover, if $n = 7$, then $\delta = n - 4 = 3$ and so $w_1$ must be adjacent to both $w_2$ and $w_3$ in $G$ (noting that $|N(w_1) \cap N(v)| = 1$). One can check that the function $h$ defined by $h(v) = h(w_1) = 4$, $h(x) = 3$ for $x \in N(w_1) \cap N(v)$ and $h(x) = 0$ otherwise, is a GTRDF on $G$ and hence $\gamma_{g[3R]}(G) \leq \omega(h) = 11 < 3n - 7$. Suppose next that $n = 6$.

Note that $\delta = n-4 = 2$. Let $N(v) = \{v_1, v_2\}$. Recall that $|N(w_1) \cap N(v)| = 1$. Without loss of generality, assume that $N(w_1) \cap \{v_1, v_2\} = \{v_1\}$. Moreover, since $\delta = 2$, we have $w_1w_2 \in E(G)$ or $w_1w_3 \in E(G)$. Without loss of generality, assume that $w_1w_2 \in E(G)$. We proceed further with the following claims.

**Claim 1.** *If $w_1w_3, w_2w_3 \notin E(G)$, then $\gamma_{g[3R]}(G) \leq 3n-8$.*

***Proof.*** Since $\delta = 2$, we have $N(w_3) = \{v_1, v_2\}$. If $w_2v_2 \in E(G)$, then the function $h$ defined by $h(w_3) = h(v) = 0$, $h(w_1) = h(v_2) = 2$ and $h(w_2) = h(v_1) = 3$, is a GTRDF on $G$ and so $\gamma_{g[3R]}(G) \leq \omega(h) = 10 = 3n-8$. Assume next that $w_2v_2 \notin E(G)$. Moreover, since $w_2v, w_2w_3 \notin E(G)$, $w_2w_1 \in E(G)$ and $\delta = 2$, we have $w_2v_1 \in E(G)$. Then the function $h$ defined by $h(w_2) = h(w_3) = 0$, $h(w_1) = h(v_2) = 2$ and $h(v_1) = h(v) = 3$, is a GTRDF on $G$ and so $\gamma_{g[3R]}(G) \leq \omega(h) = 10 = 3n-8$. Thus Claim 1 holds. $\square$

As shown earlier, $|N(w_i) \cap N(v)| \geq 1$ for each $i \in \{2,3\}$, this forces $|N(w_i) \cap \{v_1, v_2\}| \in \{1,2\}$.

**Claim 2.** *If $w_1w_3 \notin E(G)$ and $w_2w_3 \in E(G)$, then $\gamma_{g[3R]}(G) \leq 3n-7$ with equality if and only if $G = H_4$.*

***Proof.*** Recall that $|N(w_i) \cap \{v_1, v_2\}| \in \{1,2\}$ for each $i \in \{2,3\}$. If $|N(w_3) \cap \{v_1, v_2\}| = 1$, then define the GTRDF $h$ on $G$ by $h(w_1) = h(w_3) = 0$, $h(w_2) = h(v) = 2$ and $h(v_1) = h(v_2) = 3$ and hence $\gamma_{g[3R]}(G) \leq \omega(h) = 10 = 3n-8$. So in the following we may assume that $|N(w_3) \cap \{v_1, v_2\}| = 2$.

First, suppose that $v_1v_2 \notin E(G)$. If $v_1w_2 \in E(G)$, then define the GTRDF $h$ on $G$ by $h(w_1) = h(w_2) = h(w_3) = h(v_2) = 0$ and $h(v_1) = h(v) = 4$; and if $v_1w_2 \notin E(G)$, then define the GTRDF $h$ on $G$ by $h(w_1) = h(w_2) = h(v_2) = h(v) = 0$ and $h(w_3) = h(v_1) = 4$. In either case, we have $\gamma_{g[3R]}(G) \leq \omega(h) = 8 < 3n-7$.

Second, suppose that $v_1v_2 \in E(G)$. If $v_1w_2 \notin E(G)$, then define the GTRDF $h$ on $G$ by $h(w_2) = h(w_3) = h(v_2) = h(v) = 0$ and $h(w_1) = h(v_1) = 4$ and if $v_1w_2 \in E(G)$ and $v_2w_2 \notin E(G)$, then define the GTRDF $h$ on $G$ by $h(w_1) = h(v) = 0$, $h(w_2) = h(v_2) = 2$ and $h(w_3) = h(v_1) = 3$. In either case, we have $\gamma_{g[3R]}(G) \leq \omega(h) \leq 10 = 3n-8$. If $v_1w_2, v_2w_2 \in E(G)$, that is, if $G = H_4$, then by Proposition 11, $\gamma_{g[3R]}(G) = 3n-7$. Thus Claim 2 holds. $\square$

**Claim 3.** *If $w_1w_3 \in E(G)$ and $w_2w_3 \notin E(G)$, then $\gamma_{g[3R]}(G) \leq 3n-7$ with equality if and only if $G = H_5$.*

***Proof.*** First, suppose that $v_1v_2 \notin E(G)$. Note that $\delta = 2$. If $|N(v_1) \cap \{w_2, w_3\}| = 0$, then clearly $v_2w_2, v_2w_3 \in E(G)$ and define the GTRDF $h$ on $G$ by $h(w_2) = h(w_3) = 0$, $h(v_1) = h(v_2) = 2$ and $h(w_1) = h(v) = 3$, and if $|N(v_1) \cap \{w_2, w_3\}| = 2$, then define the GTRDF $h$ on $G$ by $h(w_1) = h(w_2) = h(w_3) = h(v_2) = 0$ and $h(v_1) = h(v) = 4$. In either case, we have $\gamma_{g[3R]}(G) \leq \omega(h) \leq 10 = 3n-8$. Suppose now that $|N(v_1) \cap \{w_2, w_3\}| = 1$. Without loss of generality, assume that $v_1w_2 \notin E(G)$ and $v_1w_3 \in E(G)$. Moreover, since $w_2w_3, w_2v \notin E(G)$, $w_2w_1 \in E(G)$ and $\delta = 2$, we have $v_2w_2 \in E(G)$. Observe that the function $h$ defined by $h(w_1) = h(v_1) = 0$, $h(w_2) = h(v) = 2$ and $h(w_3) = h(v_2) = 3$, is a GTRDF on $G$ and so $\gamma_{g[3R]}(G) \leq \omega(h) = 10 = 3n - 8$.

