Discussiones Mathematicae
Graph Theory 45 (2025) 809–822
https://doi.org/10.7151/dmgt.2556

OPTIMAL PEBBLING OF COMPLETE BINARY TREES AND A META-FIBONACCI SEQUENCE

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Abstract

In 1999, Fu and Shiue published a paper on optimal pebblings of complete m-ary trees. Among other things, they produced OPCBT, an integer linear program that produces an optimal pebbling of a complete binary tree. Building upon their work, we give an explicit representation of the optimal pebbling number of a complete binary tree. Among other things, we show that the sequence of optimal pebbling numbers of complete binary trees indexed by their heights is related to the Conolly sequence, a type of meta-Fibonacci sequence.

Keywords: optimal pebbling, complete binary tree, meta-Fibonacci sequence.

2020 Mathematics Subject Classification: 05C57, 05C05.

1. Introduction

1.1. Prologue

In 1999, Fu and Shiue [11] developed an integer linear program that produces an optimal pebbling of a complete binary tree. In this paper, we build upon their work and show that the sequence of optimal pebbling numbers of the complete binary trees indexed by their heights is related to the Conolly sequence, a type of meta-Fibonacci sequence.

Let G be a graph with vertex set V and edge set E. A *pebbling configuration* (or just *configuration*) on G is a function $f : V \to \mathbb{N} \cup \{0\}$. For $v \in V$, we think of $f(v)$ as the number of *pebbles* at v and $f(G) = \sum_{w \in V} f(w)$ as the number of pebbles on G. For each positive integer p, let $\mathscr{F}_p(G)$ denote the collection of configurations on G containing p pebbles.

A *pebbling move* on G consists of removing two pebbles from a vertex and placing a single pebble at an adjacent vertex. In effect, to move a pebble, we must pay a pebble. A configuration *pebbles* G provided that given any vertex v, there exists a sequence of pebbling moves (possibly empty) that brings a pebble to v. The *optimal pebbling number* of G is

$$\pi^*(G) = \min\{p : \exists f \in \mathscr{F}_p(G) \text{ such that } f \text{ pebbles } G\}.$$

An *optimal pebbling* of G is a configuration $f \in \mathscr{F}_{\pi^*(G)}(G)$ that pebbles G. In this paper, we study the optimal pebbling numbers and optimal pebblings of complete binary trees.

A *tree* is an undirected graph in which any two vertices are connected by exactly one path. We adopt the following terminology regarding trees. A *rooted tree* is a tree with a distinguished vertex, called the *root* of the tree. Let T be a rooted tree with vertex set V, edge set E, and root r. Given a vertex $v \in V$, the distance from r to v, denoted by $d(r, v)$, is the number of edges in the path from r to v. For each nonnegative integer k, the kth *level* of T is the set of vertices $L_k = \{v \in V : d(r, v) = k\}$. A *leaf* of T is a vertex with degree one. The *children* of a non-leaf vertex v are the vertices in the next highest level that are adjacent to v. A *complete binary tree* is a rooted tree in which each non-leaf vertex has two children and every leaf vertex is at the same level.[1] The *height* of a complete binary tree is the distance from the root to any leaf. Hereafter T_h denotes a complete binary tree of height h.

To describe our results, we first introduce a sequence of partial sums. Given a list of lists $\mathscr{L}_1, \ldots, \mathscr{L}_i$, let $\text{Join}[\mathscr{L}_1, \ldots, \mathscr{L}_i]$ be the list obtained by concatenating (in order) $\mathscr{L}_1$ through $\mathscr{L}_i$. We define a list of numbers composed entirely of 1s and 5s. We begin with $A_1 = (5)$. We define successive lists recursively: for each $k \geq 2$, let $A_k = \text{Join}[A_{k-1}, A_{k-1}, (1)]$. Some examples of the lists $\{A_k\}$ are collected in Table 1.

Let A be the direct limit of this sequence of lists. Let $a_0 = 0$ and, for $n \geq 1$, let a_n denote the nth element of the list A. Let $\{s_n\}$ denote the sequence of partial sums of $\{a_n\}$; see Table 2.

