

SOME UPPER BOUNDS ON RAMSEY NUMBERS INVOLVING C_4

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Abstract

We obtain some new upper bounds on the Ramsey numbers of the form

$$R(\underbrace{C_4, \dots, C_4}_m, G_1, \dots, G_n),$$

where $m \geq 1$ and G_1, \dots, G_n are arbitrary graphs. We focus on the cases of G_i 's being complete graph K_k , star $K_{1,k}$ or book B_k , where $B_k = K_2 + kK_1$. If $k \geq 2$, then our main upper bound theorem implies that

$$R(C_4, B_k) \leq R(C_4, K_{1,k}) + \left\lceil \sqrt{R(C_4, K_{1,k})} \right\rceil + 1.$$

Our techniques are used to obtain new upper bounds in several concrete cases, including: $R(C_4, K_{11}) \leq 43$, $R(C_4, K_{12}) \leq 51$, $R(C_4, K_3, K_4) \leq 29$, $R(C_4, K_4, K_4) \leq 66$, $R(C_4, K_3, K_3, K_3) \leq 57$, $R(C_4, C_4, K_3, K_4) \leq 75$, $R(C_4, C_4, K_4, K_4) \leq 177$, and $R(C_4, B_{17}) \leq 28$.

Keywords: Ramsey numbers, cycles.

2020 Mathematics Subject Classification: 05C55.

1. INTRODUCTION

For n given graphs H_1, H_2, \dots, H_n , the Ramsey number $R(H_1, H_2, \dots, H_n)$ is the smallest integer R such that if we arbitrarily color the edges of a complete graph of order R with n colors, then it contains a monochromatic copy of H_i in color i , for some $1 \leq i \leq n$.

We will use the following notations from [13]: K_k is a complete graph on k vertices, the graph kG is formed by k vertex-disjoint copies of G , $G \cup H$ stands for vertex-disjoint union of graphs, and the join graph $G + H$ is obtained by adding all of the edges between vertices of G and H to $G \cup H$. C_k is a cycle on k vertices, P_k is a path on k vertices, $K_{1,k} = K_1 + kK_1$ is a star on $k + 1$ vertices, and $B_k = K_2 + kK_1$ is a book on $k + 2$ vertices.

An (H_1, \dots, H_n) -coloring of the edges of K_N is a coloring using n colors, such that it does not contain any monochromatic copy of H_i in color i , for any i , $1 \leq i \leq n$. Note that if such coloring exists, then $N < R(H_1, \dots, H_n)$. In the case of 2 colors, we will interpret graphs G as colorings in which the edges of G are assigned the first color, and the nonedges are assigned the second color.

Let G be a graph or a coloring of edges and let $V(G)$ denote the vertex set of G . For $v \in V(G)$, $G - v$ is the graph or the coloring induced by $V(G) \setminus \{v\}$. If G is a coloring using n colors and $v \in V(G)$, then $d_i(v)$ is the number of edges in color i incident to v in G . If G is an (H_1, \dots, H_n) -coloring, $1 \leq i \leq n$, $v \in V(G)$ and $u_i \in V(H_i)$, then an elementary property of Ramsey colorings implies that $d_i(v) \leq R(H_1, \dots, H_{i-1}, H_i - u_i, H_{i+1}, \dots, H_n) - 1$. Numerous results on 2-color and multicolor Ramsey numbers involving C_4 are summarized in the dynamic survey [13], mainly in Sections 3.3 (note that $C_4 = K_{2,2}$), 4, and 6 [4,5,6,7].

The main goal of this paper is to derive some new upper bounds on the Ramsey numbers of the form $R(\underbrace{C_4, \dots, C_4}_m, G_1, \dots, G_n)$, where $m \geq 1$ and G_1, \dots, G_n

are arbitrary graphs. The main result, Theorem 6, is obtained in Section 2. Then, in Sections 3 and 4 we focus on the cases of G_i 's being complete graph, star or book. Also in these sections several new concrete upper bounds are presented.

2. MAIN RESULT

The main objective of this section is to obtain Theorem 6 claiming a new upper bound on the Ramsey numbers of the form $R(\underbrace{C_4, \dots, C_4}_m, G_1, \dots, G_n)$, with only relatively mild technical constraints. We need some auxilliary results, which will be presented first.

