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STAR-CRITICAL RAMSEY NUMBERS AND REGULAR RAMSEY NUMBERS FOR STARS

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Abstract

Let G be a graph, H be a subgraph of G, and let $G - H$ be the graph obtained from G by removing a copy of H. Let $K_{1,n}$ be the star on $n+1$ vertices. Let $t \geq 2$ be an integer and $H_1, \ldots, H_t$ and H be graphs, and let $H \to (H_1, \ldots, H_t)$ denote that every t coloring of $E(H)$ yields a monochromatic copy of H_i in color i for some $i \in [t]$. The Ramsey number $r(H_1, \ldots, H_t)$ is the minimum integer N such that $K_N \to (H_1, \ldots, H_t)$. The star-critical Ramsey number $r_*(H_1, \ldots, H_t)$ is the minimum integer k such that $K_N - K_{1,N-1-k} \to (H_1, \ldots, H_t)$ where $N = r(H_1, \ldots, H_t)$. Let $rr(H_1, \ldots, H_t)$ be the regular Ramsey number for $H_1, \ldots, H_t$, which is the minimum integer r such that if G is an r-regular graph on $r(H_1, \ldots, H_t)$ vertices, then $G \to (H_1, \ldots, H_t)$. Let $m_1, \ldots, m_t$ be integers larger than one, exactly k of which are even. In this paper, we prove that if $k \geq 2$ is even, then $r_*(K_{1,m_1}, \ldots, K_{1,m_t}) = \sum_{i=1}^{t} m_i - t + 1 - \frac{k}{2}$ which disproves a conjecture of Budden and DeJonge in 2022. Furthermore, we prove that

$$rr(K_{1,m_1}, \ldots, K_{1,m_t}) = \begin{cases} \sum_{i=1}^{t} m_i - t, & k \geq 2 \text{ is even}, \\ \sum_{i=1}^{t} m_i - t + 1, & \textit{otherwise}. \end{cases}$$

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1. Introduction

Let $V(G)$ and $E(G)$ be the vertex set and the edge set of G, respectively. Let $K_{1,n}$ be the star on $n+1$ vertices. Let $t \geq 2$ be an integer and $H, H_1, \ldots, H_t$ be graphs, and let $H \to (H_1, \ldots, H_t)$ denote that every t coloring of $E(H)$ yields a monochromatic copy of H_i in color i for some $i \in [t]$. The Ramsey number $r(H_1, \ldots, H_t)$ is the minimum integer N such that $K_N \to (H_1, \ldots, H_t)$. In 1972, Harary [5] determined the value of $r(K_{1,n}, K_{1,m})$. And Burr and Roberts extended it.

Theorem 1 (Burr and Roberts [3]). *If $m_1, \ldots, m_t$ are integers larger than one, exactly k of which are even, then*

$$r(K_{1,m_1}, \ldots, K_{1,m_t}) = \begin{cases} \sum_{i=1}^{t} m_i - t + 1, & k \geq 2 \text{ is even,} \\ \sum_{i=1}^{t} m_i - t + 2, & \text{otherwise.} \end{cases}$$

Let G be a graph, H be a subgraph of G, and $G - H$ be the graph obtained from G by removing a copy of H, i.e., $V(G-H) = V(G)$ and $E(G-H) = E(G) - E(H)$. In 2011, Hook and Isaak [6] introduced the star-critical Ramsey number $r_*(H_1, \ldots, H_t)$ which is the minimum integer k such that $K_N - K_{1,N-1-k} \to (H_1, \ldots, H_t)$ where $N = r(H_1, \ldots, H_t)$. In 2022, Budden and DeJonge considered the star-critical Ramsey number for stars and conjectured the following.

Conjecture 2 (Budden and DeJonge [2]). *If $m_1, \ldots, m_t$ are integers larger than one, exactly k of which are even, then*

$$r_*(K_{1,m_1}, \ldots, K_{1,m_t}) = \begin{cases} \sum_{i=1}^{t} m_i - t, & k \geq 2 \text{ is even,} \\ 1, & \text{otherwise.} \end{cases}$$

They proved their conjecture for all cases except that $k > 2$ is even.

For a positive integer r, call a graph G r-regular graph if every vertex of $V(G)$ has degree exactly r. Let $rr(H_1, \ldots, H_t)$ be the regular Ramsey number for $H_1, \ldots, H_t$, which is the minimum integer r such that if G is an r-regular graph on $N = r(H_1, \ldots, H_t)$ vertices, then $G \to (H_1, \ldots, H_t)$. The following holds by the definition of Ramsey number, star-critical Ramsey number and regular Ramsey number.

