

EDGE COLORING OF PRODUCTS OF SIGNED GRAPHS

ROBERT JANCZEWSKI^a, KRZYSZTOF TUROWSKI^{1,b}

AND

BARTŁOMIEJ WRÓBLEWSKI^{2,a}

^a*Department of Algorithms and System Modeling
Faculty of Electronics, Telecommunication and Informatics
Gdańsk University of Technology
ul. Narutowicza 11/12, Gdańsk, Poland*

^b*Theoretical Computer Science Department
Faculty of Mathematics and Computer Science
Jagiellonian University
ul. Łojasiewicza 6, Kraków, Poland*

e-mail: skalar@eti.pg.gda.pl
krzysztof.szymon.turowski@gmail.com
bart.wroblew@gmail.com

Abstract

In 2020, Behr defined the problem of edge coloring of signed graphs and showed that every signed graph (G, σ) can be colored using exactly $\Delta(G)$ or $\Delta(G) + 1$ colors, where $\Delta(G)$ is the maximum degree in graph G .

In this paper, we focus on products of signed graphs. We recall the definitions of the Cartesian, tensor, strong, and corona products of signed graphs and prove results for them. In particular, we show that (1) the Cartesian product of Δ -edge-colorable signed graphs is Δ -edge-colorable, (2) the tensor product of a Δ -edge-colorable signed graph and a signed tree requires only Δ colors and (3) the corona product of almost any two signed graphs is Δ -edge-colorable. We also prove some results related to the coloring of products of signed paths and cycles.

Keywords: signed graphs, edge coloring of signed graphs, graph products.

2020 Mathematics Subject Classification: 05C15.

¹Krzysztof Turowski's research was funded in whole by Polish National Science Center 2020/39/D/ST6/00419 grant. For the purpose of Open Access, the author has applied a CC-BY public copyright license to any Author Accepted Manuscript (AAM) version arising from this submission.

²Corresponding author.

1. INTRODUCTION

This article focuses on simple, finite, and undirected graphs. A graph G is defined by its vertices $V(G)$ and edges $E(G)$. Their respective quantities are denoted by $n(G)$ and $m(G)$. The degree of a vertex v in G is given by $\deg_G(v)$ (or sometimes just $\deg(v)$, when G is unambiguous from the context), and $\Delta(G)$ indicates the maximum degree among all vertices in G .

An *incidence* consists of a vertex v paired with an edge e , with v acting as an endpoint of e . This pairing is denoted $v:e$, and the set of all such incidences for a graph G is denoted by $I(G)$. Any other terms and definitions are consistent with those put forth by Diestel [3].

A concept of signed graphs emerged in the 1950s, introduced by Harary [4]. They represent an extended form of simple graphs, devised to better model social relations. A *signed graph* is a pair (G, σ) , where G is a graph and $\sigma: E(G) \rightarrow \{\pm 1\}$ is a function. Here, G is an *underlying graph*, and σ denotes its *signature*. An underlying graph of a signed graph S is sometimes also denoted by $|S|$. An edge $e \in E(G)$ is called *positive* (or *negative*) if $\sigma(e) = 1$ (or $\sigma(e) = -1$, respectively). Cycles in (G, σ) are called *positive* or *negative* based on the product of their edge signs. A signed graph having only positive cycles is *balanced*. In all other cases, it is *unbalanced*.

In the context of a signed graph (G, σ) , switching refers to an operation on a subset $V' \subseteq V(G)$, yielding a new signed graph (G, σ') . For any given edge uv in $E(G)$, the following is the result of switching

$$\sigma'(uv) = \begin{cases} -\sigma(uv), & \text{if } V' \text{ includes exactly one of } u, v, \\ \sigma(uv), & \text{otherwise.} \end{cases}$$

If a graph S' can be obtained from graph S by switching some vertices of S , we call them *switching equivalent* signed graphs.

Additionally, by *decomposition* of a signed graph $S = (G, \sigma)$ we denote a sequence of signed graphs $S_1 = (G_1, \sigma_1), \dots, S_k = (G_k, \sigma_k)$ such that $E(G) = \bigcup_{i=1}^k E(G_i)$, $E(G_i) \cap E(G_j) = \emptyset$ for any $1 \leq i < j \leq k$, and $\sigma_i(e) = \sigma(e)$ for all $e \in E(G_i)$ and for all $i \in \{1, \dots, k\}$. It turns out that it is often useful to decompose signed graphs into (properly chosen) sequences of signed cycles and/or matchings.

There were several coloring problems on graphs considered in the literature, starting with works of Harary [5] and Zaslavsky [15, 16, 17]. In particular, there is a generalized definition of the chromatic number, for which on the one hand the equivalent of Vizing's theorem was obtained by Máčajová *et al.* [12], but also it was proved that the Four Color Theorem for signed planar graphs does not hold [11]. In addition, there were also studied other problems on signed graphs

such as circular coloring [10], choosability [7, 13], or chromatic spectrum [8, 9]. For an overview of these research directions in the recent years see [14].

Recently, Behr [1] defined the problem of edge coloring of signed graphs in such a way that it generalizes the well-known edge coloring problem. Let k be a positive integer and

$$M_k = \begin{cases} \{0, \pm 1, \dots, \pm l\}, & \text{if } k = 2l + 1, \\ \{\pm 1, \dots, \pm l\}, & \text{if } k = 2l. \end{cases}$$

A k -edge-coloring of a signed graph $S = (G, \sigma)$ is a function $f: I(G) \rightarrow M_k$ such that $f(u:uv) = -\sigma(uv)f(v:uv)$ for each edge $uv \in E(G)$ and $f(u:uv_1) \neq f(u:uv_2)$ for edges $uv_1 \neq uv_2$. The chromatic index of a signed graph S is denoted by $\chi'(S)$ and defined as the smallest k for which graph S has a k -edge-coloring. It is a well-known fact that all the switching equivalent signed graphs have exactly the same chromatic index.

Behr [1] also proved the generalized Vizing's theorem, also called the Behr's theorem.

Theorem 1 [1]. $\Delta(|S|) \leq \chi'(S) \leq \Delta(|S|) + 1$ for every signed graphs S .

Behr [1] introduced a *class ratio* $\mathcal{C}(G)$ of graph G as the number of signatures $\sigma: E(G) \rightarrow \{\pm 1\}$ such that a signed graph (G, σ) can be colored using $\Delta(G)$ colors, divided by the number of all possible signatures that can be defined on $E(G)$, that is, $2^{m(G)}$. The class ratio is a rational number satisfying $0 \leq \mathcal{C}(G) \leq 1$.

Additionally, Behr [1] proved some basic results concerning the problem of edge coloring of paths and cycles.

Theorem 2 [1]. Every signed path $S = (P_k, \sigma)$ can be colored using exactly $\Delta(|S|)$ colors.

It is easy to observe that a signed graph of a maximum degree 1 can be colored using exactly one color—0.

Theorem 3 [1]. A signed cycle $S = (C_k, \sigma)$ can be colored using $\Delta(|S|)$ colors if and only if S is balanced. Otherwise, it requires $\Delta(|S|) + 1$ colors.

It is also known the following.

Theorem 4 [6]. Every signed cactus S that is not an unbalanced cycle can be colored using exactly $\Delta(|S|)$ colors.

Corollary 5. Every signed tree S can be colored using exactly $\Delta(|S|)$ colors.

If $\chi'(S) = \Delta(|S|)$, we say S is Δ -edge-colorable.

It follows from Theorem 2 and Theorem 3 that a signed graph of maximum degree 2 can be colored using 2 colors if each of its connected components is either a path or a balanced cycle. In most of the proofs, we show that a given signed graph has a decomposition into subgraphs of maximum degree 2 or graphs of maximum degree 2 and a graph of maximum degree 1. Then it is clear that the subgraphs can be properly colored using either 2 or 1 colors and the colors used to color specific incidences do not matter for the proofs. We always guarantee that the sets of colors used to color different subgraphs are disjoint.

In this paper, we focus on the products of signed graphs and determine the value of the chromatic index of products of some signed graphs. In Section 2 we define what the Cartesian product of signed graphs is and prove that a Cartesian product of two Δ -edge-colorable signed graphs is also Δ -edge-colorable. We also find the chromatic index of products of a signed path and a signed cycle, and products of two signed cycles. In Section 3 we define a tensor product of signed graphs and prove that a tensor product of a Δ -edge-colorable signed graph and a signed tree is Δ -edge-colorable. In Section 4 we define a strong product of signed graphs and prove that a strong product of two signed paths can be colored using Δ colors. In Section 5 we deal with the concept of corona products of signed graphs by first giving its definition and then proving that the chromatic index of a corona product of any two non-trivial signed graphs is exactly equal to Δ .

Throughout the paper for simplicity we will often switch between signed graphs and their underlying graphs e.g. refer to $V(S)$, $E(S)$, $n(S)$, $m(S)$, $\Delta(S)$ instead of $V(|S|)$ etc. In the same manner, whenever we will refer to a sequence H_1, \dots, H_k as a decomposition of S , we will be denoting both signed graphs or their underlying graphs, depending on the particular context.

2. CARTESIAN PRODUCTS

Let $S_1 = (G_1, \sigma_1)$ and $S_2 = (G_2, \sigma_2)$ be signed graphs. The Cartesian product of graphs S_1 , S_2 is a signed graph $S = (G, \sigma)$, such that $V(S) = V(S_1) \times V(S_2)$ and there is an edge $e = (u, u')(v, v')$ in S , where $u, v \in V(S_1)$, $u', v' \in V(S_2)$, if and only if one of the following conditions holds.

1. $u = v$, $u'v' \in E(S_2)$. In such case $\sigma(e) = \sigma_2(u'v')$;
2. $uv \in E(S_1)$, $u' = v'$. In such case $\sigma(e) = \sigma_1(uv)$.

We denote S by $S_1 \square S_2$. Clearly, $|S| = G_1 \square G_2$.

It is easy to observe that S consists of $n(S_1)$ disjoint copies of graph S_2 and $n(S_2)$ disjoint copies of graph S_1 . It is also clear that $\deg_S((u, u')) = \deg_{S_1}(u) + \deg_{S_2}(u')$. It follows that $\Delta(S) = \Delta(S_1) + \Delta(S_2)$.

Theorem 6. *Let $S_1 = (G_1, \sigma_1)$, $S_2 = (G_2, \sigma_2)$ be edge-disjoint signed graphs and $S = (G, \sigma) = S_1 \square S_2$. If $\chi'(S_1) = \Delta(S_1)$ and $\chi'(S_2) = \Delta(S_2)$, then $\chi'(S) = \Delta(S)$.*

Proof. Let $V(S_i) = \{v_1^i, \dots, v_{n_i}^i\}$, $E(S_i) = \{e_1^i, \dots, e_{m_i}^i\}$ for $i \in \{1, 2\}$.

