

## THE DIRECTED UNIFORM HAMILTON-WATERLOO PROBLEM INVOLVING EVEN CYCLE SIZES

FATİH YETGİN<sup>1</sup>, UĞUR ODABAŞI<sup>2</sup>

AND

SIBEL ÖZKAN<sup>1</sup>

<sup>1</sup>*Department of Mathematics  
Gebze Technical University  
Kocaeli, 41400, Turkey*

<sup>2</sup>*Department of Engineering Sciences  
Istanbul University-Cerrahpasa  
Istanbul, 34320, Turkey*

**e-mail:** fyetgin@gtu.edu.tr  
ugur.odabasi@iuc.edu.tr  
s.ozkan@gtu.edu.tr

### Abstract

In this paper, factorizations of the complete symmetric digraph  $K_v^*$  into uniform factors consisting of directed even cycle factors are studied as a generalization of the undirected Hamilton-Waterloo Problem. It is shown, with a few possible exceptions, that  $K_v^*$  can be factorized into two nonisomorphic factors, where these factors are uniform factors of  $K_v^*$  involving  $K_2^*$  or directed  $m$ -cycles, and directed  $m$ -cycles or  $2m$ -cycles for even  $m$ .

**Keywords:** The Directed Hamilton-Waterloo Problem, 2-factorizations, directed cycle factorizations.

**2020 Mathematics Subject Classification:** 05C51, 05C70.

### 1. INTRODUCTION

In this paper, edges and arcs are denoted by using curly braces and parentheses, respectively. Throughout this paper, we denote by  $K_{(x:y)}$  a complete equipartite graph having  $y$  parts of size  $x$  each. Also, for a simple graph  $G$ , we use  $G^*$  to denote the symmetric digraph with vertex set  $V(G^*) = V(G)$  and arc set

$E(G^*) = \bigcup_{\{x,y\} \in E(G)} \{(x,y), (y,x)\}$ . Hence,  $K_v^*$  and  $K_{(x:y)}^*$  respectively denote the complete symmetric digraph of order  $v$  and the complete symmetric equipartite digraph with  $y$  parts of size  $x$ . We also use  $(x,y)^*$  to denote the double arc which consists of  $(x,y)$  and  $(y,x)$ .

A  $k$ -factor of a graph  $G$  is a  $k$ -regular spanning subgraph of  $G$ . A  $k$ -factorization of a graph  $G$  is a partition of the edge set of  $G$  into  $k$ -factors; in other words, it is a partition (decomposition) of the edge set of  $G$  into edge-disjoint  $k$ -factors. It is easy to see that a 2-factor consists of an Hamilton cycle, which is a cycle that visits each vertex exactly, or union of vertex-disjoint cycles. If a 2-factor consists only of cycles (directed cycles) of length  $m$ , it is called a  $C_m$ -factor ( $\vec{C}_m$ -factor). Furthermore, in the special case where  $m = 2$ , this factor becomes a  $K_2$ -factor. There are two well-studied 2-factorization problems. The Oberwolfach Problem asks for the existence of a decomposition of  $K_{2v+1}$  or  $K_{2v} - I$  (i.e.,  $K_{2v}$  with the edges of the 1-factor  $I$  removed) into copies of a given 2-factor  $F$ . The uniform version of the Oberwolfach Problem in which there is only one type of cycle in the factor  $F$  has been mostly solved, see [4, 5, 24, 31]. In the Hamilton-Waterloo Problem, there are two types of 2-factors. The uniform version of the Hamilton-Waterloo Problem asks for a 2-factorization of  $K_v$  (or for even  $v$ , 2-factorization of  $K_v - I$ ) in which  $r$  of its 2-factors consist of only  $m$ -cycles and the remaining  $s$  of its 2-factors consist of only  $n$ -cycles, and we will denote it by  $\text{HWP}(v; m^r, n^s)$ . Any of its solutions will be referred to as a  $\{C_m^r, C_n^s\}$ -factorization of  $K_v$  (or  $K_v - I$  for even  $v$ ).

Initially, small cases such as  $(m, n) \in \{(4, 6), (4, 8), (4, 16), (8, 16), (3, 5), (3, 15), (5, 15)\}$  are studied and solved with a few exceptions by Adams *et al.* [2], and later the cases where the cycle sizes are non-constant are investigated. The Hamilton-Waterloo Problem is nearly completely solved when both  $m$  and  $n$  are simultaneously either even or odd [10, 11, 15, 16]. When the parity of  $m$  and  $n$  is different, one of the cycle sizes is usually fixed. For instance, the cases  $(m, n) \in \{(3, v), (3, 3x), (4, n)\}$  have been studied, see [6, 20, 25, 29]. For more recent results on this problem, we refer the reader to [12, 13, 14]. Also, there exists an asymptotic solution (for sufficiently large  $v$ ) [21] for the general form of the Oberwolfach and the Hamilton-Waterloo Problems. However, this asymptotic solution does not provide an explicit lower bound that guarantees the solvability of the problem. In [34], Traetta constructs solutions to the Oberwolfach Problem whenever  $F$  contains a cycle of length greater than an explicit lower bound.

The concept of factor and factorization can be applied to digraphs and one can consider the directed version of the Oberwolfach and Hamilton-Waterloo Problems. In the directed version of these problems, factorization of the complete symmetric digraph  $K_v^*$  into directed cycle factors is studied. The Directed Uniform Oberwolfach Problem is denoted by  $\text{OP}^*(m^k)$  where each 2-factor is composed of  $k$  directed  $m$ -cycles.

The following theorem summarizes the previous results on the Directed Oberwolfach Problem that will be used in this paper.

**Theorem 1** [1, 3, 7, 9, 17, 18, 26, 33]. *Let  $m$  and  $k$  be nonnegative integers. Then,  $\text{OP}^*(m^k)$  has a solution if and only if  $(m, k) \notin \{(3, 2), (4, 1), (6, 1)\}$ .*

The directed Oberwolfach Problem for complete symmetric equipartite digraphs and uniform-length cycles was solved by Francetić and Šajna in [19].

