

## DECOMPOSITION OF COMPLETE GRAPHS INTO FORESTS WITH SIX EDGES

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### Abstract

Let  $G$  be a forest with six edges. We prove that  $G$  decomposes the complete graph  $K_n$  if and only if  $n \equiv 0, 1, 4$ , or  $9 \pmod{12}$ , unless  $n = 9$  and  $G$  is one of nine exceptional forests.

**Keywords:** graph decomposition, forests,  $\rho$ -labeling.

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### 1. INTRODUCTION

For some graph  $G$ , a  $G$ -decomposition of a graph  $H$  is a set  $\mathcal{G} = \{G_1, G_2, \dots, G_t\}$  of pairwise edge-disjoint subgraphs of  $H$ , each of which is isomorphic to  $G$ , such that  $E(H) = \bigcup_{i=1}^t E(G_i)$ . The elements of  $\mathcal{G}$  are called  $G$ -blocks.

Finding  $G$ -decompositions is a classical combinatorial problem that spans the related fields of graph theory, design theory, and error-correcting codes. The problem was given new life in the 1960s by Kotzig and Rosa who produced a new method for decomposing graphs now known as graph labeling.

We only consider  $G$ -decompositions of the complete graph  $K_n$ , so from here on, we refer to  $G$ -decompositions of  $K_n$  simply as  $G$ -decompositions. Determining the set of all integers  $n$  such that a  $G$ -decomposition exists is known as the decomposition spectrum for  $G$ . We are particularly interested in solving this spectrum problem for all small graphs. The decomposition spectrum for graphs with less than six edges is completely determined. In this article, we complete the decomposition spectrum for graphs with six edges. An overview of the results for graphs with a small number of edges is given in [9], and we reproduce an updated version of it here.

### 1.1. Graphs with at most four edges

For graphs with one or two edges, the cases are trivial. Graphs with three edges are only  $K_3$ ,  $K_{1,3}$ ,  $P_4$ ,  $P_2 \cup P_3$ , or  $3K_2$ . For  $K_3$  we have a Steiner Triple System, and it has been solved by Kirkman [14].  $K_{1,3}$  was classified by Cain [5],  $P_4$  was settled by Bermond [2],  $P_2 \cup P_3$  was solved by Bermond, Huang, Rosa and Sotteau [3], and  $3K_2$  was classified by de Werra [6].

There are five types of connected graphs with four edges. Kotzig covered the  $C_4$  case [15], and Bermond and Schönheim the cases containing  $K_3$  ( $K_3$  with a pendant edge and  $K_3 \cup P_2$  [4]). The path  $P_5$  and the tree with a unique vertex of degree three were settled by Huang and Rosa [12], and the star  $K_{1,4}$  by Yamamoto *et al.* [18].

The forests with six vertices, that is,  $2P_3$ ,  $P_4 \cup P_2$  and  $K_{1,3} \cup K_2$  were classified by Yin and Gong [19].

The case  $P_3 \cup 2P_2$  appears to be folklore, and follows from its  $\sigma^+$ - or 1-rotational  $\sigma^+$ -labeling. The matching  $4K_2$  was classified by de Werra [6].

### 1.2. Graphs with five edges

Graphs with five vertices were studied by Bermond, Huang, Rosa and Sotteau [3]. Huang and Rosa [12] covered the cases for all trees with up to nine vertices. Disconnected graphs containing a cycle were settled by Yin and Gong except for  $K_3 \cup 2K_2$  [19]. The matching with five edges was solved by de Werra [6], while the remaining forests are treated in [10].

### 1.3. Graphs with six edges

The only graph on four vertices with six edges is  $K_4$ , which has been covered by Hanani [11]. The graphs on five vertices have been completely settled by Bermond, Huang, Rosa and Sotteau [3] except for  $\overline{P_5}$ . This case was solved by Kang and Wang [13].

All graphs with six edges and six vertices decompose were settled by Yin and Gong [19].

Graphs with six edges and more than six vertices are either trees (treated in [12]) or disconnected. This leaves only disconnected unicyclic graphs with six edges, which were settled in [1], and forests with six edges. As previously noted, the matching with six edges was solved by de Werra in [6]. We settle the remaining forests with six edges in this article by proving the following theorem.

**Theorem 1.** *Let  $G$  be a forest with exactly six edges. There exists a  $G$ -decomposition of  $K_n$  if and only if  $n > 4$  and  $n \equiv 0, 1, 4, \text{ or } 9 \pmod{12}$ , unless  $n = 9$  and  $G$  is one of the nine exceptional forests listed below.*

- $K_{1,5} \cup K_2$
- $K_{1,4} \cup 2K_2$
- $K_{1,4} \cup P_3$
- $K_{1,3} \cup 3K_2$
- $P_4 \cup 3K_2$
- $2P_3 \cup 2K_2$
- $P_3 \cup 4K_2$
- $6K_2$
- $2K_{1,3}$

#### 1.4. Catalog and necessary conditions

Suppose  $G$  is a graph with six edges and a  $G$ -decomposition exists. Then 6 must divide  $|E(K_n)| = \binom{n}{2}$ . This implies  $n \equiv 0, 1, 4, \text{ or } 9 \pmod{12}$ . We will prove these conditions are sufficient, up to a few small exceptions, when  $G$  is a forest by using labeling techniques based on Rosa's in the sections that follow [17].

Rosa's revolutionary approach to decomposing complete graphs is based on assigning lengths to each edge of the complete graph  $K_{2m+1}$  as follows. Let  $V(K_{2m+1}) = \{0, 1, \dots, 2m\}$  and define the *length* of edge  $uv \in E(K_{2m+1})$  as  $\ell(uv) = \min\{|u-v|, 2m+1-|u-v|\}$ . This length function partitions the  $m(2m+1)$  edges of  $K_{2m+1}$  into  $2m+1$  edges of each length in  $\{1, 2, \dots, m\}$ . If  $G$  is a graph with  $m$  edges, the problem of finding a  $G$ -decomposition of  $K_{2m+1}$  now reduces to finding an injective assignment of labels  $f : V(G) \rightarrow \{0, 1, \dots, 2m\}$  such that  $\{\ell(uv) : uv \in E(G)\} = \{1, 2, \dots, m\}$ . This is because once such an assignment of labels is found, it may be cyclically increased by 1 modulo  $2m+1$ , to produce  $2m+1$  edge-disjoint copies of  $G$  which uses  $m(2m+1)$  edges of  $K_{2m+1}$ . Since this operation we call *clicking* preserves edge length, we see that no edge has been repeated, and we have therefore constructed a  $G$ -design of  $K_{2m+1}$ .

The forests with six edges and at least two components are cataloged in Figures 1 and 2. We use the naming convention  $G(k; e_1, e_2, \dots, e_k)_t$  to denote the  $t^{\text{th}}$  forest with  $k$  connected components and  $e_i$  edges in the  $i^{\text{th}}$  connected component for  $1 \leq i \leq k$ . To simplify the names, we will use exponential notation for components of the same size.

## 2. DECOMPOSITIONS OF $K_9$

In this section, we characterize all forests with six edges that decompose  $K_9$ . There are nine such forests that do not decompose  $K_9$ .

**Lemma 2.** *Let  $G$  be a forest with exactly six edges. If  $G$  is isomorphic to one of the graphs below, then  $G$  does not decompose  $K_9$ .*

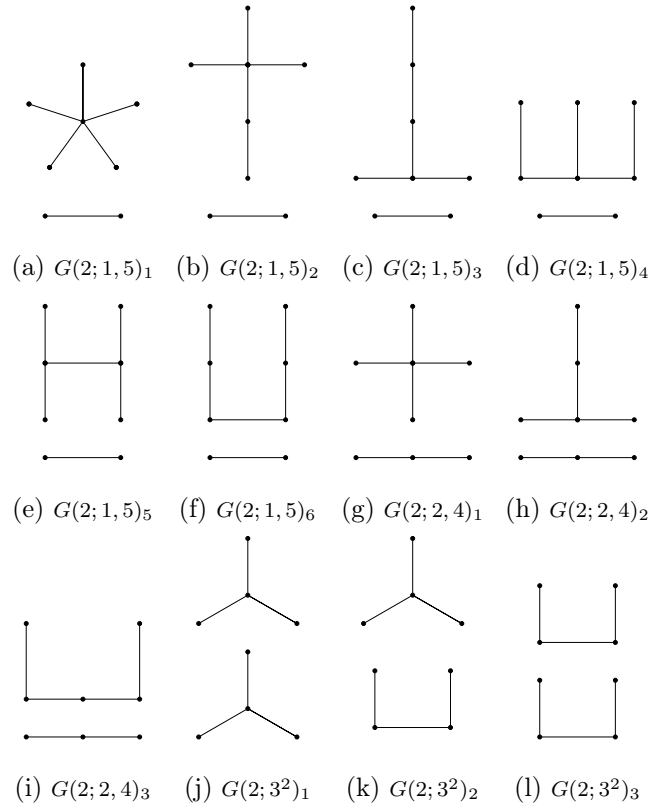


Figure 1. Forests with six edges and two connected components.

