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THE STRUCTURE OF 2-MATCHING CONNECTED GRAPHS

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Abstract

Constructive characterizations of classical connectivity have been studied by many researchers. Whitney states that a graph is 2-connected if and only if it is a cycle or can be obtained from a cycle by repeatedly adding ears. Tutte asserts that a graph is 3-connected if and only if it is a wheel or can be obtained from a wheel by repeatedly adding edges and splitting vertices of degree more than three. Slater determines that the class of 4-connected graphs is the class of graphs obtained from K_5 by finite sequences of edge addition, 4-soldering, 4-vertex-splitting, 4-edge-splitting, and 3-fold-4-vertex-splitting. For an integer k, a graph G is *k-matching connected* if $G - V(M)$ is a connected nontrivial graph for each matching M with $|M| \leq k-1$, where $V(M)$ is the set of vertices covered by M. In this paper, we present that a graph G is 2-matching connected if and only if either $G \in \{C_4, K_4\}$ or it can be obtained from a C_4 by repeatedly applying the following three operations.

(i) Adding a new edge uv for a pair of nonadjacent vertices u and v where $\{u, v\}$ is not a vertex cut of G.

(ii) Adding an ear uwv, where w is a new vertex and u, v are a pair of nonadjacent vertices of G.

(iii) Replacing a vertex z by two new adjacent vertices z_1 and z_2, and making z_i adjacent to every vertex of N_i such that $d(z_i) \geq 2$ for $i \in \{1, 2\}$, where $N_1 \cup N_2 = N(z)$ and $N_1 \cap N_2 = \emptyset$.

Keywords: matching, connectivity, matching-connectivity, graph characterization.

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1. Introduction

Throughout the paper, all graphs are assumed to be finite, undirected and simple. We refer to the book [1] for undefined notation and terminology in this paper.

The constructive characterization of graphs is an important tool to study the properties of graphs. Particularly, constructive characterizations of classical connectivity have been studied by many famous researchers. These characterizations provide insights into the structural properties of connected graphs and help establish relationships between connectivity and other graph parameters. Here are a few notable constructive characterizations of classical connectivity.

Whitney [13] gave a constructive characterization of 2-connected graphs.

Theorem 1 (Whitney [13]). *A graph is* 2*-connected if and only if it is a cycle or can be obtained from a cycle by repeatedly adding ears.*

Tutte [12] gave a constructive characterization of 3-connected graphs.

Theorem 2 (Tutte [12]). *A graph is* 3*-connected if and only if it is a wheel or can be obtained from a wheel by repeatedly adding edges and splitting vertices of degree at least four.*

Slater [11] gave a constructive characterization of 4-connected graphs.

Theorem 3 (Slater [11]). *The class of* 4*-connected graphs is the class of graphs obtained from* K_5 *by finite sequences of edge addition,* 4*-soldering,* 4*-vertex-splitting,* 4*-edge-splitting, and* 3*-fold-*4*-vertex-splitting.*

Politof and Satyanarayana [10] gave a constructive characterization of quasi 4-connected graphs.

Theorem 4 (Politof and Satyanarayana [10]). *Every quasi* 4*-connected graph can be obtained from a wheel on at most six vertices, or a prism or a Möbius ladder by repeatedly* (i) *adding edges,* (ii) *splitting vertices, and/or* (iii) *replacing a triangle containing vertices of degree at least four by the graph obtained from* K_4 *by deleting an edge.*

To compensate for some shortcomings, Harary [5] introduced the concept of conditional connectivity. Let G be a connected undirected graph, and let $\mathcal{P}$ be a graph-theoretic property. Harary defined the conditional connectivity $\kappa(G;\mathcal{P})$ as the minimum cardinality of a vertex set, if it exists, whose deletion disconnects G, but every remaining component would still have the property $\mathcal{P}$. Following this idea, Esfahanian and Hakimi [2, 3] generalized the notion of connectivity by introducing the concept of restricted connectivity.

A vertex set S of a graph G is called a *vertex cut* if $G-S$ is disconnected. A vertex cut S of a graph G is called a *restricted vertex cut* if $N_G(x) \nsubseteq S$ for each $x \in V(G)$. The *restricted connectivity* of a connected graph G is defined as the minimum cardinality of a restricted vertex cut of G, if it exists.

Fàbrega and Fiol introduced g-extra connectivity [4]. A vertex cut S of a graph G is called a *g-extra cut* if every remaining component of $G-S$ has at least $g+1$ vertices. The *g-extra connectivity of* G of a connected graph G is defined as the minimum cardinality of a g-extra cut of G, if it exists.

Latifi, Hegde and Pour introduced *R_g-connectivity* [6]. A vertex cut S of a graph G is called a *R_g-cut* if each vertex of the remaining components has at least g vertices in $G-S$. The R_g-connectivity of G, denoted by $\kappa^g(G)$ is the minimum cardinality of all R_g-cuts of G.

We can see that Esfahanian and Hakimi, Fàbrega and Fiol, and Latifi, Hegde, and Pour generalized the concept of connectivity by adding conditions for components. While, Oellermann [9] generalized the concept of connectivity by adding the condition that the induced subgraph by vertex cut is connected. A vertex cut S of a graph G is called a *connected cutset* if $G[S]$ is connected. The *connected cutset connectivity* of G is the minimum cardinality of a connected cut set.

Recently, Lin *et al.* [8] put forward the concepts of structure connectivity and substructure connectivity by adding conditions for cut set.

Definition [7, 8]**.** Let $\mathcal{F}$ be a family of vertex-disjoint subgraphs of a graph G, and define

$$V(\mathcal{F}) = \bigcup_{F\in\mathcal{F}} V(F).$$

(i) If every member of $\mathcal{F}$ is a connected subgraph of G, and $G-V(\mathcal{F})$ is disconnected or a trivial graph, then $\mathcal{F}$ is a *subgraph-cut* of G.

Let H be a connected subgraph of G.