When it comes to the Directed Hamilton-Waterloo Problem, here  $K_v^*$  is decomposed into two types of directed 2-factors. If these factors consist of directed cycles of sizes  $m$  and  $n$ , respectively, the notation  $\text{HWP}^*(v; m^r, n^s)$  is used to denote the Directed Uniform Hamilton-Waterloo Problem.

In [35], the necessary conditions for the existence of a solution to the Directed Hamilton-Waterloo Problem are given.

**Lemma 2** [35]. *If  $\text{HWP}^*(v; m^r, n^s)$  has a solution, then the following statements hold:*

- (i) *if  $r > 0$ ,  $v \equiv 0 \pmod{m}$ ,*
- (ii) *if  $s > 0$ ,  $v \equiv 0 \pmod{n}$ ,*
- (iii)  *$r + s = v - 1$ .*

Additionally, the cases  $(m, n) \in \{(3, 5), (3, 15), (5, 15), (4, 6), (4, 8), (4, 12), (4, 16), (6, 12), (8, 16)\}$  are solved with a few possibly exceptions in [35].

In [30], factorizations of  $K_v$  into  $K_2$ -factors and  $C_m$ -factors are studied, and also new solutions to  $\text{HWP}(2m; m^r, (2m)^s)$  are given. Here, the problem of decomposing  $K_v^*$  into  $K_2^*$ -factors and  $\vec{C}_m$ -factors will be examined where  $\vec{C}_m$  is the directed cycle of order  $m$ . Since  $K_2^*$  can be considered as  $\vec{C}_2$ , this problem can be included in the  $\text{HWP}^*(v; 2^r, m^s)$ . Afterwards,  $\text{HWP}^*(v; m^r, (2m)^s)$  will be studied.

In Section 2, we give some basic definitions and present some preliminary results that will be used in the next sections. In Section 3, we focus on finding solutions to  $\text{HWP}^*(v; 2^r, m^s)$  for even  $m$  with  $r + s = v - 1$ . Also a solution is denoted as a  $\{(K_2^*)^r, \vec{C}_m^s\}$ -factorization of  $K_v^*$ . In Section 4, we will concentrate on solving  $\text{HWP}^*(v; m^r, (2m)^s)$  for even  $m$  with  $r + s = v - 1$ . Here are our main results.

**Theorem 3.** *Let  $r, s$  be nonnegative integers, and let  $m \geq 4$  be even. Then  $\text{HWP}^*(v; 2^r, m^s)$  has a solution if  $m|v$ ,  $r + s = v - 1$ ,  $s \neq 1$ ,  $(r, v) \neq (0, 6)$ ,  $(m, r, v) \neq (4, 0, 4)$ , and one of the following conditions holds:*

- (i)  *$m > 4$ ,  $s \neq 3$  and  $m \equiv 0 \pmod{4}$ ,*
- (ii)  *$m > 4$ ,  $m \equiv 2 \pmod{4}$ , and  $s \neq 3$  when  $\frac{v}{m}$  is odd,*
- (iii)  *$m = 4$  and  $v \equiv 0, 8, 16 \pmod{24}$ ,*

- (iv)  $m = 4$ ,  $v \equiv 12 \pmod{24}$  and  $s \notin \{3, 5\}$ ,
- (v)  $m = 4$ ,  $v \equiv 4, 20 \pmod{24}$  and  $r$  is odd.

**Theorem 4.** *Let  $r, s$  be nonnegative integers, and let  $m \geq 4$  be even. Then  $\text{HWP}^*(v; m^r, (2m)^s)$  has a solution if and only if  $m|v$ ,  $r + s = v - 1$  and  $v \geq 4$  except for  $(s, v, m) \in \{(0, 4, 4), (0, 6, 3), (0, 6, 6)\}$ , and except possibly when  $s \in \{1, 3\}$ .*

## 2. PRELIMINARY RESULTS

First, let us start with some definitions and notations that we will use throughout the paper.

Let  $G$  be a graph and  $G_0, G_1, \dots, G_{k-1}$  be  $k$  vertex disjoint copies of  $G$  with  $v_i \in V(G_i)$  for each  $v \in V(G)$ . Let  $G[k]$  denote the graph with vertex set  $V(G[k]) = V(G_0) \cup V(G_1) \cup \dots \cup V(G_{k-1})$  and edge set  $E(G[k]) = \{\{u_i, v_j\} : \{u, v\} \in E(G) \text{ and } 0 \leq i, j \leq k-1\}$ . It is easy to see that there is an  $H[k]$ -factorization of  $G[k]$  if the graph  $G$  has an  $H$ -factorization. Note that  $K_{(x:y)} \cong K_y[x]$ .

If  $G_1$  and  $G_2$  are two edge-disjoint graphs with  $V(G_1) = V(G_2)$ , then we use  $G_1 \oplus G_2$  to denote the graph on the same vertex set with  $E(G_1 \oplus G_2) = E(G_1) \cup E(G_2)$ . We will denote the vertex disjoint union of  $\alpha$  copies of  $G$  by  $\alpha G$ .

The above definitions can be extended to digraphs. Let  $D$  be a digraph and  $D_0, D_1, \dots, D_{k-1}$  be  $k$  vertex disjoint copies of  $D$  with  $v_i \in V(D_i)$  for each  $v \in V(D)$ . Then  $D[k]$  has the vertex set  $V(D[k]) = V(D_0) \cup V(D_1) \cup \dots \cup V(D_{k-1})$  and arc set  $E(D[k]) = \{(u_i, v_j) : (u, v) \in E(D) \text{ and } 0 \leq i, j \leq k-1\}$ .

Let  $G$  be a digraph and  $R(G)$  denote the digraph on the same vertex set as  $G$  but the arcs are taken in opposite directions.