Lemma 1 (Sedrakyan's inequality [4]). *For any real numbers a_1, \dots, a_m and*

positive real numbers b_1, \dots, b_m , we have

$$\sum_{k=1}^m \frac{a_k^2}{b_k} \geq \frac{(\sum_{k=1}^m a_k)^2}{\sum_{k=1}^m b_k}.$$

Note that if $b_k = 1$ for all k , $1 \leq k \leq m$, then Lemma 1 reduces to:

Corollary 2. $\sum_{k=1}^m a_k^2 \geq \frac{(\sum_{k=1}^m a_k)^2}{m}.$

A simple argument, involving just the basic definition of Ramsey numbers, leads to the next lemma.

Lemma 3. $R(P_3, H_1, \dots, H_n) + 1 \leq R(C_4, H_1 \cup K_1, \dots, H_n \cup K_1)$

$$= \max\{R(C_4, H_1, \dots, H_n), |V(H_1)| + 1, \dots, |V(H_n)| + 1\}.$$

Proof. Let $N = R(P_3, H_1, \dots, H_n) - 1$. Consider any (P_3, H_1, \dots, H_n) -coloring of K_N . By adding a new vertex adjacent to all of K_N and using the first color for the new edges, a $(C_4, H_1 \cup K_1, \dots, H_n \cup K_1)$ -coloring of K_{N+1} is obtained. Thus, $N + 1 < R(C_4, H_1 \cup K_1, \dots, H_n \cup K_1)$ and the first part of the lemma is obtained. Next, observe that any graph G containing H_n contains $H_n \cup K_1$ as well, if $|V(G)| > |V(H_n)|$. Thus, $R(C_4, H_1, \dots, H_{n-1}, H_n \cup K_1) = \max\{R(C_4, H_1, \dots, H_{n-1}, H_n), |V(H_n)| + 1\}$. We complete the proof by using the same argument for all colors. ■

Lemma 4. Let $m \geq 1$ and $n \geq 0$. Consider n graphs, G_1, \dots, G_n . For each color i with $1 \leq i \leq n$, let $G'_i = G_i - w_i$, where $w_i \in V(G_i)$, and let r_i 's be integers such that

$$r_i \geq R(P_3, \underbrace{C_4, \dots, C_4}_{m-1}, G_1, \dots, G_{i-1}, G'_i, G_{i+1}, \dots, G_n).$$

Let $R = R(P_3, \underbrace{C_4, \dots, C_4}_{m-1}, G_1, \dots, G_n)$. Then, we have

$$(1) \quad R \leq \sum_{i=1}^n r_i - n + 3 + \frac{m^2 - m}{2} + \left\lceil \sqrt{\frac{(m^2 - m)^2}{4} + (m - 1)^2 \left(\sum_{i=1}^n r_i - n + 1 \right)} \right\rceil.$$

Proof. Let $N = R - 1$ and G be a $(P_3, \underbrace{C_4, \dots, C_4}_{m-1}, G_1, \dots, G_n)$ -coloring of the edges of K_N . Let $v_0 \in V(G)$ such that $\sum_{i=2}^m d_i(v_0) = \min_{v \in V(G)} \{\sum_{i=2}^m d_i(v)\}$.

In order to avoid a P_3 of the first color, we have $d_1(v_0) \leq 1$. If $1 \leq i \leq n$, in order to prevent a G_i of color $i + m$, we need $d_{i+m}(v_0) \leq r_i - 1$. Hence, we arrive at the relation

$$(2) \quad N = 1 + \sum_{i=1}^{m+n} d_i(v_0) \leq 2 + \sum_{i=2}^m d_i(v_0) + \sum_{i=1}^n (r_i - 1) = 2 - n + \sum_{i=2}^m d_i(v_0) + \sum_{i=1}^n r_i.$$

If $m = 1$, then $R = N + 1 \leq 3 - n + \sum_{i=1}^n r_i$, and the result is obtained.

Now, let us assume that $m \geq 2$.