(ii) If every element in $\mathcal{F}$ is isomorphic to H and $\mathcal{F}$ is a subgraph-cut, then $\mathcal{F}$ is an *H-structure-cut.* The *H-structure-connectivity* of G is

$$\kappa(G;H) = \min\left\{|\mathcal{F}| :\ \mathcal{F} \text{ is an } H\text{-structure-cut of } G\right\}.$$

Thus $\kappa(G;H)$ is the minimum number of vertex-disjoint subgraphs of G, each of which is isomorphic to H and whose deletion results in a disconnected graph or a trivial graph. By the definition of $\kappa(G;H)$, if $H \cong K_1$, then $\kappa(G;K_1)$ is the classic vertex connectivity of a graph G, and so $\kappa(G;H)$ can be viewed as a generalization of the vertex connectivity of G. In [7], authors call P_2-structure-cut and P_2-structure-connectivity as *matching cut* and *matching connectivity*, respectively, and write $\kappa(G;P_2)$ as $\kappa_M(G)$.

We call P_2-structure-connectivity as matching connectivity not only because the induced subgraph of each P_2-structure-cut has a perfect matching but also because the matching connectivity of a graph is closely related with its matching. We can check that if a graph of even order has no matching cut, then it is k-extendable, where a graph G is k-extendable if each k-matching of G can be extended to a perfect matching. By using matching extendable theory, authors [7] showed that $\kappa_M(G)$ is well-defined unless that $G \cong K_{2n}$ or $G \cong K_{n,n}$, and studied the relationship between $\kappa(G)$ and $\kappa_M(G)$ and came to the conclusion that $\kappa(G)/2 \leq \kappa_M(G) \leq \kappa(G)$ if G is neither $K_{n,n}$ nor K_{2n}.

For an integer k, a graph G is *k-matching connected* if $G - V(M)$ is a connected nontrivial graph for each matching M with $|M| \leq k-1$, where $V(M)$ is the set of vertices covered by M. Thus, a graph G of order at least $2k+1$ is k-matching connected if and only if either $\kappa_M(G) \geq k$ or $G \in \{K_{2n}, K_{n,n}\}$, where n is an integer larger than k.

Inspired by the study of Tutte [12], Slater[11], Politof and Satyanarayana [10], and Whitney [13], we give a constructive characterization of 2-matching connected graphs. The rest of this paper is organized as follows: In Section 2, we summarize some basics that will be utilized in our paper. In Section 3, we provide a set of three graph operations such that a graph is 2-matching connected if and only if either $G \in \{C_4, K_4\}$ or it can be obtained from a C_4 by repeatedly applying these operations. In Section 4, we conclude our work and propose the future research problem.

2. Preliminaries

In this section, we summarize some basics that will be utilized in our arguments.

Let G be a graph with the vertex set $V(G)$ and the edge set $E(G)$. The *degree* $d_G(v)$ of v is the number of edges incident with v in G. For a vertex $v \in V(G)$, let $N_G(v) = \{u \in V(G) : uv \in E(G)\}$, the *neighborhood* of v in G. For a vertex subset $X \subseteq V(G)$, we define $N_G(X) = \bigcup_{x \in X} N_G(x) \setminus X$. We often write $d(v)$ for $d_G(v)$ and $N(v)$ for $N_G(v)$, when G is understood from the context. For a vertex set $S \subseteq V(G)$ (or a subgraph S of G), we use $N_S(v)$ to denote the set $N_G(v) \cap S$. Further, we use $G[S]$ to denote the graph induced by S whose vertex set is S and whose edge set consists of all edges of G which have both ends in S. We denote by $G - S$ the graph $G[V(G) - S]$. For an edge set $E' \subseteq E(G)$, we denote by $G - E'$ the graph with vertex set $V(G)$ and edge set $E(G) \setminus E(G')$. We simple write $G - v$ and $G - e$ rather than $G - \{v\}$ and $G - \{e\}$ for $v \in V(G)$ and $e \in E(G)$, respectively. For a graph G with $A, B \subseteq V(G)$, let $E_G[A, B]$ denote the set of edges with one end in A and the other end in B.

A vertex set $S \subseteq V(G)$ is called a *vertex cut* of a connected graph G, when $G-S$ is disconnected or trivial. The connectivity of G, denoted by $\kappa(G)$, is defined as the minimum size of a vertex cut set S of G. An (x, y)-path P_{xy} is a path with start at x and end at y. For an (x, y)-path P_{xy}, denote by $V_{in}(P_{xy})$ the set of all internal vertices of P_{xy}, that is, $V_{in}(P_{xy}) = V(P_{xy}) \setminus \{x, y\}$. Two (x, y)-paths P, Q are *internally disjoint* if they have no internal vertices in common, that is, if $V(P) \cap V(Q) = \{x, y\}$. The *local connectivity* between distinct vertices x and y is the maximum number of pairwise internally disjoint (x, y)-paths, denoted $p(x, y)$; the local connectivity is undefined when $x = y$. A nontrivial graph G is *k-connected* if $p(u, v) \geq k$ for any two distinct vertices u and v in G. By Menger's Theorem, $\kappa(G) := \min\{p(u, v) : u, v \in V, u \neq v\}$ if G is not a complete graph. The following theorem is well known.

Theorem 5 (Bondy and Murty [1]). *In every finite undirected graph the number of vertices with odd degree is always even.*

For a graph G, the *number of components* of G is denoted by $c(G)$. The *contraction* of an edge $e = uv$ in G removes u and v from G, and replaces them by a new vertex, which is made adjacent to precisely those vertices that were adjacent to at least one of the vertices u and v. If a contraction creates multiple edges, we reduce their multiplicity to one and keep the graph simple. The resulting graph is denoted by G/e. The following theorem states that the operation of contracting an edge in a graph does not change the number of components in the graph.

Theorem 6 (Bondy and Murty [1]). *For any $e \in E(G)$, $c(G/e) = c(G)$.*

3. Main Results

This section introduces our main results. First, we define three operations which will be used in the arguments.

Definition. Let $G = (V(G), E(G))$ be a graph. Define three operations as follows (see Figure 1).

Operation *I*. Adding a new edge uv for a pair of nonadjacent vertices u and v where $\{u, v\}$ is not a vertex cut of G.

Operation *II*. Adding an ear uwv, where w is a new vertex and u, v are a pair of nonadjacent vertices of G.