Second, suppose that $v_1v_2 \in E(G)$. Note that $|N(w_i) \cap \{v_1, v_2\}| \in \{1, 2\}$ for each $i \in \{2, 3\}$. If $|N(w_2) \cap \{v_1, v_2\}| = |N(w_3) \cap \{v_1, v_2\}| = 1$, then define the GTRDF $h$ on $G$ by $h(w_2) = h(w_3) = 0$, $h(w_1) = h(v) = 2$ and $h(v_1) = h(v_2) = 3$ and hence $\gamma_{g[3R]}(G) \leq \omega(h) = 10 = 3n-8$. So in the following we may assume that $|N(w_3) \cap \{v_1, v_2\}| = 2$ (the case $|N(w_2) \cap \{v_1, v_2\}| = 2$ is similar). If $w_2v_1 \notin E(G)$, then define the GTRDF $h$ on $G$ by $h(w_1) = h(w_3) = h(v_1) = h(v) = 0$ and $h(w_2) = h(v_2) = 4$; and if $w_2v_1 \in E(G)$ and $w_2v_2 \notin E(G)$, then define the GTRDF $h$ on $G$ by $h(w_3) = h(v_2) = 0$, $h(w_1) = h(v) = 2$ and $h(w_2) = h(v_1) = 3$. In either case, we have $\gamma_{g[3R]}(G) \leq \omega(h) \leq 10 = 3n - 8$. If $w_2v_1, w_2v_2 \in E(G)$, that is, if $G = H_5$, then by Proposition 11, $\gamma_{g[3R]}(G) = 3n - 7$. Thus Claim 3 holds. □

**Claim 4.** *If $w_1w_3, w_2w_3 \in E(G)$, then $\gamma_{g[3R]}(G) \leq 3n - 7$ with equality if and only if $G = H_6$.*

***Proof.*** Note that $|N(w_i) \cap \{v_1, v_2\}| \in \{1, 2\}$ for each $i \in \{2, 3\}$. Thus if $|N(w_3) \cap \{v_1, v_2\}| = 1$, then define the GTRDF $h$ on $G$ by $h(w_1) = h(w_3) = 0$, $h(w_2) = h(v) = 2$ and $h(v_1) = h(v_2) = 3$; if $|N(w_3) \cap \{v_1, v_2\}| = 2$ and $|N(w_2) \cap \{v_1, v_2\}| = 1$, then define the GTRDF $h$ on $G$ by $h(w_1) = h(w_2) = 0$, $h(w_3) = h(v) = 2$ and $h(v_1) = h(v_2) = 3$; and if $|N(w_3) \cap \{v_1, v_2\}| = |N(w_2) \cap \{v_1, v_2\}| = 2$ and $v_1v_2 \notin E(G)$, then define the GTRDF $h$ on $G$ by $h(w_1) = h(w_2) = h(w_3) = h(v_2) = 0$ and $h(v_1) = h(v) = 4$. In the above cases, we have $\gamma_{g[3R]}(G) \leq \omega(h) \leq 10 = 3n-8$. If $|N(w_3) \cap \{v_1, v_2\}| = |N(w_2) \cap \{v_1, v_2\}| = 2$ and $v_1v_2 \in E(G)$, that is, if $G = H_6$, then by Proposition 11, $\gamma_{g[3R]}(G) = 3n - 7$. Thus Claim 4 holds. □

This completes the proof. ■

We next consider the graphs $G$ on $n$ vertices different from $K_n - e$ with $\mathrm{diam}(G) = 2$ and $\delta \in \{n - 3, n - 2\}$.

**Lemma 13.** *For any connected graph $G$ on $n$ vertices different from $K_n - e$ with $\mathrm{diam}(G) = \omega(G) = 2$ and $\delta \in \{n - 3, n - 2\}$,*

$$\gamma_{g[3R]}(G) \leq 3n - 4.$$

*Furthermore, the following hold.*

(a) $\gamma_{g[3R]}(G) = 3n - 4$ *if and only if* $G = K_{1,3}$ *or* $\overline{G} = 2P_2$.

(b) $\gamma_{g[3R]}(G) \neq 3n - 5$.

(c) $\gamma_{g[3R]}(G) = 3n - 6$ *if and only if* $G = C_5$.

(d) $\gamma_{g[3R]}(G) = 3n - 7$ *if and only if* $\overline{G} = P_2 \cup C_3$.

***Proof.*** If $n = 3$, then since $\mathrm{diam}(G) = 2$, we have $G = P_3 = K_3 - e$, a contradiction. So in the following we may assume that $n \geq 4$. Let $v$ be a vertex of $G$ having minimum degree $\delta$. Since $G$ has diameter two, we have that every vertex in $V(G)\backslash N[v]$ is adjacent to some vertex in $N(v)$. Moreover, since $\omega(G) = 2$, we have that $G[N(v)]$ is empty.

We first assume that $\delta = n-2$. If $n = 4$, then $\delta = 2$, implying that $G$ is the cycle $C_4$ (noting that the complement of $C_4$ is $2P_2$) and $\gamma_{g[3R]}(G) = 8 = 3n-4$. If $n \geq 5$, then since $G[N(v)]$ is empty, we have $d(x) \leq 2$ for each $x \in N(v)$, a contradiction to the fact that $d(x) \geq \delta = n-2 \geq 3$.