[1] This definition of a complete binary tree is not universally recognized. Our usage follows Cormen, *et al.* [7] and, notably, Fu and Shiue [11].

n	A_n
1	(5)
2	$(5, 5, 1)$
3	$(5, 5, 1, 5, 5, 1, 1)$
4	$(5, 5, 1, 5, 5, 1, 1, 5, 5, 1, 5, 5, 1, 1, 1)$

Table 1. The lists A_1 through A_4.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
a_n	0	5	5	1	5	5	1	1	5	5	1	5	5	1	1	1
s_n	0	5	10	11	16	21	22	23	28	33	34	39	44	45	46	47

Table 2. Some terms of $\{a_n\}$ and $\{s_n\}$.

1.2. Results

In this paper, we present three results. For each positive integer ℓ, let

$$s^{-1}(\ell) = \max\{k \geq 0 : s_k \leq \ell\}, \tag{1}$$

a left inverse of the sequence s. First, we show that $\pi^*(T_h) = 2^h - s^{-1}(2^h)$. As part of our proof, we introduce a simple method to generate an optimal pebbling of T_h through the μ-expansion of 2^h. The proof of this initial result is divided into two distinct sections: an upper-bound argument in Section 3 and a lower-bound argument in Section 4. In Section 5, we show that the sequence $\{s_n\}$ is related to the Conolly sequence, a type of meta-Fibonacci sequence. The first two terms of the Conolly sequence are $c_1 = 1$ and $c_2 = 2$. Thereafter, for $n \geq 2$, the sequence satisfies the nested recurrence relation

$$c_n = c(n) = c(n - c(n-1)) + c(n - 1 - c(n-2)). \tag{2}$$

The Conolly sequence $\{c_n\}$ is an offset of entry A046699 in [25]. We show that $s_n = 4c_n + n$. In Section 6, we develop an asymptotic expansion of $s^{-1}(2^h)$, which augments the concluding remarks of Fu and Shiue [11]. In particular, we show that, as $h \to \infty$,

$$s^{-1}(2^h) = \frac{1}{3}(2^h) - \frac{1}{3}(h+1) - \frac{1}{3}\alpha(h)\log_2(h+1) + O(1). \tag{3}$$

The function $\alpha(h)$, which appears in the third-order term of the expansion, is bounded between -1 and $+1$ and satisfies $\liminf \alpha(h) = -1$ and $\limsup \alpha(h) = 1$ as $h \to \infty$. Finally, in Section 7, we propose two problems for further study.

1.3. Background and related work

The paper of Fu and Shiue [11] concerns the optimal pebbling number of a complete m-ary tree, and our results lean heavily on their work. For each integer m such that $m \geq 2$, a complete m-ary tree is a rooted tree in which each non-leaf vertex has m children and each leaf vertex is at the same level. For $m \geq 3$, they show that placing 2^h pebbles at the root produces an optimal pebbling of a complete m-ary tree of height h. Most of their paper is devoted to describing optimal pebblings of complete binary trees, which are more subtle and intricate, depositing pebbles at various levels within the tree. Among other things, they show that optimal pebblings can always be found within the class of symmetric configurations. Symmetric configurations are homogeneous within levels and, except for the root, are even. Fu and Shiue provide a necessary and sufficient condition for a symmetric configuration to pebble a complete binary tree and they develop OPCBT, an integer linear program that produces an optimal pebbling. Roughly speaking, OPCBT is a bottom-up program: an optimal pebbling is revealed in successive steps, starting from the leaves and terminating at the root. One aspect of this approach is that $\pi^*(T_h)$ is not known until the program terminates and the configuration is examined. By comparison, our method demonstrates that $\pi^*(T_h) = 2^h - s^{-1}(2^h)$ and that an optimal pebbling can be found through the μ-expansion of 2^h.

While graph pebbling grew out of problems in combinatorial number theory and group theory, it was formally introduced in its present form by Chung [6] in her analysis of the pebbling number of the hypercube. The optimal pebbling number of a graph was introduced later by Pachter, Snevily, and Voxman [20]. The paper of Hurlbert [17] is an excellent survey of graph pebbling.

