Discussiones Mathematicae
Graph Theory 45 (2025) 845–856
https://doi.org/10.7151/dmgt.2557

SOME UPPER BOUNDS ON RAMSEY NUMBERS INVOLVING C_4

Luis Boza

Departamento de Matemática Aplicada I
Universidad de Sevilla
e-mail: boza@us.es

and

Stanisław P. Radziszowski

Department of Computer Science
Rochester Institute of Technology
e-mail: spr@cs.rit.edu

Abstract

We obtain some new upper bounds on the Ramsey numbers of the form

$$R(\underbrace{C_4, \ldots, C_4}_{m}, G_1, \ldots, G_n),$$

where $m \geq 1$ and $G_1, \ldots, G_n$ are arbitrary graphs. We focus on the cases of G_i's being complete graph K_k, star $K_{1,k}$ or book B_k, where $B_k = K_2 + kK_1$. If $k \geq 2$, then our main upper bound theorem implies that

$$R(C_4, B_k) \leq R(C_4, K_{1,k}) + \left\lceil \sqrt{R(C_4, K_{1,k})} \right\rceil + 1.$$

Our techniques are used to obtain new upper bounds in several concrete cases, including: $R(C_4, K_{11}) \leq 43$, $R(C_4, K_{12}) \leq 51$, $R(C_4, K_3, K_4) \leq 29$, $R(C_4, K_4, K_4) \leq 66$, $R(C_4, K_3, K_3, K_3) \leq 57$, $R(C_4, C_4, K_3, K_4) \leq 75$, $R(C_4, C_4, K_4, K_4) \leq 177$, and $R(C_4, B_{17}) \leq 28$.

Keywords: Ramsey numbers, cycles.

2020 Mathematics Subject Classification: 05C55.

1. Introduction

For n given graphs $H_1, H_2, \ldots, H_n$, the Ramsey number $R(H_1, H_2, \ldots, H_n)$ is the smallest integer R such that if we arbitrarily color the edges of a complete graph of order R with n colors, then it contains a monochromatic copy of H_i in color i, for some $1 \leq i \leq n$.

We will use the following notations from [13]: K_k is a complete graph on k vertices, the graph kG is formed by k vertex-disjoint copies of G, $G \cup H$ stands for vertex-disjoint union of graphs, and the join graph $G + H$ is obtained by adding all of the edges between vertices of G and H to $G \cup H$. C_k is a cycle on k vertices, P_k is a path on k vertices, $K_{1,k} = K_1 + kK_1$ is a star on $k + 1$ vertices, and $B_k = K_2 + kK_1$ is a book on $k + 2$ vertices.

An $(H_1, \ldots, H_n)$-coloring of the edges of K_N is a coloring using n colors, such that it does not contain any monochromatic copy of H_i in color i, for any i, $1 \leq i \leq n$. Note that if such coloring exists, then $N < R(H_1, \ldots, H_n)$. In the case of 2 colors, we will interpret graphs G as colorings in which the edges of G are assigned the first color, and the nonedges are assigned the second color.

Let G be a graph or a coloring of edges and let $V(G)$ denote the vertex set of G. For $v \in V(G)$, $G - v$ is the graph or the coloring induced by $V(G) \setminus \{v\}$. If G is a coloring using n colors and $v \in V(G)$, then $d_i(v)$ is the number of edges in color i incident to v in G. If G is an $(H_1, \ldots, H_n)$-coloring, $1 \leq i \leq n$, $v \in V(G)$ and $u_i \in V(H_i)$, then an elementary property of Ramsey colorings implies that $d_i(v) \leq R(H_1, \ldots, H_{i-1}, H_i - u_i, H_{i+1}, \ldots, H_n) - 1$. Numerous results on 2-color and multicolor Ramsey numbers involving C_4 are summarized in the dynamic survey [13], mainly in Sections 3.3 (note that $C_4 = K_{2,2}$), 4, and 6 [4,5,6,7].

The main goal of this paper is to derive some new upper bounds on the Ramsey numbers of the form $R(\underbrace{C_4, \ldots, C_4}_{m}, G_1, \ldots, G_n)$, where $m \geq 1$ and $G_1, \ldots, G_n$ are arbitrary graphs. The main result, Theorem 6, is obtained in Section 2. Then, in Sections 3 and 4 we focus on the cases of G_i's being complete graph, star or book. Also in these sections several new concrete upper bounds are presented.