- $G(2; 1, 5)_1$
- $G(2; 2, 4)_1$
- $G(2; 3, 3)_1$
- $G(3; 1^2, 4)_1$
- $G(4; 1^3, 3)_1$
- $G(4; 1^3, 3)_2$
- $G(4; 1^2, 2^2)_1$
- $G(5; 1^4, 2)_1$
- $G(6; 1^6)_1$

**Proof.** Suppose  $\mathcal{G} = \{G_1, G_2, \dots, G_6\}$  is a  $G(2, 1, 5)_1$ -decomposition of  $K_9$ . By the pigeonhole principle, there exists a vertex  $v \in V(K_9)$  such that  $v$  does not appear as the center of the star  $K_{1,5}$  in any  $G_i$  for  $1 \leq i \leq 6$ . But then,  $\deg(v) \leq 6$  since all other vertices have degree 1. But of course every vertex in  $V(K_9)$  has degree 8, so we have a contradiction. Essentially the same argument proves that  $G(3; 1^2, 4)_1$  cannot decompose  $K_9$ . If  $G \cong G(2; 2, 4)_1$  or  $G(2; 3, 3)_1$ , the

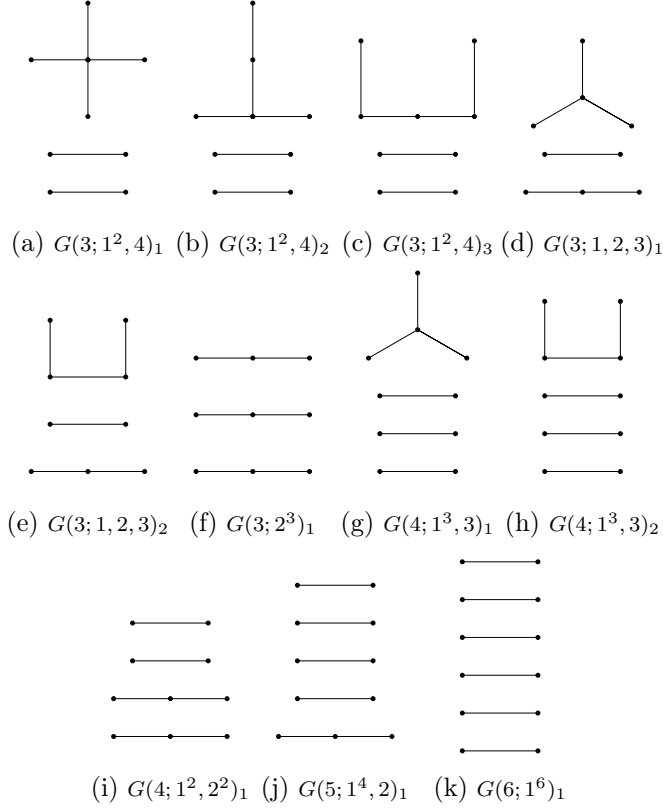


Figure 2. Forests with six edges and more than two connected components.

decomposition was shown not to exist by Meszka [16].

If  $G$  is isomorphic to one of the five remaining forests on the list, then  $G$  contains 10 or more vertices, so of course a  $G$ -decomposition of any graph with 9 vertices cannot exist. ■

The next theorem shows that the nine forests from Lemma 2 are the only ones that do not decompose  $K_9$ .

**Theorem 3.** *A forest with exactly six edges decomposes  $K_9$  if and only if it is not isomorphic to one of the graphs below.*

- $G(2; 1, 5)_1$
- $G(2; 2, 4)_1$
- $G(2; 3, 3)_1$
- $G(3; 1^2, 4)_1$
- $G(4; 1^3, 3)_1$
- $G(4; 1^3, 3)_2$

- $G(4; 1^2, 2^2)_1$
- $G(5; 1^4, 2)_1$
- $G(6; 1^6)_1$

**Proof.** Let  $G$  be a forest with six edges. If  $G$  is isomorphic to one of the nine graphs listed in the theorem statement, the decomposition does not exist by Lemma 2. For the remaining forests, we construct a  $G$ -decomposition as follows.

Let  $V(K_9) = \{0, 1, \dots, 8\}$  and assign each edge  $uv \in E(K_9)$  a length

$$\ell(uv) = \min\{|u - v|, 9 - |u - v|\}$$

and a color

$$c(uv) = \begin{cases} \text{blue} & \text{if } u + v \equiv 0 \pmod{3}, \\ \text{red} & \text{if } u + v \equiv 1 \pmod{3}, \\ \text{green} & \text{if } u + v \equiv 2 \pmod{3}. \end{cases}$$

Observe that  $K_9$  contains exactly nine edges of each of the lengths  $\{1, 2, 3, 4\}$ . Furthermore, among the nine edges of each length, there are exactly three of each color.

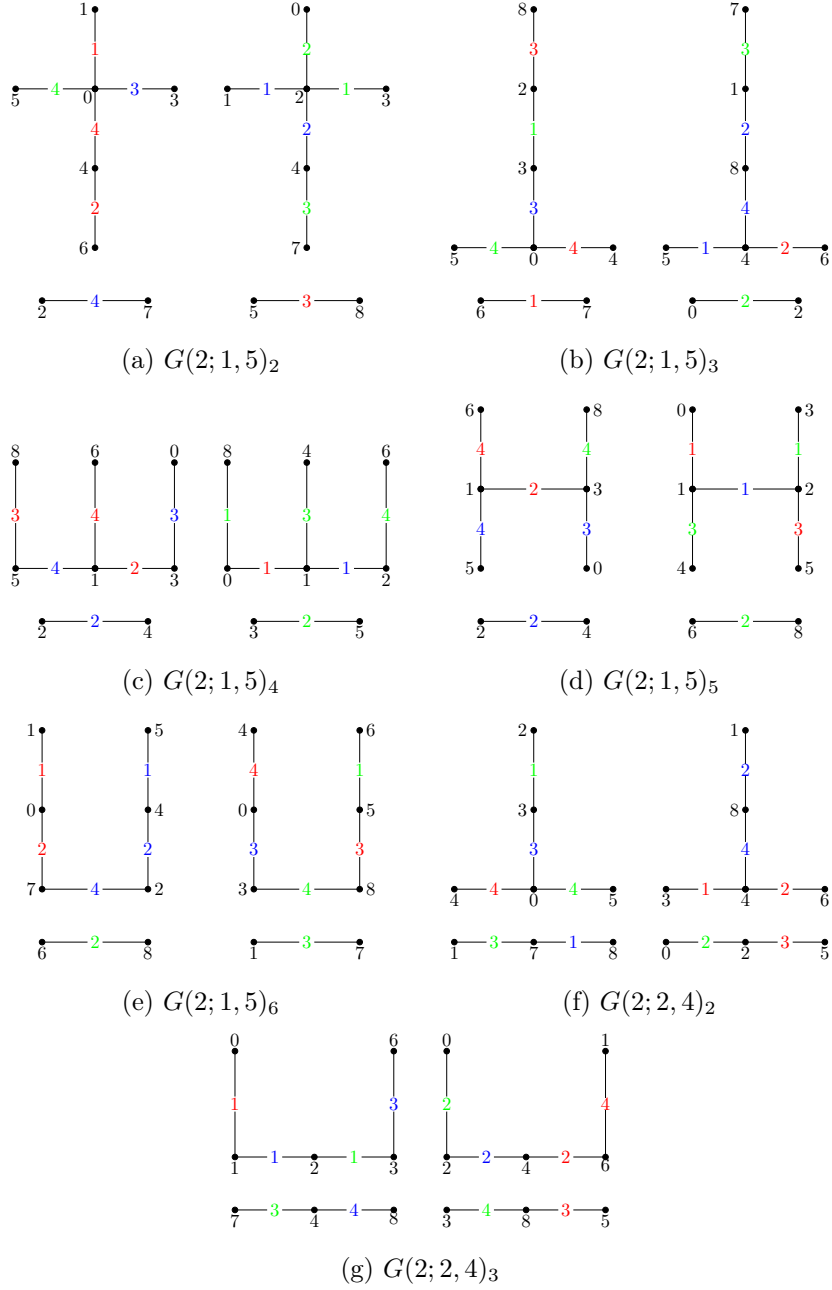
Let  $G_1$  and  $G_2$  be the blocks in the appropriate subfigure of Figure 3. Click  $G_1$  by 3 to form blocks  $G_3$  and  $G_5$ . Repeat this procedure with  $G_2$ , forming blocks  $G_4$  and  $G_6$ . Notice that clicking by 3 preserves both length and color. Therefore, it suffices for the reader to check that the edges in blocks  $G_1$  together with  $G_2$  fill the color spectrum for each edge length 1, 2, 3 and 4. ■