Following a reasoning in [5, 12], for each color $i \in \{2, \dots, m\}$, since there is no C_4 of color i , for any pair of vertices $u_1, u_2 \in V(G)$, there is at most one vertex connected to both u_1 and u_2 by edges of color i . Since each vertex $v \in V(G)$ is the common neighbor in color i of exactly $\binom{d_i(v)}{2}$ pairs of vertices in $V(G)$, we have that $\sum_{v \in V(G)} \binom{d_i(v)}{2} \leq \binom{N}{2}$, and

$$\sum_{v \in V(G)} \left(\sum_{i=2}^m d_i(v)^2 - \sum_{i=2}^m d_i(v) \right) = \sum_{i=2}^m \sum_{v \in V(G)} d_i(v)(d_i(v) - 1) \leq (m-1)N(N-1).$$

Then, by Corollary 2, for any $v \in V(G)$ we have $\sum_{i=2}^m d_i(v)^2 \geq \frac{(\sum_{i=2}^m d_i(v))^2}{m-1}$, and thus

$$\begin{aligned} (m-1)N(N-1) &\geq \sum_{v \in V(G)} \left(\sum_{i=2}^m d_i(v)^2 - \sum_{i=2}^m d_i(v) \right) \\ &\geq \sum_{v \in V(G)} \left(\frac{(\sum_{i=2}^m d_i(v))^2}{m-1} - \sum_{i=2}^m d_i(v) \right) = \sum_{v \in V(G)} \left(\sum_{i=2}^m d_i(v) \right) \left(\frac{\sum_{i=2}^m d_i(v)}{m-1} - 1 \right) \\ &\geq N \sum_{i=2}^m d_i(v_0) \left(\frac{\sum_{i=2}^m d_i(v_0)}{m-1} - 1 \right) = N \left(\frac{(\sum_{i=2}^m d_i(v_0))^2}{m-1} - \sum_{i=2}^m d_i(v_0) \right). \end{aligned}$$

Hence, using (2), we obtain

$$\begin{aligned} \frac{(\sum_{i=2}^m d_i(v_0))^2}{m-1} - \sum_{i=2}^m d_i(v_0) &\leq (m-1)(N-1) \\ &\leq (m-1) \left(1 - n + \sum_{i=2}^m d_i(v_0) + \sum_{i=1}^n r_i \right) \end{aligned}$$

and

$$\left(\sum_{i=2}^m d_i(v_0) \right)^2 - (m-1) \sum_{i=2}^m d_i(v_0) \leq (m-1)^2 \left(1 - n + \sum_{i=2}^m d_i(v_0) + \sum_{i=1}^n r_i \right),$$

which implies

$$\left(\sum_{i=2}^m d_i(v_0)\right)^2 - m(m-1) \sum_{i=2}^m d_i(v_0) - (m-1)^2 \left(1 - n + \sum_{i=1}^n r_i\right) \leq 0.$$

Consequently, seeing the latter as a quadratic in $\sum_{i=2}^m d_i(v_0)$, we have that

$$\sum_{i=2}^m d_i(v_0) \leq \frac{m^2 - m}{2} + \sqrt{\frac{(m^2 - m)^2}{4} + (m-1)^2 \left(\sum_{i=1}^n r_i - n + 1\right)}.$$

Thus, by (2),

$$R \leq \sum_{i=1}^n r_i - n + 3 + \frac{m^2 - m}{2} + \sqrt{\frac{(m^2 - m)^2}{4} + (m-1)^2 \left(\sum_{i=1}^n r_i - n + 1\right)}.$$

Since R is an integer, the result is obtained. \blacksquare

Using Lemmas 3 and 4, we obtain the next (and last) lemma.

Lemma 5. *Let $m \geq 1$ and $n \geq 0$. Consider any graphs G_1, \dots, G_n . For each color i , $1 \leq i \leq n$, let $G'_i = G_i - w_i$, where $w_i \in V(G_i)$, and let r_i 's be integers such that*

$$r_i \geq R(\underbrace{C_4, \dots, C_4}_m, G_1, \dots, G_{i-1}, G'_i, G_{i+1}, \dots, G_n).$$

Assume further that $R(\underbrace{C_4, \dots, C_4}_m, G_1, \dots, G_n) > \max_{1 \leq i \leq n} \{|V(G_i)|\}$ and $G_i \neq K_2$ for some $i \in \{1, \dots, n\}$ if $m = 1$. Then we have

$$(3) \quad \begin{aligned} & R(P_3, \underbrace{C_4, \dots, C_4}_{m-1}, G_1, \dots, G_n) \\ & \leq \sum_{i=1}^n r_i - n + \frac{m^2 + m}{2} + \left\lceil m \sqrt{\frac{(m+1)^2}{4} + \sum_{i=1}^n r_i - n} \right\rceil. \end{aligned}$$