2. Main Result

The main objective of this section is to obtain Theorem 6 claiming a new upper bound on the Ramsey numbers of the form $R(\underbrace{C_4, \ldots, C_4}_{m}, G_1, \ldots, G_n)$, with only relatively mild technical constraints. We need some auxilliary results, which will be presented first.

Lemma 1 (Sedrakyan's inequality [4])**.** *For any real numbers $a_1, \ldots, a_m$ and* *positive real numbers $b_1, \ldots, b_m$, we have*

$$\sum_{k=1}^{m} \frac{a_k^2}{b_k} \geq \frac{\left(\sum_{k=1}^{m} a_k\right)^2}{\sum_{k=1}^{m} b_k}.$$

Note that if $b_k = 1$ for all k, $1 \leq k \leq m$, then Lemma 1 reduces to:

Corollary 2. $\sum_{k=1}^{m} a_k^2 \geq \dfrac{(\sum_{k=1}^{m} a_k)^2}{m}$.

A simple argument, involving just the basic definition of Ramsey numbers, leads to the next lemma.

Lemma 3. $R(P_3, H_1, \ldots, H_n) + 1 \leq R(C_4, H_1 \cup K_1, \ldots, H_n \cup K_1)$

$$= \max\{R(C_4, H_1, \ldots, H_n), |V(H_1)| + 1, \ldots, |V(H_n)| + 1\}.$$

Proof. Let $N = R(P_3, H_1, \ldots, H_n) - 1$. Consider any $(P_3, H_1, \ldots, H_n)$-coloring of K_N. By adding a new vertex adjacent to all of K_N and using the first color for the new edges, a $(C_4, H_1 \cup K_1, \ldots, H_n \cup K_1)$-coloring of K_{N+1} is obtained. Thus, $N + 1 < R(C_4, H_1 \cup K_1, \ldots, H_n \cup K_1)$ and the first part of the lemma is obtained. Next, observe that any graph G containing H_n contains $H_n \cup K_1$ as well, if $|V(G)| > |V(H_n)|$. Thus, $R(C_4, H_1, \ldots, H_{n-1}, H_n \cup K_1) = \max\{R(C_4, H_1, \ldots, H_{n-1}, H_n),\ |V(H_n)| + 1\}$. We complete the proof by using the same argument for all colors. ■

Lemma 4. *Let $m \geq 1$ and $n \geq 0$. Consider n graphs, $G_1, \ldots, G_n$. For each color i with $1 \leq i \leq n$, let $G_i' = G_i - w_i$, where $w_i \in V(G_i)$, and let r_i's be integers such that*

$$r_i \geq R(P_3, \underbrace{C_4, \ldots, C_4}_{m-1}, G_1, \ldots, G_{i-1}, G_i', G_{i+1}, \ldots, G_n).$$

Let $R = R(P_3, \underbrace{C_4, \ldots, C_4}_{m-1}, G_1, \ldots, G_n)$. Then, we have

$$(1)\quad R \leq \sum_{i=1}^{n} r_i - n + 3 + \frac{m^2 - m}{2} + \left\lfloor \sqrt{\frac{\left(m^2 - m\right)^2}{4} + (m-1)^2 \left(\sum_{i=1}^{n} r_i - n + 1\right)} \right\rfloor.$$

Proof. Let $N = R - 1$ and G be a $(P_3, \underbrace{C_4, \ldots, C_4}_{m-1}, G_1, \ldots, G_n)$-coloring of the edges of K_N. Let $v_0 \in V(G)$ such that $\sum_{i=2}^{m} d_i(v_0) = \min_{v \in V(G)}\{\sum_{i=2}^{m} d_i(v)\}$.

In order to avoid a P_3 of the first color, we have $d_1(v_0) \leq 1$. If $1 \leq i \leq n$, in order to prevent a G_i of color $i+m$, we need $d_{i+m}(v_0) \leq r_i - 1$. Hence, we arrive at the relation

$$(2) \quad N = 1 + \sum_{i=1}^{m+n} d_i(v_0) \leq 2 + \sum_{i=2}^{m} d_i(v_0) + \sum_{i=1}^{n} (r_i - 1) = 2 - n + \sum_{i=2}^{m} d_i(v_0) + \sum_{i=1}^{n} r_i.$$

If $m = 1$, then $R = N + 1 \leq 3 - n + \sum_{i=1}^{n} r_i$, and the result is obtained.