We define functions $\eta_i^V: V(S) \rightarrow V(S_i)$ for $i \in \{1, 2\}$, such that $\eta_1^V(v_i^1, v_j^2) = v_i^1$ and $\eta_2^V(v_i^1, v_j^2) = v_j^2$. Analogously, we define functions $\eta_i^E: E(S) \rightarrow E(S_i)$ for $i \in \{1, 2\}$, such that $\eta_1^E((v_i^1, v_j^2), (v_k^1, v_l^2)) = v_i^1 v_k^1$, $\eta_2^E((v_i^1, v_j^2), (v_i^1, v_k^2)) = v_j^2 v_k^2$ and functions $\eta_i^I: I(S) \rightarrow I(S_i)$ for $i \in \{1, 2\}$, such that $\eta_i^I(v:e) = \eta_i^V(v):\eta_i^E(e)$.

Let c_i be an arbitrary optimal edge coloring of graph S_i , $i \in \{1, 2\}$.

We consider two cases.

1. At least one of $\Delta(S_1)$, $\Delta(S_2)$ is even. Without loss of generality, we assume that $\Delta(S_2)$ is even. Let c be a function such that

$$c(v:e) = \begin{cases} c_1(\eta_1^I(v:e)), & \text{if } e = (v_i^1, v_j^2), (v_k^1, v_l^2), \\ c_2(\eta_2^I(v:e)) + \left\lfloor \frac{\Delta(S_1)}{2} \right\rfloor, & \text{if } e = (v_i^1, v_j^2), (v_i^1, v_k^2) \text{ and } c_2(\eta_2^I(v:e)) > 0, \\ c_2(\eta_2^I(v:e)) - \left\lfloor \frac{\Delta(S_1)}{2} \right\rfloor, & \text{if } e = (v_i^1, v_j^2), (v_i^1, v_k^2) \text{ and } c_2(\eta_2^I(v:e)) < 0. \end{cases}$$

The equation covers all possible incidences because function c_2 is never equal to 0 since $\Delta(S_2)$ is even and c_2 is a $\Delta(S_2)$ -edge coloring of graph S_2 .

We observe that all the incidences from the copies of graph S_1 get the same colors in c as the respective incidences in c_1 . The copies are disjoint so the incidences are colored correctly. We denote those colors by C_1 . $C_1 = \{\pm 1, \dots, \pm \frac{\Delta(S_1)}{2}\}$ if $\Delta(S_1)$ is even or $C_1 = \{0, \pm 1, \dots, \pm \frac{\Delta(S_1)}{2}\}$ if $\Delta(S_1)$ is odd.

We observe that incidences from the copies of graph S_2 get colors from the set $C_2 = \left\{1 + \left\lfloor \frac{\Delta(S_1)}{2} \right\rfloor, \dots, \frac{\Delta(S_2)}{2} + \left\lfloor \frac{\Delta(S_1)}{2} \right\rfloor\right\} \cup \left\{-1 - \left\lfloor \frac{\Delta(S_1)}{2} \right\rfloor, \dots, -\frac{\Delta(S_2)}{2} - \left\lfloor \frac{\Delta(S_1)}{2} \right\rfloor\right\}$ in c . The copies are disjoint and the colors used to color their incidences are equal to colors from c_2 shifted by either $\left\lfloor \frac{\Delta(S_1)}{2} \right\rfloor$ or $-\left\lfloor \frac{\Delta(S_1)}{2} \right\rfloor$, so the incidences are colored correctly.

It is clear that $C_1 \cap C_2 = \emptyset$ since $\max\{|x|: x \in C_1\} < \min\{|x|: x \in C_2\}$. It follows that c is an edge coloring of graph S . We observe that c uses exactly $2 \left(\frac{\Delta(S_2)}{2} + \left\lfloor \frac{\Delta(S_1)}{2} \right\rfloor \right) = \Delta(S_1) + \Delta(S_2) = \Delta(S)$ colors when $\Delta(S_1)$ is even and exactly $1 + 2 \left(\frac{\Delta(S_2)}{2} + \left\lfloor \frac{\Delta(S_1)}{2} \right\rfloor \right) = 1 + 2 \left(\frac{\Delta(S_2)}{2} + \frac{\Delta(S_1)-1}{2} \right) = \Delta(S_1) + \Delta(S_2) = \Delta(S)$ colors when $\Delta(S_1)$ is odd.

2. Both $\Delta(S_1)$ and $\Delta(S_2)$ are odd. We observe that if we followed the previous case, both c_1 and c_2 would use color 0 so the previous reasoning does not directly apply in this case since there would be adjacent incidences both colored with color 0. We also observe that $\Delta(S)$ is even, so if there exists a

$\Delta(S)$ -coloring of graph S , it does not use color 0. We show how to modify the reasoning so it also applies here.

Let c be a function analogous to the function c from the previous case. We only extend it such that $c(v:e) = 0$ if $e = (v_i^1, v_j^2)(v_i^1, v_k^2)$ and $c_2(\eta_2^I(v:e)) = 0$. That way, it has a value for all the possible incidences of graph S . We observe that c may not be an edge coloring of S , because there might be adjacent incidences sharing the same vertex with color 0 assigned to both of them.

Let H be a subgraph of S such that $uv \in E(H)$ if and only if $c(u:uv) = 0$ and $c(v:uv) = 0$, that is, it contains exactly the edges of S such that their incidences get color 0 in c . Note that from the definition of function c , it follows that if for some edge uv it is true that $c(u:uv) = 0$, it is also true that $c(v:uv) = 0$. It is clear that H is not an empty graph, because there exists at least one incidence in S_i such that it gets color 0 in coloring c_i , for $i \in \{1, 2\}$.

Let us consider an arbitrary connected component H' of graph H . Without loss of generality, we assume that $(v_i^1, v_j^2)(v_k^1, v_j^2) \in E(H')$. We denote that edge by e_1 . It is clear that H' does not contain edge $(v_i^1, v_j^2)(v_p^1, v_j^2)$ for $p \neq k$ and edge $(v_i^1, v_j^2)(v_k^1, v_j^2)$ for $t \neq i$ for otherwise, there would be adjacent edges in S_1 both with color 0 assigned in c_1 . We define set E' consisting of two edges: $e_2 = (v_i^1, v_j^2)(v_i^1, v_p^2)$ and $e_3 = (v_k^1, v_p^2)(v_k^1, v_j^2)$. There are two cases to consider.

- (a) $E' \cap E(H') = \emptyset$. Then $m(H') = 1$ so H' is a matching.
- (b) $E' \cap E(H') \neq \emptyset$. We observe that $E' \subset E(H')$ because $\eta_2^E(e_2) = \eta_2^E(e_3)$. Let $e_4 = (v_i^1, v_p^2)(v_k^1, v_p^2)$. Since $\eta_1^E(e_1) = \eta_1^E(e_4)$ and e_4 is adjacent to e_2 and e_3 , it follows that $e_4 \in E(H')$.

It is easy to observe that H' contains only edges e_1, \dots, e_4 . Otherwise, edges colored by either c_1 or c_2 with color 0 would not be matchings, so either c_1 or c_2 would not be a proper coloring. We observe that edges of graph H' span a cycle C_4 and since $\sigma(e_1) = \sigma(e_4)$, $\sigma(e_2) = \sigma(e_3)$, the cycle is balanced.

It follows that all the connected components of graph H are either matchings or balanced cycles C_4 , so H can be colored using 2 colors. We change the values of function c such that incidences previously colored by color 0 are now colored using colors $\pm \left(\left\lfloor \frac{\Delta(S_1)}{2} \right\rfloor + \left\lfloor \frac{\Delta(S_2)}{2} \right\rfloor + 1 \right)$. After this change, c is an edge coloring of graph S . Moreover, c uses exactly $2 \left(\left\lfloor \frac{\Delta(S_1)}{2} \right\rfloor + \left\lfloor \frac{\Delta(S_2)}{2} \right\rfloor + 1 \right) = 2 \left(\frac{\Delta(S_1)-1}{2} + \frac{\Delta(S_2)-1}{2} + 1 \right) = \Delta(S_1) + \Delta(S_2) = \Delta(S)$, so c is a proper $\Delta(S)$ -edge coloring of graph S . ■

Since every signed path is Δ -edge-colorable, it follows the corollary.

Corollary 7. *Let $S_1 = (P_r, \sigma_1)$, $S_2 = (P_s, \sigma_2)$ and $S = S_1 \square S_2$. Then it follows that $\chi'(S) = \Delta(S)$.*

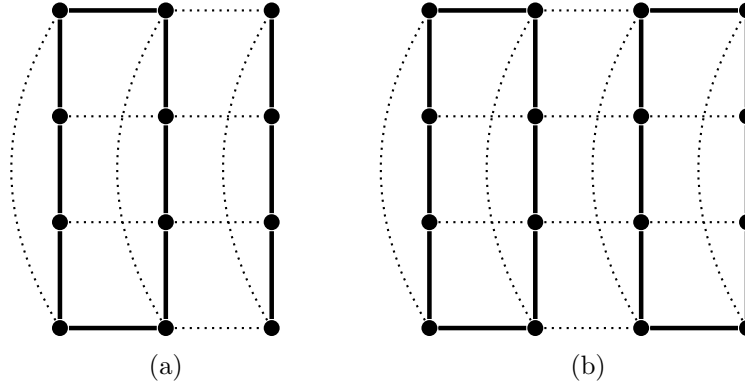


Figure 1. Example signed graphs considered in Theorem 8 — (a) $(P_3 \square C_4, \sigma)$, (b) $(P_4 \square C_4, \sigma)$. Edges belonging to graph H_1 are marked with bold lines in both cases.

Next, we deal with Cartesian products of paths and cycles. Obviously, since $P_1 \square C_s = C_s$ and the chromatic index of signed cycles is already known (see Theorem 3), we can restrict our attention to longer paths.

Theorem 8. *Let $S_1 = (P_r, \sigma_1)$, $S_2 = (C_s, \sigma_2)$ and $S = (P_r \square C_s, \sigma) = S_1 \square S_2$. If $r > 1$, then $\chi'(S) = \Delta(S)$.*

Proof. We first observe that $\Delta(S) = 3$ if $r = 2$, otherwise $\Delta(S) = 4$.

Let $V(S_1) = \{v_1^1, \dots, v_r^1\}$ and $V(S_2) = \{v_1^2, \dots, v_s^2\}$.