Proof. Let $RHS(1)$ denote the right-hand side of inequality (1) in Lemma 4, and let $RHS(3)$ denote the right-hand side of inequality (3). In order to prove this lemma, by Lemma 4, it suffices to show that $RHS(3) \geq RHS(1)$. In the proof below, among other steps, we will use an easy observation that for any positive integer k , it is true that $\lceil \sqrt{k+1} \rceil = \lfloor \sqrt{k} \rfloor + 1$.

If $m \geq 2$ then $RHS(3) =$

$$\begin{aligned}
& \sum_{i=1}^n r_i - n + \frac{m^2 + m}{2} + 1 + \left\lfloor \sqrt{\frac{m^2(m+1)^2}{4} + m^2 \left(\sum_{i=1}^n r_i - n \right) - 1} \right\rfloor \\
& \geq \sum_{i=1}^n r_i - n + 1 + \frac{m^2 - m}{2} + m + \left\lfloor \sqrt{\frac{(m^2 - m)^2}{4} + m^3 + (m-1)^2 \left(\sum_{i=1}^n r_i - n \right) - 1} \right\rfloor \\
& = \sum_{i=1}^n r_i - n + (1 + m) + \frac{m^2 - m}{2} \\
& + \left\lfloor \sqrt{\frac{(m^2 - m)^2}{4} + (m-1)^2 \left(\sum_{i=1}^n r_i - n + 1 \right) + (m^2 + 2)(m-1)} \right\rfloor \geq RHS(1).
\end{aligned}$$

If $m = 1$, let i_0 be an integer such that $G_{i_0} \neq K_2$, so that $r_{i_0} \geq 2$ and $\sum_{i=1}^n r_i - n \geq 1$. Then

$$RHS(3) = \sum_{i=1}^n r_i - n + 1 + \left\lfloor \sqrt{1 + \sum_{i=1}^n r_i - n} \right\rfloor \geq \sum_{i=1}^n r_i - n + 3 = RHS(1),$$

where in the latter the RHS 's were simplified using $m = 1$. ■

Now, we are ready to present our main result.

Theorem 6. *Let $m \geq 1$ and $n \geq 0$. Consider n graphs, G_1, \dots, G_n . For each color i with $1 \leq i \leq n$, let $G'_i = G_i - w_i$, where $w_i \in V(G_i)$, and let r_i 's be integers such that*

$$r_i \geq R(\underbrace{C_4, \dots, C_4}_m, G_1, \dots, G_{i-1}, G'_i, G_{i+1}, \dots, G_n).$$

Assume further that $R = R(\underbrace{C_4, \dots, C_4}_m, G_1, \dots, G_n) > \max_{1 \leq i \leq n} \{|V(G_i)|\}$ and $G_i \neq K_2$ for some $i \in \{1, \dots, n\}$ if $m = 1$. Then, we have

$$R \leq \sum_{i=1}^n r_i - n + 1 + \frac{m^2 + m}{2} + \left\lfloor m \sqrt{\frac{(m+1)^2}{4} + \sum_{i=1}^n r_i - n} \right\rfloor.$$

Proof. Set $N = R - 1$, and let G be a $(\underbrace{C_4, \dots, C_4}_m, G_1, \dots, G_n)$ -coloring of the edges of K_N . Let $v_0 \in V(G)$ such that $\sum_{i=1}^m d_i(v_0) = \min_{v \in V(G)} \{\sum_{i=1}^m d_i(v)\}$.