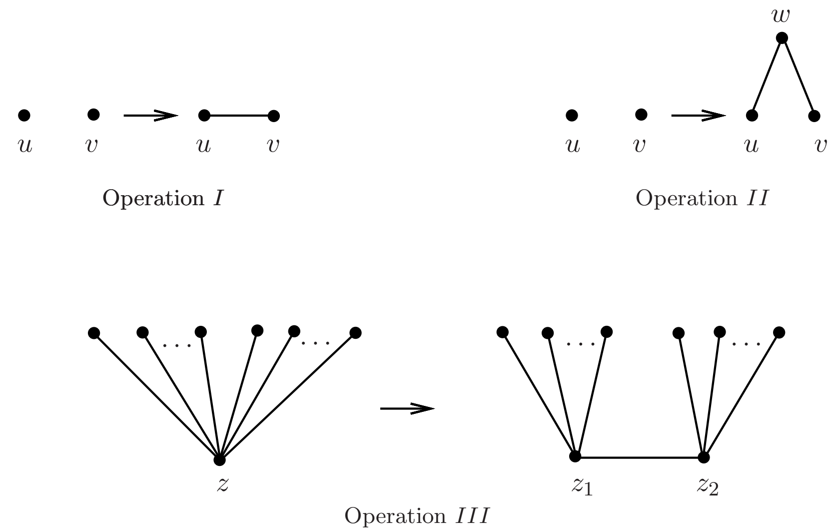

Figure 1. The illustration of Operation I, Operation II and Operation III.

Operation *III*. Replacing a vertex z by two new adjacent vertices z_1 and z_2, and making z_i adjacent to every vertex of N_i such that $d(z_i) \geq 2$ for $i \in \{1, 2\}$, where $N_1 \cup N_2 = N(z)$ and $N_1 \cap N_2 = \emptyset$.

Our main result is as follows.

Theorem 7. *A graph G is 2-matching connected if and only if either $G \in \{C_4, K_4\}$ or it can be obtained from a C_4 by repeatedly applying* Operations *I, II, III.*

Now, we present some basic results to carry out our main work.

Observation 8. *Let G be a connected graph on at least four vertices. If G is 2-matching connected, then G has no cut vertex.*

Obviously, C_4 and K_4 are 2-matching connected graphs. So the sufficiency of Theorem 7 holds by the following lemmas.

Lemma 9. *If G can be obtained from a 2-matching connected graph H by applying* Operation *I, then G is also 2-matching connected.*

Proof. Suppose, by way of contradiction, that G can be obtained from a 2-matching connected graph H by applying Operation I on $\{u, v\} \subseteq V(H)$, but G is not 2-matching connected. Let st be a matching cut of G. Since H is 2-matching connected, st is not a matching cut of H, thus $\{u, v\} \cap \{s, t\} \neq \emptyset$. We also have $\{u, v\} \neq \{s, t\}$; otherwise, $\{u, v\}$ is a vertex cut of H, contrary to the definition of Operation I. So $|\{u, v\} \cap \{s, t\}| = 1$, without loss of generality, assume that uu' is a matching cut of G with $u' \in N_G(u) \setminus v$. Then $G - u - u' = H - u - u'$ is connected, a contradiction. ■

Lemma 10. *If G can be obtained from a 2-matching connected graph H by applying* Operation II, *then G is also 2-matching connected.*

Proof. Suppose, by way of contradiction, that G can be obtained from a 2-matching connected graph H by applying Operation II on $\{u, v\} \subseteq V(H)$ (adding an ear uwv, where $w \notin V(H)$ is the new vertex), but G is not 2-matching connected. Let st be a matching cut of G. We can check that $w \notin \{s, t\}$. For example, if uw is a matching cut of G, then $H - u = G - u - w$ is disconnected, contrary to Observation 8. Since $uv \notin E(G)$, at least one of u, v not in $\{s, t\}$, without loss of generality, assume that $u \notin \{s, t\}$. Note that $G - s - t - w = H - s - t$ is connected since H is 2-matching connected. Moreover, w is adjacent to u in $G - s - t$, where $u \in V(G - s - t - w)$, thus $G - s - t$ is connected, which is a contradiction. ■

Lemma 11. *If G can be obtained from a 2-matching connected graph H by applying* Operation III, *then G is also 2-matching connected.*

Proof. Suppose, by way of contradiction, that G can be obtained from a 2-matching connected graph H by applying Operation III on z (denote by z_1, z_2 the vertices resulting from the operation), but G is not 2-matching connected. Let st be a matching cut of G. Then $\{s, t\} \cap \{z_1, z_2\} \neq \emptyset$; otherwise, st is a matching cut of H, contrary to the fact that H is 2-matching connected. We can also get that $\{s, t\} \neq \{z_1, z_2\}$; otherwise, z is a cut vertex of H, contrary to Observation 8. Without loss of generality, assume that $\{s, t\} \cap \{z_1, z_2\} = z_1$ and $st = z_1 z_1'$ is a matching cut of G with $z_1' \in N_G(z_1) \subseteq N_H(z)$. As H is 2-matching connected, we have $G - z_1 - z_2 - z_1' = H - z - z_1'$ is connected. Since $d_G(z_2) \geq 2$ and $z_1' \notin N_G(z_2)$, the vertex z_2 has at least one neighbor in $G - z_1 - z_2 - z_1'$. Hence $G - z_1 - z_1'$ is connected, a contradiction. ■

For a 2-matching connected graph G with order n and size m, $G \in \{C_4, K_4\}$ as $n \leq 4$. To prove the necessity of Theorem 7, we only need to consider each 2-matching connected graph with at least five vertices.

Lemma 12. *Each 2-matching connected graph G on five vertices can be obtained from a C_4 by repeatedly applying* Operations I, II, III.

Proof. Let G be a connected graph with $\kappa_M(G) \geq 2$ and $V(G) = \{u_1, u_2, u_3, u_4, u_5\}$. We argue by contradiction and assume that G cannot be obtained from a C_4 by repeatedly applying Operations I, II, III. Then Claim 13 holds.