We first observe that $\sigma((v_i^1, v_j^2)(v_{i+1}^1, v_j^2)) = \sigma((v_i^1, v_k^2)(v_{i+1}^1, v_k^2))$ for $1 \leq i < r$, $1 \leq j, k \leq s$. Moreover, $\sigma((v_i^1, v_k^2)(v_i^1, v_{k+1}^2)) = \sigma((v_j^1, v_k^2)(v_j^1, v_{k+1}^2))$ for $1 \leq i, j \leq r$, $1 \leq k < s$ and $\sigma((v_i^1, v_1^2)(v_i^1, v_s^2)) = \sigma((v_j^1, v_1^2)(v_j^1, v_s^2))$ for $1 \leq i, j \leq r$.

We will show the decomposition of S into graphs H_1, H_2 such that both of them can be colored using 2 colors. Let us first describe the structure of H_1 : $V(H_1) = V(S)$, $E(H_1) = \{(v_i^1, v_j^2)(v_{i+1}^1, v_j^2): 1 \leq i \leq r, 1 \leq j < s\} \cup \{(v_{2i-1}^1, v_j^2)(v_{2i}^1, v_j^2): 1 \leq i \leq \lfloor \frac{r}{2} \rfloor, j \in \{1, s\}\}$ (see also examples of graph H_1 in Figure 1). It follows that H_1 contains $\lfloor \frac{r}{2} \rfloor$ cycles and possibly a single path when r is odd. It is clear that all the cycles are balanced, as they contain an even number of pairs of edges with the same signs (it follows from the previous observation). Then it becomes clear that H_1 contains balanced cycles and possibly a path, so H_1 can be colored using 2 colors.

It is easy to verify that H_2 is then defined as $V(H_2) = V(S)$ with $E(H_2) = E_1 \cup E_2 \cup E_3$, where $E_1 = \{(v_i^1, v_1^2)(v_i^1, v_s^2): 1 \leq i \leq r\}$, $E_2 = \{(v_i^1, v_j^2)(v_{i+1}^1, v_j^2): 1 \leq i < r, 1 < j < s\}$ and $E_3 = \{(v_{2i}^1, v_j^2)(v_{2i+1}^1, v_j^2): 1 \leq i < \lfloor \frac{r}{2} \rfloor, j \in \{1, s\}\}$. It is clear that sets E_1, E_2, E_3 are disjoint. We observe that E_2 can be partitioned into $s-2$ paths of length $r-1$. $E_1 \cup E_3$ can be partitioned into 1 or 2 edges (when r is odd or even, respectively) and $\lfloor \frac{r-1}{2} \rfloor$ cycles. The cycles are balanced since

each of them consists of two pairs of edges with the same signs. We consider two cases.

1. $r = 2$. H_2 has a decomposition into the following components: two disjoint edges and $s - 2$ disjoint paths of length 1, which are just disjoint edges. H_2 has a decomposition into disjoint edges, so it is a matching and can be colored using color 0. Since H_1 is colored using 2 colors, 0 is not used for H_1 and can be used to color H_2 . It follows that S can be colored using 3 colors, and $\Delta(S) = 3$.

2. $r > 2$. H_2 has a decomposition into the following components: one or two disjoint edges (when r is odd or even, respectively), and $s - 2$ disjoint paths and balanced cycles. It follows that H_2 can be colored using 2 colors, so S can be colored using 4 colors in total and $\Delta(S) = 4$.

It follows that in all cases S can be colored using $\Delta(S)$ colors in total. \blacksquare

Lemma 9. Let $S_1 = (C_{2r}, \sigma_1)$, $S_2 = (C_{2s}, \sigma_2)$ and $S = (G, \sigma) = S_1 \square S_2$. Then it holds that $\chi'(S) = \Delta(S)$.

Proof. First, let us note that $\Delta(S) = 4$. Let $V(S_1) = \{v_1^1, \dots, v_{2r}^1\}$ and $V(S_2) = \{v_1^2, \dots, v_{2s}^2\}$.

We will show the decomposition of S into graphs H_1, H_2 such that both of them can be colored using 2 colors regardless of the edge signs. We first describe the structure of H_1 . Obviously, $V(H_1) = V(S)$. Graph H_1 contains the following configurations of edges.

1. Edges $e_1 = (v_1^1, v_1^2)(v_1^1, v_{2s}^2)$, $e_2 = (v_1^1, v_{2s}^2)(v_{2r}^1, v_{2s}^2)$, $e_3 = (v_{2r}^1, v_{2s}^2)(v_{2r}^1, v_1^2)$, $e_4 = (v_{2r}^1, v_1^2)(v_1^1, v_1^2)$. It is easy to observe that these edges span a cycle (see Figure 2(a)) and it is balanced regardless of the signs in S_1 and S_2 since $\sigma(e_1) = \sigma(e_3)$ and $\sigma(e_2) = \sigma(e_4)$.

2. Edges $e_1^i = (v_{2i}^1, v_1^2)(v_{2i}^1, v_{2s}^2)$, $e_2^i = (v_{2i}^1, v_{2s}^2)(v_{2i+1}^1, v_{2s}^2)$, $e_3^i = (v_{2i+1}^1, v_{2s}^2)(v_{2i+1}^1, v_1^2)$, $e_4^i = (v_{2i+1}^1, v_1^2)(v_{2i}^1, v_1^2)$ for $1 \leq i < r$. Again, it is easy to observe that $e_1^i, e_2^i, e_3^i, e_4^i$ span a cycle (see Figure 2(b)) and it is balanced since $\sigma(e_1^i) = \sigma(e_3^i)$ and $\sigma(e_2^i) = \sigma(e_4^i)$ for $1 \leq i < r$. All the cycles are vertex-disjoint.

3. Edges $e_1^i = (v_1^1, v_{2i}^2)(v_1^1, v_{2i+1}^2)$, $e_2^i = (v_1^1, v_{2i+1}^2)(v_{2r}^1, v_{2i+1}^2)$, $e_3^i = (v_{2r}^1, v_{2i+1}^2)(v_{2r}^1, v_{2i}^2)$, $e_4^i = (v_{2r}^1, v_{2i}^2)(v_1^1, v_{2i}^2)$ for $1 \leq i < s$. We note that $e_1^i, e_2^i, e_3^i, e_4^i$ span a cycle (see Figure 2(c)) and it is balanced since $\sigma(e_1^i) = \sigma(e_3^i)$ and $\sigma(e_2^i) = \sigma(e_4^i)$ for $1 \leq i < s$. All the cycles are vertex-disjoint.

4. Edges $e_1^{i,j} = (v_{2i}^1, v_{2j}^2)(v_{2i}^1, v_{2j+1}^2)$, $e_2^{i,j} = (v_{2i}^1, v_{2j+1}^2)(v_{2i+1}^1, v_{2j+1}^2)$, $e_3^{i,j} = (v_{2i+1}^1, v_{2j+1}^2)(v_{2i+1}^1, v_{2j}^2)$, $e_4^{i,j} = (v_{2i+1}^1, v_{2j}^2)(v_{2i}^1, v_{2j}^2)$ for $1 \leq i < r, 1 \leq j < s$. Again we observe that $e_1^{i,j}, e_2^{i,j}, e_3^{i,j}, e_4^{i,j}$ span a cycle (see Figure 2(d)) and it is balanced since $\sigma(e_1^{i,j}) = \sigma(e_3^{i,j})$ and $\sigma(e_2^{i,j}) = \sigma(e_4^{i,j})$ for $1 \leq i < r, 1 \leq j < s$. All the cycles are vertex-disjoint.

It is easy to observe that all the cycles listed above are vertex-disjoint, so graph H_1 has a decomposition into balanced cycles and can be colored using colors

± 1 . We note that the decomposition consists of $1 + (r-1) + (s-1) + (r-1)(s-1) = rs$ cycles, so $m(H_1) = 4rs = \frac{m(S)}{2}$ since $m(S) = 4(2r)(2s)/2 = 8rs$. Clearly, H_1 is 2-regular.

$V(H_2) = V(S)$, $E(H_2) = E(S) \setminus E(H_1)$. It follows that graph H_2 contains edges $e_1^{i,j} = (v_{2i-1}^1, v_{2j-1}^2)(v_{2i-1}^1, v_{2j}^2)$, $e_2^{i,j} = (v_{2i-1}^1, v_{2j}^2)(v_{2i}^1, v_{2j}^2)$, $e_3^{i,j} = (v_{2i}^1, v_{2j}^2)(v_{2i}^1, v_{2j-1}^2)$, $e_4^{i,j} = (v_{2i}^1, v_{2j-1}^2)(v_{2i-1}^1, v_{2j-1}^2)$ for $1 \leq i \leq r$, $1 \leq j \leq s$. It is easy to observe that $e_1^{i,j}$, $e_2^{i,j}$, $e_3^{i,j}$, $e_4^{i,j}$ span a cycle and it is balanced since $\sigma(e_1^{i,j}) = \sigma(e_3^{i,j})$ and $\sigma(e_2^{i,j}) = \sigma(e_4^{i,j})$ for $1 \leq i \leq r$, $1 \leq j \leq s$. We also observe that H_1 and H_2 are edge-disjoint. Graph H_2 has a decomposition into vertex-disjoint balanced cycles, so it can be colored using colors ± 2 . H_2 contains rs cycles, so $m(H_2) = 4rs = \frac{m(S)}{2}$.

Clearly, $S = H_1 \cup H_2$ so S can be colored using 4 colors. It follows that $\chi'(S) = 4 = \Delta(S)$. ■

Now let us proceed to products of cycles, where at least one of the cycles is odd. To prove the results, we need to recall a lemma.

Lemma 10 [6]. *Let S be a $2r$ -regular signed graph. $\chi'(S) = \Delta(S)$ if and only if S admits a decomposition into exactly r spanning edge-disjoint 2-regular balanced subgraphs.*

Lemma 11. *Let S be a $2r$ -regular signed graph. If S contains an odd number of negative edges, then $\chi'(S) = \Delta(S) + 1$.*

Proof. We assume that $\chi'(S) = \Delta(S)$. From Lemma 10 it follows that S admits a decomposition into r spanning edge-disjoint 2-regular graphs such that their connected components are balanced cycles. We observe that at least one of the graphs must contain an odd number of negative edges and at least one of its subgraphs (which is a cycle) must contain an odd number of negative edges. Obviously, such a cycle is not balanced, so we reach a contradiction. ■

With these prerequisites in hand, we can proceed with the lemma.