The optimal pebbling numbers of some classes of graphs have been studied. For example, the optimal pebbling numbers have been determined for caterpillars [23], the squares of paths and cycles [27], spindle graphs [12], staircase graphs [13], and grid graphs [14, 26]. The optimal pebbling numbers have been studied for products of graphs [16], graphs with a given diameter [15], and graphs with a given minimum degree [10].

In recent years, a variety of adaptations and analogs of optimal pebbling have emerged. For example, the optimal pebbling number of a graph has been extended in a variety of ways by restricting the capacity of a configuration or placing additional requirements on a configuration; see, for example, [5,21,22,24]. Graph rubbling is a cognate of pebbling; interested readers in graph rubbling should consult [1–4].

2. The M and μ-Expansions

For each positive integer i, let $M_i = 2^i - 1$ denote the ith Mersenne number.

Given a positive integer n, let $\ell = \max\{i : M_i \leq n\}$ and write $n = M_\ell + r$, where $0 \leq r \leq M_\ell$. If $r = 0$, then we stop and write $n = M_\ell$. If $r = M_\ell$, then we stop and write $n = 2M_\ell$. If else, then we continue this process with r. In this way, we can write $n = \varepsilon_1 M_1 + \cdots + \varepsilon_\ell M_\ell$, where $\varepsilon_i \in \{0, 1, 2\}$ for each $i \in [\ell]$, and if $\varepsilon_j = 2$ for some $j \in [\ell]$, then $\varepsilon_i = 0$ for all $i \in [j-1]$. We call the sum on the right side of equation the *M-expansion* of n. Let $\langle n \rangle_M = (\varepsilon_1, \ldots, \varepsilon_\ell)$ denote the coefficient list of the M-expansion of n. For example, $\langle 47 \rangle_M = (1, 0, 0, 1, 1)$ and $\langle 157 \rangle_M = (0, 0, 0, 2, 0, 0, 1)$. The M-expansions (written in reverse order) of the natural numbers are the canonical skew-binary numbers; see entry A169683 in [25].

For each positive integer i, let $\mu_i = 2^{i+1} + 2^i - 1$; see entry A083329 in [25]. The μ-expansion of a positive integer is developed in a parallel fashion. Let n be a positive integer. If $n \leq 4$, then we stop. If $n \geq 5$, we let $\ell = \max\{i : \mu_i \leq n\}$ and write $n = \mu_\ell + r$, where $0 \leq r \leq \mu_\ell$. If $r = 0$, then we stop and write $n = \mu_\ell$. If $r = \mu_\ell$, then we stop and write $n = 2\mu_\ell$. If else, then we continue this process with r. In this way, we can write $n = r + \varepsilon_1\mu_1 + \cdots + \varepsilon_\ell\mu_\ell$, where $r \in \{0, 1, 2, 3, 4\}$, $\varepsilon_i \in \{0, 1, 2\}$ for each $i \in [\ell]$, and if $\varepsilon_j = 2$ for some $j \in [\ell]$, then $r = 0$ and $\varepsilon_i = 0$ for all $i \in [j-1]$. We call the sum on the right side of the equation the *μ-expansion* of n. When $r = 0$, we let $\langle n \rangle_\mu = (\varepsilon_1, \ldots, \varepsilon_\ell)$ denote the coefficient list of the μ-expansion of n. For example, $409 = 3 + \mu_3 + \mu_7$, $140 = 2\mu_2 + \mu_3 + \mu_5$, and $\langle 140 \rangle_\mu = (0, 2, 1, 0, 1)$.

Lemma 1. *For each positive integer n, $\langle s_n \rangle_\mu = \langle n \rangle_M$.*

Proof. We begin by proving a provisional form of this theorem; namely, for each positive integer k, $s_{M_k} = \mu_k$. This is true for $k = 1$ by inspection: $s_{M_1} = s_1 = 5 = \mu_1$. Let k be a positive integer. Recall that the list A_{k+1} contains M_{k+1} terms and has the form

$$A_{k+1} = \mathrm{Join}[A_k, A_k, (1)]. \tag{4}$$

Thus $s_{M_{k+1}} = 2s_{M_k} + 1$. By induction, it follows that $s_{M_k} = \mu_k$.