Now, let us assume that $m \geq 2$.

Following a reasoning in [5, 12], for each color $i \in \{2, \ldots, m\}$, since there is no C_4 of color i, for any pair of vertices $u_1, u_2 \in V(G)$, there is at most one vertex connected to both u_1 and u_2 by edges of color i. Since each vertex $v \in V(G)$ is the common neighbor in color i of exactly $\binom{d_i(v)}{2}$ pairs of vertices in $V(G)$, we have that $\sum_{v \in V(G)} \binom{d_i(v)}{2} \leq \binom{N}{2}$, and

$$\sum_{v \in V(G)} \left(\sum_{i=2}^{m} d_i(v)^2 - \sum_{i=2}^{m} d_i(v) \right) = \sum_{i=2}^{m} \sum_{v \in V(G)} d_i(v)(d_i(v) - 1) \leq (m-1)N(N-1).$$

Then, by Corollary 2, for any $v \in V(G)$ we have $\sum_{i=2}^{m} d_i(v)^2 \geq \dfrac{(\sum_{i=2}^{m} d_i(v))^2}{m-1}$, and thus

$$\begin{aligned}
&(m-1)N(N-1) \geq \sum_{v \in V(G)} \left(\sum_{i=2}^{m} d_i(v)^2 - \sum_{i=2}^{m} d_i(v) \right) \\
&\geq \sum_{v \in V(G)} \left(\frac{(\sum_{i=2}^{m} d_i(v))^2}{m-1} - \sum_{i=2}^{m} d_i(v) \right) = \sum_{v \in V(G)} \left(\sum_{i=2}^{m} d_i(v) \right) \left(\frac{\sum_{i=2}^{m} d_i(v)}{m-1} - 1 \right) \\
&\geq N \sum_{i=2}^{m} d_i(v_0) \left(\frac{\sum_{i=2}^{m} d_i(v_0)}{m-1} - 1 \right) = N \left(\frac{(\sum_{i=2}^{m} d_i(v_0))^2}{m-1} - \sum_{i=2}^{m} d_i(v_0) \right).
\end{aligned}$$

Hence, using (2), we obtain

$$\begin{aligned}
\frac{(\sum_{i=2}^{m} d_i(v_0))^2}{m-1} - \sum_{i=2}^{m} d_i(v_0) &\leq (m-1)(N-1) \\
&\leq (m-1) \left(1 - n + \sum_{i=2}^{m} d_i(v_0) + \sum_{i=1}^{n} r_i \right)
\end{aligned}$$

and

$$\left(\sum_{i=2}^{m} d_i(v_0) \right)^2 - (m-1) \sum_{i=2}^{m} d_i(v_0) \leq (m-1)^2 \left(1 - n + \sum_{i=2}^{m} d_i(v_0) + \sum_{i=1}^{n} r_i \right),$$

which implies

$$\left(\sum_{i=2}^{m} d_i(v_0)\right)^2 - m(m-1)\sum_{i=2}^{m} d_i(v_0) - (m-1)^2\left(1 - n + \sum_{i=1}^{n} r_i\right) \leq 0.$$

Consequently, seeing the latter as a quadratic in $\sum_{i=2}^{m} d_i(v_0)$, we have that

$$\sum_{i=2}^{m} d_i(v_0) \leq \frac{m^2-m}{2} + \sqrt{\frac{\left(m^2-m\right)^2}{4} + (m-1)^2\left(\sum_{i=1}^{n} r_i - n + 1\right)}.$$

Thus, by (2),

$$R \leq \sum_{i=1}^{n} r_i - n + 3 + \frac{m^2-m}{2} + \sqrt{\frac{\left(m^2-m\right)^2}{4} + (m-1)^2\left(\sum_{i=1}^{n} r_i - n + 1\right)}.$$

Since R is an integer, the result is obtained. ∎

Using Lemmas 3 and 4, we obtain the next (and last) lemma.