Lemma 12. *Let $S_1 = (C_{2r}, \sigma_1)$, $S_2 = (C_{2s+1}, \sigma_2)$ and $S = (G, \sigma) = S_1 \square S_2$. Then $\mathcal{C}(S) = 1/2$.*

Proof. We observe that S is a 4-regular graph. Let us consider 4 cases.

1. Both S_1 , S_2 are balanced. Then $\chi'(S_1) = 2$ and $\chi'(S_2) = 2$ so it follows from Theorem 6 that $\chi'(S) = \Delta(S)$.

2. Both S_1 , S_2 are unbalanced. S_1 and S_2 have $2k+1$ and $2l+1$ negative edges, respectively. S has $(2s+1)(2k+1) + 2r(2l+1) = 2(2ks + s + k + 2lr + r) + 1$ negative edges. Since S is 4-regular, it follows from Lemma 11 that $\chi'(S) = \Delta(S) + 1$.

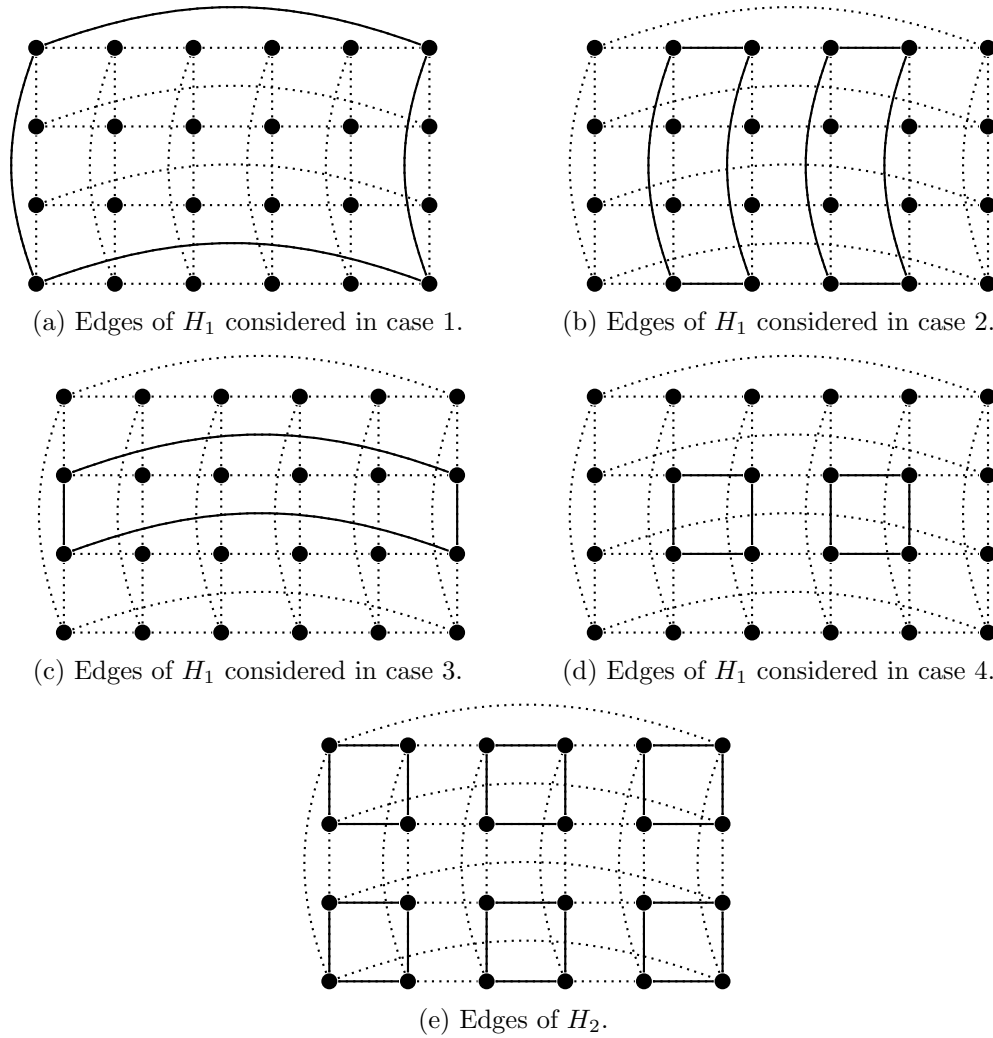


Figure 2. Example graph $(C_6 \square C_4, \sigma)$ considered in Lemma 9. Edges of respective graphs are marked with solid lines.

3. S_1 is unbalanced and S_2 is balanced. S_1 and S_2 have $2k+1$ and $2l$ negative edges, respectively. S has $(2s+1)(2k+1)+2r2l = 2(2ks+s+k+2lr)+1$ negative edges. Again since S is 4-regular, it follows from Lemma 11 that $\chi'(S) = \Delta(S)+1$.

4. S_1 is balanced and S_2 is unbalanced. Let $V(S_1) = \{v_1^1, \dots, v_{2r}^1\}$ and $V(S_2) = \{v_1^2, \dots, v_{2s+1}^2\}$.

We will show the decomposition of S into graphs H_1, H_2 such that both of them can be colored using 2 colors regardless of the edge signs. We first describe the structure of H_1 . Obviously, $V(H_1) = V(S)$. Graph H_1 contains edges:

- (a) $e_1^i = (v_{2i-1}^1, v_{2s+1}^2)(v_{2i-1}^1, v_1^2)$,
- (b) $e_2^i = (v_{2i}^1, v_{2s+1}^2)(v_{2i-1}^1, v_{2s+1}^2)$,
- (c) $e_3^i = (v_{2i}^1, v_1^2)(v_{2i}^1, v_{2s+1}^2)$,
- (d) $e_4^i = (v_{2i-1}^1, v_{2s}^2)(v_{2i}^1, v_{2s}^2)$,
- (e) $e_{2s-1}^i = (v_{2i-1}^1, v_1^2)(v_{2i-1}^1, v_{2s}^2), \dots, e_{2s-1}^i = (v_{2i-1}^1, v_{2s-1}^2)(v_{2i-1}^1, v_{2s}^2)$,
- (f) $e_{2s-1}^i = (v_{2i}^1, v_1^2)(v_{2i}^1, v_{2s}^2), \dots, e_{2s-1}^i = (v_{2i}^1, v_{2s}^2)(v_{2i}^1, v_{2s-1}^2)$,

for $1 \leq i \leq r$. We observe that those edges span cycles $\gamma_i = (e_1^i, e_2^i, e_3^i, e_{2s-1}^i, \dots, e_{2s-1}^i, e_4^i, e_{2s}^i, \dots, e_1^i)$ for $1 \leq i \leq r$. All of the cycles γ_i are vertex-disjoint and are balanced since $\sigma(e_1^i) = \sigma(e_3^i)$, $\sigma(e_2^i) = \sigma(e_4^i)$ and $\sigma(e_j^i) = \sigma(e_{j'}^i)$ for $1 \leq j \leq 2s-1$. It follows that graph H_1 has a decomposition into balanced cycles, so it can be colored using colors ± 1 .

Graph H_2 is such a graph that $V(H_2) = V(S)$ and $E(H_2) = E(S) \setminus E(H_1)$. It follows that graph contains the following configurations of edges.

- (a) Edges $e_1^i = (v_1^1, v_i^2)(v_2^1, v_i^2), \dots, e_{2r-1}^i = (v_{2r-1}^1, v_i^2)(v_{2r}^1, v_i^2)$, $e_{2r}^i = (v_{2r}^1, v_i^2)(v_1^1, v_i^2)$ for $1 \leq i \leq 2s-1$. The edges span $2s-1$ vertex-disjoint cycles and they are balanced since they are the copies of graph S_1 in S .
- (b) Edges $e_1' = (v_1^1, v_{2s}^2)(v_1^1, v_{2s+1}^2)$, $e_2' = (v_1^1, v_{2s+1}^2)(v_{2r}^1, v_{2s+1}^2)$, $e_3' = (v_{2r}^1, v_{2s+1}^2)(v_{2r}^1, v_{2s}^2)$, $e_4' = (v_{2r}^1, v_{2s}^2)(v_1^1, v_{2s}^2)$. Obviously, the edges span a cycle and it is balanced since $\sigma(e_1') = \sigma(e_3')$, $\sigma(e_2') = \sigma(e_4')$.
- (c) Edges $e_1^i = (v_{2i}^1, v_{2s}^2)(v_{2i+1}^1, v_{2s}^2)$, $e_2^i = (v_{2i+1}^1, v_{2s}^2)(v_{2i+1}^1, v_{2s+1}^2)$, $e_3^i = (v_{2i+1}^1, v_{2s+1}^2)(v_{2i}^1, v_{2s+1}^2)$, $e_4^i = (v_{2i}^1, v_{2s+1}^2)(v_{2i}^1, v_{2s}^2)$ for $1 \leq i \leq s-1$. The edges span $s-1$ vertex-disjoint cycles and they are balanced since $\sigma(e_1^i) = \sigma(e_3^i)$, $\sigma(e_2^i) = \sigma(e_4^i)$ for $1 \leq i \leq s-1$.

Clearly, the cycles are vertex-disjoint, and graph H_2 has a decomposition into balanced cycles so H_2 can be colored using colors ± 2 . It is easy to observe that graphs H_1, H_2 are edge-disjoint and are a decomposition of graph S . It follows that S can be colored using colors $\pm 1, \pm 2$, so $\chi'(S) = \Delta(S)$.

The case 1 corresponds to exactly $1/4$ of possible signatures of S and so the case 4, so it follows that $\mathcal{C}(S) = 2/4 = 1/2$. ■

Lemma 13. *Let $S_1 = (C_{2r+1}, \sigma_1)$, $S_2 = (C_{2s+1}, \sigma_2)$ and $S = (G, \sigma) = S_1 \square S_2$. If both S_1, S_2 are unbalanced, then there exists a signed graph S'' such that it is a switching equivalent of S and all of its edges are negative.*

Proof. Let $V(S_1) = \{v_1^1, \dots, v_{2r+1}^1\}$ and $V(S_2) = \{v_1^2, \dots, v_{2s+1}^2\}$.