Let n be a positive integer and let $\ell = \max\{i : n \geq M_i\}$. Then $n = M_\ell + r$, where $0 \leq r \leq M_\ell$. Referring once again to equation (4), we see that $s_{M_\ell + r}$ is the sum of the first $M_\ell + r$ terms in $A_{\ell+1}$, read left to right. Clearly this is the sum of the terms of A_ℓ plus the first r terms of list A_ℓ, that is, $s_n = s_{M_\ell} + s_r$. If $r = 0$, then $n = M_\ell$ and $s_n = \mu_\ell$, and if $r = M_\ell$, then $n = 2M_\ell$ and $s_n = 2\mu_\ell$. In either case, we are done. Otherwise $0 < r < M_\ell$ and we continue by developing the μ-expansion of s_r as above. ∎

Lemma 2. *Let λ be a positive integer and let $\lambda = r + \varepsilon_1\mu_1 + \cdots + \varepsilon_k\mu_k$ be the μ-expansion of λ. Then $s^{-1}(\lambda) = \varepsilon_1 M_1 + \cdots + \varepsilon_k M_k$.*

Proof. Let $\ell = \varepsilon_1 M_1 + \cdots + \varepsilon_k M_k$. By Lemma 1, $\lambda = r + s_\ell$. In particular, $s_\ell \le \lambda$. To finish our proof, we will show $s_{\ell+1} > \lambda$.

If $r = 0$ in the μ-expansion of λ, then $s_\ell = \lambda$. Since s is strictly increasing, $s_{\ell+1} > \lambda$. On the other hand, if $r \neq 0$, then $\varepsilon_1 \in \{0, 1\}$; consequently, the M-expansion of $\ell+1$ is $(\varepsilon_1+1)M_1+\cdots+\varepsilon_k M_k$, and, by Lemma 1, $s_{\ell+1} = \lambda+5-r > \lambda$, as was to be shown. ∎

3. The Upper Bound of $\pi^*(T_h)$

A configuration f of T_h is *homogenous* provided that $f(v) = f(w)$ whenever the vertices v and w are at the same level. When f is a homogenous pebbling of T_h, we write $f = (f_0, \ldots, f_h)$, where f_i is the number of pebbles at each vertex at level i. Let

$$\Gamma(f) = 2f_0 + \sum_{i=1}^{h} \mu_i f_i. \tag{5}$$

In accord with Fu and Shiue, a homogeneous pebbling $f = (f_0, \ldots, f_h)$ of T_h is *symmetric* provided that f_i is even for $i \ge 1$. In their Lemma 3.2, Fu and Shiue [11] provide a necessary and sufficient condition for a symmetric configuration to pebble T_h. This lemma can be recast as follows:

Lemma 3 (Fu and Shiue)**.** *The symmetric configuration $f = (f_0, \ldots, f_h)$ pebbles T_h if and only if $\Gamma(f) \ge 2^{h+1}$.*

For each positive integer h, we define a special symmetric configuration on T_h. Let $f^1 = (2, 0)$ and $f^2 = (4, 0, 0)$. For $h \ge 3$, let

$$2^h = r + \varepsilon_1 \mu_1 + \cdots + \varepsilon_{h-2}\mu_{h-2}, \tag{6}$$

be the μ-expansion of 2^h and let

$$f^h = (r, 2\varepsilon_1, 2\varepsilon_2, \ldots, 2\varepsilon_{h-2}, 0, 0).$$

Theorem 4. *For each positive integer h, f^h pebbles T_h and $f^h(T_h) = 2^h - s^{-1}(2^h)$.*

Proof. First we show that $\Gamma(f^h) = 2^{h+1}$ for each positive integer h, which, by virtue of Lemma 3, shows that f^h pebbles T_h. By inspection $\Gamma(f^1) = 2^2$ and $\Gamma(f^2) = 2^3$. For $h \ge 3$, $\Gamma(f^h) = 2^{h+1}$ by the μ-expansion formula, equation (6).