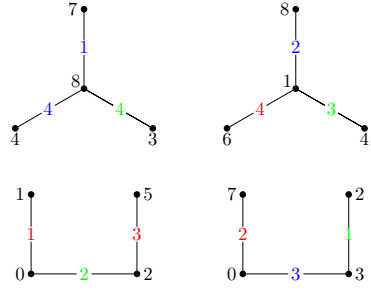
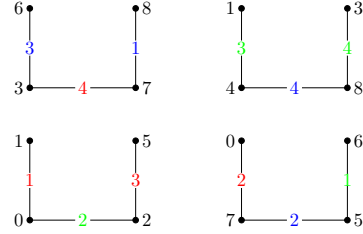
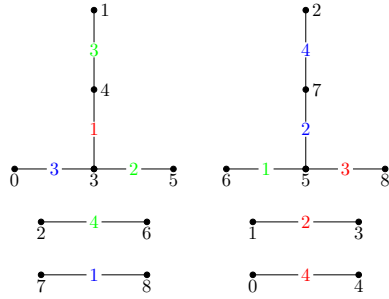
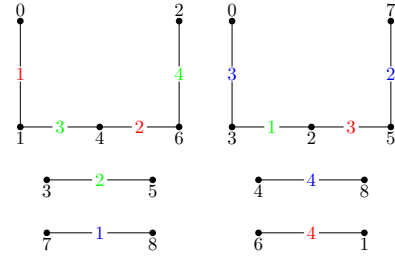
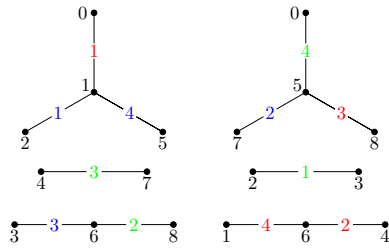
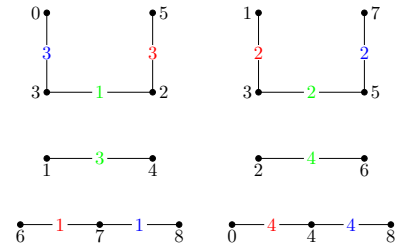
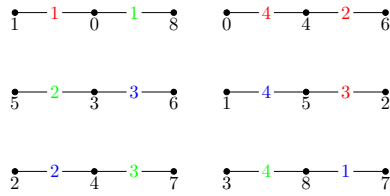
### 3. DECOMPOSITIONS OF $K_n$ FOR $n \equiv 0$ OR $1 \pmod{12}$

Freyberg and Tran introduced the following labeling in [8]. Let  $G$  be a bipartite graph with  $m$  edges and bipartition  $V(G) = A \cup B$ . A  $\sigma^{+-}$ -labeling of  $G$  is an injection  $f : V(G) \rightarrow \{0, 1, \dots, 2m - 2\}$  which induces a bijective length function  $\ell : E(G) \rightarrow \{1, 2, \dots, m\}$  where  $\ell(uv) = f(v) - f(u)$  for every edge  $uv \in E(G)$  with  $u \in A$  and  $v \in B$ , and  $f$  has the additional property that  $f(u) - f(v) \neq m$  for all  $u \in A$  and  $v \in B$ .

Readers familiar with the Rosa-type labelings may recognize that a  $\sigma^{+-}$ -labeling is a restricted  $\rho$ -labeling (see [7] for a nice overview). The usefulness of a  $\sigma^{+-}$ -labeling is punctuated by the following.

**Theorem 4** (Freyberg and Tran, [8]). *Let  $G$  be a bipartite graph with  $m$  edges and a  $\sigma^{+-}$ -labeling such that the edge of length  $m$  is a pendant edge  $e$ . Then there exists a  $G$ -decomposition of  $K_{2mx}$  and  $K_{2mx+1}$  for every positive integer  $x$ .*


 Figure 3. Blocks for  $G$ -decompositions of  $K_9$ .

(h)  $G(2; 3, 3)_2$ (i)  $G(2; 3, 3)_3$ (j)  $G(3; 1, 1, 4)_2$ (k)  $G(3; 1, 1, 4)_3$ (l)  $G(3; 1, 2, 3)_1$ (m)  $G(3; 1, 2, 3)_2$ (n)  $G(3; 2, 2, 2)_3$ Figure 3. (Cont.) Blocks for  $G$ -decompositions of  $K_9$ .



This leads to the main result of this section.

**Theorem 5.** *Let  $G$  be a forest with exactly six edges. Then  $G$  decomposes  $K_n$  whenever  $n \equiv 0$  or  $1 \pmod{12}$ .*

**Proof.** If  $G$  is a matching, then the proof was given in [6]. If not, then the proof is by Theorem 4 and the  $\sigma^{+-}$  labeling of  $G$  given in Figures 4 and 5. Red-labeled vertices belong to  $A$  while black-labeled vertices belong to  $B$ . The length of each edge is displayed in blue. ■

#### 4. DECOMPOSITIONS OF $K_n$ FOR $n \equiv 4$ OR $9 \pmod{12}$ AND $n > 9$

If  $n$  is odd, we let  $V(K_n) = \{0, 1, \dots, n-1\}$  and define the length of an edge  $uv \in E(K_n)$  as  $\ell(uv) = \min\{|u-v|, n-|u-v|\}$ , just as in the proof of Theorem 3. If  $\ell(uv) = n-|u-v|$ , we call  $uv$  a *wrap-around edge*. It follows that  $K_n$  contains exactly  $n$  edges of each length from the set  $\{1, 2, \dots, \frac{n-1}{2}\}$ . On the other hand, if  $n$  is even we let  $V(K_n) = \{0, 1, \dots, n-2\} \cup \{\infty\}$  and for each  $uv \in E(K_n)$ , define  $\ell(uv) = \min\{|u-v|, n-1-|u-v|\}$  whenever  $u, v \in \{0, 1, \dots, n-2\}$ , and  $\ell(uv) = \infty$  otherwise. In this case,  $K_n$  has exactly  $n-1$  edges of each length from  $\{1, 2, \dots, \frac{n-2}{2}, \infty\}$ . We prove the following theorems using this observation of edge length distribution.

**Theorem 6.** *Let  $G$  be a forest with six edges and an integer  $n = 12k+9$  be given. If  $k \geq 1$ , then there exists a  $G$ -decomposition of  $K_n$ .*

**Proof.** Let  $G \cong G(2; 1, 5)_1$ . Label blocks  $G_1, G_2, \dots, G_{k+2}$  as in Figure 29. Notice the blocks have the following properties:

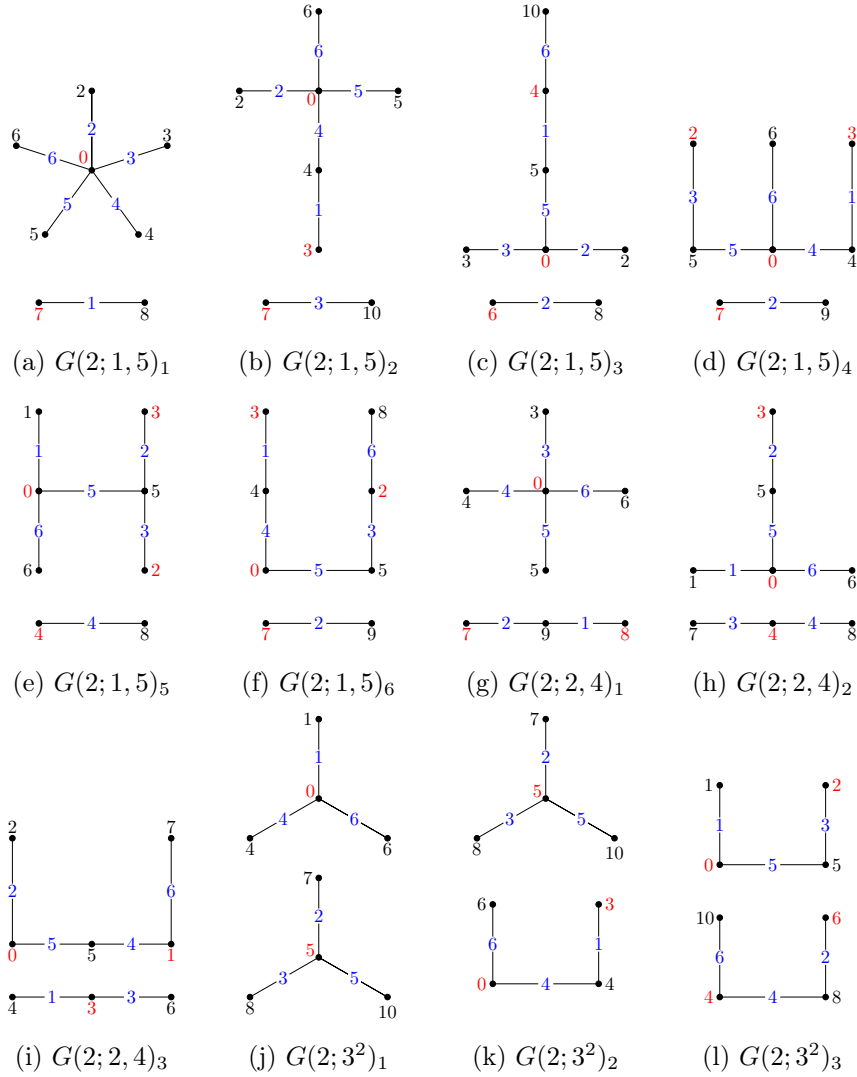
1. The blocks contain no wrap-around edges.
2.  $G_1$  contains exactly three edges of length 1.
3.  $G_1$  contains exactly three edges of length 7.

Let  $x_0y_0, x_1y_1$ , and  $x_2y_2$  be any three edges of the same length in  $G_1$ . Then  $x_i + y_i \equiv i \pmod{3}$  (see subscripts of the vertex labels in  $G_1$ ).

4.  $\bigcup_{i=2}^{k+1} G_i$  contains exactly one edge of each length  $\{\infty, 2, 3, \dots, 6k+1\} \setminus \{7\}$ .

Clicking  $G_1 \cup G_2$  by 3 and  $G_i$  by 1 for  $3 \leq i \leq k+2$  yields  $n$  edges of each length. Property 1 preserves the length of the edges in  $G_1 \cup G_2$ , for all  $i$ . Property 3 above ensures no edge repeats when clicking by 3. Hence, we have described a  $G$ -decomposition of  $K_n$ .