Suppose next that $\delta = n-3$. Note that every vertex in $V(G)\backslash N[v]$ is adjacent to some vertex in $N(v)$ and $\omega(G) = 2$. Thus if $n = 4$, then $\delta = 1$, implying that $G$ is the graph $K_{1,3}$ and therefore $\gamma_{g[3R]}(G) = 8 = 3n-4$; and if $n = 5$, then $\delta = 2$, implying that $G \in \{C_5, K_{2,3}\}$ and so $\gamma_{g[3R]}(K_{2,3}) = 8 = 3n-7$ (noting that the complement of $K_{2,3}$ is $P_2 \cup C_3$). Now let $G = C_5 = w_1w_2\cdots w_5w_1$ and let $h = (V_0^h, V_1^h, \ldots, V_4^h)$ be a $\gamma_{g[3R]}(G)$-function. If there exists some vertex, say $w_1 \in V_0^h$, then by the definition of $\gamma_{g[3R]}(G)$-function, we have $h(w_2)+h(w_5) \geq 4$ and $h(w_3)+h(w_4) \geq 4$. If at least one of $h(w_2)+h(w_5)$ and $h(w_3)+h(w_4)$ equals at least 5, then $\gamma_{g[3R]}(G) = \omega(h) \geq 9$. Now let $h(w_2)+h(w_5) = h(w_3)+h(w_4) = 4$. Since $w_1 \in V_0^h$, we may assume that $w_2 \in V_4^h$ and $w_5 \in V_0^h$. This forces that $w_4 \in V_4^h$ and $w_3 \in V_0^h$, a contradiction to our assumption that $h$ is a $\gamma_{g[3R]}(G)$-function. Thus $V_0^h = \emptyset$. Suppose that there exists some vertex, say $w_1 \in V_1^h$. Since $h$ is a GTRDF on $G$, we have that either $\{w_2, w_5\} \cap (V_3^h \cup V_4^h) \neq \emptyset$ or $w_2, w_5 \in V_2^h$, and so $h(w_2)+h(w_5) \geq 4$ (noting that $V_0 = \emptyset$). Similarly, since $h$ is also a GTRDF on $\overline{G}$, we have $h(w_3)+h(w_4) \geq 4$. Thus $\gamma_{g[3R]}(G) = \omega(h) \geq 9$. Next, assume that $V_1^h = \emptyset$. Moreover, since $V_0^h = \emptyset$, we have $\gamma_{g[3R]}(G) = \omega(h) = 2|V_2^h| + 3|V_3^h| + 4|V_4^h| \geq 2(|V_2^h| + |V_3^h| + |V_4^h|) = 10$. In each case, we have $\gamma_{g[3R]}(G) \geq 9$. On the other hand, one can check that the function $g$ defined by $g(w_1) = 1$ and $g(x) = 2$ otherwise is a GTRDF on $G$ and hence $\gamma_{g[3R]}(G) \leq 9$. This forces $\gamma_{g[3R]}(G) = 9 = 3n-6$.

Assume next that $n \geq 6$. Since $G[N(v)]$ is empty, we have that for each $x \in N(v)$, $3 \geq d(x) \geq \delta = n-3 \geq 3$, implying that $d(x) = 3$ and $n = 6$. Let $V(G) \setminus N[v] = \{w_1, w_2\}$. It is clear that $N(x) = \{v, w_1, w_2\}$ for each $x \in N(v)$. Further, since $\omega(G) = 2$, we have $w_1w_2 \notin E(G)$. This forces $G = K_{3,3}$ and $\gamma_{g[3R]}(G) = 8 = 3n-10$. This completes the proof. ■

In order to state the following results, we shall introduce some additional notations. For any graph $G$ with $\mathrm{diam}(G) = 2$ and $\omega(G) \geq 3$, let $X$ denote a subset of the vertex set of $G$ such that $G[X]$ is a clique with $\omega(G)$ vertices and let $Y = V(G)\backslash X$.

**Observation 14.** *Let $G$ be a connected graph on $n$ vertices with* $\mathrm{diam}(G) = 2$, $\omega(G) \geq 3$ *and* $\delta \in \{n-3, n-2\}$. *Then for each* $v \in Y$, $|X\backslash N(v)| \in \{1, 2\}$.

***Proof.*** If there exists some vertex, say $v$, in $Y$ such that $|X\backslash N(v)| = 0$, then since $G[X]$ is a clique, we have that $G[\{v\} \cup X]$ is also a clique with $|X|+1 = \omega(G)+1$ vertices, a contradiction. Moreover, if there exists some vertex, say $v$, in $Y$ such that $|X\backslash N(v)| \geq 3$,then $\delta \leq d(v) \leq n - |X\backslash N(v)| - 1 \leq n - 4$, a contradiction. Thus $|X\backslash N(v)| \in \{1, 2\}$ for each $v \in Y$, as desired. ■

**Proposition 15.** *Let $G$ be a connected graph on $n \geq 5$ vertices. If $\overline{G} \in \{C_3 \cup (n-3)K_1, P_4 \cup (n-4)K_1\}$, then $\gamma_{g[3R]}(G) = 3n - 5$.*

***Proof.*** For each $\overline{G} \in \{C_3 \cup (n-3)K_1, P_4 \cup (n-4)K_1\}$, let $h$ be a $\gamma_{g[3R]}(\overline{G})$-function. First, suppose that $\overline{G} = C_3 \cup (n-3)K_1$, where $C_3 = w_1w_2w_3w_1$. Since $h$ is a TRDF on $\overline{G}$, we have that $\sum_{x \in V(C_3)} h(x) \geq 4$ and $h(x) \geq 3$ for all other vertices $x$ of $\overline{G}$. Thus $\gamma_{g[3R]}(\overline{G}) = \omega(h) \geq 4+3(n-3) = 3n-5$. On the other hand, one can check that the function $g$ defined by $g(w_1) = 4$, $g(w_2) = g(w_3) = 0$ and $g(x) = 3$ otherwise, is a GTRDF on $\overline{G}$ and hence $\gamma_{g[3R]}(\overline{G}) \leq 4+3(n-3) = 3n-5$. This forces $\gamma_{g[3R]}(G) = \gamma_{g[3R]}(\overline{G}) = 3n - 5$.

Second, suppose that $\overline{G} = P_4 \cup (n-4)K_1$, where $P_4 = w_1w_2w_3w_4$. Since $h$ is a TRDF on $\overline{G}$, we have $h(w_1) + h(w_2) \geq 3$, $h(w_3) + h(w_4) \geq 3$ and $h(x) \geq 3$ for all other vertices $x$ of $\overline{G}$. If $h(w_1) + h(w_2) = h(w_3) + h(w_4) = 3$, then it is easy to see that $h(w_1) = h(w_4) = 3$ and $h(w_2) = h(w_3) = 0$, a contradiction to our assumption that $h$ is a TRDF on $\overline{G}$. Thus at least one of $h(w_1) + h(w_2)$ and $h(w_3) + h(w_4)$ equals at least 4, implying that $\gamma_{g[3R]}(\overline{G}) \geq 4 + 3 + 3(n-4) = 3n - 5$. On the other hand, one can check that the function $g$ defined by $g(w_1) = g(w_2) = 2$, $g(w_3) = 0$ and $g(x) = 3$ otherwise, is a GTRDF on $\overline{G}$ and hence $\gamma_{g[3R]}(\overline{G}) \leq 4 + 3(n-3) = 3n - 5$. Thus $\gamma_{g[3R]}(G) = \gamma_{g[3R]}(\overline{G}) = 3n - 5$. ■

**Lemma 16.** *Let $G$ be a connected graph on $n$ vertices different from $K_n - e$ with* $\mathrm{diam}(G) = 2$, *$\omega(G) \geq 3$ and $\delta \in \{n-3, n-2\}$. If $G[Y]$ is an empty graph with $|Y| \geq 2$, then $\overline{G} \in \{C_3 \cup (n-3)K_1 \ (n \geq 5), P_4 \cup (n-4)K_1 \ (n \geq 5)\}$ and $\gamma_{g[3R]}(G) = 3n - 5$.*