Fact 3. $1 \leq r_*(H_1, \ldots, H_t) \leq rr(H_1, \ldots, H_t) \leq r(H_1, \ldots, H_t) - 1$.

In this paper, we first disprove Conjecture 2 for the remaining cases by proving the following.

Theorem 4. *Let $m_1, \ldots, m_k$ be even integers and let $m_{k+1}, \ldots, m_t$ be odd integers larger than one. If $k \geq 2$ is even, then*

$$r_*(K_{1,m_1}, \ldots, K_{1,m_t}) = \sum_{i=1}^{t} m_i - t + 1 - \frac{k}{2}.$$

Then we consider the regular Ramsey number for stars.

Theorem 5. *Let $m_1, \dots, m_t$ be integers larger than one, exactly k of which are even. Then*

$$rr(K_{1,m_1}, \dots, K_{1,m_t}) = \begin{cases} \sum_{i=1}^{t} m_i - t, & k \geq 2 \text{ is even,} \\ \sum_{i=1}^{t} m_i - t + 1, & \text{otherwise.} \end{cases}$$

Notations and definitions. Let $G \cup H$ be the union of vertex-disjoint copies of G and H. A matching of G is a 1-regular subgraph of G, and a maximum matching of G is a matching with the maximum size taken over all matching of G. For a positive integer f, an f-factor of graph G is an f-regular subgraph of G on $V(G)$.

2. Star-Critical Ramsey Number for Stars

We first introduce a decomposition of a complete graph by Harary in 1969.

Theorem 6 (Harary [4]). *K_{2n} can be decomposed into $(2n-1)$ edge-disjoint 1-factors, and K_{2n+1} can be decomposed into n edge-disjoint 2-factors.*

There is a stronger result for the complete graph on odd vertices.

Theorem 7 (Lucas [7], Bollabás [1]). *K_{2n+1} can be decomposed into n edge-disjoint Hamiltonian cycles.*

Corollary 8. *Let n be a positive integer. If n is even, then for all $r \in [n-1]$, there is an r-regular graph on n vertices. If n is odd, then for all even $r \in [n-1]$, there is an r-regular graph on n vertices.*

Corollary 9. *Let n be a positive integer, $s \leq n-1$ be a positive integer, and G be a graph on n vertices without $K_{1,s}$. Then the following holds. If n is odd and s is even, then $e(G) \leq \frac{1}{2}[(s-1)n-1]$. Otherwise, $e(G) \leq \frac{1}{2}(s-1)n$. Furthermore, the upper bound is the best.*

Proof. Since G is $K_{1,s}$-free, every vertex has degree at most $s-1$. And thus, $e(G) \leq \frac{1}{2}(s-1)n$. If n is odd and s is even, then there is no $(s-1)$-regular graph on n vertices since the sum of the degree of each graph is even. By Corollary 8, there exists an $(s-2)$-regular graph on n vertices. Thus, at most $n-1$ vertices have degree $s-1$ and at least one vertex has degree $s-2$. Consequently, $e(G) \leq \frac{1}{2}[(s-1)(n-1)+s-2]$.

If n is odd and s is even, then by Theorem 7, let C be a Hamiltonian cycle on n vertices and, let H' be an $(s-2)$-regular graph on n vertices such that C and H' are edge-disjoint. Let C' be a maximum matching of C and let $H = H' \cup C'$.

Note that H is a graph on n vertices containing $n-1$ vertices with degree $s-1$ and one vertex with degree $s-2$. Thus, $e(H) = \frac{1}{2}[(s-1)(n-1)+s-2]$.

If either n is even or s is odd, then by Corollary 8, there exists an $(s-1)$-regular graph H on n vertices. Thus, $e(H) = \frac{1}{2}(s-1)n$. ∎

Now, we are ready to prove our first result.

Theorem 4. *Let $m_1, \ldots, m_k$ be even integers and let $m_{k+1}, \ldots, m_t$ be odd integers larger than one. If $k \geq 2$ is even, then*

$$r_*(K_{1,m_1}, \ldots, K_{1,m_t}) = \sum_{i=1}^{t} m_i - t + 1 - \frac{k}{2}.$$

Proof. Let $N = \sum_{i=1}^{t} m_i - t + 1$ and $r_* = r_*(K_{1,m_1}, \ldots, K_{1,m_t})$. Let $V = V(K_{N-1})$ and let v be a vertex. Let H be the graph obtained by joining v and r_* vertices of V. Color $E(H)$ with t colors arbitrarily. Let H_i be the graph induced by all edges with color i in H for every $i \in [t]$.