Next we show that $f^h(T_h) = 2^h - s^{-1}(2^h)$ for each positive integer h. By inspection, $f^1(T_1)$ and $f^2(T_2)$ have the prescribed size since both $s^{-1}(2^1)$ and $s^{-1}(2^2)$ are equal to 0. For $h \ge 3$ the number of pebbles in f^h is

$$f^h(T_h) = r + 2(2\varepsilon_1) + 2^2(2\varepsilon_2) + \cdots + 2^{h-2}(2\varepsilon_{h-2}).$$

For each positive integer i, $\mu_i - M_i = 2^{i+1}$; accordingly,

$$\begin{aligned} f^h(T^h) &= r + \varepsilon_1(\mu_1 - M_1) + \cdots + \varepsilon_{h-2}(\mu_{h-2} - M_{h-2}) \\ &= (r + \varepsilon_1\mu_1 + \cdots + \varepsilon_{h-2}\mu_{h-2}) - (\varepsilon_1 M_1 + \cdots + \varepsilon_{h-2}M_{h-2}) \\ &= 2^h - s^{-1}(2^h), \end{aligned}$$

where we used equation (6) and Lemma 2 to obtain the last line in the chain. ■

4. THE LOWER BOUND OF $\pi^*(T_h)$

Here is the main result of this section.

Theorem 5. *Let h be a positive integer. A configuration with fewer than $2^h - s^{-1}(2^h)$ pebbles cannot pebble T_h.*

Consider the sequence $\{b_k\}$ defined by $b_1 = 2$ and $b_k = 2\mu_{k-1} = 3(2^k) - 2$ for $k \geq 2$. A simple but important attribute of this sequence is contained in the next lemma, which we state without proof but follows by induction.

Lemma 6. *For $i \geq 2$, $3b_1 + b_2 + \cdots + b_i = 2b_i - 2i$.*

Let h be a positive integer, let m be an integer, $1 \leq m < 2^{h+2}$, and let $m = \delta_0 2^0 + \delta_1 2^1 + \cdots + \delta_{h+1}2^{h+1}$ be the binary expansion of m, where $\delta_i \in \{0,1\}$ for each index i. Let $\phi_m^h = (\delta_0 + 2\delta_1, 2\delta_2, \ldots, 2\delta_{h+1})$ denote a special configuration on T_h consisting of m pebbles. For simplicity, let $\gamma_m^h = \Gamma(\phi_m^h)$. Owing to the definition of the sequence $\{b_k\}$,

$$\gamma_m^h = (\delta_0 + 2\delta_1)b_1 + \delta_2 b_2 + \cdots + \delta_{h+1}b_{h+1}. \tag{7}$$

Lemma 7. *Let h and m be positive integers such that $\gamma_m^h < 2^{h+1}$. Then $\pi^*(T_h) > m$.*

Proof. For the sake of a contradiction, let us assume that T_h can be pebbled by a configuration consisting of m pebbles. Then, by Theorem 3.4 of Fu and Shiue [11], there is a symmetric configuration f^* that pebbles T_h and consists of m pebbles.

The list $\phi_m^h = (\delta_0 + 2\delta_1, 2\delta_2, \ldots, 2\delta_{h+1})$ is a symmetric configuration on T_h containing m pebbles. Since $2\mu_i < \mu_{i+1}$ for each $i \geq 1$, ϕ_m^h maximizes Γ among the set of symmetric configurations f on T_h containing m pebbles. As a consequence, $\Gamma(f^*) \leq \gamma_m^h < 2^{h+1}$. By Lemma 3, f^* does not pebble T_h, which is a contradiction. Thus, there does not exist a configuration that pebbles T_h and consists of m pebbles. ■

Finally, we present the proof of Theorem 5, the main result of this section.

Proof. The result is true for $h = 1$ by inspection. Let $h \geq 2$ be given and let $N_h = 2^h - s^{-1}(2^h)$. We show that $\gamma^h_{N_h-1} < 2^{h+1}$, which, according to Lemma 7, proves the theorem.

Let $2^h = r + \varepsilon_1\mu_1 + \cdots + \varepsilon_{h-2}\mu_{h-2}$ be the μ-expansion of 2^h. By Lemma 2, $s^{-1}(2^h) = \varepsilon_1 M_1 + \cdots + \varepsilon_{h-2}M_{h-2}$; therefore,

$$\begin{aligned} N_h &= r + \varepsilon_1(\mu_1 - M_1) + \cdots + \varepsilon_{h-2}(\mu_{h-2} - M_{h-2}) \\ &= r + \varepsilon_1 2^2 + \cdots + \varepsilon_{h-2}2^{h-1}. \end{aligned}$$

We divide the rest of the proof into two cases according to whether or not $r = 0$.