Claim 13. *$d(u) > 2$ for any $u \in V(G)$.*

Proof. The proof proceeds by contradiction. As G is connected and by Observation 8, we can get $d(u) \geq 2$ for any $u \in V(G)$. Let u_1 be a vertex of degree two with $N_G(u_1) = \{u_2, u_3\}$. Clearly $u_2u_3 \notin E(G)$, since otherwise u_2u_3 is a matching cut of G. If $\{u_2, u_3\}$ is a vertex cut of $G - u_1$, then $G - u_1 - u_2 - u_3$ is an empty graph, that is, a graph without edges. It follows that $G \cong G_1$, where G_1 is shown in Figure 2. Clearly, G can be obtained from a four cycle $u_1u_2u_4u_3u_1$ by applying Operation II on $\{u_2, u_3\}$, a contradiction. Otherwise, that is, $\{u_2, u_3\}$ is not a vertex cut of $G - u_1$, it means that u_4 is adjacent to u_5 in $G - u_1 - u_2 - u_3$. Since $u_1u_4, u_1u_5 \notin E(G)$, $2 \leq d(u_4) \leq 3$ and $2 \leq d(u_5) \leq 3$. If $d(u_4) = d(u_5) = 2$, then $G \cong G_2$, where G_2 is shown in Figure 2. If either $d(u_4) = 2$ and $d(u_5) = 3$ or $d(u_4) = 3$ and $d(u_5) = 2$, by symmetry of u_4 and u_5, suppose that $d(u_4) = 3$ and $d(u_5) = 2$, then $G \cong G_3$, where G_3 is shown in Figure 2. If $d(u_4) = d(u_5) = 3$, then $G \cong G_4$, where G_4 is shown in Figure 2. Since $\kappa_M(G) \geq 2$ then $G \cong G_2$ or $G \cong G_4$. It is easy to check that G_2 can be obtained from a four cycle by applying Operation III, and G_4 can be obtained from G_1 by applying Operation I on $\{u_4, u_5\}$. Thus G can be obtained from C_4 by repeatedly applying Operations I, II, III, a contradiction. So Claim 13 holds. □

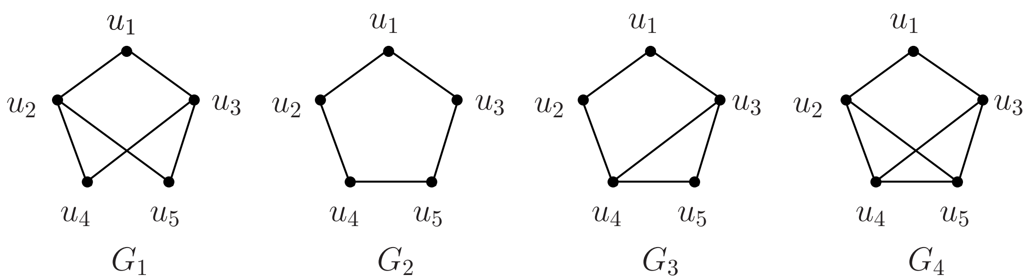

Figure 2. The illustration of Claim 13 in Lemma 12.

By Claim 13, we have $d(u) \geq 3$ for any $u \in V(G)$. By Theorem 5, G has 0 or 2 or 4 vertices with odd degree. Thus G is isomorphic to one of the graphs in Figure 3.

G_5 can be obtained from a four cycle $u_1u_2u_3u_5u_1$ by applying Operation II on $\{u_2, u_5\}$ (adding an ear $u_2u_4u_5$), and then applying Operations I on $\{u_1, u_4\}$ and $\{u_1, u_3\}$, respectively. Moreover, G_6 can be obtained from G_5 by applying Operation I on $\{u_2, u_5\}$, and G_7 can be obtained from G_6 by applying Operation I on $\{u_3, u_4\}$. Thus G can be obtained from a C_4 by Operations I, II, a contradiction. Hence, for each 2-matching connected graph G on five vertices, G can be obtained from a C_4 by repeatedly applying Operations I, II, III. ■

Now we choose a counterexample G so that (1) its order is minimum, and (2)

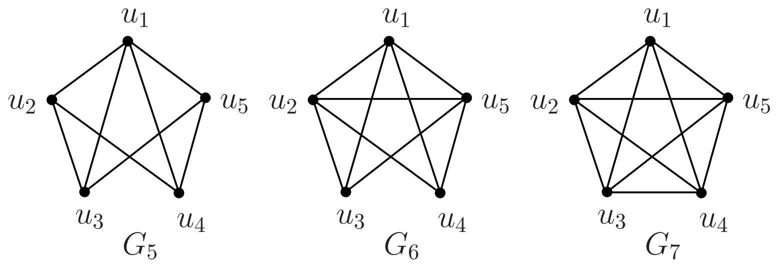

Figure 3. All kinds of G with $d(u) \geq 3$ for any $u \in V(G)$ of Lemma 12.

its size is minimum subject to (1). That is, G is a counterexample minimizing $(|V(G)|, |E(G)|)$. Then we want to show that G does not exist. We use $[n]$ to denote the set $\{1, 2, \ldots, n\}$ in the following argument.

For any $xy \in E(G)$, we say $uv \in E(G)$ is a *bad edge* with respect to xy if $G - xy - u - v$ is disconnected. It is easy to check that $\{x, y\} \cap \{u, v\} = \emptyset$. Let $F(xy)$ be the set of all bad edges with respect to xy. For any two nonadjacent vertices x and y, the graph obtained from G by adding a new edge xy joining x and y is denoted by $G + xy$.