Lemma 5. *Let $m \geq 1$ and $n \geq 0$. Consider any graphs $G_1, \ldots, G_n$. For each color i, $1 \leq i \leq n$, let $G_i' = G_i - w_i$, where $w_i \in V(G_i)$, and let r_i's be integers such that*

$$r_i \geq R(\underbrace{C_4, \ldots, C_4}_{m}, G_1, \ldots, G_{i-1}, G_i', G_{i+1}, \ldots, G_n).$$

Assume further that $R(\underbrace{C_4, \ldots, C_4}_{m}, G_1, \ldots, G_n) > \max_{1\leq i\leq n}\{|V(G_i)|\}$ and $G_i \neq K_2$ for some $i \in \{1, \ldots, n\}$ if $m = 1$. Then we have

$$\begin{aligned} &R(P_3, \underbrace{C_4, \ldots, C_4}_{m-1}, G_1, \ldots, G_n) \\ &\leq \sum_{i=1}^{n} r_i - n + \frac{m^2+m}{2} + \left\lceil m\sqrt{\frac{(m+1)^2}{4} + \sum_{i=1}^{n} r_i - n}\,\right\rceil. \end{aligned} \tag{3}$$

Proof. Let $RHS(1)$ denote the right-hand side of inequality (1) in Lemma 4, and let $RHS(3)$ denote the right-hand side of inequality (3). In order to prove this lemma, by Lemma 4, it suffices to show that $RHS(3) \geq RHS(1)$. In the proof below, among other steps, we will use an easy observation that for any positive integer k, it is true that $\lceil\sqrt{k+1}\rceil = \left\lfloor\sqrt{k}\right\rfloor + 1$.

If $m \geq 2$ then $RHS(3) =$

$$\sum_{i=1}^{n} r_i - n + \frac{m^2+m}{2} + 1 + \left\lfloor \sqrt{\frac{m^2(m+1)^2}{4} + m^2\left(\sum_{i=1}^{n} r_i - n\right) - 1} \right\rfloor$$

$$\geq \sum_{i=1}^{n} r_i - n + 1 + \frac{m^2-m}{2} + m + \left\lfloor \sqrt{\frac{(m^2-m)^2}{4} + m^3 + (m-1)^2\left(\sum_{i=1}^{n} r_i - n\right) - 1} \right\rfloor$$

$$= \sum_{i=1}^{n} r_i - n + (1+m) + \frac{m^2-m}{2}$$

$$+ \left\lfloor \sqrt{\frac{(m^2-m)^2}{4} + (m-1)^2\left(\sum_{i=1}^{n} r_i - n + 1\right) + (m^2+2)(m-1)} \right\rfloor \geq RHS(1).$$

If $m = 1$, let i_0 be an integer such that $G_{i_0} \neq K_2$, so that $r_{i_0} \geq 2$ and $\sum_{i=1}^{n} r_i - n \geq 1$. Then

$$RHS(3) = \sum_{i=1}^{n} r_i - n + 1 + \left\lfloor \sqrt{1 + \sum_{i=1}^{n} r_i - n} \right\rfloor \geq \sum_{i=1}^{n} r_i - n + 3 = RHS(1),$$

where in the latter the RHS's were simplified using $m = 1$. ■

Now, we are ready to present our main result.

Theorem 6. *Let $m \geq 1$ and $n \geq 0$. Consider n graphs, $G_1, \ldots, G_n$. For each color i with $1 \leq i \leq n$, let $G'_i = G_i - w_i$, where $w_i \in V(G_i)$, and let r_i's be integers such that*

$$r_i \geq R(\underbrace{C_4, \ldots, C_4}_{m}, G_1, \ldots, G_{i-1}, G'_i, G_{i+1}, \ldots, G_n).$$

Assume further that $R = R(\underbrace{C_4, \ldots, C_4}_{m}, G_1, \ldots, G_n) > \max_{1 \leq i \leq n}\{|V(G_i)|\}$ and $G_i \neq K_2$ for some $i \in \{1, \ldots, n\}$ if $m = 1$. Then, we have

$$R \leq \sum_{i=1}^{n} r_i - n + 1 + \frac{m^2+m}{2} + \left\lfloor m\sqrt{\frac{(m+1)^2}{4} + \sum_{i=1}^{n} r_i - n} \right\rfloor.$$

Proof. Set $N = R - 1$, and let G be a $(\underbrace{C_4, \ldots, C_4}_{m}, G_1, \ldots, G_n)$-coloring of the edges of K_N. Let $v_0 \in V(G)$ such that $\sum_{i=1}^{m} d_i(v_0) = \min_{v \in V(G)}\{\sum_{i=1}^{m} d_i(v)\}$.