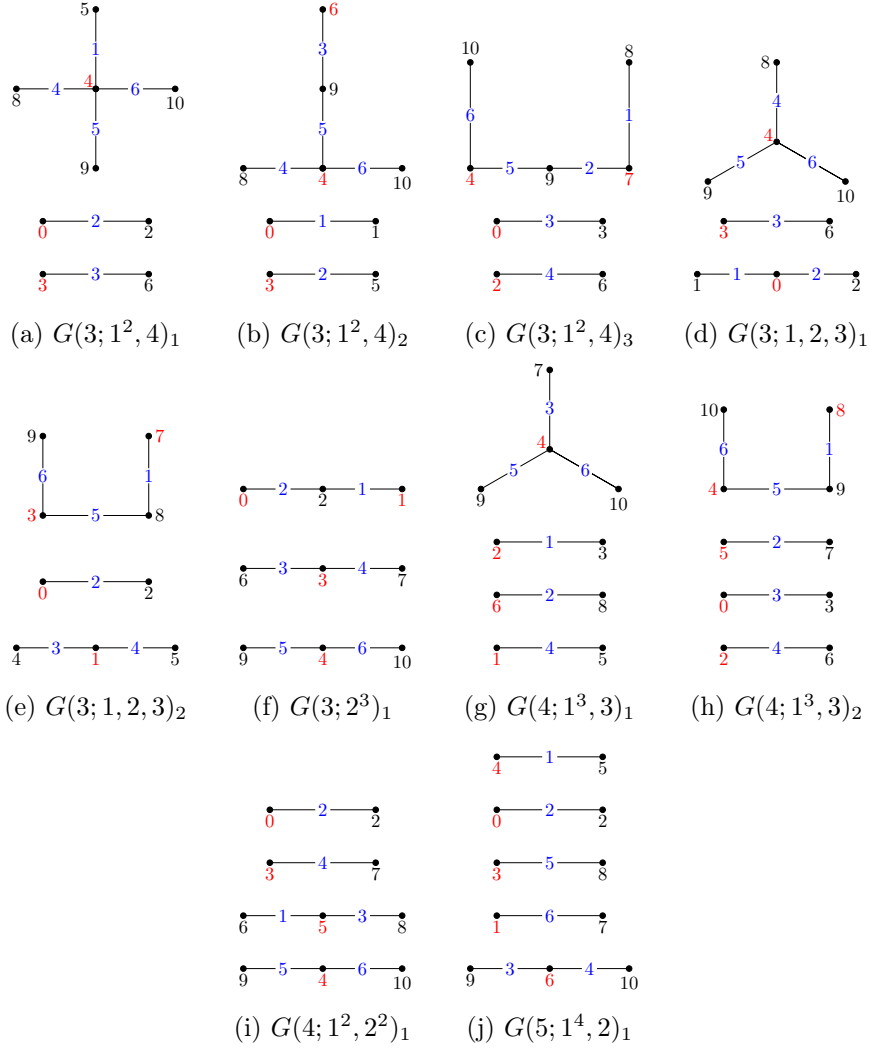
The proof follows in the same way for the remaining forests. See Figures 30 through 51 for the labelings. ■

Figure 4.  $\sigma^{+-}$ -labeling of forests with two components.

The proof of the next theorem is very similar.

**Theorem 7.** *Let  $G$  be a forest with six edges and an integer  $n = 12k + 4$  be given. If  $k \geq 1$ , then there exists a  $G$ -decomposition of  $K_n$ .*

**Proof.** Let  $G$  be a forest with six edges and  $G \not\cong G(2; 1, 5)_1$ . Label blocks  $G_1, G_2, \dots, G_{k+1}$  as in the appropriate figure (see Figures 7 through 28). Notice the blocks have the following properties.


 Figure 5.  $\sigma^{+-}$ -labeling of forests with more than two components.

Clicking  $G_1$  by 3 and  $G_i$  by 1 for  $2 \leq i \leq k+1$  yields  $n-1$  edges of each length. Property 4 above ensures no edge has been repeated in clicking  $G_1$  by 3. Property 1 preserves the length of the edges  $G_1$  for all  $k$ . Hence, we have described a  $G$ -decomposition of  $K_n$ .

The construction for the remaining forest  $G(2; 1, 5)_1$ , follows in the same way except that there are four starter blocks instead of two (see Figure 6). Click block  $G_i$  by 3 for  $1 \leq i \leq 4$ , and by 1 for  $5 \leq i \leq k+3$  to obtain the decomposition. ■

## 5. CONCLUSION

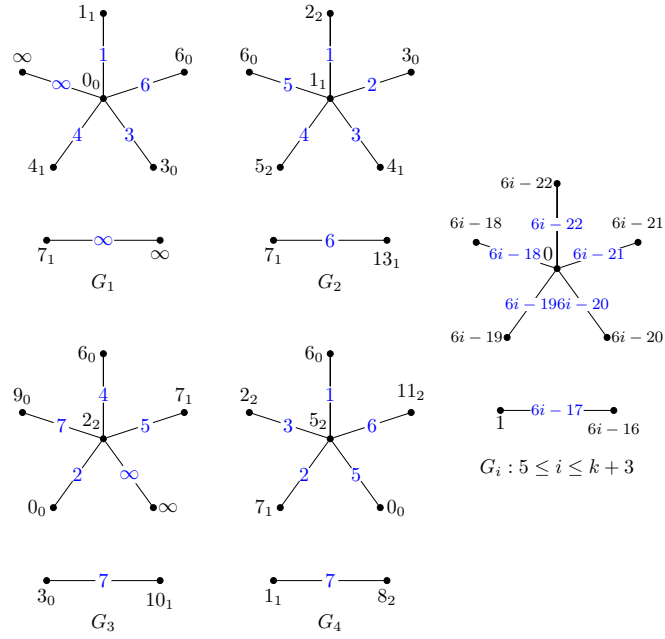
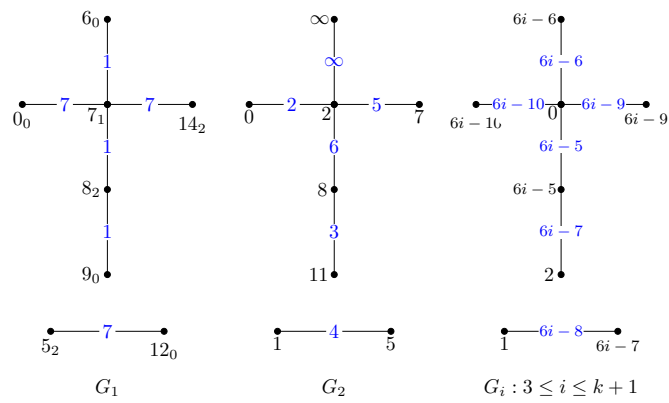
The proof of Theorem 1 follows directly from Theorems 3, 5, 6, and 7. Therefore, we have solved the  $K_n$  decomposition spectrum problem for forests with six edges. As a result, we now have a complete answer for which  $n$  and  $G$  there exists a  $G$ -decomposition of  $K_n$  where  $G$  is any graph with six or less edges.