Lemma 14. *For any $uv \in F(xy)$, each (x, y)-path in $G - xy$ contains at least one vertex of u and v.*

Proof. If not, that is, there exists an (x, y)-path P_{xy} in $G - xy - u - v$. Since $G - u - v$ is connected and $G - xy - u - v$ is disconnected, xy is a cut edge of $G - u - v$, which contradicts the fact that $P_{xy} + xy$ is a cycle in $G - u - v$. ■

Lemma 15. *For any $e \in E(G)$, $\kappa_M(G - e) = \kappa_M(G/e) = 1$.*

Proof. The proof proceeds by contradiction. Let $xy \in E(G)$ such that $G - xy$ is 2-matching connected. Since $G - V(xy)$ is connected then $\{x, y\}$ is not a vertex cut of $G - xy$ so that G can be obtained from $G - xy$ by applying Operation I on $\{x, y\}$. Furthermore, $G - xy$ can be obtained from a C_4 by repeatedly applying Operations I, II, III, since otherwise $G - xy$ is a counterexample with smaller size than G. Hence G can be obtained from a C_4 by repeatedly applying Operations I, II, III, which is contradict to our assumption.

Let $uv \in E(G)$ such that G/uv is 2-matching connected and z be the vertex resulting from the contraction of uv. Then $N_{G/uv}(z) = (N_G(u) \cup N_G(v)) \setminus \{u, v\}$. Note that $|N_G(u) \cap N_G(v)| < 3$; otherwise, suppose there exist $u_1, u_2, u_3 \in N_G(u) \cap N_G(v)$. Since $\kappa_M(G - uv) = 1$, suppose st is a matching cut of $G - uv$, it follows that st is a bad edge with respect to uv in G. There is at least one of u_1, u_2, u_3 not in $\{s, t\}$, without loss of generality, assume that $u_1 \notin \{s, t\}$, then $P = uu_1v$ is a (u, v)-path such that $V(P) \cap \{s, t\} = \emptyset$, contrary to Lemma 14. We can check that $N_G(u) \cap N_G(v) \neq \emptyset$; otherwise, G can be obtained from G/uv

by applying Operation III on z by Lemma 11. Thus G can be obtained from a C_4 by repeatedly applying Operations I, II, III because G/uv can be obtained from a C_4 by repeatedly applying Operations I, II, III by the induction hypothesis. We can get that $|N_G(u) \cap N_G(v)| \neq 1$; otherwise, let u_v be the vertex such that $N_G(u) \cap N_G(v) = \{u_v\}$ and let G' be the subgraph of G such that G' can be obtained from G/uv by applying Operation III on z (see Figure 4). Hence $|E(G')| = |E(G)| - 1$. But G' is 2-matching connected by Lemma 11, which is contradict to $\kappa_M(G - e) = 1$ for any $e \in E(G)$.

Suppose $N_G(u) \cap N_G(v) = \{u_1, u_2\}$. Let st be a matching cut of $G - uv$. If $N_G(u) = N_G(v) = \{u_1, u_2\}$, then $u_1u_2 \notin E(G)$; otherwise u_1u_2 is a matching cut of G. So at least one of u_1, u_2 is not in $\{s, t\}$, without loss of generality, suppose that $u_1 \notin \{s, t\}$, then $P = uu_1v$ is a path connecting u and v in $G - uv$ such that $V(P) \cap \{s, t\} = \emptyset$, contrary to Lemma 14. If either $N_G(u) \neq \{u_1, u_2\}$ or $N_G(v) \neq \{u_1, u_2\}$, without loss of generality, assume that $N_G(v) \neq \{u_1, u_2\}$. Let G' be a subgraph of G such that G' can be obtained from G/uv by applying Operation III on z and $u_1, u_2 \in N_{G'}(u)$ (see Figure 5). Clearly, $G' + vu_1 = G - vu_2$. Since $G' - \{v, u_1\} = G - v - u_1$ then $\{v, u_1\}$ is not a vertex cut of G'. Thus $G' + vu_1$ is 2-matching connected by Lemma 9, contrary to $\kappa_M(G - vu_2) = 1$. ∎

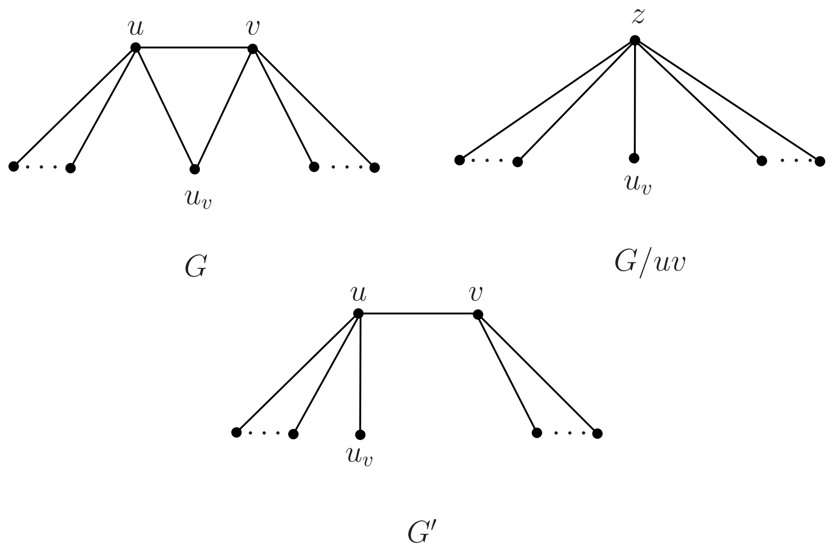

Figure 4. The illustration of Operation III in paragraph 2 of Lemma 15.

Lemma 16. *For any $u \in V(G)$ in which u has two nonadjacent neighbors, $\kappa_M(G - u) = 1$.*

Proof. The proof proceeds by contradiction. Let $u \in V(G)$ such that $G - u$ is 2-matching connected, in which u has two nonadjacent neighbors u_1, u_2 in

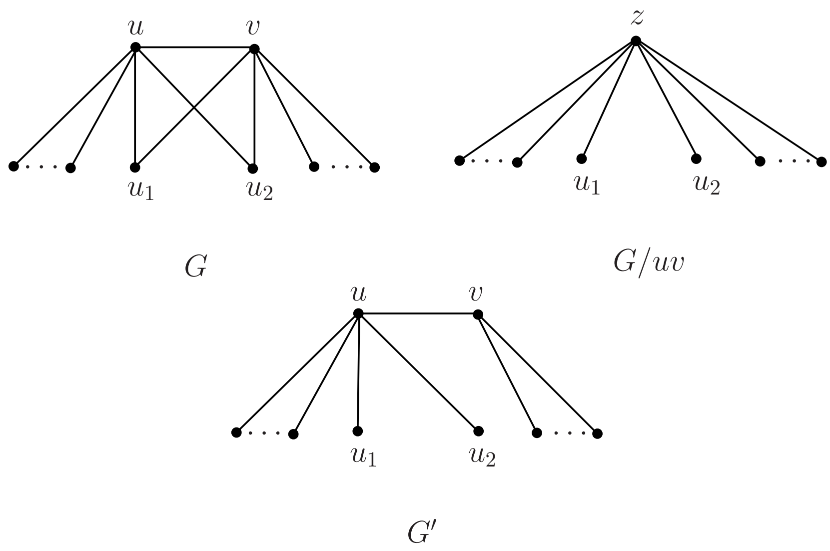

Figure 5. The illustration of Operation III in paragraph 3 of Lemma 15.