For $1 \le i \le n$, in order to avoid G_i of color $i+m$, we must have $d_{i+m}(v_0) \le r_i - 1$. Hence, we also have

$$N = 1 + \sum_{i=1}^{m+n} d_i(v_0) \le 1 - n + \sum_{i=1}^{m} d_i(v_0) + \sum_{i=1}^{n} r_i. \tag{4}$$

For each $i \in \{1, \ldots, m\}$, the number of P_3's in color i cannot exceed $\binom{N}{2}$, since otherwise they would force a C_4 in color i. Thus, as noted in the proof of Lemma 4, $\sum_{v\in V(G)} \binom{d_i(v)}{2} \le \binom{N}{2}$. If $\sum_{v\in V(G)} \binom{d_1(v)}{2} = \binom{N}{2}$, then by the Friendship Theorem [5], which states that in any graph in which any two vertices have precisely one common neighbor, then there is a vertex which is adjacent to all other vertices. In that case, let u be the vertex adjacent to all the others with edges of the first color. $G - u$ is a $(P_3, \underbrace{C_4, \ldots, C_4}_{m-1}, G_1, \ldots, G_n)$-coloring of K_{N-1}, so $R - 2 = N - 1 \ge R(P_3, \underbrace{C_4, \ldots, C_4}_{m-1}, G_1, \ldots, G_n) - 1$, and by Lemma 5, the result follows.

Similarly, the same argument applies if $\sum_{v\in V(G)} \binom{d_i(v)}{2} = \binom{N}{2}$ for some $i \le m$. Therefore, we can assume that $\sum_{v\in V(G)} d_i(v)(d_i(v)-1) < N(N-1)$ for all i and

$$\sum_{v\in V(G)} \left(\sum_{i=1}^{m} d_i(v)^2 - \sum_{i=1}^{m} d_i(v) \right) = \sum_{i=1}^{m} \sum_{v\in V(G)} d_i(v)(d_i(v)-1) < mN(N-1).$$

Then, by Corollary 2, for any $v \in V(G)$ we have $m\sum_{i=1}^{m} d_i(v)^2 \ge (\sum_{i=1}^{m} d_i(v))^2$, and further

$$\begin{aligned} &mN(N-1) > \sum_{v\in V(G)} \left(\sum_{i=1}^{m} d_i(v)^2 - \sum_{i=1}^{m} d_i(v) \right) \\ &\ge \sum_{v\in V(G)} \left(\frac{(\sum_{i=1}^{m} d_i(v))^2}{m} - \sum_{i=1}^{m} d_i(v) \right) = \sum_{v\in V(G)} \left(\sum_{i=1}^{m} d_i(v) \right) \left(\frac{\sum_{i=1}^{m} d_i(v)}{m} - 1 \right) \\ &\ge N \sum_{i=1}^{m} d_i(v_0) \left(\frac{\sum_{i=1}^{m} d_i(v_0)}{m} - 1 \right) = N \left(\frac{(\sum_{i=1}^{m} d_i(v_0))^2}{m} - \sum_{i=1}^{m} d_i(v_0) \right). \end{aligned}$$

Therefore, by (4), we see that

$$\frac{(\sum_{i=1}^{m} d_i(v_0))^2}{m} - \sum_{i=1}^{m} d_i(v_0) < m(N-1) \le m \left(-n + \sum_{i=1}^{m} d_i(v_0) + \sum_{i=1}^{n} r_i \right)$$

and

$$\left(\sum_{i=1}^{m} d_i(v_0) \right)^2 - m(m+1) \sum_{i=1}^{m} d_i(v_0) - m^2 \left(-n + \sum_{i=1}^{n} r_i \right) < 0,$$

and hence

$$\sum_{i=1}^{m} d_i(v_0) \leq \frac{m^2+m}{2} + \sqrt{\frac{\left(m^2+m\right)^2}{4} + m^2\left(\sum_{i=1}^{n} r_i - n\right) - 1}.$$

Consequently, by (4),

$$R = N+1 \leq 2 + \sum_{i=1}^{n} r_i - n + \frac{m^2+m}{2} + \sqrt{\frac{\left(m^2+m\right)^2}{4} + m^2\left(\sum_{i=1}^{n} r_i - n\right) - 1}.$$