***Proof.*** It follows from Observation 14 that $|X\backslash N(v)| \in \{1, 2\}$ for each $v \in Y$. Moreover, since $G[Y]$ is an empty graph with $|Y| \geq 2$, we have that for each $v \in Y$,

$$\delta \leq d(v) = n - |Y| - |X\backslash N(v)| \leq n - 3$$

and since $\delta \in \{n-3, n-2\}$, we obtain $\delta = n - 3$, $|Y| = 2$ and $|X\backslash N(v)| = 1$. Further, we note that $n = |X| + |Y| = \omega(G) + 2 \geq 5$. Let $Y = \{v_1, v_2\}$. It is clear that $|X\backslash N(v_1)| = |X\backslash N(v_2)| = 1$. Therefore, if $X\backslash N(v_1) = X\backslash N(v_2)$, then $\overline{G} = C_3 \cup (n-3)K_1$ $(n \geq 5)$ and if $X\backslash N(v_1) \neq X\backslash N(v_2)$, then $\overline{G} = P_4 \cup (n-4)K_1$ $(n \geq 5)$. Thus by Proposition 15, $\gamma_{g[3R]}(G) = 3n - 5$. ■

By the method similar to Proposition 15, we can verify the following proposition. The details are omitted.

**Proposition 17.** *Let $G$ be a connected graph on $n$ vertices.*

(a) *If $\overline{G} \in \{P_3 \cup K_1, 2P_2 \cup (n-4)K_1 \ (n \geq 5)\}$, then $\gamma_{g[3R]}(G) = 3n-4$.*

(b) *If $\overline{G} \in \{C_4 \cup K_1, P_3 \cup (n-3)K_1 \ (n \geq 5), P_4 \cup (n-4)K_1 \ (n \geq 5)\}$, then $\gamma_{g[3R]}(G) = 3n-5$.*

(c) *If $\overline{G} \in \{3P_2 \cup (n-6)K_1 \ (n \geq 6), C_4 \cup (n-4)K_1 \ (n \geq 6)\}$, then $\gamma_{g[3R]}(G) = 3n-6$.*

(d) *If $\overline{G} \in \{P_5 \cup (n-5)K_1 \ (n \geq 5), P_2 \cup P_3 \cup (n-5)K_1 \ (n \geq 5), P_2 \cup P_4 \cup (n-6)K_1 \ (n \geq 6)\}$, then $\gamma_{g[3R]}(G) = 3n-7$.*

**Lemma 18.** *Let $G$ be a connected graph on $n$ vertices different from $K_n - e$ with* $\operatorname{diam}(G) = 2$, $\omega(G) \geq 3$ *and* $\delta \in \{n-3, n-2\}$. *If $G[Y]$ is a clique, then*

$$\gamma_{g[3R]}(G) \leq 3n-4.$$

*Furthermore, the following hold.*

(a) $\gamma_{g[3R]}(G) = 3n-4$ *if and only if* $\overline{G} \in \{P_3 \cup K_1, 2P_2 \cup (n-4)K_1 \ (n \geq 5)\}$.

(b) $\gamma_{g[3R]}(G) = 3n-5$ *if and only if* $\overline{G} \in \{C_4 \cup K_1, P_3 \cup (n-3)K_1 \ (n \geq 5), P_4 \cup (n-4)K_1 \ (n \geq 5)\}$.

(c) $\gamma_{g[3R]}(G) = 3n-6$ *if and only if* $\overline{G} \in \{3P_2 \cup (n-6)K_1 \ (n \geq 6), C_4 \cup (n-4)K_1 \ (n \geq 6)\}$.

(d) $\gamma_{g[3R]}(G) = 3n-7$ *if and only if* $\overline{G} \in \{P_5 \cup (n-5)K_1 \ (n \geq 5), P_2 \cup P_3 \cup (n-5)K_1 \ (n \geq 5), P_2 \cup P_4 \cup (n-6)K_1 \ (n \geq 6)\}$.

***Proof.*** Note that $G$ is a graph different from $K_n - e$ with $\delta \in \{n-3, n-2\}$. Thus if $|Y| = 1$, then since $G[X]$ is a clique, we have that $G$ is the graph obtained from the complete graph $K_n$ by deleting two adjacent edges, implying that $\overline{G} = P_3 \cup (n-3)K_1$ ($n = |X| + |Y| = \omega(G) + 1 \geq 4$) and hence by Proposition 17, $\gamma_{g[3R]}(G) = 3n-4$ when $n = 4$ and $\gamma_{g[3R]}(G) = 3n-5$ when $n \geq 5$. So in the following we may assume that $|Y| \geq 2$. Let $Y = \{v_1, v_2, \ldots, v_{n-\omega(G)}\}$. By Observation 14, we have $|X \backslash N(v_i)| \in \{1, 2\}$ for each $i \in \{1, 2, \ldots, |Y|\}$.

**Claim 1.** *If $|Y| = 2$, then $\gamma_{g[3R]}(G) \leq 3n-4$. Moreover,*

(a) $\gamma_{g[3R]}(G) = 3n-4$ *if and only if* $\overline{G} = 2P_2 \cup (n-4)K_1 \ (n \geq 5)$,

(b) $\gamma_{g[3R]}(G) = 3n-5$ *if and only if* $\overline{G} \in \{C_4 \cup K_1, P_4 \cup (n-4)K_1 \ (n \geq 5)\}$,

(c) $\gamma_{g[3R]}(G) = 3n-6$ *if and only if* $\overline{G} = C_4 \cup (n-4)K_1 \ (n \geq 6)$,

(d) $\gamma_{g[3R]}(G) = 3n-7$ *if and only if* $\overline{G} \in \{P_5 \cup (n-5)K_1 \ (n \geq 5), P_2 \cup P_3 \cup (n-5)K_1 \ (n \geq 5)\}$.

***Proof.*** Clearly, $n = |X| + |Y| = \omega(G) + 2 \geq 5$. Recall that $|X \backslash N(v_i)| \in \{1, 2\}$ for each $i \in \{1, 2\}$. First, assume that $|X \backslash N(v_1)| = |X \backslash N(v_2)| = 1$. If $X \backslash N(v_1) =$

$X\backslash N(v_2)$, then since both $G[X]$ and $G[Y]$ are cliques, we have that $G[\{v_1, v_2\} \cup (N(v_1) \cap X)]$ is also a clique with $|X| + 1 = \omega(G) + 1$ vertices, a contradiction. Therefore $X\backslash N(v_1) \neq X\backslash N(v_2)$, implying that $\overline{G} = 2P_2 \cup (n-4)K_1$ $(n \geq 5)$ and so by Proposition 17, $\gamma_{g[3R]}(G) = 3n - 4$.