Let us define some special factors and cycles that will be used throughout this article. Let  $F_m$  be a 1-factor of  $K_m$  with edge set  $E(F_m) = \{\{0, m/2\}, \{i, m-i\} : 1 \leq i \leq (m/2) - 1\}$  and let  $\mathcal{C} = (0, 1, 2, m-1, 3, m-2, \dots, \frac{m}{2}-1, \frac{m}{2}+2, \frac{m}{2}, \frac{m}{2}+1)$  be an  $m$ -cycle, which are the same as in Wlacki's construction. Also, we define a factor  $F_m^*$  as a  $K_2^*$ -factor of  $K_m^*$  with  $E(F_m^*) = \{(0, m/2)^*, (i, m-i)^* : 1 \leq i \leq (m/2) - 1\}$  and  $\mathcal{C}^*$  is the symmetric version of the  $\mathcal{C}$ .

Using the above factors and cycles, we can define  $\Gamma_m$  and  $\Gamma_m^*$  as  $\mathcal{C}[2] \oplus F_m[2]$  and  $\mathcal{C}^*[2] \oplus F_m^*[2]$ , respectively. We use these notations for the rest of the paper.

Let  $A$  be a finite additive group and let  $S$  be a subset of  $A$ , where  $S$  does not contain the identity of  $A$ . The Directed Cayley graph  $\vec{X}(A; S)$  on  $A$  with connection set  $S$  is a digraph with  $V(\vec{X}(A; S)) = A$  and  $E(\vec{X}(A; S)) = \{(x, y) : x, y \in A, y - x \in S\}$ .

Let  $m$  be an even integer and the vertex set of  $K_{2m}^*$  be  $\mathbb{Z}_{2m}$ . Let  $I_{2m}^*$  be a  $K_2^*$ -factor of  $K_{2m}^*$  with  $E(I_{2m}^*) = \{(i, m+i)^* : 0 \leq i \leq m-1\}$  and define the bijective function  $f : \mathbb{Z}_{2m} \rightarrow \mathbb{Z}_2 \times \mathbb{Z}_m$  with

$$f(i) = \begin{cases} (0, i) & \text{if } i < m, \\ (1, i) & \text{if } i \geq m. \end{cases}$$

Then  $E(I_{2m}^*)$  can be restated as a set  $\{(0, i), (1, i)^* : 0 \leq i \leq m-1\}$  on  $\mathbb{Z}_2 \times \mathbb{Z}_m$  using this bijective function.

We will represent  $C_m^*[2]$  and  $C_m^*[2] \oplus I_{2m}^*$  as the directed Cayley graphs  $\vec{X}(\mathbb{Z}_2 \times \mathbb{Z}_m, S)$  and  $\vec{X}(\mathbb{Z}_2 \times \mathbb{Z}_m, S \cup \{(1, 0)\})$  where  $S = \{(0, 1), (1, 1), (0, -1), (1, -1)\}$ .

Also, the arc set of  $F_m^*$  which is denoted by  $E(F_m^*)$ , can be expressed as  $\{(0, 0), (0, m/2)^*, ((0, i), (0, m-i))^* : 1 \leq i \leq (m/2) - 1\}$  using above bijective function. Thus, we can represent the vertex set and the edge set of  $\Gamma_m^*$  as  $V(\Gamma_m^*) = \mathbb{Z}_2 \times \mathbb{Z}_m$  and  $E(\Gamma_m^*) = \bigcup_{j=0}^{m-1} \{((i, j), (i, j+1))^*, ((i, j), (i+1, j+1))^*\} \cup E(F_m^*)$  for  $i = 0, 1$ , respectively.

Häggkvist used  $G[2]$  to build 2-factorizations that include even cycles [22].

**Lemma 5** (Häggkvist Lemma). *Let  $G$  be a path or a cycle with  $n$  edges and let  $H$  be a 2-regular graph on  $2n$  vertices with all components even cycle. Then  $G[2] \cong G' \oplus G''$  where  $G' \cong G'' \cong H$ . Therefore,  $G[2]$  has an  $H$ -decomposition.*

The following proposition, which is useful for transferring the results of undirected graphs to digraph and symmetric digraph, states that if we have an  $H$ -factorization of the undirected graph  $G$ , then using this factorization an  $H^*$ -factorization of  $G^*$  can be obtained.

**Proposition 6.** *Let  $G$  be a graph and let  $H$  be a subgraph of  $G$ . If  $G$  has an  $H$ -factorization, then  $G^*$  has an  $H^*$ -factorization.*

It is known that  $K_{2x}$  has a 1-factorization [32]. Therefore, as a natural consequence of Proposition 6, the following proposition can be stated.

**Proposition 7.** *The complete symmetric digraph  $K_{2x}^*$  has a  $K_2^*$ -factorization for every integer  $x \geq 1$ .*

The following result of Liu on equipartite graph has been helpful in solving the Oberwolfach and Hamilton-Waterloo Problems. We will use this result to obtain a  $\vec{C}_m$ -factorization of  $K_{(x:y)}^*$ .

**Theorem 8** [27]. *The complete equipartite graph  $K_{(x:y)}$  has a  $C_m$ -factorization for  $m \geq 3$  and  $x \geq 2$  if and only if  $m \mid xy$ ,  $x(y-1)$  is even,  $m$  is even if  $y = 2$  and  $(x, y, m) \neq (2, 3, 3), (6, 3, 3), (2, 6, 3), (6, 2, 6)$ .*

The necessary and sufficient condition for the existence of a 1-factorization of a complete equipartite graph  $K_{(x:y)}$  is given by Hoffman and Rodger [23].

**Theorem 9** [23]. *The complete equipartite graph  $K_{(x:y)}$  has a 1-factorization if and only if  $xy$  is even.*

The following lemma is a straightforward consequence of Proposition 6 and Theorem 9.

**Lemma 10.** *The complete symmetric equipartite digraph  $K_{(x:y)}^*$  has a  $K_2^*$ -factorization if and only if even  $xy$ .*

The following two well-known results of Walecki imply that  $K_m$  (a 1-factor removed graph of  $K_m$  when  $m$  is even) decomposes into Hamilton cycles. We will use these results and Proposition 6 to factorize  $K_m^*$  into symmetric Hamilton cycles in Section 3.