Since R is an integer, we have

$$\begin{aligned} R &\leq 2 + \sum_{i=1}^{n} r_i - n + \frac{m^2+m}{2} + \left\lfloor \sqrt{\frac{\left(m^2+m\right)^2}{4} + m^2\left(\sum_{i=1}^{n} r_i - n\right) - 1} \right\rfloor \\ &= \sum_{i=1}^{n} r_i - n + 1 + \frac{m^2+m}{2} + \left\lceil m\sqrt{\frac{\left(m+1\right)^2}{4} + \sum_{i=1}^{n} r_i - n} \right\rceil, \end{aligned}$$

and the result follows. ■

Note that if $m \geq 2$ and $n = 0$, then the bound in Theorem 6 coincides with the known result $R(\underbrace{C_4, \ldots, C_4}_{m}) \leq m^2 + m + 1$ [2, 9].

3. Complete Graphs

In this section, we focus attention on concrete upper bounds for the Ramsey numbers of the form $R(\underbrace{C_4, \ldots, C_4}_{m}, G_1, \ldots, G_n)$, where all G_i's are complete graphs, for $1 \leq i \leq n$. We gather our results in Table 1, in which the new upper bounds are shown in the last column. Proposition 7 below provides the upper bound in Case #3, while all other cases are derived in the proof of Theorem 8.

Proposition 7. $R(C_4, K_3, K_4) \leq 29$.

Proof. First, we note that $R(K_3, K_4) = 9$ [8] and $R(C_4, K_9) = 30$ [10]. Hence, if there exists any (C_4, K_3, K_4)-coloring G of K_{29}, then by merging the last two colors of G we obtain a (C_4, K_9)-coloring, i.e., a C_4-free graph G' on 29 vertices with maximum independent set of order at most 8. All such graphs were obtained in [10], and up to isomorphism there are 267 of them.

Case #	Ramsey number	$m, n, \sum_{i=1}^{n} r_i$	lower bound	old upper bound	new upper bound
1	$R(C_4, K_{11})$	$1, 1, 36$	40 [14]	44 [10]	43
2	$R(C_4, K_{12})$	$1, 1, 43$	43 (*)	52 [10]	51
3	$R(C_4, K_3, K_4)$	Proposition 7	27 [3]	32 [16]	29
4	$R(C_4, K_4, K_4)$	$1, 2, 58$	52 [16]	71 [11]	66
5	$R(C_4, K_3, K_3, K_3)$	$1, 3, 51$	49 [1]	59 [11]	57
6	$R(C_4, C_4, K_3, K_4)$	$2, 2, 57$	43 [3]	76 [16]	75
7	$R(C_4, C_4, K_4, K_4)$	$2, 2, 150$	87 [16]	179 [16]	177

Table 1. New bounds on Ramsey numbers of C_4 versus complete graphs described in Section 3: parameters, lower bounds and old and new upper bounds. (*) Lower bound 43 in case #2 is easily obtained by adding vertex-disjoint K_3 to the lower bound witness graph in case #1. In all cases, except case #3, the new upper bound is obtained by using Theorem 6.

We verified by computations that for every such graph (one of 267 possible graphs), its non-edges cannot be partitioned into a K_3-free graph and a K_4-free graph. Thus, no (C_4, K_3, K_4)-coloring of K_{29} exists, and the bound in the proposition holds. ■

Theorem 8. *The upper bounds in the last column of Table* 1 *hold.*

Proof. Proposition 7 proves the bound in Case #3. The upper bounds in all other cases are obtained by applying Theorem 6 with some additional simple steps, as described below.

#1. It is known that $R(C_4, K_{10}) = 36$ [10]. Theorem 6 with $m = n = 1$, $G_1 = K_{11}$ and $r_1 = 36$ gives $R(C_4, K_{11}) \leq 36 + 1 + \sqrt{36} = 43$.

#2. Let $r_1 = 43$, so by Case #1 we have $r_1 \geq R(C_4, K_{11})$. With $m = n = 1$ and $G_1 = K_{12}$, we obtain $R(C_4, K_{12}) \leq r_1 + 1 + \lceil\sqrt{r_1}\rceil = 44 + 7 = 51$.