Since S_1 is unbalanced, it can be switched to graph S'_1 such that all its edges are negative. Let X^1 denote one of the possible sets of vertices that are switched to get S_1 to S'_1 . Let $X_i^1 = \{(u, v_i^2) : u \in X^1\}$, for $1 \leq i \leq 2s+1$. We note that for all i it holds that $X_i^1 \subseteq V(S)$. Let $S' = (G, \sigma')$ be a graph formed from S

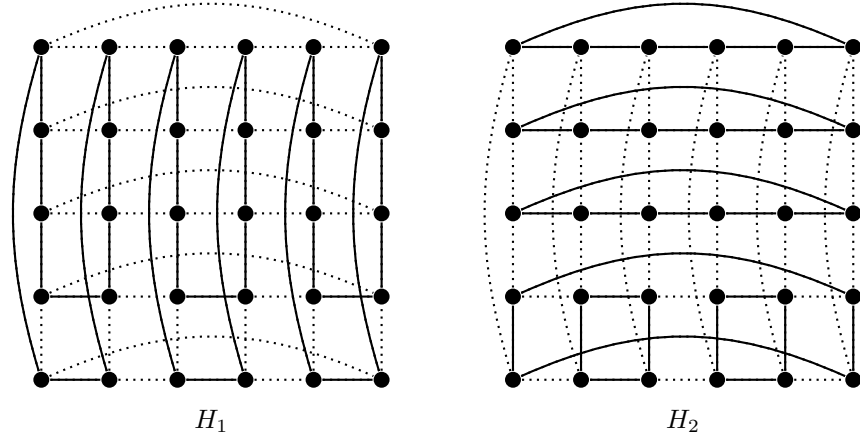


Figure 3. Example graph $(C_6 \square C_5, \sigma)$ considered in Lemma 12. Edges of H_1 and H_2 are marked with solid lines.

after switching all the vertices from $\bigcup_{i=1}^{2s+1} X_i^1$. It is easy to observe that all the edges $(v_i^1, v_j^2)(v_k^1, v_j^2)$, $i \neq k$ of S' are negative.

We can also show that for all the edges $e = (v_i^1, v_j^2)(v_i^1, v_k^2)$ ($j \neq k$) it holds that $\sigma'(e) = \sigma(e)$. We prove that by contradiction. Let us assume that there is an edge $e = (v_i^1, v_j^2)(v_i^1, v_k^2)$ such that $\sigma'(e) \neq \sigma(e)$. It follows that one of the below cases is true.

1. $(v_i^1, v_j^2) \in X_j^1$ and $(v_i^1, v_k^2) \notin X_k^1$. It follows from the first statement that $v_i^1 \in X^1$ and from the second one that $v_i^1 \notin X^1$, a contradiction.
2. $(v_i^1, v_j^2) \notin X_j^1$ and $(v_i^1, v_k^2) \in X_k^1$. Clearly, a contradiction.

It follows that edge e changed its sign either zero or two times when switching graph S to S' so the sign did not change. It follows that no edge $(v_i^1, v_j^2)(v_i^1, v_k^2)$ ($j \neq k$) changed the sign.

Analogously, we can define sets $X^2 \subseteq V(S_2)$, X_1^2, \dots, X_{2r+1}^2 and show that switching vertices $\bigcup_{i=1}^{2r+1} X_i^2$ in S' changes all the signs of edges in all copies of the graph S_2 to negative while not changing any signs of all the other edges so it results in a desired signed graph S'' with all the edges negative. And since being a switching equivalent is a transitive relation, S and S'' are switching equivalent. ■

Lemma 14. Let $S_1 = (C_{2r+1}, \sigma_1)$, $S_2 = (C_{2s+1}, \sigma_2)$ and $S = (C_{2r+1} \square C_{2s+1}, \sigma) = S_1 \square S_2$. Then, $\mathcal{C}(S) = 1/4$.

Proof. We observe that S is 4-regular. We consider 3 possible cases.

1. Both S_1 , S_2 are balanced. Then $\chi'(S_1) = 2 = \Delta(S_1)$ and $\chi'(S_2) = 2 = \Delta(S_2)$. It follows from Theorem 6 that $\chi'(S) = \Delta(S)$.

2. Exactly one of S_1, S_2 is balanced. Without loss of generality, we assume S_1 is balanced and S_2 is unbalanced, so S_1 has $2k$ negative edges and S_2 has $2l+1$ of them. We observe that S contains exactly $2k(2s+1) + (2l+1)(2r+1) = 2(2ks + k + 2lr + l + r) + 1$ negative edges. It follows from Lemma 11 that $\chi'(S) = \Delta(S) + 1$.

3. Both S_1, S_2 are unbalanced. It follows from Lemma 13 that S can be switched to graph S' with all edges negative. We assume that S' can be colored using $\Delta(S')$ colors. It follows from Lemma 10 that S' has a decomposition into exactly 2 spanning edge-disjoint 2-regular balanced subgraphs H_1, H_2 . We observe that $n(S') = (2r+1)(2s+1) = 2(2rs + r + s) + 1$ and $m(H_1) = m(H_2) = n(S')$. Graphs H_1, H_2 have odd numbers of edges. All the edges of S' are negative so both H_1 and H_2 also have odd numbers of negative edges. Thus it follows from Lemma 11 that $\chi'(H_1) = \Delta(H_1) + 1$ and $\chi'(H_2) = \Delta(H_2) + 1$, so none of H_1, H_2 are balanced, a contradiction. So S' does not have such a decomposition into H_1, H_2 and $\chi'(S') = \Delta(S') + 1$. Since S and S' are switching equivalent, $\chi'(S) = \Delta(S) + 1$.

The first case corresponds to exactly $1/4$ of possible signatures of S and it is the only case with $\chi'(S)$ being equal to $\Delta(S)$, so it follows that $\mathcal{C}(S) = 1/4$. ■

Now we covered all the possible Cartesian products of a signed cycle and a signed path and can provide a general theorem.

Theorem 15. *Let $S_1 = (C_r, \sigma_1)$, $S_2 = (C_s, \sigma_2)$ be signed cycles and $S = S_1 \square S_2$. Then*

$$\mathcal{C}(S) = \begin{cases} 1, & \text{if both } r, s \text{ are even,} \\ 1/2, & \text{if one of } r, s \text{ is even and one is odd,} \\ 1/4, & \text{if both } r, s \text{ are odd.} \end{cases}$$

Proof. The first case ($\mathcal{C}(S) = 1$) follows directly from Lemma 9, the second one ($\mathcal{C}(S) = 1/2$)—from Lemma 12, and the third one ($\mathcal{C}(S) = 1/4$) — from Lemma 14. ■

3. TENSOR PRODUCTS

Let $S_1 = (G_1, \sigma_1)$, $S_2 = (G_2, \sigma_2)$ be signed graphs. The tensor product of graphs S_1, S_2 is a signed graph $S = (G, \sigma)$ such that $V(S) = V(S_1) \times V(S_2)$ and there is an edge $e = (u, u')(v, v')$ in S , where $u, v \in V(S_1)$, $u', v' \in V(S_2)$, if and only if $uv \in E(S_1)$ and $u'v' \in E(S_2)$. Then $\sigma(e) = \sigma_1(uv)\sigma_2(u'v')$.

We denote S by $S_1 \times S_2$. Clearly, $|S| = G_1 \times G_2$.

First, let us recall a simple relation between the maximum degree of graphs and their tensor product.

Lemma 16. *For arbitrary graphs G_1, G_2 it holds that $\Delta(G_1 \times G_2) = \Delta(G_1)\Delta(G_2)$.*

Proof. From the definition of a tensor product it follows that $\deg_{G_1 \times G_2}((u, u')) = \deg_{G_1}(u)\deg_{G_2}(u')$ for any two vertices $u \in V(G_1)$, $u' \in V(G_2)$. Let u, u' be arbitrary vertices of maximum degree in G_1, G_2 , respectively. It is clear that (u, u') is a vertex of maximum degree in $G_1 \times G_2$ and its degree is equal to $\deg_{G_1}(u)\deg_{G_2}(u') = \Delta(G_1)\Delta(G_2)$. ■

Now we proceed with main results of this section, starting with tensor products of arbitrary graphs and paths on 2 vertices.

Theorem 17. *Let $S_1 = (G_1, \sigma_1)$ be an arbitrary signed graph, $S_2 = (P_2, \sigma_2)$ be a signed path, and $S = S_1 \times S_2$. If $\chi'(S_1) = \Delta(S_1)$, then $\chi'(S) = \Delta(S)$.*

Proof. Since $\Delta(S_2) = 1$, it follows from Lemma 16 that $\Delta(S) = \Delta(S_1)$. We also observe that $m(S) = 2m(S_1)$. Let c_1 be an arbitrary optimal coloring of S_1 , using exactly $\Delta(S_1)$ colors (denoted by R_1). For every pair of colors $\pm\alpha$ with $\alpha \geq 0$ (note that α might be possibly equal to 0) we consider graph S_1^α such that $S_1^\alpha \subseteq S_1$ and it contains only the edges colored with $\pm\alpha$ in c_1 . By S^α we denote a subgraph of S such that $S^\alpha = S_1^\alpha \times S_2$. It is easy to observe that $\bigcup_{\pm\alpha \in R_1} S^\alpha = S$ and graphs $S^{|\alpha|}, S^{|\beta|}$ are edge-disjoint for any $\alpha, \beta \in R_1$ such that $|\alpha| \neq |\beta|$.

We observe that S_1^α is a graph of a maximum degree at most 2 with all the connected components balanced. We show the coloring c of S by showing how to color incidences of S^α using only colors used by incidences of S_1^α . Let H be any connected component of graph S_1^α . We consider two cases.

1. H is a path. Let $P = H \times S_2$, $P \subseteq S^\alpha$. It is easy to observe that P consists of two vertex-disjoint paths isomorphic to H , so they can be colored using colors $\pm\alpha$ in the coloring c .

2. H is a cycle C_r . It follows that H is balanced, so has an even number of negative edges. We consider two cases separately.

(a) $r = 2k$. We observe that since H has an even number of edges and an even number of negative edges, it also has an even number of positive edges. Let $C = H \times S_2$. C consists of two vertex-disjoint cycles (C', σ') , (C'', σ'') such that $n(C') = n(C'') = r$.

We consider two cases.

- The only edge of S_2 is positive. Every edge in C' has the same sign as the corresponding edge in H , so there is the same number of negative edges in C' as in H , thus C' is balanced.
- The only edge of S_2 is negative. Every edge in C' has the opposite sign as the corresponding edge in H , so there is the same number of negative edges in

C' as the number of positive edges in H . H has an even number of positive edges, so C' is balanced.

The considerations above are exactly the same for C'' , so we infer that both cycles are balanced and can be colored in c using colors $\pm\alpha$ for some $\alpha > 0$.

(b) $r = 2k + 1$. Let $C = H \times S_2$. It is easy to observe that C is a cycle with $2r$ nodes and there are r pairs of edges with the same signs in C . It follows that C is a balanced cycle, so can be colored in c using colors $\pm\alpha$ for some $\alpha > 0$.