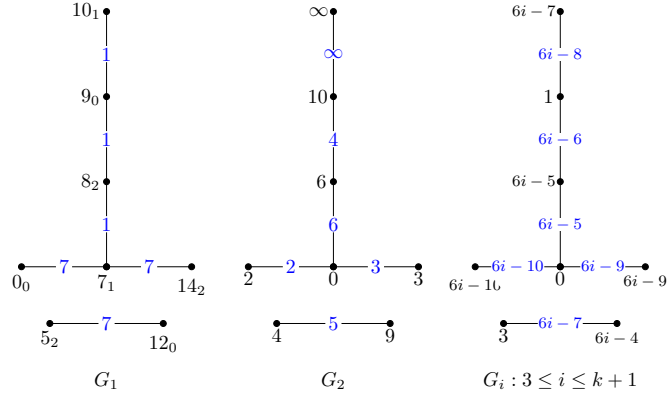
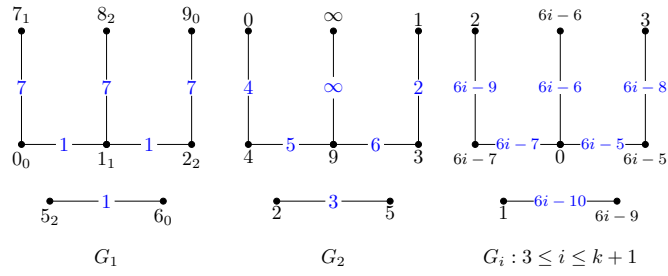
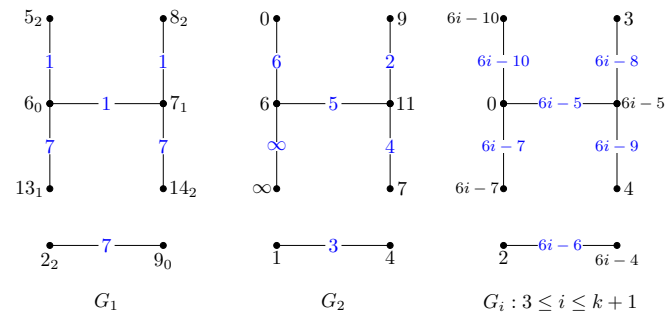
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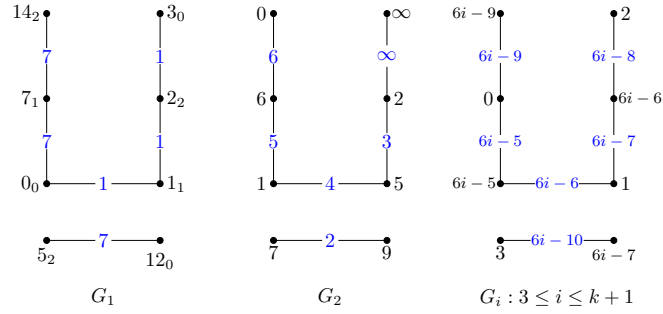
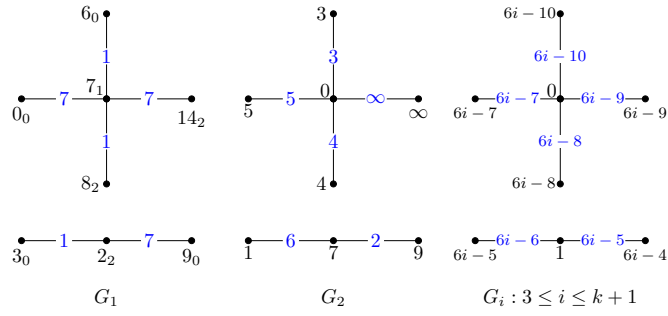
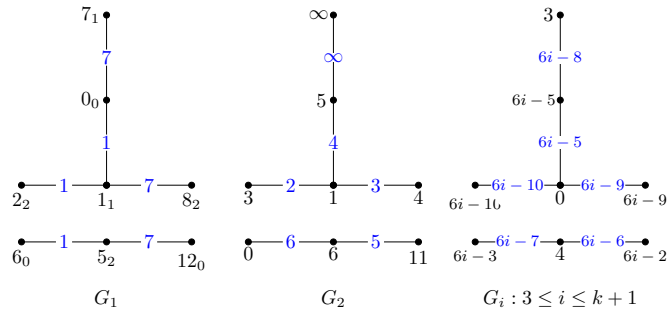
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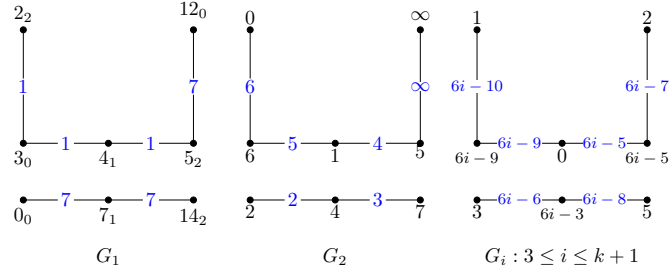
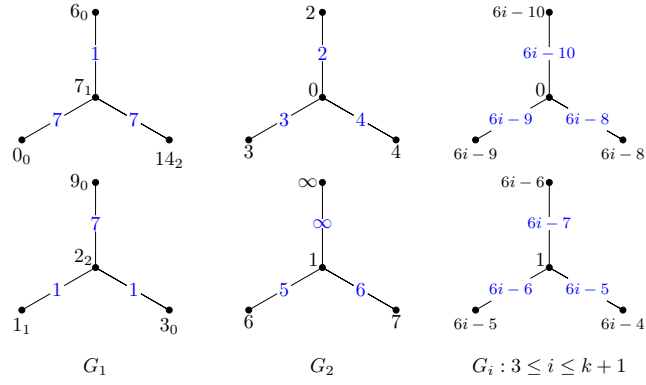
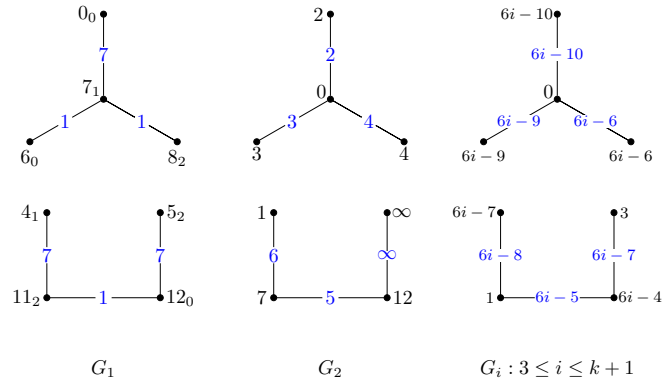
## 6. APPENDIX

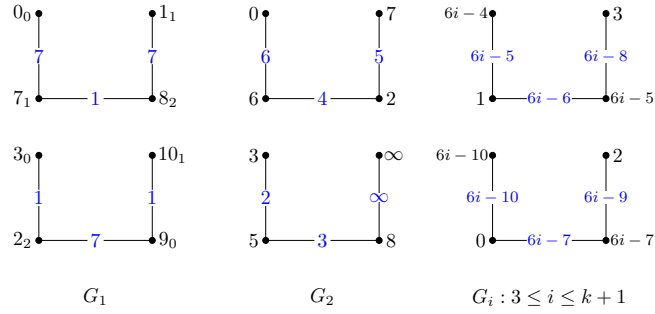
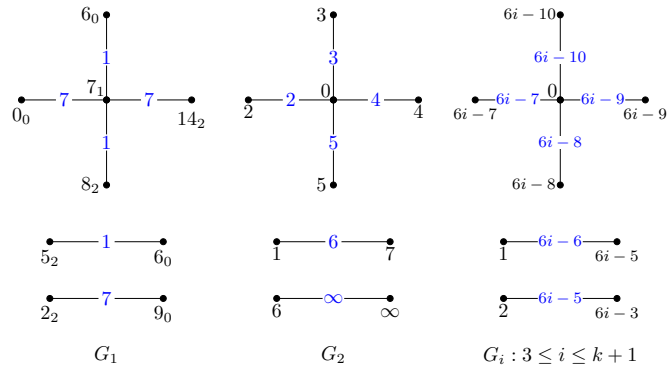
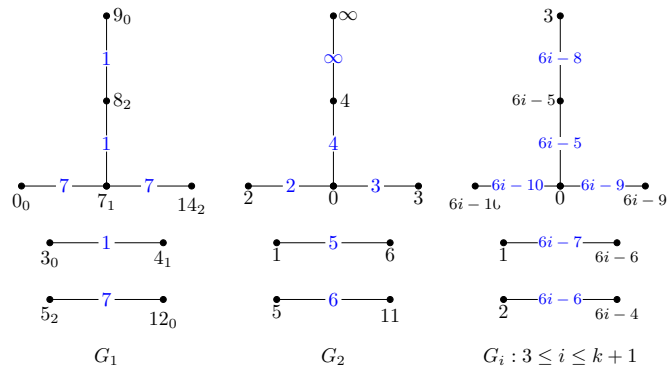
6.1. Labelings for  $n \equiv 4 \pmod{12}$ Figure 6.  $G(2; 1, 5)_1$ Figure 7.  $G(2; 1, 5)_2$

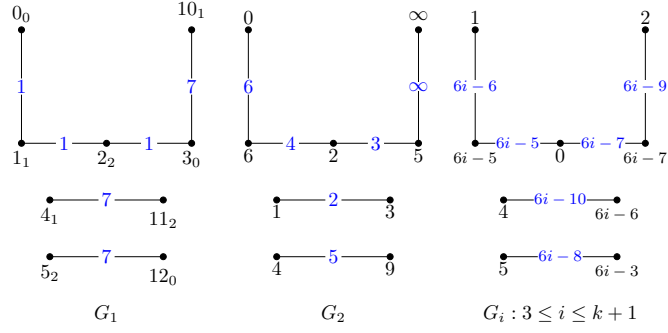
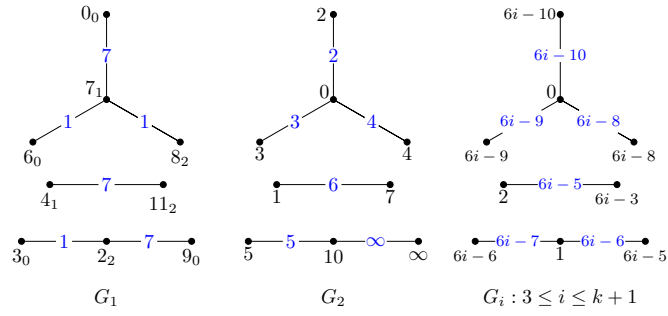
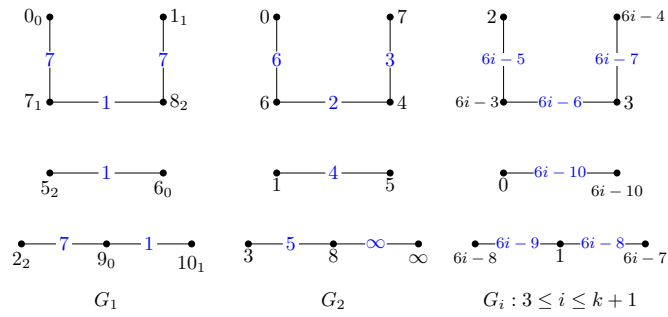