First, let us suppose that $r = 0$ and that $\varepsilon_i \neq 2$ for each $i \in [h-2]$ in the μ-expansion of 2^h. Then $\varepsilon_1 2^2 + \cdots + \varepsilon_{h-2}2^{h-1}$ is the binary expansion of N_h and therefore

$$\begin{aligned} \gamma^h_{N_h} &= \varepsilon_1 b_2 + \varepsilon_2 b_3 + \cdots + \varepsilon_{h-2}b_{h-1} \\ &= 2(\varepsilon_1\mu_1 + \varepsilon_2\mu_2 + \cdots + \varepsilon_{h-2}\mu_{h-2}) \\ &= 2^{h+1}. \end{aligned}$$

Since the sequence $\{\gamma^h_k\}$ is strictly increasing, $\gamma^h_{N_h-1} < 2^{h+1}$.

Next, let us suppose that $r = 0$ but that $\varepsilon_j = 2$ for some $j \in [h-2]$ in the μ-expansion of 2^h. In particular, this implies that $\varepsilon_i \in \{0, 1\}$ for each integer $i \in \{j+1, \ldots, h-2\}$. Therefore, the binary expansion of $N_h - 1$ is $1 + 2 + 2^2 + \cdots + 2^{j+1} + \varepsilon_{j+1}2^{j+2} + \cdots + \varepsilon_{h-2}2^{h-1}$, hence

$$\gamma^h_{N_h-1} = 3b_1 + b_2 + \cdots + b_{j+1} + \varepsilon_{j+1}b_{j+2} + \cdots + \varepsilon_{h-2}b_{h-1}.$$

By Lemma 6,

$$\begin{aligned} \gamma^h_{N_h-1} &= 2b_{j+1} + \varepsilon_{j+1}b_{j+2} + \cdots + \varepsilon_{h-2}b_{h-1} - (j+1)2 \\ &= 2(2\mu_j + \varepsilon_{j+1}\mu_{j+1} + \cdots + \varepsilon_{h-2}\mu_{h-2}) - (j+1)2 \\ &= 2^{h+1} - (j+1)2, \end{aligned}$$

which shows that $\gamma^h_{N_h-1} < 2^{h+1}$.

Lastly, let us assume that $r \in \{1, 2, 3, 4\}$ in the μ-expansion of 2^h. This implies that $\varepsilon_i \in \{0, 1\}$ for each $i \in [h-2]$. Consequently,

$$N_h - 1 = (r-1) + \varepsilon_1 2^2 + \varepsilon_2 2^3 + \cdots + \varepsilon_{h-2}2^{h-1}$$

and

$$\begin{aligned} \gamma^h_{N_h-1} &= 2(r-1) + \varepsilon_1 b_2 + \varepsilon_2 b_3 + \cdots + \varepsilon_{h-2}b_{h-1} \\ &= 2\left((r-1) + \varepsilon_1\mu_1 + \varepsilon_2\mu_3 + \cdots + \varepsilon_{h-2}\mu_{h-2}\right) \\ &= 2^{h+1} - 2, \end{aligned}$$

which shows that $\gamma^h_{N_h-1} < 2^{h+1}$, completing our proof. ∎

5. Connection with the Connolly Sequence

Recall from Section 1 that the Conolly sequence $\{c_n\}$ satisfies the recurrence relation (2) with initial conditions $c_1 = 1$ and $c_2 = 2$. We will prove the following theorem.

Theorem 8. *For each positive integer n, $s_n = 4c_n + n$.*

Proof. We begin by defining a list of numbers composed entirely of 0s and 1s. Let $D_1 = (1)$. We define successive lists recursively: for each integer k, $k \geq 2$, let $D_k = \text{Join}[D_{k-1}, D_{k-1}, (0)]$. The lists D_1 through D_4 are collected in Table 3.

k	D_k
1	(1)
2	(1, 1, 0)
3	(1, 1, 0, 1, 1, 0, 0)
4	(1, 1, 0, 1, 1, 0, 0, 1, 1, 0, 1, 1, 0, 0, 0)

Table 3. The lists D_1 through D_4.