For $1 \leq i \leq n$, in order to avoid G_i of color $i+m$, we must have $d_{i+m}(v_0) \leq r_i - 1$. Hence, we also have

$$(4) \quad N = 1 + \sum_{i=1}^{m+n} d_i(v_0) \leq 1 - n + \sum_{i=1}^m d_i(v_0) + \sum_{i=1}^n r_i.$$

For each $i \in \{1, \dots, m\}$, the number of P_3 's in color i cannot exceed $\binom{N}{2}$, since otherwise they would force a C_4 in color i . Thus, as noted in the proof of Lemma 4, $\sum_{v \in V(G)} \binom{d_i(v)}{2} \leq \binom{N}{2}$. If $\sum_{v \in V(G)} \binom{d_i(v)}{2} = \binom{N}{2}$, then by the Friendship Theorem [5], which states that in any graph in which any two vertices have precisely one common neighbor, then there is a vertex which is adjacent to all other vertices. In that case, let u be the vertex adjacent to all the others with edges of the first color. $G - u$ is a $(P_3, \underbrace{C_4, \dots, C_4}_{m-1}, G_1, \dots, G_n)$ -coloring of K_{N-1} ,

so $R - 2 = N - 1 \geq R(P_3, \underbrace{C_4, \dots, C_4}_{m-1}, G_1, \dots, G_n) - 1$, and by Lemma 5, the result follows.

Similarly, the same argument applies if $\sum_{v \in V(G)} \binom{d_i(v)}{2} = \binom{N}{2}$ for some $i \leq m$. Therefore, we can assume that $\sum_{v \in V(G)} d_i(v)(d_i(v) - 1) < N(N - 1)$ for all i and

$$\sum_{v \in V(G)} \left(\sum_{i=1}^m d_i(v)^2 - \sum_{i=1}^m d_i(v) \right) = \sum_{i=1}^m \sum_{v \in V(G)} d_i(v)(d_i(v) - 1) < mN(N - 1).$$

Then, by Corollary 2, for any $v \in V(G)$ we have $m \sum_{i=1}^m d_i(v)^2 \geq (\sum_{i=1}^m d_i(v))^2$, and further

$$\begin{aligned} mN(N - 1) &> \sum_{v \in V(G)} \left(\sum_{i=1}^m d_i(v)^2 - \sum_{i=1}^m d_i(v) \right) \\ &\geq \sum_{v \in V(G)} \left(\frac{(\sum_{i=1}^m d_i(v))^2}{m} - \sum_{i=1}^m d_i(v) \right) = \sum_{v \in V(G)} \left(\sum_{i=1}^m d_i(v) \right) \left(\frac{\sum_{i=1}^m d_i(v)}{m} - 1 \right) \\ &\geq N \sum_{i=1}^m d_i(v_0) \left(\frac{\sum_{i=1}^m d_i(v_0)}{m} - 1 \right) = N \left(\frac{(\sum_{i=1}^m d_i(v_0))^2}{m} - \sum_{i=1}^m d_i(v_0) \right). \end{aligned}$$

Therefore, by (4), we see that

$$\frac{(\sum_{i=1}^m d_i(v_0))^2}{m} - \sum_{i=1}^m d_i(v_0) < m(N - 1) \leq m \left(-n + \sum_{i=1}^m d_i(v_0) + \sum_{i=1}^n r_i \right)$$

and

$$\left(\sum_{i=1}^m d_i(v_0) \right)^2 - m(m + 1) \sum_{i=1}^m d_i(v_0) - m^2 \left(-n + \sum_{i=1}^n r_i \right) < 0,$$

and hence

$$\sum_{i=1}^m d_i(v_0) \leq \frac{m^2 + m}{2} + \sqrt{\frac{(m^2 + m)^2}{4} + m^2 \left(\sum_{i=1}^n r_i - n \right) - 1}.$$

Consequently, by (4),

$$R = N + 1 \leq 2 + \sum_{i=1}^n r_i - n + \frac{m^2 + m}{2} + \sqrt{\frac{(m^2 + m)^2}{4} + m^2 \left(\sum_{i=1}^n r_i - n \right) - 1}.$$

Since R is an integer, we have

$$\begin{aligned} R &\leq 2 + \sum_{i=1}^n r_i - n + \frac{m^2 + m}{2} + \left\lfloor \sqrt{\frac{(m^2 + m)^2}{4} + m^2 \left(\sum_{i=1}^n r_i - n \right) - 1} \right\rfloor \\ &= \sum_{i=1}^n r_i - n + 1 + \frac{m^2 + m}{2} + \left\lfloor m \sqrt{\frac{(m+1)^2}{4} + \sum_{i=1}^n r_i - n} \right\rfloor, \end{aligned}$$

and the result follows. ■

Note that if $m \geq 2$ and $n = 0$, then the bound in Theorem 6 coincides with the known result $R(\underbrace{C_4, \dots, C_4}_m) \leq m^2 + m + 1$ [2, 9].