Note that N is odd. If H_i does not contain K_{1,m_i} for every $i \in [t]$, then by Corollary 9,

$$\begin{aligned} e(H) =& \sum_{i=1}^{k} e(G_i) + \sum_{i=k+1}^{t} e(G_i) \\ \leq& \sum_{i=1}^{k} \frac{1}{2}[(m_i-1)N-1] + \sum_{i=k+1}^{t} \frac{1}{2}(m_i-1)N \\ =& \frac{1}{2}N(N-1) - \frac{k}{2}. \end{aligned}$$

Thus, if $e(H) \geq \frac{1}{2}N(N-1) - \frac{k}{2} + 1$, then by the pigeonhole principle, there is either $i_0 \in [k]$ such that $e(H_{i_0}) \geq \frac{1}{2}[(m_i-1)N+1]$ or $i_0 \in [t]\backslash[k]$ such that $e(H_{i_0}) \geq \frac{1}{2}(m_i-1)N+1$. By Corollary 9 again, H_{i_0} contains a copy of $K_{1,m_{i_0}}$. Consequently,

$$r_* \leq \frac{1}{2}N(N-1) - \frac{k}{2} + 1 - \binom{N-1}{2} = N - \frac{k}{2}.$$

Let G be the graph obtained by joining v and $N - \frac{k}{2} - 1$ vertices of V, which will be chosen later. Let G_i be the graph induced by all edges with color i in G. We will color $E(G)$ with t colors such that G_i does not contain K_{1,m_i} for every $i \in [t]$.

By Theorem 7, K_N can be decomposed into $\frac{N-1}{2}$ edge-disjoint Hamiltonian cycles and denote them by $C_{i,j}$ where $i \in [t]$ and j satisfies the following: if $i \in [\frac{k}{2}]$, then $j \in [\frac{m_i}{2}]$; if $i \in [k]\backslash[\frac{k}{2}]$, then $j \in [\frac{m_i}{2}]\backslash[1]$; if $i \in [t]\backslash[k]$, then $j \in [\frac{m_i-1}{2}]$. For every $i \in [\frac{k}{2}]$, let $u_iv \in E(C_{i,1})$ and let $P_{i,1}$ be the graph obtained from $C_{i,1}$ by removing u_iv. Since $C_{i,1}$ is a Hamiltonian cycle, $P_{i,1}$ is a Hamiltonian path. Furthermore, since N is odd, there are two edge-disjoint maximum matchings in $P_{i,1}$ covering all but one vertex. Denote them by M_i and $M_{i+\frac{k}{2}}$, and we may assume that $v \in V(M_i)$ and $u_i \in V(M_{i+\frac{k}{2}})$. Finally, for every $i \in [k]$, let $G_i = M_i \cup \bigcup_{j=2}^{m_i/2} C_{i,j}$, and for every $i \in [t]\backslash[k]$, let $G_i = \bigcup_{j=1}^{(m_i-1)/2} C_{i,j}$.

Note that for every $i \in \left[\frac{k}{2}\right]$, all vertices of G_i have degree $m_i - 1$ except that u_i has degree $m_i - 2$; for every $i \in [k]\backslash\left[\frac{k}{2}\right]$, all vertices of G_i have degree $m_i - 1$ except that v has degree $m_i - 2$; for every $i \in [t]\backslash[k]$, all vertices of G_i have degree $m_i - 1$. Thus, for every $i \in [t]$, G_i does not contain K_{1,m_i}.

Note that $d_G(v) = N - 1 - \frac{k}{2}$ and $G[V] = K_{N-1}$. Thus, $r_* \geq N - \frac{k}{2}$ and we finish the proof. ■

3. REGULAR RAMSEY NUMBER FOR STARS

In this section, we will prove a more general result, and Theorem 5 is a direct corollary.

Theorem 10. *Let $m_1, \ldots, m_k$ be even integers, $m_{k+1}, \ldots, m_t$ be odd integers larger than one, and let $n \geq r(K_{1,m_1}, \ldots, K_{1,m_t})$ be an integer. Let $g(n)$ be the minimum integer such that if G is a $g(n)$-regular graph on n vertices, then $G \to (K_{1,m_1}, \ldots, K_{1,m_t})$. If n is odd and $k \geq 2$ is even, then $g(n) = \sum_{i=1}^{t} m_i - t$. Otherwise, $g(n) = \sum_{i=1}^{t} m_i - t + 1$.*

Proof. Note that if $\Delta(G) \geq \sum_{i=1}^{t} m_i - t + 1$, then by the pigeonhole principle, there is a monochromatic copy of K_{1,m_i} in color i for some $i \in [t]$. Thus, $g(n) \leq \sum_{i=1}^{t} m_i - t + 1$.