We note that if $0 \in R_1$, then S^0 is a matching, that is, a collection of paths P_2 . In this case all $H \times S_2$ consist of two vertex-disjoint paths P_2 , which of course can be colored with color 0.

It follows that graph S^α can be colored using colors $\pm\alpha$ and since $\bigcup_{\pm\alpha \in R_1} S^\alpha = S$, it follows that S can be colored using colors R_1 , so $\chi'(S) = \Delta(S)$ and $\chi'(S) = \Delta(S)$. ■

Now let us proceed to the tensor products of arbitrary graphs with trees.

Lemma 18. *For an arbitrary graph G and a tree T it holds that $\Delta(G \times T) = \Delta(T)\Delta(G \times P_2)$.*

Proof. It follows from Lemma 16 that $\Delta(G \times P_2) = \Delta(G)$ and $\Delta(G \times T) = \Delta(T)\Delta(G)$, so $\Delta(G \times T) = \Delta(T)\Delta(G \times P_2)$. ■

Lemma 19. *Let G_1, G_2 be arbitrary graphs, P_2 be a path and $G = G_1 \times G_2$, $G_P = G_1 \times P_2$. Then G has a decomposition into exactly $m(G_2)$ copies of G_P . Moreover, every copy may be assigned a corresponding edge in G_2 in such a way that two copies have any vertices in common if and only if their respective edges are adjacent in G_2 .*

Proof. The lemma follows directly from the definition of a tensor product. We first consider the case $G_2 = P_2$. In this case it is clear that G has a decomposition into $m(G_2) = 1$ copies of G_P , since $G = G_P$. That single copy has a corresponding edge in G_2 —the only edge of G_2 .

We observe that when a new edge is added to G_2 , there appears a new copy of G_P in G , denoted by H' , so intuitively we assign the new edge in G_2 as a corresponding edge for H' . It is easy to observe that H' has some vertices in common with other copy H'' if and only if the corresponding edges of H', H'' are adjacent in G_2 . It follows from the definition of a tensor product that if copies have vertices in common, there are exactly $\frac{n(G_P)}{2}$ of them. ■

Theorem 20. *Let T be a tree, $S_1 = (G_1, \sigma_1)$, $S_2 = (T, \sigma_2)$ and $S = (G_1 \times T, \sigma) = S_1 \times S_2$. If $\chi'(S_1) = \Delta(S_1)$, then $\chi'(S) = \Delta(S)$.*

Proof. Let $H = G_1 \times P_2$. It follows from Lemma 19 that $|S|$ has a decomposition into exactly $m(T)$ copies of graph H . Moreover, copies of H are connected in the same way as edges of T . It means that each edge of T has a respective copy of H in S and two copies of H share a set of vertices if and only if their respective edges are adjacent in T .

We observe that graph S can be switched to a switching equivalent in such a way that all the copies of H in the latter signed graph have exactly the same signs. That way we can consider them as copies of some $S' = (H, \sigma')$ — and without loss of generality we treat these copies of S' as a decomposition of S .

Let c' be an arbitrary optimal edge-coloring of S' . Since $S' = (G_1 \times P_2, \sigma')$, from Theorem 17 we know that c' uses exactly $\Delta(S')$ colors. Let $k = \lfloor \frac{\Delta(S')}{2} \rfloor$. Note that c' uses colors $\{\pm 1, \dots, \pm k\}$ and additionally color 0 in case when $\Delta(S')$ is odd.

We define sets $R_1, \dots, R_{\Delta(T)}$ such that $R_i = \{\pm((i-1)k+1), \dots, \pm ik\}$ if c' does not use color 0 (i.e., when $\Delta(S')$ is even), otherwise $R_i = \{0\} \cup \{\pm((i-1)k+1), \dots, \pm ik\}$. We observe that these sets contain colors that are shifted in such a way that $R_1 \cup \dots \cup R_{\Delta(T)}$ contains all the colors $\pm 1, \dots, \pm k\Delta(T)$ and moreover $R_i \cap R_j \subseteq \{0\}$ for any $i \neq j$.

We observe that any of the copies of graph S' in S could be colored using any of the $R_1, \dots, R_{\Delta(T)}$ sets of colors and the coloring of that single copy would be correct.

Additionally, let c'' be an arbitrary optimal edge-coloring of a tree T (without considering its signs). It is a well-known fact that such a coloring uses $\Delta(T)$ colors, namely $\{1, \dots, \Delta(T)\}$.

Now, we will construct a coloring c of S . We first color separately every copy S'_i of S' that we obtained from decomposition of S . We remind that by Lemma 19 each such copy has an edge of T assigned to it, which we will denote by e_i . This way, we color S'_i using only colors from the $R_{c''(e_i)}$ set and respecting the only requirement that we color all the copies such that corresponding edges of different copies get either the color 0 or the colors with a difference equal to some multiple of k —which by construction of the colors sets is always possible.

We observe that since c'' is a correct coloring of T , c is *almost* a correct coloring of S —because there may arise conflicts in coloring c , but occurring only on edges colored with 0, since it is the only color that might be shared between different sets R_i, R_j . It follows that in the case when $\Delta(S')$ is even, color 0 is not used in c at all and c is a proper coloring. Then c uses exactly $\Delta(T)\Delta(S') = \Delta(S)$ colors.

Consider now the case when $\Delta(S')$ is odd. Clearly, color 0 is used in the colorings of all the copies of S' in S . By $X \subseteq S$ we denote a subgraph containing all the edges colored by c with color 0. We will show the procedure of recoloring incidences of X to get a correct coloring of S . We observe that X is an acyclic

graph. If there was a cycle C_r in X , then either of two cases would need to hold.

1. There is a copy of S' in S with two adjacent edges both belonging to X . It follows that they both were colored with 0 by c . Clearly, such a coloring of that copy is incorrect since the set of edges colored with color 0 must span a matching, so we reach a contradiction.

2. There is no copy of S' in S with two adjacent edges both belonging to X . It means that any two adjacent edges of C_r in X belong to different copies that share some vertices. We can list these copies: S'_1, \dots, S'_r , clearly $V(S'_i) \cap V(S'_{i+1}) \neq \emptyset$ for $1 \leq i < r$. Since the cycle must be closed, $V(S'_1) \cap V(S'_r) \neq \emptyset$. We remind that each copy S'_i has a corresponding edge e_i in T , so it follows that there is a cycle in T with edges e_1, \dots, e_r . Clearly, a contradiction.

From the first case above it is also easy to observe that $\Delta(X) = \Delta(T)$. We note that c uses exactly $\Delta(T)(\Delta(S') - 1)$ colors other than 0. We note that $\Delta(T)(\Delta(S') - 1)$ is even since $\Delta(S')$ is odd. Since signed forests are always Δ -edge-colorable [6], we can easily color X using a symmetric set of $\Delta(T)$ colors that are different than all the colors used by c on the edges of S that do not belong to X . That way the new coloring uses exactly $\Delta(T)(\Delta(S') - 1) + \Delta(T) = \Delta(T)\Delta(S') = \Delta(S)$ colors. ■

Since every path is a tree and every signed path can be edge-colored using Δ colors, it follows the corollary.

Corollary 21. *Let $S_1 = (P_r, \sigma_1)$, $S_2 = (P_s, \sigma_2)$ and $S = S_1 \times S_2$. If $r, s > 1$, then $\chi'(S) = \Delta(S)$.*

This, together with Theorem 2 and the fact that $P_r \times P_1$ are exactly graphs with no edges proves that the tensor products of paths always belong to class 1^\pm .

4. STRONG PRODUCTS

Let $S_1 = (G_1, \sigma_1)$, $S_2 = (G_2, \sigma_2)$ be signed graphs. The strong product of graphs S_1, S_2 is a signed graph $S = (G, \sigma)$, such that $V(S) = V(S_1) \times V(S_2)$ and there is an edge $e = (u, u')(v, v')$ in S , where $u, v \in V(S_1)$, $u', v' \in V(S_2)$, if and only if one of the following conditions holds.

1. $u = v, u'v' \in E(S_2)$. In such case $\sigma(e) = \sigma_2(u'v')$,
2. $uv \in E(S_1), u' = v'$. In such case $\sigma(e) = \sigma_1(uv)$,
3. $uv \in E(S_1)$ and $u'v' \in E(S_2)$. In such case $\sigma(e) = \sigma_1(uv)\sigma_2(u'v')$.

We denote S by $S_1 \boxtimes S_2$. Clearly, $|S| = G_1 \boxtimes G_2$. It is clear that $S_1 \boxtimes S_2$ has a decomposition into $S_1 \square S_2$ and $S_1 \times S_2$. We observe that $\deg_S(u, u') = \deg_{S_1 \square S_2}(u, u') + \deg_{S_1 \times S_2}(u, u') = \deg_{S_1}(u) + \deg_{S_2}(u') + \deg_{S_1}(u) \deg_{S_2}(u')$, thus it follows that $\Delta(S) = \Delta(S_1 \square S_2) + \Delta(S_1 \times S_2) = \Delta(S_1) + \Delta(S_2) + \Delta(S_1)\Delta(S_2)$.

Lemma 22. *Let S_1, S_2 be signed graphs and $H_1 = S_1 \square S_2$, $H_2 = S_1 \times S_2$, $S = S_1 \boxtimes S_2$. If at least one of $\{\Delta(H_1), \Delta(H_2)\}$ is even, $\chi'(H_1) = \Delta(H_1)$ and $\chi'(H_2) = \Delta(H_2)$, then $\chi'(S) = \Delta(S)$.*

Proof. Without loss of generality, we assume that $\Delta(H_2)$ is even.

Let c_1, c_2 be arbitrary optimal colorings of graphs H_1, H_2 , respectively. c_1 uses colors $\pm 1, \dots, \pm \lfloor \frac{\Delta(H_1)}{2} \rfloor$ and possibly color 0 if $\Delta(H_1)$ is odd, and c_2 uses colors $\pm 1, \dots, \pm \frac{\Delta(H_2)}{2}$. It is clear that $H_1, H_2 \subseteq S$ and $H_1 \cup H_2 = S$. Let c be an edge coloring of S such that

$$c(u:uv) = \begin{cases} c_1(u:uv), & \text{if } uv \in E(H_1), \\ c_2(u:uv) + \lfloor \frac{\Delta(H_1)}{2} \rfloor, & \text{if } uv \in E(H_2) \text{ and } c_2(u:uv) > 0, \\ c_2(u:uv) - \lfloor \frac{\Delta(H_1)}{2} \rfloor, & \text{if } uv \in E(H_2) \text{ and } c_2(u:uv) < 0. \end{cases}$$

We observe that incidences of H_1 are colored by c using the same colors as by c_1 , so their coloring is proper. The set of colors used to color incidences of graph H_2 has been shifted from $\pm 1, \dots, \pm \frac{\Delta(H_2)}{2}$ to $\pm \left(\lfloor \frac{\Delta(H_1)}{2} \rfloor + 1 \right), \dots, \pm \left(\lfloor \frac{\Delta(H_1)}{2} \rfloor + \frac{\Delta(H_2)}{2} \right)$. Clearly, incidences of H_2 are properly colored in c . It is easy to observe that the sets of colors used to color incidences of H_1 and H_2 in c are disjoint. We consider two cases.