G. Let G' be the subgraph of G such that G' can be obtained from $G-u$ by applying Operation II on $\{u_1,u_2\}$ (adding an ear u_1uu_2). It follows from Lemma 10 that G' is 2-matching connected. Let $G_1,G_2,\ldots,G_k$ be a sequence of subgraphs of G such that G_i can be obtained from G_{i-1} by adding e_i, where $E(G)\setminus E(G')=\{e_1,e_2,\ldots,e_k\}$, $i\in[k]$, $G_0=G'$ and $G_k=G$. Then there exists G_j such that $\kappa_M(G_j)=1$ for some $j\in[k]$; otherwise, G can be obtained from G' by repeatedly applying Operation I. Let st be a matching cut of G_j. If $st\in E(G')$, then G_j-s-t is connected because G' is 2-matching connected. Otherwise, $st\in E(G)\setminus E(G')$, then $G_j-s-t=G-s-t$ is connected since G is 2-matching connected, contrary to the hypothesis and establish the lemma. ■

Lemma 17. *$d_G(v)>2$ for any $v\in V(G)$.*

Proof. The proof proceeds by contradiction. From Observation 8, it follows that $d_G(u)\geq 2$ for any $u\in V(G)$. Pick $v\in V(G)$ such that $d_G(v)=2$ and x,y are neighbors of v in G. Clearly, $xy\notin E(G)$. We only need to consider the following two cases.

Case 1. $\{x,y\}$ is not a vertex cut of $G-v$. Let $G':=G/xv$ and denote by v_x the vertex resulting from the contraction of xv. By Lemma 15, $\kappa_M(G')=1$. Let st be a matching cut of G'. In this case, $G'-v_x-y=G-v-x-y$ is connected. We can check that $\{v_x,y\}\cap\{s,t\}\neq\emptyset$; otherwise, $G'-s-t$ is connected because $c(G'-s-t)=c(G-s-t)=1$ by Theorem 6. Without loss of generality, suppose that $v_xx_1=st$ with $x_1\in N_G(x)\setminus v$. Since $G-x-x_1$ is connected and y is the only neighbor of v in $G - x - x_1$, $G' - v_x - x_1 = G - x - x_1 - v$ is connected, a contradiction.

Case 2. $\{x, y\}$ is a vertex cut of $G - v$. Let $G' := G - v$. We have $\kappa_M(G') = 1$ by Lemma 16. Let $G_1, G_2, \ldots, G_q$ be the components of $G' - x - y$. Note that $N_{G'}(x) \cap V(G_i) \neq \emptyset$ for any $i \in [q]$; otherwise, vy is a matching cut of G, a contradiction. By the symmetry of x and y, we also get that $N_{G'}(y) \cap V(G_i) \neq \emptyset$ for any $i \in [q]$. Thus we can find two internally disjoint (x, y)-paths P, P' in G' where $V_{in}(P) \subseteq V(G_1)$ and $V_{in}(P') \subseteq V(G_2)$. Let st be a matching cut of G'. We claim that $\{s, t\} \cap \{x, y\} = \emptyset$. Or else, say by symmetry that $xx_1 = st$ with $x_1 \in N_{G'}(x)$, then $G' - x - x_1 = G - x - x_1 - v$ is connected because $G - x - x_1$ is connected and y is the only neighbor of v in $G - x - x_1$, a contradiction. Since $G - s - t$ is connected, there exists a (u, v)-path P_u for any $u \in V(G - s - t - v)$ in $G - s - t$. Thus $P_u - v$ is either a (u, x)-path or a (u, y)-path because $N_G(v) = \{x, y\}$. Moreover, $\{s, t\} \cap V(P) = \emptyset$ or $\{s, t\} \cap V(P') = \emptyset$, thus there exists an (x, y)-path in $G' - s - t$. Hence, $G' - s - t$ is connected, which contradicts to the fact that st is a matching cut in G'. ■

Lemma 18. *For each $e = uv \in E(G)$, there is no pair of internal disjoint (u, v)-paths P_{uv} and P'_{uv} such that $E_G[V_{in}(P_{uv}), V_{in}(P'_{uv})] = \emptyset$.*

Proof. If not, pick $uv \in E(G)$ such that there exists a pair of internal disjoint (u, v)-paths P_{uv} and P'_{uv} such that $E_G[V_{in}(P_{uv}), V_{in}(P'_{uv})] = \emptyset$. It follows from Lemma 15 that $\kappa_M(G - uv) = 1$. Let st be a matching cut of $G - uv$. Since $E_G[V_{in}(P_{uv}), V_{in}(P'_{uv})] = \emptyset$, we have that $\{s, t\} \cap V_{in}(P_{uv}) = \emptyset$ or $\{s, t\} \cap V_{in}(P'_{uv}) = \emptyset$, contrary to Lemma 14. ■

Lemma 19. *For any $e \in E(G)$, $G - e$ has no cut vertex.*

Proof. If not, pick $xy \in E(G)$ and $v \in V(G)$ such that v is a cut vertex of $G - xy$. Because xy is a cut edge of $G - v$, so $G - xy - v$ has two components G_1, G_2, assume that $x \in V(G_1), y \in V(G_2)$. If $vx \in E(G)$, then G_1 is trivial; otherwise, vx is a matching cut of G. It follows that $d_G(x) = 2$, contrary to Lemma 17. Thus $vx \notin E(G)$. Similarly we can check that $vy \notin E(G)$. Let $G' := G/xy$ and z be the new vertex resulting from contraction of xy (see Figure 6).