**Lemma 11** [28]. *For all odd  $m \geq 3$ ,  $K_m$  decomposes into  $(\frac{m-1}{2})$  Hamilton cycles.*

**Lemma 12** [28]. *For all even  $m \geq 4$ ,  $K_m - F_m$  has an Hamilton cycle decomposition with prescribed cycles  $\{\mathcal{C}, \sigma(\mathcal{C}), \sigma^2(\mathcal{C}), \dots, \sigma^{\frac{m-4}{2}}(\mathcal{C})\}$  for  $\sigma = (0)(1, 2, 3, \dots, m-2, m-1)$ .*

Lemmata 13 and 14 show the existence of the  $\{C_m^r, C_{2m}^s\}$ -factorization of the  $C_m[2]$  and  $(\mathcal{C} \oplus F_m)[2]$  for  $r + s = 2$  and  $r + s = 3$ , respectively. They will be used to find a  $\{(K_2^*)^r, \vec{C}_m^s\}$ -factorization of the  $C_m^*[2]$  for  $r \in \{0, 2, 4\}$ ,  $r + s = 4$  and a  $\vec{C}_{2m}$ -factorization of  $\Gamma_m^* = C^*[2] \oplus F_m^*[2]$ .

**Lemma 13** [30]. *Let  $m$  be an integer with  $m \geq 3$ . Then  $C_m[2]$  has a  $\{C_m^r, C_{2m}^s\}$ -factorization for nonnegative integers  $r$  and  $s$  with  $r + s = 2$  except when  $m$  is odd and  $r = 2$ , and except possibly when  $m$  is even and  $r = 1$ .*

**Lemma 14** [30]. *Let  $m \geq 4$  be an even integer and  $\Gamma_m = \mathcal{C}[2] \oplus F_m[2]$ . Then  $\Gamma_m$  has a*

- (i)  $C_{2m}$ -factorization,
- (ii)  $C_m$ -factorization when  $m \equiv 0 \pmod{4}$ , and
- (iii)  $\{C_m^2, C_{2m}^1\}$ -factorization when  $m \equiv 2 \pmod{4}$ .

**Lemma 15** [18]. *Let  $m \geq 4$  be an even integer and  $x$  be a positive integer. Then  $K_{(\frac{mx}{2}:2)}^*$  has a  $\vec{C}_m$ -factorization.*

**Theorem 16** [8]. *The complete symmetric equipartite digraph  $K_{(x:y)}^*$  has a  $\vec{C}_3$ -factorization if and only if  $3 \mid xy$  and  $(x, y) \neq (1, 6)$  with possible exceptions  $(x, y) = (x, 6)$ , where  $x \notin \{m : m \text{ is divisible by a prime less than } 17\}$ .*

Theorem 8 states that  $K_{(x:y)}$  has a  $C_m$ -factorization with a few exceptions. This result will be used to show that  $K_{(x:y)}^*$  has a  $\vec{C}_m$ -factorization. However, some of the exceptions in the undirected version do not exist in the symmetric version. It is shown that there is actually a solution for these exceptions in the symmetric version. Francetić and Šajna gave the following general result for the  $\vec{C}_t$ -factorization of  $K_{(x:y)}^*$ . The necessity part of this theorem is a consequence of Lemma 15, Proposition 6 and Theorems 8 and 16.

**Theorem 17** [19]. *Let  $x, y$ , and  $t$  be integers greater than 1, and let  $g = \gcd(y, t)$ . Assume one of the following conditions holds:*

- (i)  $x(y-1)$  is even; or
- (ii)  $g \notin \{1, 3\}$ ; or
- (iii)  $g = 1$ , and  $y \equiv 0 \pmod{4}$  or  $y \equiv 0 \pmod{6}$ ; or
- (iv)  $g = 3$ , and if  $y = 6$ , then  $x$  is divisible by a prime  $p \leq 37$ .

*Then the complete symmetric equipartite digraph  $K_{(x:y)}^*$  has a  $\vec{C}_t$ -factorization if and only if  $t \mid xy$  and  $t$  is even in case  $y = 2$ .*

The following theorem presents a solution for the Directed Hamilton-Waterloo Problem for small even cycle factors. It will also help us in solving  $\text{HWP}^*(v; m^r, 2m^s)$  in Section 4, when  $m = 4$ .

**Theorem 18** [35]. *For nonnegative integers  $r$  and  $s$ ,  $\text{HWP}^*(v; m^r, n^s)$  has a solution for  $(m, n) \in \{(4, 6), (4, 8), (4, 12), (4, 16), (6, 12), (8, 16)\}$  if and only if  $r + s = v - 1$  and  $\text{lcm}(m, n) \mid v$ .*

Using Lemmata 11 and 12,  $K_{\frac{m}{2}}$  and  $K_{\frac{m}{2}} - F_{\frac{m}{2}}$  factorize into  $\left(\frac{m-2}{4}\right) C_{\frac{m}{2}}$  cycles and  $\left(\frac{m-4}{4}\right) C_{\frac{m}{2}}$  cycles, respectively. Also,  $K_m$  is isomorphic to  $K_{\frac{m}{2}}[2] \oplus I_m$ . Hence, using Proposition 6, Lemmata 11 and 12, we will obtain a  $\left\{(C_{\frac{m}{2}}^*[2])^{\frac{m-6}{4}}, C_{\frac{m}{2}}^*[2] \oplus I_m^*\right\}$ -factorization and a  $\left\{(C_{\frac{m}{2}}^*[2])^{\frac{m-12}{4}}, C_{\frac{m}{2}}^*[2] \oplus I_m^*, \Gamma_{\frac{m}{2}}^*\right\}$ -factorization of  $K_m^*$  depending on whether  $m \equiv 0$  or  $2 \pmod{4}$ . Later, we will use these factorizations to obtain a  $\{(K_2^*)^r, \vec{C}_m^s\}$ -factorization of  $K_{mx}^*$ . Furthermore, we will need to have a  $\{(K_2^*)^r, \vec{C}_{2m}^s\}$ -factorization of  $C_m^*[2]$  in order to factorize  $K_{mx}^*$  into  $K_2^*$ -factors and  $\vec{C}_m$ -factors.