3. COMPLETE GRAPHS

In this section, we focus attention on concrete upper bounds for the Ramsey numbers of the form $R(\underbrace{C_4, \dots, C_4}_m, G_1, \dots, G_n)$, where all G_i 's are complete graphs,

for $1 \leq i \leq n$. We gather our results in Table 1, in which the new upper bounds are shown in the last column. Proposition 7 below provides the upper bound in Case #3, while all other cases are derived in the proof of Theorem 8.

Proposition 7. $R(C_4, K_3, K_4) \leq 29$.

Proof. First, we note that $R(K_3, K_4) = 9$ [8] and $R(C_4, K_9) = 30$ [10]. Hence, if there exists any (C_4, K_3, K_4) -coloring G of K_{29} , then by merging the last two colors of G we obtain a (C_4, K_9) -coloring, i.e., a C_4 -free graph G' on 29 vertices with maximum independent set of order at most 8. All such graphs were obtained in [10], and up to isomorphism there are 267 of them.

Case #	Ramsey number	$m, n, \sum_{i=1}^n r_i$	lower bound	old upper bound	new upper bound
1	$R(C_4, K_{11})$	1, 1, 36	40 [14]	44 [10]	43
2	$R(C_4, K_{12})$	1, 1, 43	43 (*)	52 [10]	51
3	$R(C_4, K_3, K_4)$	Proposition 7	27 [3]	32 [16]	29
4	$R(C_4, K_4, K_4)$	1, 2, 58	52 [16]	71 [11]	66
5	$R(C_4, K_3, K_3, K_3)$	1, 3, 51	49 [1]	59 [11]	57
6	$R(C_4, C_4, K_3, K_4)$	2, 2, 57	43 [3]	76 [16]	75
7	$R(C_4, C_4, K_4, K_4)$	2, 2, 150	87 [16]	179 [16]	177

Table 1. New bounds on Ramsey numbers of C_4 versus complete graphs described in Section 3: parameters, lower bounds and old and new upper bounds. (*) Lower bound 43 in case #2 is easily obtained by adding vertex-disjoint K_3 to the lower bound witness graph in case #1. In all cases, except case #3, the new upper bound is obtained by using Theorem 6.

We verified by computations that for every such graph (one of 267 possible graphs), its non-edges cannot be partitioned into a K_3 -free graph and a K_4 -free graph. Thus, no (C_4, K_3, K_4) -coloring of K_{29} exists, and the bound in the proposition holds. ■

Theorem 8. *The upper bounds in the last column of Table 1 hold.*

Proof. Proposition 7 proves the bound in Case #3. The upper bounds in all other cases are obtained by applying Theorem 6 with some additional simple steps, as described below.

#1. It is known that $R(C_4, K_{10}) = 36$ [10]. Theorem 6 with $m = n = 1$, $G_1 = K_{11}$ and $r_1 = 36$ gives $R(C_4, K_{11}) \leq 36 + 1 + \sqrt{36} = 43$.

#2. Let $r_1 = 43$, so by Case #1 we have $r_1 \geq R(C_4, K_{11})$. With $m = n = 1$ and $G_1 = K_{12}$, we obtain $R(C_4, K_{12}) \leq r_1 + 1 + \lceil \sqrt{r_1} \rceil = 44 + 7 = 51$.

#4. Let $r_1 = r_2 = 29$, so by Case #3 we have $r_1, r_2 \geq R(C_4, K_3, K_4)$. With $m = 1$, $n = 2$ and $G_1 = G_2 = K_4$, we obtain

$$R(C_4, K_4, K_4) \leq r_1 + r_2 + \lceil \sqrt{r_1 + r_2 - 1} \rceil = 58 + \lceil \sqrt{57} \rceil = 66.$$

#5. It is known that $R(C_4, K_3, K_3) = 17$ [6]. Let $r_i = 17$ and $G_i = K_3$ for $1 \leq i \leq 3$. With $m = 1$ and $n = 3$, we have

$$R(C_4, K_3, K_3, K_3) \leq r_1 + r_2 + r_3 - 1 + \lceil \sqrt{r_1 + r_2 + r_3 - 2} \rceil = 50 + \sqrt{49} = 57.$$