According to Lemma 15, we have $\kappa_M(G') = 1$. Suppose that st is a matching cut of G', then $z \in \{s, t\}$; otherwise $G' - s - t = (G - s - t)/xy$ is connected by Theorem 6. Without loss of generality, we assume that $zz_x = st$ with $z_x \in N_G(x)$. Let $X_1, X_2, \ldots, X_p$ be the components of $G_1 - x - z_x$. We say that v has a neighbor belonging to X_i in G for any $i \in [p]$; otherwise, $G - x - z_x$ is disconnected, contrary to G is 2-matching connected. Let $Y_1, Y_2, \ldots, Y_q$ be the components of $G_2 - y$. We say that v has a neighbor belonging to Y_j in G for any $j \in [q]$; otherwise, y is a cut vertex of G. Hence $G' - z - z_x = G - z_x - x - y$ is connected, which contradicts to the fact that $st = zz_x$ is a matching cut in G'. ■

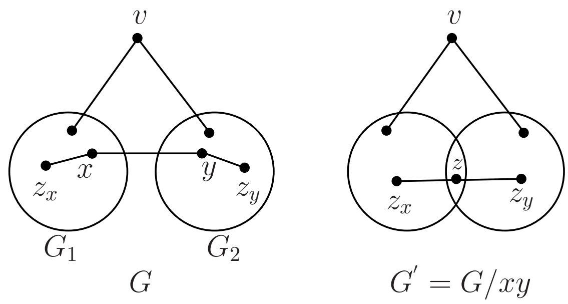

Figure 6. The illustration of Lemma 19.

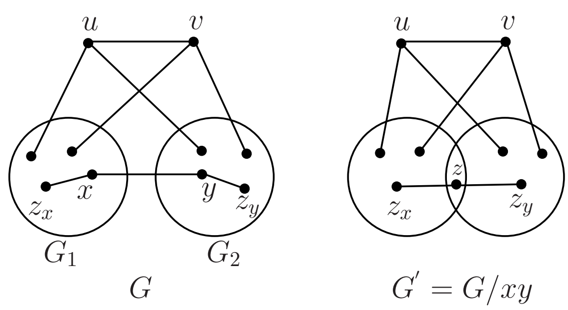

Figure 7. The illustration of Claim 21 of Lemma 20.

Lemma 20. *For any* $xy \in E(G), uv \in F(xy)$, $G[\{u,v,x,y\}] \cong C_4$.

Proof. Since $G-u-v$ is connected and $G-xy-u-v$ is disconnected, we suppose that $G-xy-u-v$ has two components G_1 and G_2 with $x \in V(G_1), y \in V(G_2)$. The following Claims 21–23 hold.

Claim 21. $E_G[\{u,v\},\{x,y\}] \neq \emptyset$.

Proof. Suppose, by way of contradiction, that $E_G[\{u,v\},\{x,y\}] = \emptyset$. Now consider the graph $G' := G/xy$ and denote by z the vertex resulting from the contraction of $xy \in E(G)$ (see Figure 7). By Lemma 15, $\kappa_M(G') = 1$. Let st be a matching cut of G'. Then $z \in \{s,t\}$; otherwise $G'-s-t = (G-s-t)/xy$ is connected by Theorem 6. Without loss of generality, we suppose that $zz_x = st$ with $z_x \in N_G(x)$. Let $X_1, X_2, \ldots, X_p$ be the components of $G_1 - x - z_x$. Then $N_G(X_i) \cap \{u,v\} \neq \emptyset$ for any $i \in [p]$; otherwise, $G-x-z_x$ is disconnected, a contradiction. Suppose that $Y_1, Y_2, \ldots, Y_q$ are the components of $G_2 - y$. Then $N_G(Y_j) \cap \{u,v\} \neq \emptyset$ for any $j \in [q]$; otherwise, y is a cut vertex of G. Furthermore, $uv \in E(G - z_x - x - y)$, thus $G' - z - z_x = G - z_x - x - y$ is connected, which contradicts to the fact that $st = zz_x$ is a matching cut in G'. □

Claim 22. (i) *If* $uy \in E(G)$*, then* $N_{G_2}(u) = y$.
(ii) *If* $ux \in E(G)$*, then* $N_{G_1}(u) = x$.
(iii) *If* $vy \in E(G)$*, then* $N_{G_2}(v) = y$.
(iv) *If* $vx \in E(G)$*, then* $N_{G_1}(v) = x$.

Proof. Since the proofs are similar, we only show that (i) holds. If not, that is, u has another neighbor, say z, belonging to G_2 in G, then uP_{zy} is a path connecting u and y, where P_{zy} is a (z, y)-path in G_2. It follows that $V_{in}(uP_{zy}) \subseteq V(G_2) \setminus y$. We say that $N_G(u) \cap V(G_1) \neq \emptyset$; otherwise, v is a cut vertex in $G - xy$, which contradicts to Lemma 19. Thus u has a neighbor, say w, belonging to G_1 in G, then $uP_{wx}y$ is a path connecting u and y, where P_{wx} is a (w, x)-path in G_1. It follows that $V_{in}(uP_{wx}y) \subseteq V(G_1)$. Thus $E_G[V_{in}(uP_{zy}), V_{in}(uP_{wx}y)] = \emptyset$, which contradict to Lemma 18. So Claim 22 holds. □

Claim 23. (i) $uy \in E(G)$ *if and only if* $vx \in E(G)$.
(ii) $ux \in E(G)$ *if and only if* $vy \in E(G)$.