If n is even or n is odd and $k = 0$, then by Corollary 8 and Theorem 6, for every $i \in [t]$, there exists an $(m_i - 1)$-regular graph H_i on n vertices such that they are edge-disjoint. Note that $\bigcup_{i=1}^{t} H_i$ is a $\left(\sum_{i=1}^{t} m_i - t\right)$-regular graph such that H_i does not contain K_{1,m_i} for every $i \in [t]$. Thus, $g(n) \geq \sum_{i=1}^{t} m_i - t + 1$.

In the following, assume that n is odd and $k > 0$.

Case 1. k is odd. By Theorem 7, K_n can be decomposed into $\frac{n-1}{2}$ edge-disjoint Hamiltonian cycles and denote them by $C_{i,j}$ where $i \in [t]$ and j satisfies the following: if $i \in [\frac{k-1}{2}]$, then $j \in [\frac{m_i}{2}]$; if $i \in [k]\backslash[\frac{k-1}{2}]$, then $j \in [\frac{m_i}{2}]\backslash[1]$; if $i \in [t]\backslash[k]$, then $j \in [\frac{m_i-1}{2}]$. For every $i \in [\frac{k-1}{2}]$, let $P_{i,1}$ be the graph obtained from $C_{i,1}$ by removing an edge u_iu_{k-i} such that $\left|\bigcup_{l=1}^{(k-1)/2}\{u_l, u_{k-l}\}\right| = k-1$. Since $C_{i,1}$ is a Hamiltonian cycle, $P_{i,1}$ is a Hamiltonian path. Furthermore, since n is odd, there are two edge-disjoint maximum matchings in $P_{i,1}$ covering all but one vertex. Denote them by M_i and M_{k-i}, and we may assume that $u_{k-i} \in V(M_i)$ and $u_i \in V(M_{k-i})$. Finally, for every $i \in [k-1]$, let $G_i = M_i \cup \bigcup_{j=2}^{m_i/2} C_{i,j}$, $G_k = \left(\bigcup_{l=1}^{\frac{k-1}{2}} u_l u_{k-l}\right) \cup \bigcup_{j=2}^{m_k/2} C_{k,j}$, and for every $i \in [t]\backslash[k]$, let $G_i = \bigcup_{j=1}^{(m_i-1)/2} C_{i,j}$.

Note that for every $i \in [k-1]$, all vertices of G_i have degree $m_i - 1$ except that u_i has degree $m_i - 2$; all vertices of G_k have degree $m_k - 2$ except that u_j has degree $m_k - 1$ for every $j \in [k-1]$; for every $i \in [t]\backslash[k]$, all vertices of G_i have degree $m_i - 1$. Thus, for every $i \in [t]$, G_i does not contain K_{1,m_i}.

Consequently, $\bigcup_{i=1}^{t} G_i$ is a $(\sum_{i=1}^{t} m_i - t - 1)$-regular graph on n vertices such that $\bigcup_{i=1}^{t} G_i \not\to (K_{1,m_1}, \ldots, K_{1,m_t})$. Note that $\sum_{i=1}^{t} m_i - t$ is odd, and by Corollary 8, there does not exist a $(\sum_{i=1}^{t} m_i - t)$-regular graph on n vertices since n is odd. Consequently, $g(n) \geq \sum_{i=1}^{t} m_i - t + 1$.

Case 2. k is even. Firstly, we improve the upper bound. Otherwise, note that $\sum_{i=1}^{t} m_i - t$ is even, and by Corollary 8, there exists a $(\sum_{i=1}^{t} m_i - t)$-regular graph H on n vertices such that $H \not\to (K_{1,m_1}, \ldots, K_{1,m_t})$. Note that m_1 is even. By Corollary 8 again and the fact that n is odd, there exists a vertex $u \in V(H)$ such that there are at most $m_1 - 2$ edges adjacent to u in color 1. Thus, at least $\sum_{i=1}^{t-1} m_i - t + 2$ edges are adjacent to u in the remaining $t-1$ colors. By the pigeonhole principle, there exists $i_0 \in [t]\backslash[1]$ such that at least m_{i_0} edges are adjacent to u in color i_0. A contradiction to $H \not\to (K_{1,m_1}, \ldots, K_{1,m_t})$. Consequently, $g(n) \leq \sum_{i=1}^{t} m_i - t$.