Let D be the limit of this sequence of lists and, for each positive integer n, let d_n denote the nth element of D. The sequence $\{d_n\}$ is the sequence of differences in the Conolly sequence; see entry A079559 in [25]. Thus, for each positive integer n, $c_n = d_1 + \cdots + d_n$. The initial terms of the sequences $\{d_n\}$ and $\{c_n\}$ are presented in Table 4.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
d_n	1	1	0	1	1	0	0	1	1	0	1	1	0	0	0
c_n	1	2	2	3	4	4	4	5	6	6	7	8	8	8	8

Table 4. The initial terms of $\{d_n\}$ and $\{c_n\}$.

Recall the sequence $\{a_n\}$ defined in Section 1; see Table 2. For each positive integer n, it is easy to see that $a_n = 4d_n + 1$ and therefore $s_n = 4c_n + n$, as was to be shown. ∎

6. Asymptotic Analysis of $s^{-1}(2^h)$

Let m be a positive integer and let $m = r + \varepsilon_1\mu_1 + \cdots + \varepsilon_\ell\mu_\ell$ be the μ-expansion of m. Define $\sigma(m) = \varepsilon_1 + \cdots + \varepsilon_\ell$. For example, the μ-expansions of 236 and 253 are $2\mu_2 + \mu_3 + \mu_6$ and $4 + \mu_2 + \mu_4 + \mu_6$, respectively. Accordingly, $\sigma(236) = 4$ and $\sigma(253) = 3$.

Theorem 9. *For each positive integer m, there exists an integer $r \in \{0,1,2,3,4\}$ such that $m - r = 3s^{-1}(m) + 2\sigma(m)$.*

Proof. Let $m = r + \varepsilon_1\mu_1 + \cdots + \varepsilon_\ell\mu_\ell$ be the μ-expansion of m. Since $\mu_i = 3M_i + 2$ for each positive integer i, $m - r = 3(\varepsilon_1 M_1 + \cdots + \varepsilon_\ell M_\ell) + 2(\varepsilon_1 + \cdots + \varepsilon_\ell)$. By Lemma 2, the right side is $3s^{-1}(m) + 2\sigma(m)$. ■

Let h be a positive integer. According to Theorem 9, there exists an integer $r \in \{0,1,2,3,4\}$ such that

$$s^{-1}(2^h) = 2^h/3 - 2\sigma(2^h)/3 - r/3. \tag{8}$$

The remainder of this section is devoted to an analysis of σ.

Lemma 10. *Let h be an integer, $h \geq 6$, and let $j^* = \min\{j : j \geq 2^{h-2j-1} - 1\}$. Then $j^* + 1 \leq \sigma(2^h) \leq h - j^*$.*

Proof. We will study equation (6), the μ-expansion of 2^h. Due to the definition of j^*, we may perform j^* successive subtractions of the sequence μ along the arithmetic sequence of indices $h-2, h-4, \ldots, h-2j^*$, which yields

$$2^h - \mu_{h-2} - \mu_{h-4} - \cdots - \mu_{h-2j^*} = 2^{h-2j^*} + j^*.$$

Since $2^{h-2j^*-1} - 1 \leq j^* \leq 2^{h-2j^*+1} - 1$, the right side of this equation is bounded below by μ_{h-2j^*-1} and above by μ_{h-2j^*}. The rest of the argument is broken into two cases.

If $2^{h-2j^*} + j^* = \mu_{h-2j*}$, then $2^h = 2\mu_{h-2j^*} + \mu_{h-2j^*+2} + \cdots + \mu_{h-2}$ and $\sigma(2^h) = j^* + 1$.