 Figure 8.  $G(2; 1, 5)_3$ 

 Figure 9.  $G(2; 1, 5)_4$ 

 Figure 10.  $G(2; 1, 5)_5$

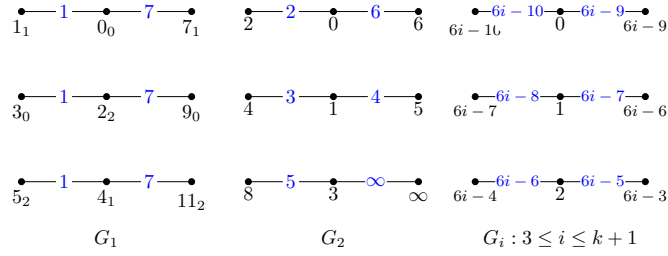
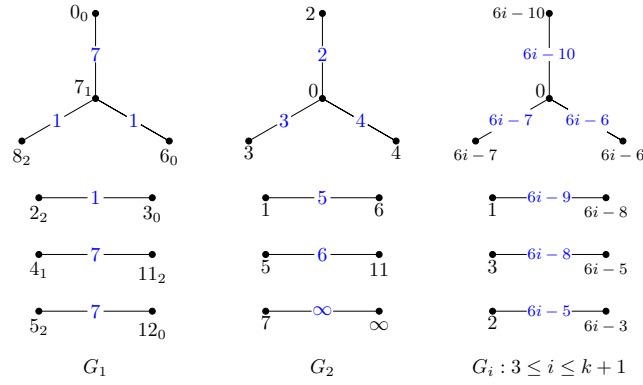
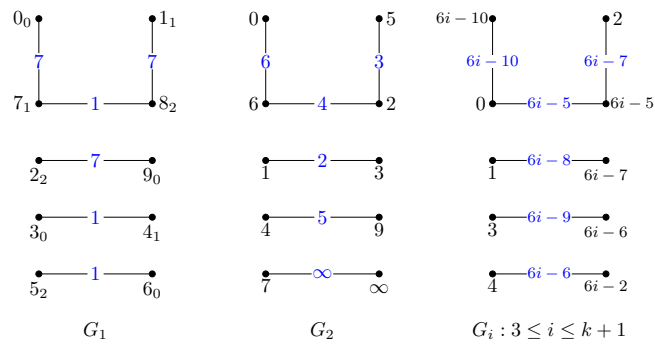
Figure 11.  $G(2; 1, 5)_6$ Figure 12.  $G(2; 2, 4)_1$ Figure 13.  $G(2; 2, 4)_2$

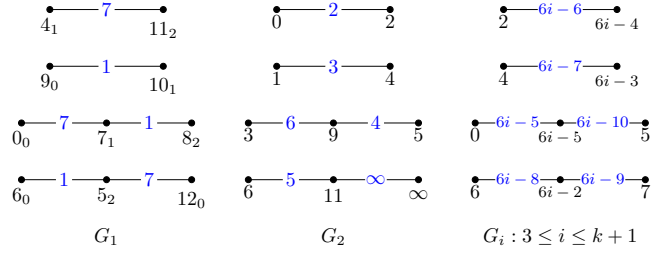
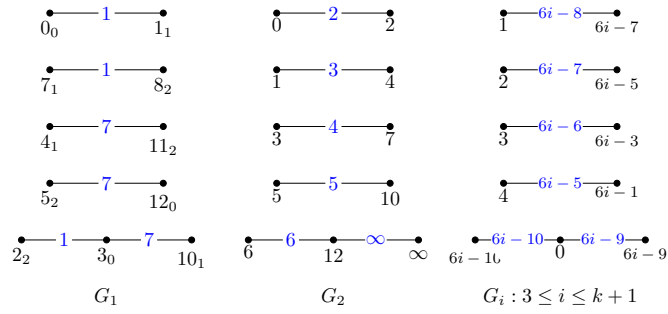
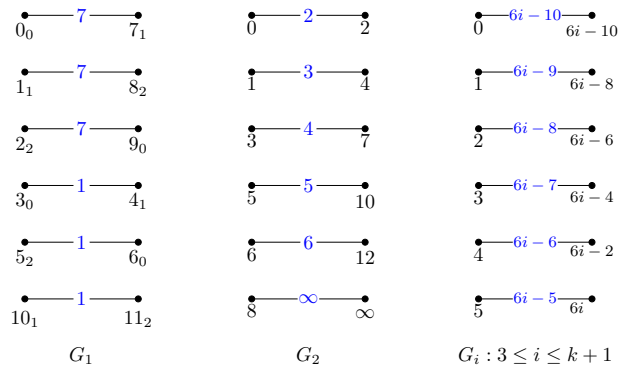



 Figure 14.  $G(2; 2, 4)_3$ 

 Figure 15.  $G(2; 3^2)_1$ 

 Figure 16.  $G(2; 3^2)_2$

Figure 17.  $G(2; 3^2)_3$ Figure 18.  $G(3; 1^2, 4)_1$ Figure 19.  $G(3; 1^2, 4)_2$


 Figure 20.  $G(3; 1^2, 4)_3$ 

 Figure 21.  $G(3; 1, 2, 3)_1$ 

 Figure 22.  $G(3; 1, 2, 3)_2$

Figure 23.  $G(3; 2^3)_1$ Figure 24.  $G(4; 1^3, 3)_1$ Figure 25.  $G(4; 1^3, 3)_2$


 Figure 26.  $G(4; 1^2, 2^2)_1$ 

 Figure 27.  $G(5; 1^4, 2)_1$ 

 Figure 28.  $G(6; 1^6)_1$

## 6.2. Labelings for $n \equiv 9 \pmod{12}$

The vertex labels for some of the blocks in the figures in the next two subsections are of the form  $i_j$ . The subscript  $j$  indicates the modulo 3 equivalence class of  $i$ . For example, if  $i = 7$ , then  $j = 1$  since  $7 \equiv 1 \pmod{3}$ .