1. $\Delta(H_1)$ is even. Graph S is colored using a set of colors $R = \left\{ \pm 1, \dots, \pm \left(\frac{\Delta(H_1)}{2} + \frac{\Delta(H_2)}{2} \right) \right\}$. Therefore $|R| = 2 \left(\frac{\Delta(H_1)}{2} + \frac{\Delta(H_2)}{2} \right) = \Delta(H_1) + \Delta(H_2) = \Delta(S)$, so $\chi'(S) = \Delta(S)$.
2. $\Delta(H_1)$ is odd. Graph S is colored using a set of colors $R = \{0, \pm 1, \dots, \pm \left(\lfloor \frac{\Delta(H_1)}{2} \rfloor + \frac{\Delta(H_2)}{2} \right)\}$. Therefore $|R| = 1 + 2 \left(\frac{\Delta(H_1)-1}{2} + \frac{\Delta(H_2)}{2} \right) = \Delta(H_1) + \Delta(H_2) = \Delta(S)$, so $\chi'(S) = \Delta(S)$. ■

Theorem 23. *Let $S_1 = (P_r, \sigma_1)$, $S_2 = (P_s, \sigma_2)$ and $S = (P_r \boxtimes P_s, \sigma) = S_1 \boxtimes S_2$. If $r > 1$ or $s > 1$, then $\chi'(S) = \Delta(S)$.*

Proof. Without loss of generality, we assume that $r \leq s$. Let $H_1 = S_1 \square S_2$, $H_2 = S_1 \times S_2$. We observe that H_1, H_2 is a decomposition of S and consider 3 cases.

1. $r = 1$ and $s \geq 2$. Graph S is a path, so can be colored using $\Delta(S)$ colors (see Theorem 2).
2. $r = 2$ and $s = 2$. Graph S is a complete graph K_4 . It follows from Corollary 7 that incidences of graph H_1 can be colored using colors ± 1 . Graph H_2 is a matching so its incidences can be colored using color 0. It follows that graph S can be colored using colors $\{0, \pm 1\}$, so $\chi'(S) = 3 = \Delta(S)$.

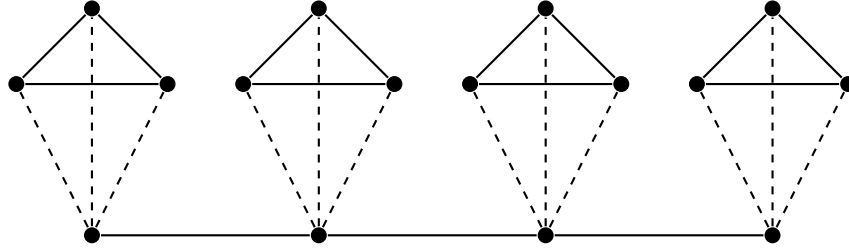


Figure 4. An example of corona product $P_4 \odot K_3$. Edges of P_4 and copies of K_3 are marked with solid lines. Edges connecting them are marked with dashed lines.

3. $r = 2$ and $s > 2$. It follows from Corollary 7 that incidences of graph H_1 can be colored using colors $\{0, \pm 1\}$. It follows from Corollary 21 that incidences of H_2 can be colored using colors ± 1 , so they can also be colored using ± 2 . It follows that graph S can be colored using colors $\{0, \pm 1, \pm 2\}$, so $\chi'(S) = 5 = \Delta(S)$.

4. $r > 2$ and $s > 2$. We observe that $\Delta(H_1) = \Delta(H_2) = 4$. It follows from Corollary 7 that $\chi'(H_1) = \Delta(H_1)$ and from Corollary 21 that $\chi'(H_2) = \Delta(H_2)$. It follows from Lemma 22 that $\chi'(S) = \Delta(S)$. ■

5. CORONA PRODUCTS

Intuitively, a corona product of graphs G_1, G_2 consists of graph G_1 , $n(G_1)$ copies of graph G_2 and edges connecting the i -th vertex of G_1 with all the vertices of the i -th copy of graph G_2 (see an example in Figure 4). More formally, the corona product of graphs G_1, G_2 is a graph $G = G_1 \odot G_2$, such that $V(G) = V(G_1) \cup \bigcup_{i=1}^{n(G_1)} V(G_2^i)$, where G_2^i denotes an i th copy of graph G_2 , and $E(G) = E(G_1) \cup \bigcup_{i=1}^{n(G_1)} E(G_2^i) \cup \{(u, v) : u \in V(G_1), v \in V(G_2^i), \text{ and } u \text{ is the } i\text{-th vertex of } G_1\}$.

In this section, we consider the edge coloring problem of signed corona products of graphs. It turns out that the signature σ of a signed corona product $S = (G_1 \odot G_2, \sigma)$ does not matter for the chromatic index of S as far as $\Delta(G_1) \geq 2$.

Theorem 24. *Let $S = (G_1 \odot G_2, \sigma)$ be a signed corona product of graphs G_1 and G_2 . If $\Delta(G_1) \geq 2$, then $\chi'(S) = \Delta(S)$.*

Proof. First, we note that $\Delta(S) = \Delta(G_1) + n(G_2)$.

Let $v_1, v_2, \dots, v_{n(G_1)}$ be any sequence of vertices in G_1 and let $G_{2,1}, G_{2,2}, \dots, G_{2,n(G_1)}$ be a sequence of copies of graph G_2 such that v_i is the only vertex connecting graph G_1 with graph $G_{2,i}$.

By S_1 we denote a subgraph of S that has all the vertices and edges of G_1 and corresponding edge signs from σ . Analogously, $S_{2,i}$ corresponds to $G_{2,i}$ with edge signs from σ .

Our main idea is to start from some edge coloring c_0 of S_1 with colors from $M_{\Delta(S)}$, and then to proceed inductively from c_{i-1} to c_i ($i = 1, \dots, n(S_1)$) by coloring a graph induced by v_i and $S_{2,i}$ while preserving the invariant that we use only colors from $M_{\Delta(S)}$. Since we will show that such coloring can be always found, from now on without loss of generality we may assume that G_1 is a regular graph.

Clearly, for c_0 the invariant holds, since first we can find an edge coloring c^\otimes of S_1 using $\Delta(S_1) + 1 \leq \Delta(S_1) + n(G_2) = \Delta(S)$ colors. Now, we have two possible cases.

- Either $M_{\Delta(S_1)+1} \subseteq M_{\Delta(S)}$, so we set c_0 as c^\otimes ,
- or $M_{\Delta(S_1)+1} \not\subseteq M_{\Delta(S)}$, thus $0 \in M_{\Delta(S_1)+1}$, but $0 \notin M_{\Delta(S)}$ – and then we define

$$c_0(v:e) = \begin{cases} \pm \left\lfloor \frac{\Delta(S)}{2} \right\rfloor, & \text{if } c^\otimes(v:e) = 0, \\ c^\otimes(v:e), & \text{otherwise.} \end{cases}$$

Note that if $0 \notin M_{\Delta(S)}$, then $\Delta(S)$ is even and the resulting coloring is correct, since $\pm \frac{\Delta(S)}{2} \notin M_{\Delta(S_1)+1}$, and we can always consistently assign signs in the formula above, as 0-edges always form a matching so any alignment of signs consistent with σ suffices.

Now, let us also fix c_i^* as an edge coloring of $S_{2,i} + v_i$ with $n(G_2) + 1 = \Delta(S_{2,i} + v_i) + 1$ colors. Since $n(G_2) + 1 < \Delta(S_1) + n(G_2) = \Delta(S)$, we can transform this coloring to c'_i as following.

- Either $M_{n(G_2)+1} \subseteq M_{\Delta(S)}$, so we set c'_i as c_i^* ,
- or $M_{n(G_2)+1} \not\subseteq M_{\Delta(S)}$, thus $0 \in M_{n(G_2)+1}$, but $0 \notin M_{\Delta(S)}$ – and then we define

$$c'_i(v:e) = \begin{cases} \pm \left\lfloor \frac{\Delta(S)}{2} \right\rfloor, & \text{if } c_i^*(v:e) = 0, \\ c_i^*(v:e), & \text{otherwise.} \end{cases}$$

As with c^\otimes and c_0 above, since $0 \notin M_{\Delta(S)}$, we know that $\Delta(S)$ is even and the resulting coloring is correct, since $\pm \left\lfloor \frac{\Delta(S)}{2} \right\rfloor \notin M_{n(G_2)+1}$.

Observe that any c'_i has the following property (I).

- If $0 \in M_{\Delta(S)}$, then there is at most one k such that there is an incidence of v_i with color k and no incidence of v_i with color $-k$.

- If $0 \notin M_{\Delta(S)}$, then there are at most two such values of k — and for one of them $\pm k$ induces a matching in G_2 .

The last part stems from the fact that the second missing color can arise only when $M_{n(G_2)+1} \not\subseteq M_{\Delta(S)}$ by using $\pm \left\lfloor \frac{\Delta(S)}{2} \right\rfloor$ in place of 0. Note that we have $\deg_{S_1}(v_i) = \Delta(S_1)$. Therefore, by our construction of c_0 the property (I) holds for all vertices v_i .

For any $i > 0$ we just need to construct c_i as a union of c_{i-1} and a proper recoloring of c'_i ensuring that all pairs of colors from c'_i are recolored to all $\pm k \in M_{\Delta(S)}$ (including a “fake pair” $0 \in M_{\Delta(S)}$ if necessary) not used by incidences already colored in c_{i-1} . Moreover, we have to ensure that these colors are used for both the incidences of v_i without introducing any conflicts with c_{i-1} .

Case 1. First, let us consider the case that $0 \in M_{\Delta(S)}$. From the property (I) we know that there exists at most one value $l \neq 0$ such that one of the incidences of v_i in c_{i-1} has color l , but none of them has color $-l$, and similarly for c'_i .