#6. It is known that $R(C_4, C_4, K_4) \leq 21$ [11] and $R(C_4, C_4, K_3, K_3) \leq 36$ [15]. Let $r_1 = 21$ and $r_2 = 36$, so that $r_1 \geq R(C_4, C_4, K_4)$ and $r_2 \geq$

$R(C_4, C_4, K_3, K_3)$. With $m = 2$, $n = 2$, $G_1 = K_3$ and $G_2 = K_4$, we have

$$\begin{aligned} R(C_4, C_4, K_3, K_4) &\leq r_1 + r_2 + 2 + \left\lceil 2\sqrt{9/4 + r_1 + r_2 - 2} \right\rceil \\ &= 59 + \left\lceil \sqrt{9 + 220} \right\rceil = 75. \end{aligned}$$

#7. Let $r_1 = r_2 = 75$, so by Case #6 we have $r_1, r_2 \geq R(C_4, C_4, K_3, K_4)$. With $m = 2$, $n = 2$ and $G_1 = G_2 = K_4$, we have

$$R(C_4, C_4, K_4, K_4) \leq r_1 + r_2 + 2 + \left\lceil 2\sqrt{9/4 + r_1 + r_2 - 2} \right\rceil = 152 + \left\lceil \sqrt{1 + 600} \right\rceil = 177. \quad \blacksquare$$

4. STARS AND BOOKS

We start this section with a classical result obtained by Parsons in 1975.

Lemma 9 [12]. *For $k \geq 2$, we have $R(C_4, K_{1,k}) \leq k + \left\lceil \sqrt{k} \right\rceil + 1$.*

Proof. The original proof was presented by Parsons, but we note that the same result is implied by our Theorem 6 using $m = n = 1$ and $r_1 = R(C_4, kK_1) = k$. \blacksquare

The next corollary puts together Theorem 6 and Lemma 9.

Corollary 10. *For $k \geq 2$, we have*

$$R(C_4, B_k) \leq R(C_4, K_{1,k}) + \left\lceil \sqrt{R(C_4, K_{1,k})} \right\rceil + 1 \leq k + \left\lceil \sqrt{k + \left\lceil \sqrt{k} \right\rceil + 1} \right\rceil + \left\lceil \sqrt{k} \right\rceil + 2.$$

Proof. Since $B_k = K_1 + K_{1,k}$, using Theorem 6 with $m = n = 1$ and $r_1 = k + \left\lceil \sqrt{k} \right\rceil + 1$ gives the first inequality. The second inequality is obtained by Lemma 9. \blacksquare

Note that for $k = q^2 - q + 1$ we have $\left\lceil \sqrt{k} \right\rceil = q$. Our result in Corollary 10, which holds for all integers $k \geq 2$, generalizes a result by Faudree, Rousseau and Sheehan [7]. In particular, the Lemma in Section 2 of [7] implies that $R(C_4, B_{17}) \leq 29$, while our Corollary 10 using Parson's [12] result $R(C_4, K_{1,17}) = 22$ gives a better bound, namely $R(C_4, B_{17}) \leq 28$.

Our last corollary about multicolor Ramsey numbers of C_4 's versus stars is also a consequence of Theorem 6.

Corollary 11. *Let $m, n, k_1, \dots, k_n \geq 1$, such that $m + \sum_{i=1}^n k_i \geq n + 2$. Then*

$$\begin{aligned} & R(\underbrace{C_4, \dots, C_4}_m, K_{1,k_1}, \dots, K_{1,k_n}) \\ & \leq 1 + \sum_{i=1}^n k_i - n + \frac{m^2 + m}{2} + \left\lceil m \sqrt{\frac{(m+1)^2}{4} + \sum_{i=1}^n k_i - n} \right\rceil. \end{aligned}$$