Second, assume that $|X\backslash N(v_1)| = |X\backslash N(v_2)| = 2$. If $|(X\backslash N(v_1)) \cap (X\backslash N(v_2))| = 2$, that is, if $\overline{G} = C_4 \cup (n-4)K_1$ $(n \geq 5)$, then by Proposition 17, $\gamma_{g[3R]}(G) = 3n - 5$ when $n = 5$ and $\gamma_{g[3R]}(G) = 3n - 6$ when $n \geq 6$. If $|(X\backslash N(v_1)) \cap (X\backslash N(v_2))| = 1$, that is, if $\overline{G} = P_5 \cup (n-5)K_1$ $(n \geq 5)$, then by Proposition 17, $\gamma_{g[3R]}(G) = 3n - 7$. If $|(X\backslash N(v_1)) \cap (X\backslash N(v_2))| = 0$, that is, if $\overline{G} = 2P_3 \cup (n-6)K_1$ $(n \geq 6)$, then clearly $\gamma_{g[3R]}(G) = 3n - 10$.

Finally, assume that one of $|X\backslash N(v_1)|$ and $|X\backslash N(v_2)|$ equals one and the other equals two. If $(X\backslash N(v_1)) \cap (X\backslash N(v_2)) = \emptyset$, that is, if $\overline{G} = P_2 \cup P_3 \cup (n-5)K_1$ $(n \geq 5)$, then $\gamma_{g[3R]}(G) = 3n - 7$ and if $(X\backslash N(v_1)) \cap (X\backslash N(v_2)) \neq \emptyset$, that is, if $\overline{G} = P_4 \cup (n-4)K_1$ $(n \geq 5)$, then $\gamma_{g[3R]}(G) = 3n - 5$. Thus Claim 1 holds. □

**Claim 2.** *If* $|Y| \geq 3$, *then* $\gamma_{g[3R]}(G) \leq 3n - 6$. *Moreover,*

(a) $\gamma_{g[3R]}(G) = 3n - 6$ *if and only if* $\overline{G} = 3P_2 \cup (n-6)K_1$ $(n \geq 6)$,

(b) $\gamma_{g[3R]}(G) = 3n - 7$ *if and only if* $\overline{G} = P_2 \cup P_4 \cup (n-6)K_1$ $(n \geq 6)$.

***Proof.*** Clearly, $n = |X| + |Y| = \omega(G) + |Y| \geq 6$. Note that $|X\backslash N(v_i)| \in \{1, 2\}$ for each $i \in \{1, 2, \ldots, |Y|\}$. We first assume that $|X\backslash N(v_i)| = 1$ for each $i \in \{1, 2, \ldots, |Y|\}$. If there exist two vertices, say $v_1$ and $v_2$, in $Y$ such that $X\backslash N(v_1) = X\backslash N(v_2)$, then $G[(X \cap N(v_1)) \cup \{v_1, v_2\}]$ is a clique with $|X| + 1 = \omega(G) + 1$ vertices, a contradiction. Therefore, we have $X\backslash N(v_i) \neq X\backslash N(v_j)$ for $i \neq j$, implying that $\overline{G} = |Y|P_2 \cup (n - 2|Y|)K_1$ $(n \geq 2|Y|)$. If $|Y| = 3$, that is, if $\overline{G} = 3P_2 \cup (n-6)K_1$ $(n \geq 6)$, then by Proposition 17, $\gamma_{g[3R]}(G) = 3n - 6$. Now let $|Y| \geq 4$. One can check that the function $h$ defined by $h(x) = 2$ for each vertex with degree one in $\overline{G}$ and $h(x) = 3$ otherwise, is a GTRDF on $\overline{G}$ and hence $\gamma_{g[3R]}(G) = \gamma_{g[3R]}(\overline{G}) \leq \omega(h) = 4|Y| + 3(n - 2|Y|) = 3n - 2|Y| \leq 3n - 8$.

So in the following we may assume that there exists a vertex, say $v_1$, in $Y$ such that $|X\backslash N(v_1)| = 2$. Let $X\backslash N(v_1) = \{w_1, w_2\}$. If $X\backslash\{w_1, w_2\} \subseteq N(v_i)$ for each $i \in \{2, 3, \ldots, |Y|\}$, then $G[V(G)\backslash\{w_1, w_2\}]$ is a clique with $n - 2 = |X| + |Y| - 2 \geq \omega(G) + 1$ vertices, a contradiction. Thus there must be two vertices, say $w_3 \in X\backslash\{w_1, w_2\}$ and $v_2 \in Y$, such that $w_3v_2 \notin E(G)$. Moreover, since $|X\backslash N(v_2)| \in \{1, 2\}$ and $X\backslash N(v_1) = \{w_1, w_2\}$, this forces $|(X\backslash N(v_1)) \cap (X\backslash N(v_2))| \in \{0, 1\}$. We distinguish several cases.

*Case* 1. $|X\backslash N(v_2)| = 2$ and $|(X\backslash N(v_1)) \cap (X\backslash N(v_2))| = 0$. Recalling that $X\backslash N(v_1) = \{w_1, w_2\}$ and $w_3v_2 \notin E(G)$, there must exist some vertex, say $w_4$, in $X\backslash\{w_1, w_2, w_3\}$ such that $X\backslash N(v_2) = \{w_3, w_4\}$. One can check that the function $h_1$ defined by $h_1(v_1) = h_1(v_2) = 4$, $h_1(w_i) = 0$ for each $i \in \{1, 2, 3, 4\}$ and

$h_1(x) = 3$ for each $x \in V(G)\backslash\{v_1, v_2, w_1, w_2, w_3, w_4\}$, is a GTRDF on $G$ and hence $\gamma_{g[3R]}(G) \leq \omega(h_1) = 3(n-6) + 8 = 3n - 10$.