#4. Let $r_1 = r_2 = 29$, so by Case #3 we have $r_1, r_2 \geq R(C_4, K_3, K_4)$. With $m = 1$, $n = 2$ and $G_1 = G_2 = K_4$, we obtain

$$R(C_4, K_4, K_4) \leq r_1 + r_2 + \left\lceil\sqrt{r_1 + r_2 - 1}\right\rceil = 58 + \left\lceil\sqrt{57}\right\rceil = 66.$$

#5. It is known that $R(C_4, K_3, K_3) = 17$ [6]. Let $r_i = 17$ and $G_i = K_3$ for $1 \leq i \leq 3$. With $m = 1$ and $n = 3$, we have

$$R(C_4, K_3, K_3, K_3) \leq r_1 + r_2 + r_3 - 1 + \left\lceil\sqrt{r_1 + r_2 + r_3 - 2}\right\rceil = 50 + \sqrt{49} = 57.$$

#6. It is known that $R(C_4, C_4, K_4) \leq 21$ [11] and $R(C_4, C_4, K_3, K_3) \leq 36$ [15]. Let $r_1 = 21$ and $r_2 = 36$, so that $r_1 \geq R(C_4, C_4, K_4)$ and $r_2 \geq$

$R(C_4, C_4, K_3, K_3)$. With $m = 2$, $n = 2$, $G_1 = K_3$ and $G_2 = K_4$, we have

$$R(C_4, C_4, K_3, K_4) \leq r_1 + r_2 + 2 + \left\lceil 2\sqrt{9/4 + r_1 + r_2 - 2} \right\rceil$$

$$= 59 + \left\lceil \sqrt{9 + 220} \right\rceil = 75.$$

#7. Let $r_1 = r_2 = 75$, so by Case #6 we have $r_1, r_2 \geq R(C_4, C_4, K_3, K_4)$. With $m = 2$, $n = 2$ and $G_1 = G_2 = K_4$, we have

$$R(C_4, C_4, K_4, K_4) \leq r_1 + r_2 + 2 + \left\lceil 2\sqrt{9/4 + r_1 + r_2 - 2} \right\rceil = 152 + \left\lceil \sqrt{1 + 600} \right\rceil = 177.$$

■

4. Stars and Books

We start this section with a classical result obtained by Parsons in 1975.

Lemma 9 [12]. *For $k \geq 2$, we have $R(C_4, K_{1,k}) \leq k + \left\lceil \sqrt{k} \right\rceil + 1$.*

Proof. The original proof was presented by Parsons, but we note that the same result is implied by our Theorem 6 using $m = n = 1$ and $r_1 = R(C_4, kK_1) = k$. ■

The next corollary puts together Theorem 6 and Lemma 9.

Corollary 10. *For $k \geq 2$, we have*

$$R(C_4, B_k) \leq R(C_4, K_{1,k}) + \left\lceil \sqrt{R(C_4, K_{1,k})} \right\rceil + 1 \leq k + \left\lceil \sqrt{k + \left\lceil \sqrt{k} \right\rceil + 1} \right\rceil + \left\lceil \sqrt{k} \right\rceil + 2.$$

Proof. Since $B_k = K_1 + K_{1,k}$, using Theorem 6 with $m = n = 1$ and $r_1 = k + \left\lceil \sqrt{k} \right\rceil + 1$ gives the first inequality. The second inequality is obtained by Lemma 9. ■

Note that for $k = q^2 - q + 1$ we have $\left\lceil \sqrt{k} \right\rceil = q$. Our result in Corollary 10, which holds for all integers $k \geq 2$, generalizes a result by Faudree, Rousseau and Sheehan [7]. In particular, the Lemma in Section 2 of [7] implies that $R(C_4, B_{17}) \leq 29$, while our Corollary 10 using Parson's [12] result $R(C_4, K_{1,17}) = 22$ gives a better bound, namely $R(C_4, B_{17}) \leq 28$.

Our last corollary about multicolor Ramsey numbers of C_4's versus stars is also a consequence of Theorem 6.