**Lemma 19.** *Let  $m \geq 4$  be an integer. Then  $C_m^*[2]$  has a  $\{(K_2^*)^r, \vec{C}_{2m}^s\}$ -factorization for  $r \in \{0, 2, 4\}$  and  $r + s = 4$ .*

**Proof.** First, note that  $C_m[2]$  has a decomposition into two  $C_{2m}$ -factors by Häggkvist Lemma and each  $C_{2m}$ -factor has a decomposition into two 1-factors.

*Case 1.* ( $r = 4$ ) Decompose  $C_m[2]$  into four 1-factors by using  $C_{2m}$ -factors. Then a  $K_2^*$ -factorization of  $C_m^*[2]$  is obtained by Proposition 6.

*Case 2.* ( $r = 2$ ) Decompose  $C_m[2]$  into one  $C_{2m}$  and two 1-factors. By Proposition 6, we get a  $\{(K_2^*)^2, C_{2m}^*\}$ -factorization of  $C_m^*[2]$  and also  $C_{2m}^*$  has a  $\vec{C}_{2m}$ -factorization with two  $\vec{C}_{2m}$ -factors. So, we obtain a  $\{(K_2^*)^2, \vec{C}_{2m}^2\}$ -factorization of  $C_m^*[2]$ .

*Case 3.* ( $r = 0$ ) Obtain a  $C_{2m}^*$ -factorization of  $C_m^*[2]$  by Proposition 6. Since  $C_{2m}^*$  has a  $\vec{C}_{2m}$ -factorization with two  $\vec{C}_{2m}$ -factors,  $C_m^*[2]$  has a  $\vec{C}_{2m}$ -factorization. ■

Since  $I_{2m}^*$  and  $F_m^*$  are  $K_2^*$ -factors, the following result can be derived from Lemma 19.

**Corollary 20.** *Let  $m \geq 4$  be an even integer. Then  $\Gamma_m^*$  has a  $\{(K_2^*)^r, \vec{C}_{2m}^s\}$ -factorization for  $r \in \{0, 2, 4, 6\}$  with  $r + s = 6$ .*

**Proof.**  $F_m^*[2]$  decomposes into two  $K_2^*$ -factors. Therefore,  $\Gamma_m^*$  has a  $\{(K_2^*)^r, \vec{C}_{2m}^s\}$ -factorization for  $r \in \{2, 4, 6\}$  with  $r + s = 6$  by Lemma 19. Also,  $\Gamma_m^*$  has a  $\vec{C}_{2m}$ -factorization by Lemma 14 and Proposition 6. ■

The following lemma is quite useful in solving the Directed Hamilton-Waterloo Problem for  $n = 2$  and even  $m$  when the values of  $r$  are even.

**Lemma 21.** *Let  $m \geq 5$  be an integer. Then  $C_m^*[2] \oplus I_{2m}^*$  has a  $\{(K_2^*)^r, \vec{C}_{2m}^s\}$ -factorization for  $r \in \{0, 1, 3, 5\}$  and  $r + s = 5$ .*

**Proof.** The cases  $r \in \{1, 3, 5\}$  can be directly obtained from Lemma 19.

When  $r = 0$ , we will examine the problem in two cases:  $m$  is odd or even.

*Case 1.* (odd  $m \geq 5$ ) Define five directed  $2m$ -cycles in  $C_m^*[2] \oplus I_{2m}^*$  as follows.  
 $\vec{C}_{2m}^{(0)} = (v_0, v_1, \dots, v_{2m-1})$  where  $v_i = (\lfloor \frac{i}{m} \rfloor, i)$ ,  
 $\vec{C}_{2m}^{(1)} = (u_0, u_1, \dots, u_{2m-1})$  where,

$$u_{2i} = \begin{cases} (0, 2i) & \text{if } 0 \leq i \leq \frac{m-1}{2}, \\ (0, -2i - 1) & \text{if } \frac{m+1}{2} \leq i \leq m - 1, \end{cases}$$

and

$$u_{2i+1} = \begin{cases} (1, 2i + 1) & \text{if } 0 \leq i \leq \frac{m-3}{2}, \\ (1, -2i - 2) & \text{if } \frac{m-1}{2} \leq i \leq m - 1, \end{cases}$$

$\vec{C}_{2m}^{(2)} = (x_0, x_1, \dots, x_{2m-1})$  where

$$x_i = \begin{cases} (0, m - \lfloor \frac{i}{2} \rfloor) & \text{if } i \equiv 0, 3 \pmod{4}, \\ (1, m - \lfloor \frac{i}{2} \rfloor) & \text{if } i \equiv 1, 2 \pmod{4}, \end{cases} \text{ for } 0 \leq i \leq 2m - 3,$$



and  $x_{2m-2} = (1, 1)$ ,  $x_{2m-1} = (0, 1)$ . Also,  $C_{2m}^{(3)} = (y_0, y_1, \dots, y_{2m-1})$  where

$$y_i = u_i + (1, 2) \text{ for } 0 \leq i \leq m-3 \text{ and } m+2 \leq i \leq 2m-1,$$

$$y_{m-2} = (1, 0), \quad y_{m-1} = (0, 1), \quad y_m = (1, 1), \quad y_{m+1} = (0, 0).$$

Finally,  $\vec{C}_{2m}^{(4)} = (z_0, z_1, \dots, z_{2m-1})$  where

$$z_i = x_i + (1, 0) \text{ for } 3 \leq i \leq 2m-4,$$

$$z_0 = (2, 0), \quad z_1 = (0, m), \quad z_2 = (0, 0), \quad z_{2m-1} = (1, 1), \quad z_{2m-2} = (1, 2), \quad z_{2m-3} = (0, 1).$$

Then,  $\{\vec{C}_{2m}^{(0)}, \vec{C}_{2m}^{(1)}, \vec{C}_{2m}^{(2)}, \vec{C}_{2m}^{(3)}, \vec{C}_{2m}^{(4)}\}$  is a  $\vec{C}_{2m}$ -factorization of  $C_m^*[2] \oplus I_{2m}^*$ .