Proof. Let w_i be the vertex of degree k_i in K_{1,k_i} . Since $k_i K_1 = K_{1,k_i} - w_i$, apply Theorem 6 with

$$r_i = R(\underbrace{C_4, \dots, C_4}_m, K_{1,k_1}, \dots, K_{1,k_{i-1}}, k_i K_1, K_{1,k_{i+1}}, \dots, K_{1,k_n}),$$

and observe that $r_i = k_i$. The result follows. \blacksquare

We note that if we consider Corollary 11 with $n = 1$, then it reduces to a result obtained in [17], which states that for $k, m \geq 1$, with $m + k > 3$, it holds that

$$R(\underbrace{C_4, \dots, C_4}_m, K_{1,k}) \leq k + \frac{m^2 + m}{2} + \left\lceil m \sqrt{k + \frac{m^2 + 2m - 3}{4}} \right\rceil.$$

REFERENCES

- [1] D. Bevan, *Personal communication to S. Radziszowski* (2002), manuscript.
- [2] F.R.K. Chung, *On triangular and cyclic Ramsey numbers with k colors*, in: Graphs Combin., R. Bari and F. Harary (Ed(s)), Lect. Notes Math. **406** (Springer, Berlin, 1974) 236–241.
- [3] J. Dybizbański and T. Dzido, *On some Ramsey numbers for quadrilaterals*, Electron. J. Combin. **18(1)** (2011) #P154.
<https://doi.org/10.37236/641>
- [4] A. Engel, *Problem-Solving Strategies* (Springer-Verlag New York, Inc, 1998).
- [5] P. Erdős, A. Rényi and V.T. Sós, *On a problem of graph theory*, Studia Sci. Math. Hungar. **1** (1966) 215–235.
- [6] G. Exoo and D.F. Reynolds, *Ramsey numbers based on C_5 -decompositions*, Discrete Math. **71** (1988) 119–127.
[https://doi.org/10.1016/0012-365X\(88\)90065-9](https://doi.org/10.1016/0012-365X(88)90065-9)
- [7] R.J. Faudree, C.C. Rousseau and J. Sheehan, *More from the Good Book*, Ninth Southeastern Conference on Combinatorics, Graph Theory, and Computing, Util. Math. Congr. Numer. **XXI** (1978) 289–299.

- [8] R.E. Greenwood and A.M. Gleason, *Combinatorial relations and chromatic graphs*, Canad. J. Math. **7** (1955) 1–7.
<https://doi.org/10.4153/CJM-1955-001-4>
- [9] R.W. Irving, *Generalised Ramsey numbers for small graphs*, Discrete Math. **9** (1974) 251–264.
[https://doi.org/10.1016/0012-365X\(74\)90008-9](https://doi.org/10.1016/0012-365X(74)90008-9)
- [10] A. Lange, I. Livinsky and S.P. Radziszowski, *Computation of the Ramsey numbers $R(C_4, K_9)$ and $R(C_4, K_{10})$* , J. Combin. Math. Combin. Comput. **97** (2016) 139–154.
- [11] B. Lidický and F. Pfender, *Semidefinite programming and Ramsey numbers* (2017, revised version 2020).
arXiv:1704.03592
- [12] T.D. Parsons, *Ramsey graphs and block designs*, I, Trans. Amer. Math. Soc. **209** (1975) 33–44.
<https://doi.org/10.1090/S0002-9947-1975-0396317-X>
- [13] S. Radziszowski, *Small Ramsey numbers*, Electron. J. Combin. (2024) DS1, revision #17.
- [14] S. Van Overberghe, *Algorithms for Computing Ramsey Numbers*, MS Thesis in Mathematics, Ghent University, Belgium (2020).
<https://github.com/Steven-VO/circulant-Ramsey>
- [15] X. Xu and S.P. Radziszowski, $28 \leq R(C_4, C_4, C_3, C_3) \leq 36$, Util. Math. **79** (2009) 253–257.
- [16] X. Xu, Z. Shao and S.P. Radziszowski, *Bounds on some Ramsey numbers involving quadrilateral*, Ars Combin. **90** (2009) 337–344.
- [17] X. Zhang, Y. Chen and T.C. Edwin Cheng, *On three color Ramsey numbers for $R(C_4, C_4, K_{1,n})$* , Discrete Math. **342** (2019) 285–291.
<https://doi.org/10.1016/j.disc.2018.09.030>

Received 4 December 2023

Revised 31 July 2024

Accepted 31 July 2024

Available online 6 September 2024