*Case* 2. $|X\backslash N(v_2)| = 2$ and $|(X\backslash N(v_1)) \cap (X\backslash N(v_2))| = 1$. Note that $X\backslash N(v_1) = \{w_1, w_2\}$ and $w_3v_2 \notin E(G)$. Without loss of generality, we may assume that $X\backslash N(v_2) = \{w_1, w_3\}$. Recall that both $G[X]$ and $G[Y]$ are cliques. Moreover, since $v_1w_1, v_2w_1 \notin E(G)$ and $\delta \in \{n-3, n-2\}$, we have $Y\backslash N(w_1) = \{v_1, v_2\}$. Thus if $v_3w_2 \notin E(G)$, then the function $h_2$ defined by $h_2(w_2) = h_2(v_2) = 4$, $h_2(w_1) = h_2(w_3) = h_2(v_1) = h_2(v_3) = 0$ and $h_2(x) = 3$ otherwise, is a GTRDF on $G$ and therefore $\gamma_{g[3R]}(G) \leq \omega(h_2) = 3(n-6) + 8 = 3n - 10$. So in the following we may assume that $v_3w_2 \in E(G)$. Moreover, since $v_3w_1 \in E(G)$ and $|X\backslash N(v_3)| \in \{1, 2\}$, this forces that there exists some vertex, say $w$, in $X\backslash\{w_1, w_2\}$ such that $wv_3 \notin E(G)$. Thus if $w = w_3$, then define the GTRDF $h_3$ on $G$ by $h_3(w_3) = h_3(v_1) = 4$, $h_3(w_1) = h_3(w_2) = h_3(v_2) = h_3(v_3) = 0$ and $h_3(x) = 3$ otherwise and if $w \neq w_3$, then define the GTRDF $h_4$ on $G$ by $h_4(w_3) = h_4(w) = h_4(v_1) = 4$, $h_4(w_1) = h_4(w_2) = h_4(v_2) = h_4(v_3) = 0$ and $h_4(x) = 3$ otherwise. In either case, it is easy to check that $\gamma_{g[3R]}(G) \leq 3n - 9$.

*Case* 3. $|X\backslash N(v_2)| = 1$. Noting that $w_3v_2 \notin E(G)$, this implies that $X\backslash N(v_2) = \{w_3\}$. Recall that $X\backslash N(v_1) = \{w_1, w_2\}$. If $v_3w_3 \notin E(G)$ (respectively, $v_3w \notin E(G)$, where $w$ is a vertex of $X\backslash\{w_1, w_2, w_3\}$), then the function $h_3$ (respectively, $h_4$) defined earlier is a GTRDF on $G$ and hence $\gamma_{g[3R]}(G) \leq 3n - 9$. Note that $|X\backslash N(v_3)| \in \{1, 2\}$. Hence we may assume that $X\backslash N(v_3) \subseteq \{w_1, w_2\}$.

First, suppose that $X\backslash N(v_3) = \{w_1, w_2\}$. Recalling that $X\backslash N(v_1) = \{w_1, w_2\}$ and $X\backslash N(v_2) = \{w_3\}$, it is easy to see that the function $h_5$ defined by $h_5(v_2) = 4$, $h_5(v_1) = h_5(v_3) = h_5(w_3) = 0$ and $h_5(x) = 3$ for each $x \in V(G)\backslash\{v_1, v_2, v_3, w_3\}$, is a GTRDF on $G$ and hence $\gamma_{g[3R]}(G) \leq \omega(h_5) = 3(n-4) + 4 = 3n - 8$.

Second, suppose that $X\backslash N(v_3) = \{w_1\}$ (the case $X\backslash N(v_3) = \{w_2\}$ is similar). Since $w_1v_1, w_1v_3 \notin E(G)$ and $\delta \in \{n-3, n-2\}$, we have $Y\backslash N(w_1) = \{v_1, v_3\}$. Let $|Y| \geq 4$. Note that $X\backslash N(v_1) = \{w_1, w_2\}$, $X\backslash N(v_2) = \{w_3\}$, $X\backslash N(v_3) = \{w_1\}$ and $Y\backslash N(w_1) = \{v_1, v_3\}$. Moreover, since $|X\backslash N(v_4)| \in \{1, 2\}$ as shown earlier, one can check that the function $h_6$ defined by $h_6(w_1) = 0$, $h_6(w_2) = h_6(w_3) = h_6(v_1) = h_6(v_2) = h_6(v_4) = 2$ and $h_6(x) = 3$ otherwise, is a GTRDF on $G$ and therefore $\gamma_{g[3R]}(G) \leq \omega(h_6) = 3(n-6) + 10 = 3n - 8$. Now let $|Y| = 3$. Recall that $X\backslash N(v_1) = \{w_1, w_2\}$, $X\backslash N(v_2) = \{w_3\}$ and $X\backslash N(v_3) = \{w_1\}$. It is easy to see that $\overline{G} = P_2 \cup P_4 \cup (n-6)K_1$ ($n = |X| + |Y| = \omega(G) + 3 \geq 6$) and so by Proposition 17, $\gamma_{g[3R]}(G) = 3n - 7$. Thus Claim 2 holds. □

This completes the proof. ■

By the method similar to Proposition 15, we can verify the following proposition. The details are omitted.

**Proposition 19.** *Let $G$ be a connected graph on $n \geq 6$ vertices. If $\overline{G} \in \{P_2 \cup C_3 \cup (n-5)K_1, P_2 \cup P_4 \cup (n-6)K_1\}$, then $\gamma_{g[3R]}(G) = 3n - 7$.*

**Lemma 20.** *Let $G$ be a connected graph on $n$ vertices different from $K_n - e$ with* $\operatorname{diam}(G) = 2$, $\omega(G) \geq 3$ *and* $\delta \in \{n-3, n-2\}$. *If $G[Y]$ is neither an empty graph nor a clique, then*

$$\gamma_{g[3R]}(G) \leq 3n - 7$$

*with equality if and only if $\overline{G} \in \{P_2 \cup C_3 \cup (n-5)K_1$ $(n \geq 6), P_2 \cup P_4 \cup (n-6)K_1$ $(n \geq 6)\}$.*

***Proof.*** It follows from Observation 14 that $|X \setminus N(v)| \in \{1, 2\}$ for each $v \in Y$. Moreover, since $G[Y]$ is neither an empty graph nor a clique, we have $|Y| \geq 3$. Thus, if $G[Y]$ is disconnected, then there must exist some vertex, say $v$, in $Y$ such that $|Y \setminus N[v]| \geq 2$ and hence $\delta \leq d(v) = n - |X \setminus N(v)| - |Y \setminus N[v]| - 1 \leq n - 4$, a contradiction. Therefore $G[Y]$ is connected. Moreover, since $G[Y]$ is not a clique, we have that $G[Y]$ has a path, say $P_3 = v_1v_2v_3$, on three vertices as an induced subgraph. Note that $v_1v_3 \notin E(G)$ and $|X \setminus N(v_i)| \in \{1, 2\}$ for each $i \in \{1, 2, 3\}$ as shown earlier. Further, since $\delta \in \{n-3, n-2\}$, we have $|X \setminus N(v_1)| = |X \setminus N(v_3)| = 1$. In fact, $\delta = n - 3$. Recall that $|X \setminus N(v_2)| \in \{1, 2\}$. According to the values of $|X \setminus N(v_2)|$, we have the following claims.