*Case 2.* (even  $m \geq 6$ ) Let  $\vec{C}_{2m}^{(0)}$  be the same as in Case 1 and define the directed  $2m$ -cycles in  $C_m^*[2] \oplus I_{2m}^*$  as follows.

$\vec{C}_{2m}^{(1)} = (x_0, x_1, \dots, x_{2m-1})$  where  $x_0 = (0, 0)$  and

$$x_i = \begin{cases} (0, m - \lfloor \frac{i+2}{2} \rfloor) & \text{if } i \equiv 1, 2 \pmod{4}, \\ (1, m - \lfloor \frac{i+2}{2} \rfloor + 1) & \text{if } i \equiv 0, 3 \pmod{4}, \end{cases} \text{ for } 1 \leq i \leq 2m-8,$$

and  $x_{2m-6+2i} = (0, 3-i)$  for  $0 \leq i \leq 2$  and  $x_{2m-7+2i} = (1, 3-i)$  for  $0 \leq i \leq 3$ . Also,  $\vec{C}_{2m}^{(2)} = (u_0, u_1, \dots, u_{2m-1})$  where  $u_0 = (0, 0)$ ,  $u_1 = (1, 0)$ ,  $u_2 = (0, m-1)$  and

$$u_i = \begin{cases} (0, m - \lfloor \frac{i-1}{2} \rfloor - 1) & \text{if } i \equiv 0, 1 \pmod{4}, \\ (1, m - \lfloor \frac{i-1}{2} \rfloor) & \text{if } i \equiv 2, 3 \pmod{4}, \end{cases} \text{ for } 3 \leq i \leq 2m-9,$$

$$u_{2m-8+j} = \begin{cases} (0, 4 - \lfloor \frac{j}{2} \rfloor) & \text{if } j \equiv 0, 2 \pmod{4}, \\ (1, 4 - \lfloor \frac{j}{2} \rfloor) & \text{if } j \equiv 1, 3 \pmod{4}, \end{cases} \text{ for } 0 \leq j \leq 7, \text{ and when } m = 6,$$

$u_3 = (1, 5)$  and we only use above piecewise function.  $\vec{C}_{2m}^{(3)} = (y_0, y_1, \dots, y_{2m-1})$  where  $y_{2i+2} = (0, m-i)$  for  $1 \leq i \leq m-4$ ,  $y_{2i+1} = (1, m-i)$  for  $1 \leq i \leq m-3$ ,  $y_0 = (0, 0)$ ,  $y_1 = (1, 1)$ ,  $y_2 = (1, 0)$ ,  $y_{2m-4} = (1, 2)$ ,  $y_{2m-3} = (0, 3)$ ,  $y_{2m-2} = (0, 2)$  and  $y_{2m-1} = (0, 1)$ .  $\vec{C}_{2m}^{(4)} = (z_0, z_1, \dots, z_{2m-1})$  where  $z_{9+2i} = (0, 4+i)$  for  $1 \leq i \leq m-5$ ,  $z_{10+2i} = (1, 4+i)$  for  $0 \leq i \leq m-6$ ,  $z_0 = (0, 0)$ ,  $z_1 = (1, m-1)$ ,  $z_2 = (1, 0)$ ,  $z_3 = (0, 1)$ ,  $z_4 = (1, 2)$ ,  $z_5 = (1, 1)$ ,  $z_6 = (0, 2)$ ,  $z_7 = (1, 3)$ ,  $z_8 = (0, 4)$ ,  $z_9 = (0, 3)$ . Then  $\{\vec{C}_{2m}^{(0)}, \vec{C}_{2m}^{(1)}, \vec{C}_{2m}^{(2)}, \vec{C}_{2m}^{(3)}, \vec{C}_{2m}^{(4)}\}$  is a  $\vec{C}_{2m}$ -factorization of  $C_m^*[2] \oplus I_{2m}^*$ . ■

By Lemma 13, we can decompose  $C_m[2]$  into two  $C_m$ -factors for even  $m$ . So, we obtain the following lemma similar to Lemma 19. Also, the following corollaries are obtained as a result of this lemma.

**Lemma 22.** *Let  $m \geq 4$  be an even integer. Then  $C_m^*[2]$  has a  $\{(K_2^*)^r, \vec{C}_m^s\}$ -factorization for  $r \in \{0, 2, 4\}$  with  $r + s = 4$ .*

**Corollary 23.** *Let  $m \geq 4$  be an even integer. Then  $C_m^*[2] \oplus I_{2m}^*$  has a  $\{(K_2^*)^r, \vec{C}_m^s\}$ -factorization for  $r \in \{1, 3, 5\}$  with  $r + s = 5$ .*

**Corollary 24.** *Let  $m \geq 4$  be an even integer. Then  $\Gamma_m^*$  has a  $\{(K_2^*)^r, \vec{C}_m^s\}$ -factorization for  $r \in \{2, 4, 6\}$  with  $r + s = 6$ .*

Recall that  $\Gamma_m^*$  is  $C_m^*[2] \oplus F_m^*[2]$ .