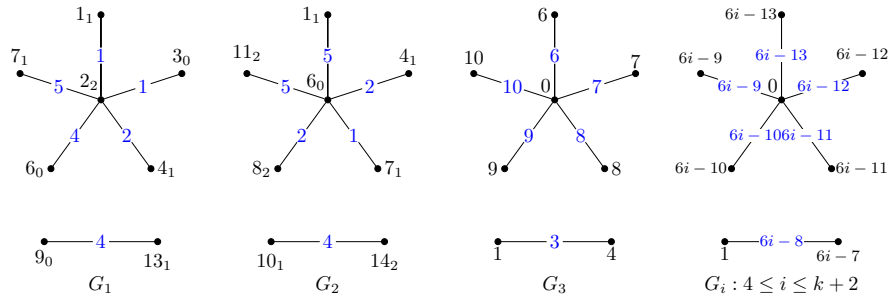


Figure 29.  $G(2; 1, 5)_1$

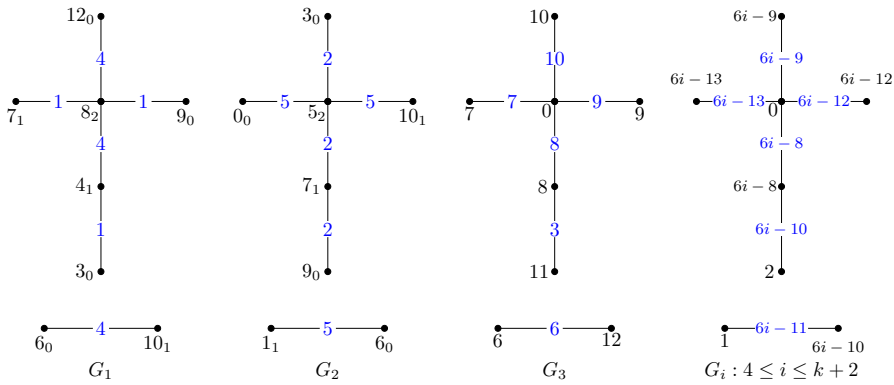
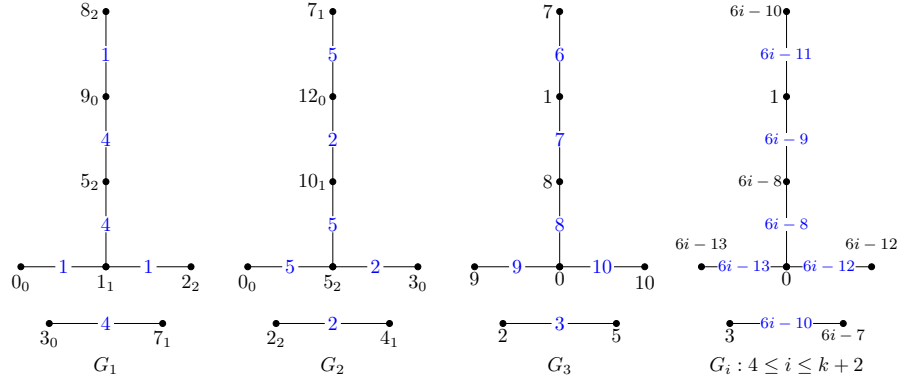
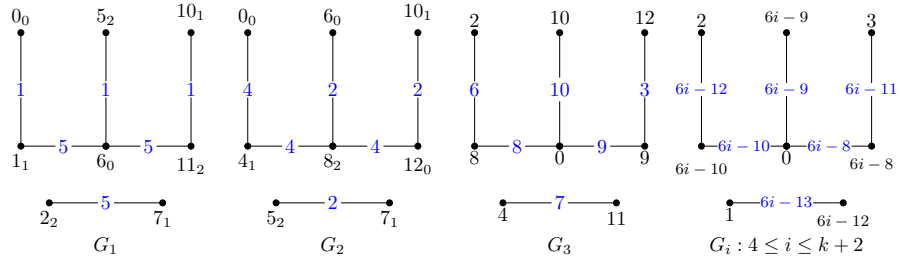
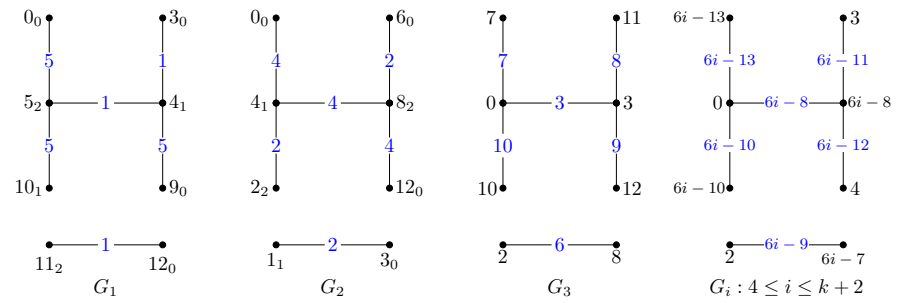
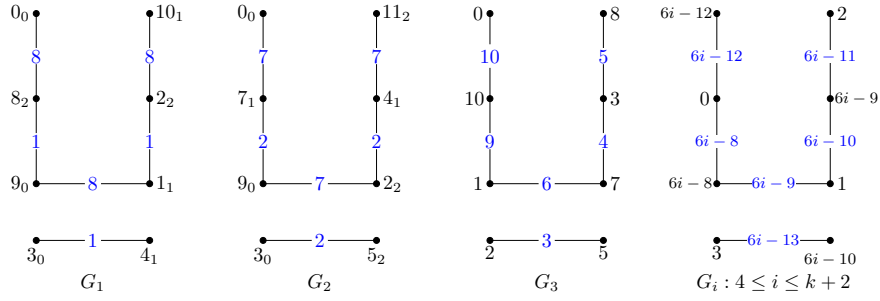
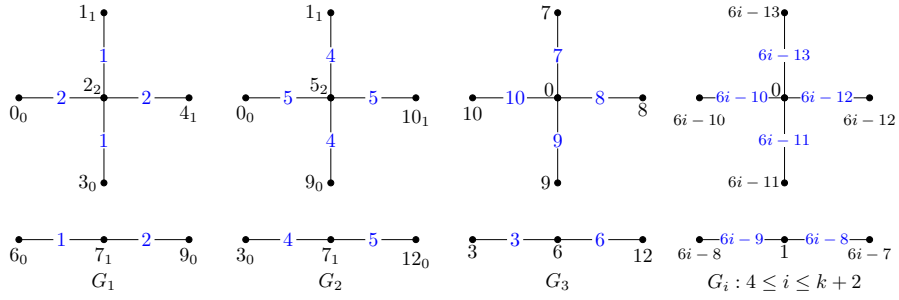
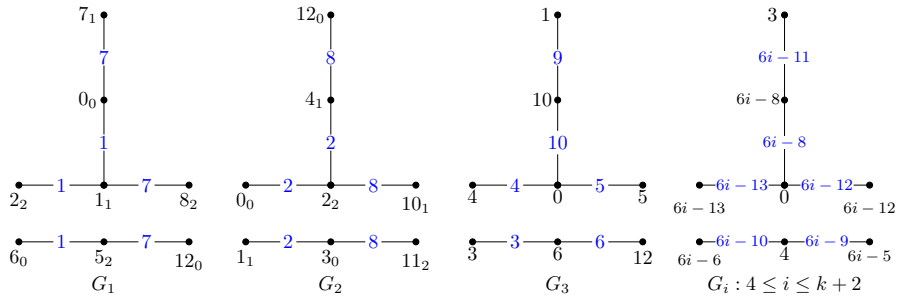
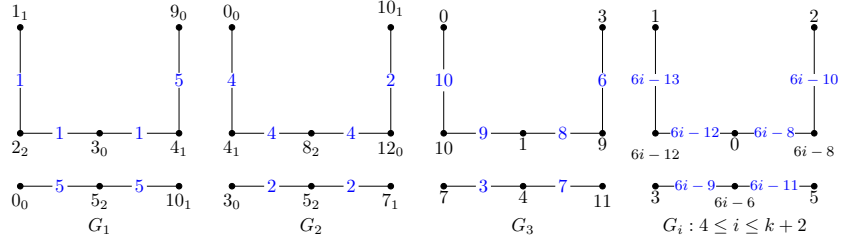
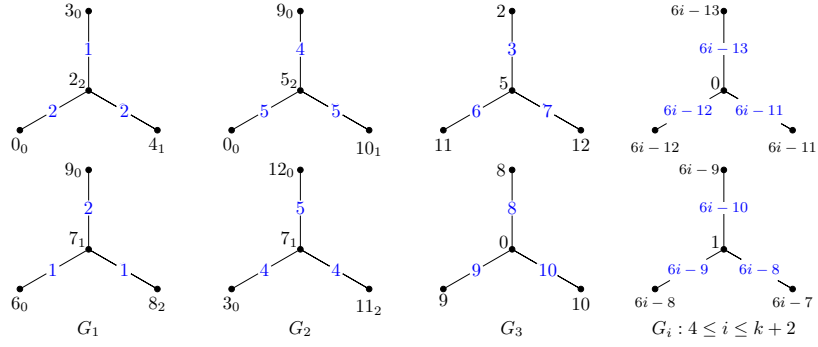
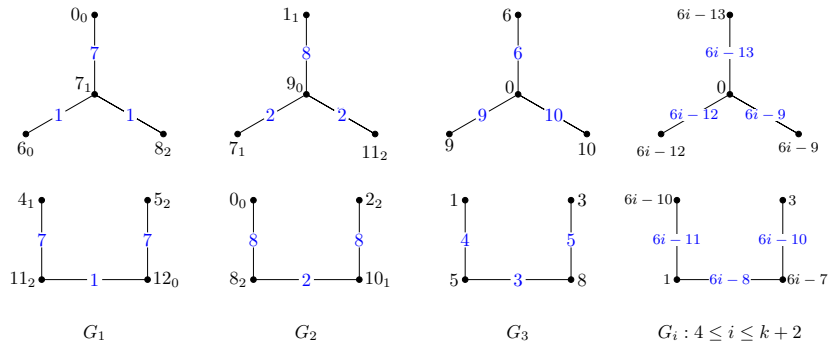


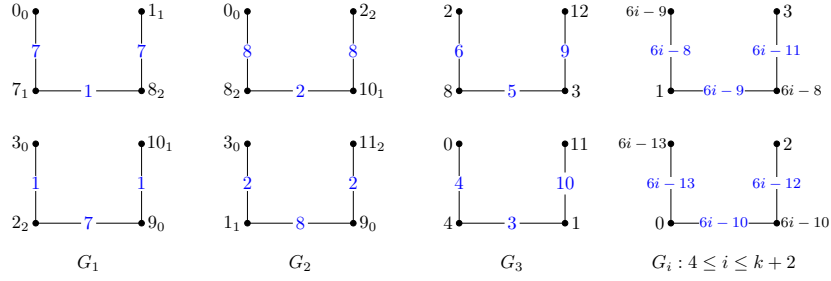
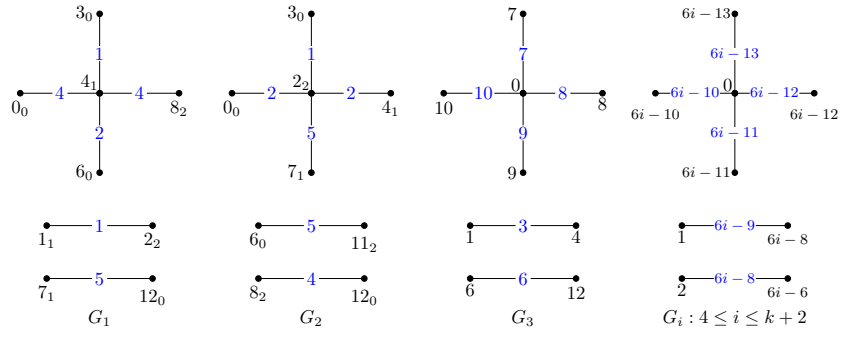
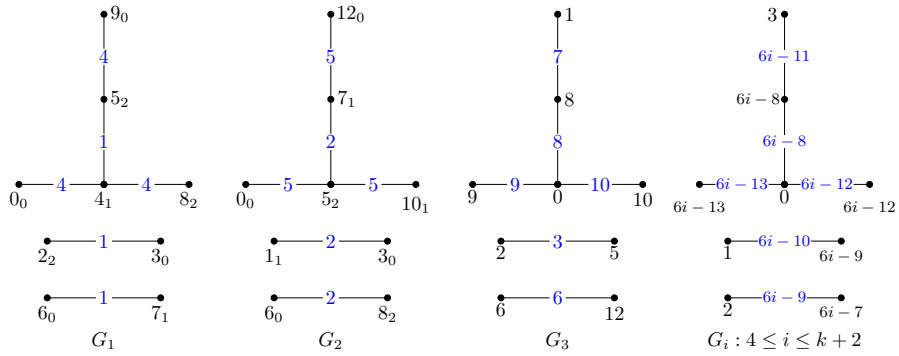
Figure 30.  $G(2; 1, 5)_2$