Case 1a. If $\Delta(S_1)$ is even, then $\Delta(S_{2,i} + v_i)$ has to be odd. Thus, c_i^* and c'_i by construction do not use color 0 at all. However, this means that the incidences of v_i in c'_i have exactly colors $\pm k'_1, \dots, \pm k'_{t'}$, and l' (i.e., there is one color without pair at v_i in c'_i) with all $k'_j \neq 0$.

Case 1a'. Suppose now that the incidences of v_i in c_{i-1} have colors $\pm k_1, \dots, \pm k_t, l$. Observe that since $\Delta(S_1)$ is even, there is some $j = 1, \dots, t$ such that $k_j = 0$. Let us map (a) colors $\pm l'$ from c'_i to $\mp l$ (reverting the signs) and (b) all colors $\pm k'_s$ from c'_i for $s = 1, \dots, t'$ to some pairs from $M_{\Delta(S)}$ different than all $\pm k_1, \dots, \pm k_t$ and $\pm l$.

Case 1a'. If, on the other hand, the incidences of v_i in c_{i-1} have colors $\pm k_1, \dots, \pm k_t$, then all such pairs of colors have to be non-zero. Moreover, $\Delta(S_1) \geq 2$ ensures that $t \geq 1$. Thus, it is sufficient to map

- colors $\pm l'$ from c'_i to $\pm k_1$ for all edges not incident to v_i ,
- the incidence of v_i with color l' in c'_i to 0,
- all colors $\pm k'_s$ from c'_i for $s = 1, \dots, t'$ to some pairs from $M_{\Delta(S)}$ different than all $\pm k_1, \dots, \pm k_t$.

Case 1b. When $\Delta(S_1)$ is odd, then c_0 by construction does not use color 0 at all, therefore all incidences of v_i in c_{i-1} have non-zero colors — in particular, they have colors $\pm k_1, \dots, \pm k_t, l$ with all $k_j \neq 0$.

Case 1b'. Suppose that the incidences of v_i in c'_i have colors $\pm k'_1, \dots, \pm k'_{t'}$, l' . Then, since $\Delta(S_{2,i} + v_i)$ is even, it has to be the case that $k'_j = 0$ for some $j = 1, \dots, t'$. In this case, we proceed as in the case 1a', additionally trivially copying incidences with color 0 from c'_i to c_i .

Case 1b". If, on the other hand, the incidences of c'_i have only colors $\pm k'_1, \dots, \pm k'_{t'}$, then $k'_j \neq 0$ for all $j = 1, \dots, t'$. Then we map

- colors $\pm k'_1$ from c'_i to $\pm k_1$ for all edges not incident to v_i ,
- the incidence of v_i with colors $\pm k'_1$ in c'_i to 0 and $-l$, respectively,
- all possible incidences with color 0 in c'_i to $\pm l$ so that they do not conflict with the one of incidences above — note that 0 might be only possibly used in $S_{2,i} + v_i$, but not at v_i in c'_i ,
- all colors $\pm k'_s$ from c'_i for $s = 2, \dots, t'$ to some pairs from $M_{\Delta(S)}$ different than all $\pm k_1, \dots, \pm k_t$ and $\pm l$.

A simple degree count for v_i shows that such recolorings are always possible: the number of colors used at v_i by both c_{i-1} and c'_i has to be equal to $\deg_S(v_i) = \Delta(S) = |M_{\Delta(S)}|$ — and it is easy to check that in each case c_i uses exactly the colors from $M_{\Delta(S)}$. Moreover, it can be verified that we map all the colors from c'_i (i.e., all appearing at v_i plus one other) and that the resulting c_i is a proper edge coloring of a signed graph since the property (I) ensures that $l, l' \neq 0$.

Note that in general for $k'_j = 0$, it is always possible to map it to a pair of values $\pm k_i$ for some $k_i \neq 0$. And the construction ensures that we would never try doing the reverse, that is, mapping a true pair of colors to 0.

Case 2. for $0 \notin M_{\Delta(S)}$ suppose that the incidences of v_i in c_{i-1} have colors $\pm k_1, \dots, \pm k_t, l_1, l_2$. Since there is no unusual, one-element pair ± 0 and $\deg_S(v_i) = \Delta(S) = |M_{\Delta(S)}|$ is even, it holds that either the incidences of v_i in c'_i have colors $\pm k'_1, \dots, \pm k'_{t'}, l'_1, l'_2$ or just $\pm k'_1, \dots, \pm k'_{t'}$. In the first case, we map $\pm l'_1$ and $\pm l'_2$ from c'_i to $\mp l_1$ and $\mp l_2$ (reverting the signs), respectively, and $\pm k'_j$ from c'_i to available pairs as above. In the second case $t' \geq 1$, so we recolor $\pm k'_1$ to $\{-l_1, -l_2\}$ and all other k'_s as before.

Similarly, when there are only l_1 and l'_1 , but not l_2 or l'_2 it is sufficient to map $\pm l'_1$ to $\mp l_1$ (reverting the signs).

And finally, the case when there is no l_1 or l_2 .

- If there is l'_1 and l'_2 , then according to the property (I) $\pm l'_2$ induces a matching in $S_{2,i} + v_i$. We can map colors $\pm l'_1$ from c'_i for all edges not incident to v_i to $\pm k_1$ and map all $\pm k'_s$ from c'_i to some available pairs, other than all $\pm k_j$ ($j = 1, \dots, t$). Finally, we have a matching plus a single edge, that is, an acyclic graph, so we can safely map it to the remaining available pair.
- If there is neither l'_1 nor l'_2 , then we just map $\pm k'_s$ to available pairs, other than all $\pm k_j$ ($j = 1, \dots, t$).

Therefore, in every case it is possible to obtain c_i from c_{i-1} which directly implies that the required $\Delta(S)$ -edge coloring of the whole signed corona product graph S always exists. ■

We also observe that a corona product of graphs G_1, G_2 with $\Delta(G_1) \leq 1$ is either a subgraph of (1) a collection of disjoint cliques (when G_1 has no edges) or (2) a collection of cliques, either disjoint, or connected in pairs with a single edge (when G_1 is a matching). However, these two problems are still open.

Conjecture 25. *Given a signed complete graph $S = (K_n, \sigma)$, show that $\chi'(S) = \Delta(S) = n - 1$.*

Conjecture 26. *Given a signed graph consisting of two identical cliques connected by a single edge $S = ((K_n \cup K_n) + e, \sigma)$, show that $\chi'(S) = \Delta(S) = n$.*

Note that in the latter if all the edge signs were consistent on both copies of K_n , then the conjecture would hold. It would be sufficient to find c' as n -edge colorings of both copies of K_n , such that α is a missing color both for v_1 in c' applied for the first K_n and for v_2 in c' applied for the second K_n for the joining edge $e = v_1v_2$. Then it is enough to return either a join of c' for both K_n with an appropriate color on both endpoints of e (if the sign of e is positive), or a join of $c', -c'$ for each K_n and a proper coloring of e (if the sign of e is negative).

REFERENCES

- [1] R. Behr, *Edge coloring signed graphs*, Discrete Math. **343** (2020) 111654.
<https://doi.org/10.1016/j.disc.2019.111654>
- [2] H. Cai, Q. Sun, G. Xu and S. Zheng, *Edge Coloring of the signed generalized Petersen graph*, Bull. Malays. Math. Sci. Soc. **45** (2022) 647–661.
<https://doi.org/10.1007/s40840-021-01212-w>
- [3] R. Diestel, *Graph Theory*, 5th Ed., Grad. Texts in Math. **173** (Springer Berlin, Heidelberg, 2017).
<https://doi.org/10.1007/978-3-662-53622-3>
- [4] F. Harary, *On the notion of balance of a signed graph*, Michigan Math. J. **2** (1953/1954) 143–146.
<https://doi.org/10.1307/mmj/1028989917>
- [5] F. Harary, *On the coloring of signed graphs*, Elem. Math. **23** (1968) 85–89.
<https://doi.org/10.5169/seals-26032>
- [6] R. Janczewski, K. Turowski and B. Wróblewski, *Edge coloring of graphs of signed class 1 and 2*, Discrete Appl. Math. **338** (2023) 311–319.
<https://doi.org/10.1016/j.dam.2023.06.029>
- [7] L. Jin, Y. Kang and E. Steffen, *Choosability in signed planar graphs*, European J. Combin. **52** (2016) 234–243.
<https://doi.org/10.1016/j.ejc.2015.10.001>
- [8] Y. Kang, *Hajós-like theorem for signed graphs*, European J. Combin. **67** (2018) 199–207.
<https://doi.org/10.1016/j.ejc.2017.08.003>

- [9] Y. Kang and E. Steffen, *The chromatic spectrum of signed graphs*, Discrete Math. **339** (2016) 2660–2663.
<https://doi.org/10.1016/j.disc.2016.05.013>
- [10] Y. Kang and E. Steffen, *Circular coloring of signed graphs*, J. Graph Theory **87** (2018) 135–148.
<https://doi.org/10.1002/jgt.22147>
- [11] F. Kardoš and J. Narboni, *On the 4-color theorem for signed graphs*, European J. Combin. **91** (2021) 103215.
<https://doi.org/10.1016/j.ejc.2020.103215>
- [12] E. Mácajová, A. Raspaud and M. Škoviera, *The chromatic number of a signed graph*, Electron. J. Combin. **23** (2016) #P1.14.
<https://doi.org/10.37236/4938>
- [13] T. Schweser and M. Stiebitz, *Degree choosable signed graphs*, Discrete Math. **340** (2017) 882–891.
<https://doi.org/10.1016/j.disc.2017.01.007>
- [14] E. Steffen and A. Vogel, *Concepts of signed graph coloring*, European J. Combin. **91** (2021) 103226.
<https://doi.org/10.1016/j.ejc.2020.103226>
- [15] T. Zaslavsky, *Signed graph coloring*, Discrete Math. **39** (1982) 215–228.
[https://doi.org/10.1016/0012-365X\(82\)90144-3](https://doi.org/10.1016/0012-365X(82)90144-3)
- [16] T. Zaslavsky, *How colorful the signed graph?*, Discrete Math. **52** (1984) 279–284.
[https://doi.org/10.1016/0012-365X\(84\)90088-8](https://doi.org/10.1016/0012-365X(84)90088-8)
- [17] T. Zaslavsky, *Balanced decompositions of a signed graph*, J. Combin. Theory Ser. B **43** (1987) 1–13.
[https://doi.org/10.1016/0095-8956\(87\)90026-8](https://doi.org/10.1016/0095-8956(87)90026-8)

Received 12 January 2024

Revised 21 June 2024

Accepted 21 June 2024

Available online 16 July 2024