Corollary 11. *Let $m, n, k_1, \ldots, k_n \geq 1$, such that $m + \sum_{i=1}^{n} k_i \geq n + 2$. Then*

$$R(\underbrace{C_4, \ldots, C_4}_{m}, K_{1,k_1}, \ldots, K_{1,k_n}) \leq 1 + \sum_{i=1}^{n} k_i - n + \frac{m^2+m}{2} + \left\lceil m\sqrt{\frac{(m+1)^2}{4} + \sum_{i=1}^{n} k_i - n} \right\rceil.$$

Proof. Let w_i be the vertex of degree k_i in K_{1,k_i}. Since $k_i K_1 = K_{1,k_i} - w_i$, apply Theorem 6 with

$$r_i = R(\underbrace{C_4, \ldots, C_4}_{m}, K_{1,k_1}, \ldots, K_{1,k_{i-1}}, k_i K_1, K_{1,k_{i+1}}, \ldots, K_{1,k_n}),$$

and observe that $r_i = k_i$. The result follows. ■

We note that if we consider Corollary 11 with $n = 1$, then it reduces to a result obtained in [17], which states that for $k, m \geq 1$, with $m + k > 3$, it holds that

$$R(\underbrace{C_4, \ldots, C_4}_{m}, K_{1,k}) \leq k + \frac{m^2+m}{2} + \left\lceil m\sqrt{k + \frac{m^2+2m-3}{4}} \right\rceil.$$

References

[1] D. Bevan, *Personal communication to S. Radziszowski* (2002), manuscript.

[2] F.R.K. Chung, *On triangular and cyclic Ramsey numbers with k colors*, in: Graphs Combin., R. Bari and F. Harary (Ed(s)), Lect. Notes Math. **406** (Springer, Berlin, 1974) 236–241.

[3] J. Dybizbański and T. Dzido, *On some Ramsey numbers for quadrilaterals*, Electron. J. Combin. **18(1)** (2011) #P154.
https://doi.org/10.37236/641

[4] A. Engel, Problem-Solving Strategies (Springer-Verlag New York, Inc, 1998).

[5] P. Erdős, A. Rényi and V.T. Sós, *On a problem of graph theory*, Studia Sci. Math. Hungar. **1** (1966) 215–235.

[6] G. Exoo and D.F. Reynolds, *Ramsey numbers based on C_5-decompositions*, Discrete Math. **71** (1988) 119–127.
https://doi.org/10.1016/0012-365X(88)90065-9

[7] R.J. Faudree, C.C. Rousseau and J. Sheehan, *More from the Good Book*, Ninth Southeastern Conference on Combinatorics, Graph Theory, and Computing, Util. Math. Congr. Numer. **XXI** (1978) 289–299.

[8] R.E. Greenwood and A.M. Gleason, *Combinatorial relations and chromatic graphs*, Canad. J. Math. **7** (1955) 1–7.
https://doi.org/10.4153/CJM-1955-001-4

[9] R.W. Irving, *Generalised Ramsey numbers for small graphs*, Discrete Math. **9** (1974) 251–264.
https://doi.org/10.1016/0012-365X(74)90008-9

[10] A. Lange, I. Livinsky and S.P. Radziszowski, *Computation of the Ramsey numbers* $R(C_4, K_9)$ *and* $R(C_4, K_{10})$, J. Combin. Math. Combin. Comput. **97** (2016) 139–154.

[11] B. Lidický and F. Pfender, *Semidefinite programming and Ramsey numbers* (2017, revised version 2020).
arXiv:1704.03592

[12] T.D. Parsons, *Ramsey graphs and block designs,* I, Trans. Amer. Math. Soc. **209** (1975) 33–44.
https://doi.org/10.1090/S0002-9947-1975-0396317-X

[13] S. Radziszowski, *Small Ramsey numbers*, Electron. J. Combin. (2024) DS1, revision #17.

[14] S. Van Overberghe, Algorithms for Computing Ramsey Numbers, MS Thesis in Mathematics, Ghent University, Belgium (2020).
https://github.com/Steven-VO/circulant-Ramsey

[15] X. Xu and S.P. Radziszowski, $28 \leq R(C_4, C_4, C_3, C_3) \leq 36$, Util. Math. **79** (2009) 253–257.

[16] X. Xu, Z. Shao and S.P. Radziszowski, *Bounds on some Ramsey numbers involving quadrilateral*, Ars Combin. **90** (2009) 337-344.

[17] X. Zhang, Y. Chen and T.C. Edwin Cheng, *On three color Ramsey numbers for* $R(C_4, C_4, K_{1,n})$, Discrete Math. **342** (2019) 285–291.
https://doi.org/10.1016/j.disc.2018.09.030

Received 4 December 2023
Revised 31 July 2024
Accepted 31 July 2024
Available online 6 September 2024