**Claim 1.** *If $|X \setminus N(v_2)| = 2$, then $\gamma_{g[3R]}(G) \leq 3n - 8$.*

***Proof.*** Note that $|X \setminus N(v_1)| = |X \setminus N(v_3)| = 1$. Therefore, $|(X \setminus N(v_2)) \cap (X \setminus N(v_i))| \in \{0, 1\}$ for each $i \in \{1, 3\}$. If $|(X \setminus N(v_2)) \cap (X \setminus N(v_i))| = 0$ for some $i \in \{1, 3\}$, then the function $h$ defined by $h(v_i) = h(v_2) = 4$, $h(x) = 0$ for each $x \in \{v_{4-i}\} \cup (X \setminus N(v_2)) \cup (X \setminus N(v_i))$ and $h(x) = 3$ otherwise, is a GTRDF on $G$ and therefore $\gamma_{g[3R]}(G) \leq \omega(h) = 3(n-6) + 8 = 3n - 10$. So in the following we may assume that $|(X \setminus N(v_2)) \cap (X \setminus N(v_1))| = |(X \setminus N(v_2)) \cap (X \setminus N(v_3))| = 1$. Without loss of generality, assume that $(X \setminus N(v_2)) \cap (X \setminus N(v_1)) = \{w_1\}$ and $(X \setminus N(v_2)) \cap (X \setminus N(v_3)) = \{w_2\}$. Note that $w_1 \neq w_2$ (for otherwise, since $v_1, v_2, v_3 \notin N(w_1)$, we have $\delta \leq d(w_1) \leq n - 4$, a contradiction). This forces $X \setminus N(v_1) = \{w_1\}$, $X \setminus N(v_2) = \{w_1, w_2\}$ and $X \setminus N(v_3) = \{w_2\}$. Then the function $h$ defined by $h(w_2) = h(v_3) = 2$, $h(v_1) = h(v_2) = 0$ and $h(x) = 3$ otherwise, is a GTRDF on $G$ and therefore $\gamma_{g[3R]}(G) \leq \omega(h) = 3(n-4) + 4 = 3n - 8$. Thus Claim 1 holds. □

**Claim 2.** *If $|X \setminus N(v_2)| = 1$, then $\gamma_{g[3R]}(G) \leq 3n - 7$ with equality if and only if $\overline{G} \in \{P_2 \cup C_3 \cup (n-5)K_1$ $(n \geq 6), P_2 \cup P_4 \cup (n-6)K_1$ $(n \geq 6)\}$.*

***Proof.*** Note that $|X \setminus N(v_1)| = |X \setminus N(v_2)| = |X \setminus N(v_3)| = 1$. If there exists some $i \in \{1, 3\}$ such that $X \setminus N(v_i) = X \setminus N(v_2)$, then $G[(N(v_2) \cap X) \cup \{v_i, v_2\}]$ is a clique with $|N(v_2) \cap X| + 2 = |X| + 1 = \omega(G) + 1$ vertices, a contradiction to the fact that the clique number is $\omega(G)$. So in the following we may assume that either $X \setminus N(v_1) = X \setminus N(v_3) \neq X \setminus N(v_2)$ or three sets $X \setminus N(v_1), X \setminus N(v_2)$ and $X \setminus N(v_3)$ are distinct. Let $Y \setminus \{v_1, v_2, v_3\} = \{v_4, v_5, \ldots, v_{n-\omega(G)}\}$ when $|Y| \geq 4$.

First, suppose that $X \setminus N(v_1) = X \setminus N(v_3) \neq X \setminus N(v_2)$. Moreover, since $|X \setminus N(v_1)| = |X \setminus N(v_2)| = |X \setminus N(v_3)| = 1$, we may assume that $X \setminus N(v_1) = X \setminus N(v_3) = \{w_1\}$ and $X \setminus N(v_2) = \{w_2\}$, where $w_1$ and $w_2$ are distinct. If $|Y| \geq 4$, then the function $h$ defined by $h(w_1) = h(w_2) = 4$, $h(v_1) = h(v_2) = h(v_3) = 0$, $h(v_4) = 2$ and $h(x) = 3$ otherwise, is a GTRDF on $G$ (noting that $|X \setminus N(v_4)| \in \{1, 2\}$ by Observation 14) and hence $\gamma_{g[3R]}(G) \leq \omega(h) = 3(n-6) + 10 = 3n - 8$. If $|Y| = 3$, then $\overline{G} = P_2 \cup C_3 \cup (n-5)K_1$ $(n = |X| + |Y| = \omega(G) + 3 \geq 6)$ and so by Proposition 19, $\gamma_{g[3R]}(G) = 3n - 7$.

Second, suppose that three sets $X \setminus N(v_1), X \setminus N(v_2)$ and $X \setminus N(v_3)$ are distinct. Moreover, since $|X \setminus N(v_1)| = |X \setminus N(v_2)| = |X \setminus N(v_3)| = 1$, we may assume that $X \setminus N(v_1) = \{w_1\}$, $X \setminus N(v_2) = \{w_2\}$ and $X \setminus N(v_3) = \{w_3\}$, where $w_1, w_2$ and $w_3$ are distinct. If $|Y| \geq 4$, then the function $h$ defined by $h(w_2) = h(w_3) = h(v_2) = h(v_3) = h(v_4) = 2$, $h(v_1) = 0$ and $h(x) = 3$ otherwise, is a GTRDF on $G$ (noting that $|X \setminus N(v_4)| \in \{1, 2\}$ by Observation 14) and therefore $\gamma_{g[3R]}(G) \leq \omega(h) = 3(n-6) + 10 = 3n-8$. If $|Y| = 3$, then $\overline{G} = P_2 \cup P_4 \cup (n-6)K_1$ $(n = |X| + |Y| = \omega(G) + 3 \geq 6)$ and so by Proposition 19, $\gamma_{g[3R]}(G) = 3n - 7$. Thus Claim 2 holds. □

This completes the proof. ■

**Theorem 21.** *For any connected graph $G$ on $n$ vertices,*

(a) $\gamma_{g[3R]}(G) \neq 3n - 3$,

(b) $\gamma_{g[3R]}(G) = 3n - 4$ *if and only if* $G \in \{P_4, K_{1,3}\}$ *or* $\overline{G} \in \{P_3 \cup K_1, 2P_2 \cup (n-4)K_1$ $(n \geq 4)\}$,