In the following, we will prove that the equality holds. By Theorem 7, K_n can be decomposed into $\frac{n-1}{2}$ edge-disjoint Hamiltonian cycles and denote them by $C_{i,j}$ where $i \in [k]$ and j satisfies the following: if $i \in [\frac{k}{2}]$, then $j \in [\frac{m_i}{2}]$; if $i \in [k-1]\backslash[\frac{k}{2}]$, then $j \in [\frac{m_i}{2}]\backslash[1]$; if $i = k$, then $j \in [\frac{m_k}{2}]\backslash[2]$; if $i \in [t]\backslash[k]$, then $j \in [\frac{m_i-1}{2}]$. For every $i \in [\frac{k}{2}]$, let $P_{i,1}$ be the graph obtained from $C_{i,1}$ by removing an edge $u_i u_{i+\frac{k}{2}}$ such that $\left|\bigcup_{l=1}^{k/2}\{u_l, u_{l+\frac{k}{2}}\}\right| = k$. Since $C_{i,1}$ is a Hamiltonian cycle, $P_{i,1}$ is a Hamiltonian path. Furthermore, since n is odd, there are two maximum matchings in $P_{i,1}$ covering all but one vertex. Denote them by M_i and $M_{i+\frac{k}{2}}$, and we may assume that $u_{i+\frac{k}{2}} \in V(M_i)$ and $u_i \in V(M_{i+\frac{k}{2}})$. Finally, for every $i \in [k-1]$, let $G_i = M_i \cup \bigcup_{j=2}^{m_i/2} C_{i,j}$, $G_k = \left(\bigcup_{l=1}^{k/2} u_l u_{l+\frac{k}{2}}\right) \cup M_k \cup \bigcup_{j=3}^{m_k/2} C_{k,j}$, and for every $i \in [t]\backslash[k]$, let $G_i = \bigcup_{j=1}^{(m_i-1)/2} C_{i,j}$.

Note that for every $i \in [k-1]$, all vertices of G_i have degree $m_i - 1$ except that u_i has degree $m_i - 2$; all vertices of G_k have degree $m_k - 3$ except that u_j has degree $m_k - 2$ for every $j \in [k-1]$; for every $i \in [t]\backslash[k]$, all vertices of G_i have degree $m_i - 1$. Thus, for every $i \in [t]$, G_i does not contain K_{1,m_i}.

Consequently, $\bigcup_{i=1}^{t} G_i$ is a $(\sum_{i=1}^{t} m_i - t - 2)$-regular graph on n vertices such that $\bigcup_{i=1}^{t} G_i \not\to (K_{1,m_1}, \ldots, K_{1,m_t})$. Note that $\sum_{i=1}^{t} m_i - t - 1$ is odd, and by

Corollary 8, there does not exist a $(\sum_{i=1}^{t} m_i - t - 1)$-regular graph on n vertices since n is odd. Consequently, $g(n) \geq \sum_{i=1}^{t} m_i - t$.

All cases have been discussed and we finish the proof. ■

4. Remark

In [8], Schelp asked that for graphs $H_1, \ldots, H_t$, if G is a graph on n vertices with $\delta(G) \geq cn$ such that $G \to (H_1, \ldots, H_t)$, then how large should c be? Let $f(n)$ be the minimum integer such that if G is a graph on n vertices with $\delta(G) \geq f(n)$, then $G \to (H_1, \ldots, H_t)$.

Theorem 11. *Let $m_1, \ldots, m_k$ be even integers, and let $m_{k+1}, \ldots, m_t$ be odd integers larger than one. If $n \geq r(K_{1,m_1}, K_{1,m_2}, \ldots, K_{1,m_t})$, then*

$$f(n) = \begin{cases} \sum_{i=1}^{t} m_i - t + 1, & k = 0 \text{ or } n \text{ is even}, \\ \sum_{i=1}^{t} m_i - t, & \text{otherwise}. \end{cases}$$

The proof of Theorem 11 is the same as the proof of Theorem 10, and we remove it.

The motivation to study regular Ramsey numbers is the following. Let us think of edges in a graph as the resources we need. A complete graph (Ramsey number) is easy to construct but needs many resources. A graph with minimum degree condition (Schelp's problem) needs fewer resources but difficult to construct. And a regular graph (regular Ramsey number) is easier to construct than a graph with minimum degree condition since we can construct by Theorem 6 and Theorem 7, and the resource is less than a complete graph. Thus, the regular Ramsey number has vast potential applications, such as coding or other fields. Under the limitation of knowledge, we are not able to apply it.

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