If $2^{h-2j^*} + j^* < \mu_{h-2j*}$, then the next subtraction in the μ-expansion of 2^h is μ_{h-2j^*-1}, which yields

$$2^h - \mu_{h-2} - \cdots - \mu_{h-2j^*} - \mu_{h-2j^*-1} = \rho, \tag{9}$$

where $\rho = j^* - 2^{h-2j^*-1} + 1$. This shows that $\sigma(2^h) \geq j^* + 1$. There are $h - 2j^* - 2$ remaining coefficients in the μ-expansion of 2^h to be determined; namely, $\varepsilon_1, \ldots, \varepsilon_{h-2j^*-2}$. Since ε_1 can equal 2 and the remaining coefficients can each equal 1, $\sigma(2^h) \leq (j^* + 1) + (h - 2j^* - 1) = h - j^*$, as was to be shown. ■

Remark 11. Let us make a few observations regarding Lemma 10, especially concerning the asymptotic behavior of $\sigma(2^h)$ as $h \to \infty$.

a. Given a height h, there are at least two ways to calculate j^*. First, $j^* = \lceil x^* \rceil$, where x^* is the root of the equation $2x + \log_2(x+1) = h - 1$. An analysis of this equation reveals that

$$j^* \geq (h-1)/2 - (1/2)\log_2((h+1)/2). \tag{10}$$

Second, for each integer k, $k \geq 3$, let $Q_k = 2^k + (k-2)$ and let $I_k = [Q_k, Q_{k+1}) \cap \mathbb{Z}$. For $h \in I_k$, $j^* = \lceil (h-k)/2 \rceil$.

b. Within each interval I_k, there are heights that satisfy the lower bound and heights that satisfy the upper bound in the lemma. For $h = Q_k$, $j^* = 2^{k-1} - 1$ and $\sigma(2^h) = 2^{k-1} = j^* + 1$. For $h = 2^{k+1} - (k+1)$, $j^* = 2^k - k$ and $\sigma(2^h) = 2^k - 1 = h - j^*$.

c. The interval $[j^* + 1, h - j^*]$ has center $(h+1)/2$ and length $h - 2j^* - 1$. Using equation (10), we may conclude that

$$|2\sigma(2^h) - (h+1)| \leq \log_2((h+1)/2). \tag{11}$$

Taking into consideration equation (8), we can write

$$3s^{-1}(2^h) = 2^h - (h+1) - \alpha(h) \log_2((h+1)/2) - r,$$

where $-1 \leq \alpha(h) \leq 1$, which gives us (3).

7. Problems for Further Study

While the bound presented in inequality (11) is sharp (at least asymptotically), it does not capture the behavior of $\sigma(2^h)$ for "most" h. Here is a possible way to improve this bound. For heights h, $h \geq 6$, let $E(h) = \sigma(2^h) - (j^* + 1)$, which we call the *excess*. The excess measures the "thickness" of an optimal pebbling at the top of the tree. For $h \in I_k \cup I_{k+1}$, $0 \leq E(h) \leq k - 1$. Within the interval $I_k \cup I_{k+1}$, there is only one height h for which $E(h) = k - 1$. Excluding this singular height, if we select a height h uniformly from $I_k \cup I_{k+1}$, then $E(h)$ appears to follow a binomial distribution with parameters $k - 2$ and $1/2$. For $h \in I_k \cup I_{k+1}$, it can be shown that $2\sigma(2^h) - h = 2\big(E(h) - (k-2)/2\big) + R$, where $R \in \{-1, 0, 1\}$. Therefore, if we sample a height uniformly from $I_k \cup I_{k+1}$, then $|E(h) - (k-2)/2| \leq 3\sqrt{(k-2)}/2$ with very high probability. We offer the following conjecture.

Conjecture 12. *For most heights $h \in I_k \cup I_{k+1}$, $|2\sigma(2^h) - h| \leq 3\sqrt{k-2} + 1$.*

It would be natural to extend the results of this paper to m-ary trees, $m \geq 3$. Under the normal pebbling rule (pay one pebble to move one pebble), Fu and Shuie showed that an optimal pebbling of an m-ary tree of height h is achieved by placing 2^h pebbles at the root; see Theorem 2.1 of [11]. For m-ary trees with $m \geq 3$, we suggest modifying the pebbling rule to disperse the pebbles away from the root in an optimal pebbling. For example, for $m = 3$, a pebbling move might consist of removing *three* pebbles from a vertex and placing a single pebble at an adjacent vertex. Some preliminary calculations show that optimal pebblings of 3-ary trees subject to this pebbling rule distribute the pebbles throughout the tree.

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Received 9 June 2023
Revised 6 July 2024
Accepted 7 July 2024
Available online 3 September 2024