Proof. Since the proofs are similar, we only show that (i) holds. Suppose, by way of contradiction, that $uy \in E(G)$ and $vx \notin E(G)$. If $N_{G_i}(v) = \emptyset$ for some $i \in [2]$, then u is a cut vertex in $G-xy$, which contradicts to Lemma 14. Thus $N_{G_i}(v) \neq \emptyset$ for any $i \in [2]$. Similarly $N_{G_i}(u) \neq \emptyset$ for any $i \in [2]$. Let P_{uv} be a (u, v)-path in G whose all internal vertices belong to G_1 and let P'_{uv} be a (u, v)-path in G whose all internal vertices belong to G_2. Since $E_G[V_{in}(P_{uv}), V_{in}(P'_{uv})] = \{xy\}$, only xy satisfies the requirement of bad edges with respect to uv by Lemma 14. It follows from Lemma 15 that $F(uv) \neq \emptyset$. Thus $F(uv) = xy$.

Let $X_1, X_2, \ldots, X_t$ be the components of $G_1 - x$, see Figure 8. If $\{u, v\} \subseteq N_G(X_i)$ for some $i \in [t]$, then there exists a (u, v)-path Q_{uv} whose all internal vertices belong to X_i such that $V(Q_{uv}) \cap \{x, y\} = \emptyset$; which contradicts to Lemma 14 and the fact that $F(uv) = xy$. Thus $|N_G(X_i) \cap \{u, v\}| \leq 1$ for each $i \in [t]$.

Let $P_{uy} = P'_{uv} + uv - uy$ be the (u, y)-path whose all internal vertex belong to $G_2 \cup \{v\}$. If $ux \in E(G)$, then uxy and P_{uy} is a pair of internal disjoint (u, y)-paths such that $E_G[V_{in}(uxy), V_{in}(P_{uy})] = \emptyset$, which is contradict to Lemma 18. Thus $ux \notin E(G)$. Pick $u_1 \in N_G(u) \setminus x$, without loss of generality, suppose $u_1 \in X_1$ and let P_{u_1x} be the (u_1, x)-path whose all internal vertex belong to X_1. It follows from $|N_G(X_i) \cap \{u, v\}| \leq 1$ that $N_G(X_1) \cap \{u, v\} = u$. Thus $uP_{u_1x}y$ and P_{uy} is a pair of internal disjoint (u, y)-paths such that $E_G[V_{in}(uP_{u_1x}y), V_{in}(P_{uy})] = \emptyset$, contrary to Lemma 18. So Claim 23 holds. □

By Claims 21–23, we have $G[\{u, v, x, y\}] \cong C_4$ or $G[\{u, v, x, y\}] \cong K_4$. Since $n > 5$, at least one of G_1 and G_2 is nontrivial, say G_1. If $G[\{u, v, x, y\}] \cong K_4$, then $N_{G_1}(v) = x$ by Claim 22. Thus ux is a matching cut of G, which contradicts to G is 2-matching connected. So $G[\{u, v, x, y\}] \cong C_4$, and Lemma 20 holds. ■

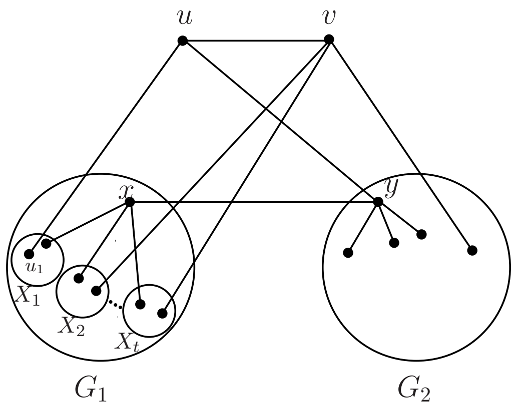

Figure 8. The illustration of Claim 23.

Proof of Theorem 7. Let T be the set of all pairs (e, e') satisfying $e \in E(G)$ and $e' \in F(e)$, and let $G_1(e, e'), G_2(e, e')$ be the components of $G - e - V(e')$ for a pair $(e, e') \in T$. Let $\xi(e, e') = \min\{|V(G_1(e, e'))|, |V(G_2(e, e'))|\}$ and $\xi(G) = \min\{\xi(e, e') : (e, e') \in T\}$.

Pick $(xy, uv) \in T$ with $\xi(xy, uv) = \xi(G)$. Let G_1, G_2 be the components of $G - xy - u - v$ such that $x \in V(G_1), y \in V(G_2)$, and $|V(G_1)| = \xi(G)$. By Lemma 20, we can assume that $uy, vx \in E(G)$ and $ux, vy \notin E(G)$. It follows from Lemma 17 that x has a neighbor x' in G_1.

Pick $st \in F(xx')$, and let G_1', G_2' be two components of $G - xx' - s - t$ such that $x' \in V(G_1'), x \in V(G_2')$. By Lemma 20, either $st \in \{uv, uy\}$ or $\{s, t\} \subseteq (V(G_1) \setminus \{x, x'\}) \cup \{u\}$.

If $st = uv$, then $|V(G_1')| \leq |V(G_1)| - 1 < |V(G_1)|$ since $V(G_1') \subseteq V(G_1) \setminus \{x\}$, which contradicts to the fact that $\xi(xx', st) < \xi(G)$. If $st = uy$, then $|V(G_1')| \leq |V(G_1)| - 1 < |V(G_1)|$ since $V(G_1') \subseteq V(G_1) \setminus \{x\}$, which contradicts to the fact that $\xi(xx', st) < \xi(G)$. If $\{s, t\} \subseteq (V(G_1) \setminus \{x, x'\}) \cup \{u\}$, then $|V(G_1')| \leq |V(G_1)| - 2 < |V(G_1)|$ since $V(G_1') \subseteq (V(G_1) \cup \{u\}) \setminus \{x, s, t\}$, which contradicts to the fact that $\xi(xx', st) < \xi(G)$. Thus G does not exist, and Theorem 7 is true. ■

4. CONCLUSION

In this paper, we give a constructive characterization of 2-matching connected graphs. For an integer k with $k \geq 3$, is this a constructive characterization of k-matching connected graphs?

Fix a positive integer k, an edge $e \notin E(G)$ is an *addible edge* with respect to a k-matching connected graph G if $G + e$ is still k-matching connected, a *non-addible edge* otherwise. We conclude this paper with the following possible problem.

Problem 24. Characterizing the class of k-matching connected graphs without addible edges.

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