**Lemma 25.** *Let  $m \geq 4$  be an even integer.  $\Gamma_m^*$  has a  $\{(K_2^*)^r, \vec{C}_m^s\}$ -factorization for  $m \equiv 2 \pmod{4}$  and  $r \in \{1, 2, 3, 4, 6\}$  with  $r + s = 6$ .*

**Proof.** The cases  $r \in \{2, 4, 6\}$  are obtained by Corollary 24.

For  $r = 1$ , we define the following  $m$ -cycles.

$$\vec{C}_m^{(0)} = (v_0, v_1, \dots, v_{m-1}) \text{ where } v_i = (0, i) \text{ for } 0 \leq i \leq m-1,$$

$$\vec{C}_m^{(1)} = (u_0, u_1, \dots, u_{m-1}) \text{ where } u_i = \begin{cases} (0, i) & \text{if } i \text{ is even,} \\ (1, i) & \text{if } i \text{ is odd,} \end{cases}$$

$$\vec{C}_m^{(2)} = (x_0, x_1, \dots, x_{m-1}) \text{ where } x_0 = (0, 0) \text{ and for } 1 \leq i \leq m-1$$

$$x_i = \begin{cases} \left( \frac{1-(-1)^i}{2}, \frac{m}{2} - \left\lfloor \frac{i}{2} \right\rfloor \right) & \text{if } i \equiv 1, 2 \pmod{4}, \\ \left( \frac{1-(-1)^i}{2}, \frac{m}{2} + \left\lfloor \frac{i}{2} \right\rfloor \right) & \text{if } i \equiv 0, 3 \pmod{4}, \end{cases}$$

$$\vec{C}_m^{(3)} = (z_0, z_1, \dots, z_{m-1}) \text{ where } z_m = (1, m-1), z_{m-1} = (0, 0) \text{ and}$$

$$z_i = \begin{cases} (0, \left\lfloor \frac{i}{2} \right\rfloor + 1) & \text{if } i \equiv 0 \pmod{4}, \\ (1, \left\lceil \frac{i}{2} \right\rceil + 1) & \text{if } i \equiv 1 \pmod{4}, \\ (0, m-i) & \text{if } i \equiv 2 \pmod{4}, \\ (1, m-i) & \text{if } i \equiv 3 \pmod{4}, \end{cases} \text{ for } 0 \leq i \leq \frac{m}{2}.$$

Let us choose the factor  $F_0$  as isomorphic to  $F_m^* \oplus (F_m^* + (1, 0))$ , then  $F_0$  becomes a  $K_2^*$ -factor. Using the above  $m$ -cycles, we obtain five  $m$ -cycle factors:  $F_1 = \vec{C}_m^{(0)} \cup (\vec{C}_m^{(0)} + (1, 0))$ ,  $F_2 = R(F_1)$ ,  $F_3 = \vec{C}_m^{(1)} \cup R(\vec{C}_m^{(1)} + (1, 0))$ ,  $F_4 = \vec{C}_m^{(2)} \cup R(\vec{C}_m^{(2)} + (1, 0))$ , and  $F_5 = \vec{C}_m^{(3)} \cup (\vec{C}_m^{(3)} + (1, 0))$ . Then  $\{F_0, F_1, F_2, F_3, F_4, F_5\}$  is a  $\{(K_2^*)^1, \vec{C}_m^5\}$ -factorization of  $\Gamma_m^*$ .

For  $r = 3$ ,  $F_1 \oplus F_2$  is a  $C_m^*$ -factor of  $\Gamma_m^*$  and has a factorization into two  $K_2^*$ -factors of  $\Gamma_m^*$  say  $F_1'$  and  $F_2'$ . Then  $\{F_0, F_1', F_2', F_3, F_4, F_5\}$  is a  $\{(K_2^*)^3, \vec{C}_m^3\}$ -factorization of  $\Gamma_m^*$ . ■

3. SOLUTIONS TO  $\text{HWP}^*(v; 2^r, m^s)$ 

Now, we can give solutions to the Directed Hamilton-Waterloo Problem for  $K_2^*$  and  $\vec{C}_m$  when even  $m$ .

**Theorem 26.** *Let  $r, s$  be nonnegative integers, and let  $m \geq 6$  be even. Then  $\text{HWP}^*(v; 2^r, m^s)$  has a solution if and only if  $m|v$ ,  $r + s = v - 1$  and  $v \geq 6$  except for  $s = 1$  or  $(r, v) = (0, 6)$ , and except possibly when at least one of the following conditions holds:*

- (i)  $s = 3$  and  $m \equiv 0 \pmod{4}$ ,
- (ii)  $s = 3$ ,  $m \equiv 2 \pmod{4}$  and  $\frac{v}{m}$  is odd.

**Proof.** Take  $(v-2)$  disjoint  $K_2^*$ -factors of  $K_v^*$ , say  $H_1^*, H_2^*, \dots, H_{v-2}^*$ . It is obvious that  $K_v^* - (H_1^* \oplus H_2^* \oplus \dots \oplus H_{v-2}^*)$  is a  $K_2^*$ -factor in  $K_v^*$ . Thus, there is no  $\{(K_2^*)^{v-2}, \vec{C}_m^1\}$ -factorization of  $K_v^*$ . Therefore, we may assume  $s \neq 1$ .

Since  $\text{HWP}^*(v; n^r, m^s)$  has a solution for  $r = 0$  except for  $(v, m) = (6, 6)$  by Theorem 1, we may assume that  $r \geq 1$ .

Let  $v = mx$  for a positive integer  $x$ . Partition the vertices of  $K_{mx}^*$  into  $2x$  sets of size  $\frac{m}{2}$ , represent each part of  $\frac{m}{2}$  vertices in  $K_{mx}^*$  with a single vertex and represent all double arcs between sets of size  $\frac{m}{2}$  as a single double arc, to get a  $K_{2x}^*$ . By Proposition 7,  $K_{2x}^*$  has a decomposition into  $(2x - 1)$   $K_2^*$ -factors. Then construct a  $K_m^*$ -factor of  $K_{mx}^*$  from one of the  $K_2^*$ -factors, and a  $K_{(\frac{m}{2}; 2)}^*$ -factor of  $K_{mx}^*$  from each of the remaining  $(2x - 2)$   $K_2^*$ -factors. Then  $K_{mx}^*$  can be factorized into a  $K_m^*$ -factor and  $(2x - 2)$   $K_{(\frac{m}{2}; 2)}^*$ -factors.