(c) $\gamma_{g[3R]}(G) = 3n - 5$ *if and only if* $\overline{G} \in \{C_4 \cup K_1, P_3 \cup (n-3)K_1$ $(n \geq 5), C_3 \cup (n-3)K_1$ $(n \geq 5), P_4 \cup (n-4)K_1$ $(n \geq 5)\}$,

(d) $\gamma_{g[3R]}(G) = 3n - 6$ *if and only if* $G \in \{H_1, C_5\}$ *or* $\overline{G} \in \{3P_2 \cup (n-6)K_1$ $(n \geq 6), C_4 \cup (n-4)K_1$ $(n \geq 6)\}$,

(e) $\gamma_{g[3R]}(G) = 3n - 7$ *if and only if* $G \in \{S(1,2), P_5\} \cup \mathcal{H} \cup \bigcup_{i=2}^{6}\{H_i\}$ *or* $\overline{G} \in \{S(1,2), H_2, H_3, P_5 \cup (n-5)K_1$ $(n \geq 5), P_2 \cup P_3 \cup (n-5)K_1$ $(n \geq 5), P_2 \cup C_3 \cup (n-5)K_1$ $(n \geq 5), P_2 \cup P_4 \cup (n-6)K_1$ $(n \geq 6)\}$,

*where $H_1$, $H_i$ $(i \in \{2,3\})$ and $H_i$ $(i \in \{4,5,6\})$ are illustrated in Figures* 2, 3 *and* 4, *respectively.*

***Proof.*** If $\operatorname{diam}(G) = 1$, then by Proposition A, $\gamma_{g[3R]}(G) = 3n$ $(n \geq 3)$. If $\operatorname{diam}(G) = 2$, then by Proposition C and Lemmas 12, 13, 16, 18 and 20, $\gamma_{g[3R]}(G) \leq 3n - 2$ and the following hold.

(a) $\gamma_{g[3R]}(G) = 3n - 2$ if and only if $G = K_n - e$ $(n \geq 3)$,

(b) $\gamma_{g[3R]}(G) \neq 3n-3$,

(c) $\gamma_{g[3R]}(G) = 3n-4$ if and only if $G = K_{1,3}$ or $\overline{G} \in \{P_3 \cup K_1, 2P_2 \cup (n-4)K_1$ $(n \geq 4)\}$,

(d) $\gamma_{g[3R]}(G) = 3n-5$ if and only if $\overline{G} \in \{C_3 \cup (n-3)K_1$ $(n \geq 5), P_4 \cup (n-4)K_1$ $(n \geq 5), C_4 \cup K_1, P_3 \cup (n-3)K_1$ $(n \geq 5)\}$,

(e) $\gamma_{g[3R]}(G) = 3n-6$ if and only if $G = C_5$ or $\overline{G} \in \{3P_2 \cup (n-6)K_1$ $(n \geq 6)$, $C_4 \cup (n-4)K_1$ $(n \geq 6)\}$,

(f) $\gamma_{g[3R]}(G) = 3n-7$ if and only if $G \in \mathcal{H} \cup \{H_4, H_5, H_6\}$ or $\overline{G} \in \{P_5 \cup (n-5)K_1$ $(n \geq 5), P_2 \cup P_3 \cup (n-5)K_1$ $(n \geq 5), P_2 \cup P_4 \cup (n-6)K_1$ $(n \geq 6), P_2 \cup C_3 \cup$ $(n-5)K_1$ $(n \geq 5)\}$.

If $\mathrm{diam}(G) = 3$, then by Lemma 10, $\gamma_{g[3R]}(G) \leq 3n-4$ and the following hold.

(a) $\gamma_{g[3R]}(G) = 3n-4$ if and only if $G = P_4$,

(b) $\gamma_{g[3R]}(G) \neq 3n-5$,

(c) $\gamma_{g[3R]}(G) = 3n-6$ if and only if $G = H_1$,

(d) $\gamma_{g[3R]}(G) = 3n-7$ if and only if $G \in \{S(1,2), H_2, H_3\}$ or $\overline{G} \in \{S(1,2),$ $H_2, H_3\}$.

If $\mathrm{diam}(G) \geq 4$, then by Lemma 8, $\gamma_{g[3R]}(G) \leq 3n-7$ with equality if and only if $G = P_5$. By considering all the above cases, the statement is trivial, which completes our proof. ■

Note that $\gamma_{g[3R]}(G) = \gamma_{g[3R]}(\overline{G})$ and if $G$ is a disconnected graph, then its complement $\overline{G}$ is a connected graph. Together with Theorem 21, we have the following characterization of any graphs with large global triple Roman domination number.

**Corollary 22.** *For any graph $G$ on $n$ vertices,*

(a) $\gamma_{g[3R]}(G) \neq 3n-3$,

(b) $\gamma_{g[3R]}(G) = 3n-4$ *if and only if one of $G$ and $\overline{G}$ belongs to* $\{P_4, K_{1,3}, P_3 \cup K_1, 2P_2 \cup (n-4)K_1$ $(n \geq 4)\}$,

(c) $\gamma_{g[3R]}(G) = 3n-5$ *if and only if one of $G$ and $\overline{G}$ belongs to* $\{C_4 \cup K_1, P_3 \cup (n-3)K_1$ $(n \geq 5), C_3 \cup (n-3)K_1$ $(n \geq 5), P_4 \cup (n-4)K_1$ $(n \geq 5)\}$,

(d) $\gamma_{g[3R]}(G) = 3n-6$ *if and only if one of $G$ and $\overline{G}$ belongs to* $\{H_1, C_5, 3P_2 \cup (n-6)K_1$ $(n \geq 6), C_4 \cup (n-4)K_1$ $(n \geq 6)\}$,

(e) $\gamma_{g[3R]}(G) = 3n-7$ *if and only if one of $G$ and $\overline{G}$ belongs to* $\{S(1,2), P_5 \cup (n-5)K_1$ $(n \geq 5), P_2 \cup P_3 \cup (n-5)K_1$ $(n \geq 5), P_2 \cup C_3 \cup (n-5)K_1$ $(n \geq 5), P_2 \cup P_4 \cup (n-6)K_1$ $(n \geq 6)\} \cup \mathcal{H} \cup \bigcup_{i=2}^{6}\{H_i\}$, *where* $H_1$, $H_i$ $(i \in \{2,3\})$ *and* $H_i$ $(i \in \{4,5,6\})$ *are illustrated in Figures* 2, 3 *and* 4, *respectively.*

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