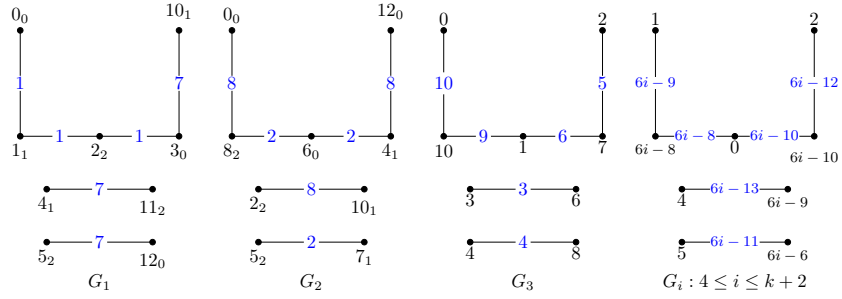
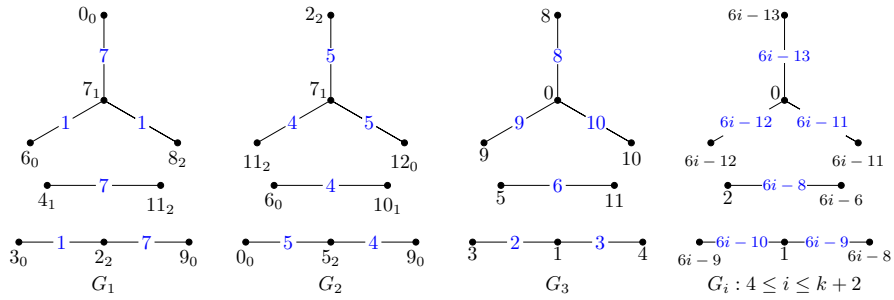
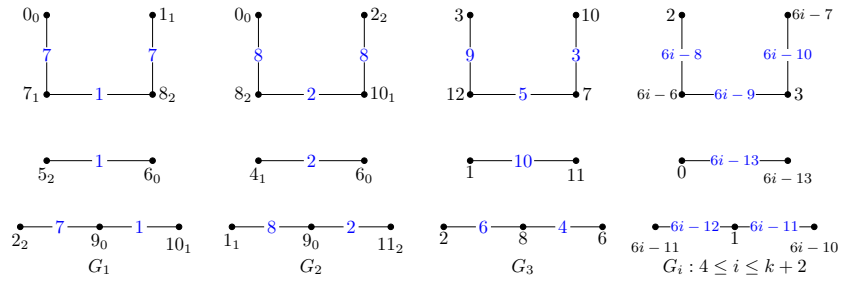

 Figure 31.  $G(2; 1, 5)_3$ 

 Figure 32.  $G(2; 1, 5)_4$ 

 Figure 33.  $G(2; 1, 5)_5$

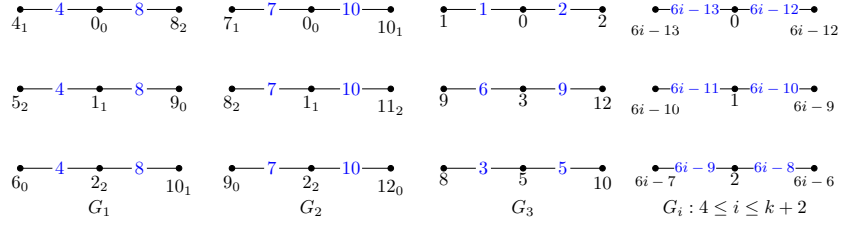
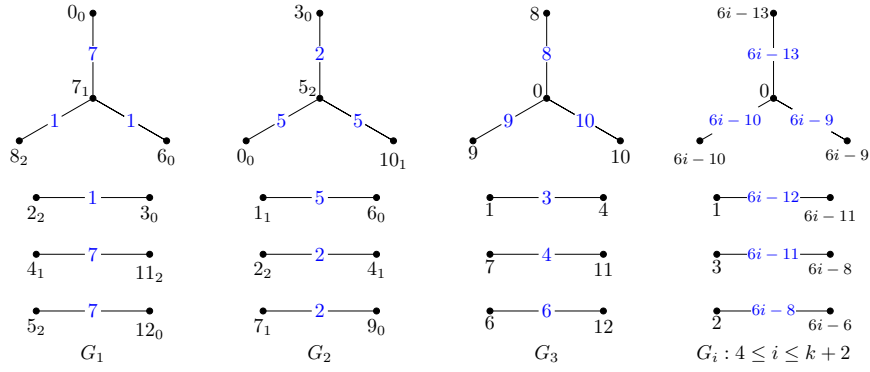
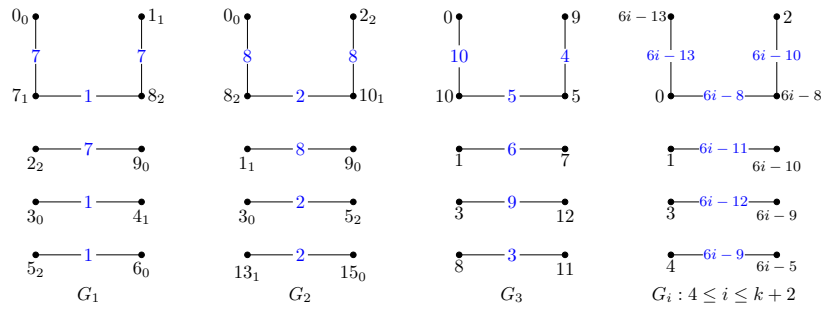
Figure 34.  $G(2; 1, 5)_6$ Figure 35.  $G(2; 2, 4)_1$ Figure 36.  $G(2; 2, 4)_2$

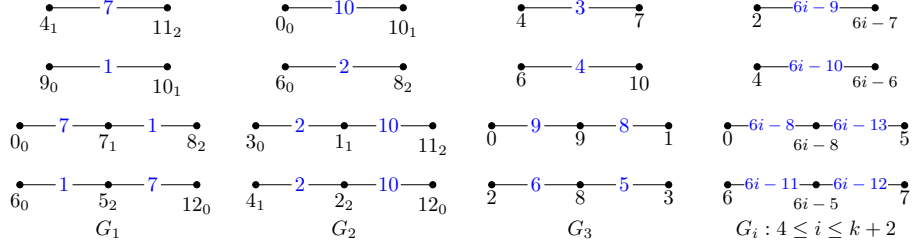
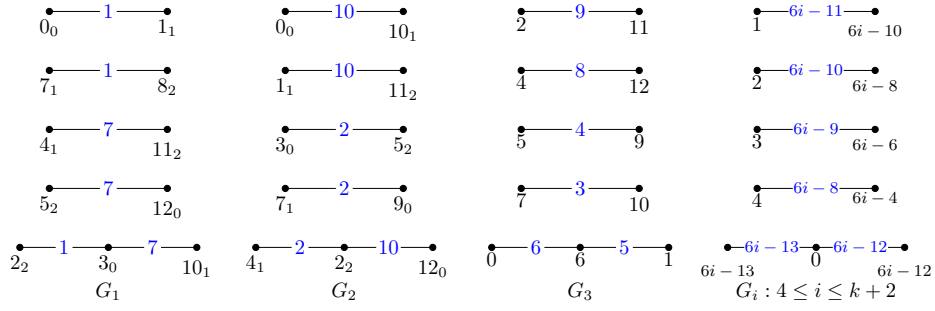
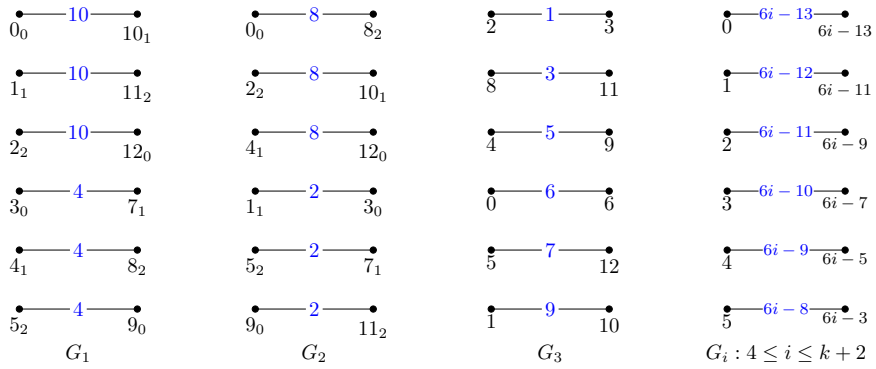



 Figure 37.  $G(2; 2, 4)_3$ 

 Figure 38.  $G(2; 3^2)_1$ 

 Figure 39.  $G(2; 3^2)_2$

Figure 40.  $G(2; 3^2)_3$ Figure 41.  $G(3; 1^2, 4)_1$ Figure 42.  $G(3; 1^2, 4)_2$


 Figure 43.  $G(3; 1^2, 4)_3$ 

 Figure 44.  $G(3; 1, 2, 3)_1$ 

 Figure 45.  $G(3; 1, 2, 3)_2$

Figure 46.  $G(3; 2^3)$ Figure 47.  $G(4; 1^3, 3)_1$ Figure 48.  $G(4; 1^3, 3)_2$


 Figure 49.  $G(4; 1^2, 2^2)_1$ 

 Figure 50.  $G(5; 1^4, 2)_1$ 

 Figure 51.  $G(6; 1^6)_1$

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