By Lemmata 10 and 15,  $K_{(\frac{m}{2}; 2)}^*$  decomposes into  $\frac{m}{2}$   $K_2^*$ -factors or  $\frac{m}{2}$   $\vec{C}_m$ -factors, respectively. As a result, we must decompose  $K_m^*$  into  $K_2^*$ -factors and  $\vec{C}_m$ -factors.

*Case 1.* (odd  $r$ ) By Lemma 12, factorize  $K_m$  into an  $F_m$ -factor and  $(\frac{m-2}{2})$   $C_m$ -factors. So,  $K_m^*$  can be factorized into an  $F_m^*$ -factor and  $(\frac{m-2}{2})$   $C_m^*$ -factors by Proposition 6.

Since  $C_m^*$  can be decomposed into two  $K_2^*$ -factors or two  $\vec{C}_m$ -factors for even  $m$ ,  $K_m^*$  has a  $\{(K_2^*)^{2r_1+1}, \vec{C}_m^{2s_1}\}$ -factorization where  $r_1 + s_1 = \frac{m-2}{2}$ .

Since  $K_{mx}^*$  has a  $\left\{K_m^*, \left(K_{(\frac{m}{2}; 2)}^*\right)^{(2x-2)}\right\}$ -factorization, placing a  $K_2^*$ -factorization on  $r_0$  of the  $K_{(\frac{m}{2}; 2)}^*$  factors for  $r_0$  even and  $0 \leq r_0 \leq 2x - 2$ , a  $\vec{C}_m$ -factorization on  $s_0$  of the  $K_{(\frac{m}{2}; 2)}^*$  where  $r_0 + s_0 = 2x - 2$ , and taking a  $\{(K_2^*)^{2r_1+1}, \vec{C}_m^{2s_1}\}$ -factorization of  $K_m^*$  give a  $\{(K_2^*)^r, \vec{C}_m^s\}$ -factorization of  $K_{mx}^*$  where  $r = \frac{m}{2}r_0 + 2r_1 + 1$  and  $s = \frac{m}{2}s_0 + 2s_1$  with  $r + s = \frac{m}{2}(r_0 + s_0) + 2(r_1 + s_1) + 1 = mx - 1 = v - 1$ .

Since any nonnegative odd integer  $1 \leq r \leq mx - 1$  can be written as  $r = \frac{m}{2}r_0 + 2r_1 + 1$  for integers  $0 \leq r_0 \leq 2x - 2$  and  $0 \leq r_1 \leq \frac{m-2}{2}$ , a solution to

$\text{HWP}^*(v; 2^r, m^s)$  exists for each odd  $r \geq 1$  and  $s \geq 1$  satisfying  $r + s = mx - 1 = v - 1$ .

*Case 2. (even  $r$ )*

(a) Assume  $m \equiv 0 \pmod{4}$ . Therefore,  $\frac{m}{2}$  is even. Each  $K_{(\frac{m}{2};2)}^*$  decompose into  $\frac{m}{2}$   $K_2^*$ -factors or  $\frac{m}{2}$   $\vec{C}_m$ -factors. For this reason we need a  $\{(K_2^*)^r, \vec{C}_m^s\}$ -factorization of  $K_m^*$  for even  $r$ .

Also,  $K_{\frac{m}{2}}^*$  can be factorized as  $\bigoplus_{i=1}^{\frac{m-4}{4}} C_i^* \oplus F_{\frac{m}{2}}^*$  where each  $C_i^*$  is isomorphic to  $C_{\frac{m}{2}}^*$ . Then,  $K_{\frac{m}{2}}^*[2] \cong \bigoplus_{i=1}^{\frac{m-4}{4}} C_i^*[2] \oplus F_{\frac{m}{2}}^*[2]$ . Also,  $K_m^*$  is isomorphic to  $K_{\frac{m}{2}}^*[2] \oplus I_m^*$ . Therefore,  $K_m^*$  has a  $\left\{ (C_{\frac{m}{2}}^*[2])^{\frac{m-12}{4}}, C_{\frac{m}{2}}^* \oplus I_m^*, \Gamma_{\frac{m}{2}}^* \right\}$ -factorization. By Lemma 19, each of  $\frac{m-12}{4}$   $C_{\frac{m}{2}}^*[2]$ -factors has a  $\{(K_2^*)^{r_0}, \vec{C}_m^{s_0}\}$ -factorization for  $r_0 \in \{0, 2, 4\}$  and  $r_0 + s_0 = 4$ . By Lemma 21,  $C_{\frac{m}{2}}^*[2] \oplus I_m^*$  has a  $\{(K_2^*)^{r_1}, \vec{C}_m^{s_1}\}$ -factorization for  $r_1 \in \{0, 1, 3, 5\}$  and  $r_1 + s_1 = 5$ . By Corollary 20,  $\Gamma_{\frac{m}{2}}^*$  has a  $\{(K_2^*)^{r_2}, \vec{C}_m^{s_2}\}$ -factorization for even  $m$  and  $r_2 \in \{0, 2, 4, 6\}$  with  $r_2 + s_2 = 6$ . Those factorizations give a  $\{(K_2^*)^{r'}, \vec{C}_m^{s'}\}$ -factorization of  $K_m^*$  where  $r' = (\frac{m-12}{4})r_0 + r_1 + r_2$  and  $s' = (\frac{m-12}{4})s_0 + s_1 + s_2$  satisfying  $r' + s' = (\frac{m-12}{4})4 + 5 + 6 = m - 1$  with  $0 \leq r', s' \leq m - 1$ . If we choose  $r_1 = 0$ , we obtain a  $\{(K_2^*)^{r'}, \vec{C}_m^{s'}\}$ -factorization of  $K_m^*$  for even  $r'$ . Since we cannot get  $r_0 = 1$ ,  $r_1 = 2$  or  $r_2 = 3$  from the above factorizations, it can be seen that  $r' = m - 4$  cannot be obtained.

Placing a  $K_2^*$ -factorization on  $r''$  of the  $K_{(\frac{m}{2};2)}^*$ -factors for  $0 \leq r'' \leq 2x - 2$ , a  $\vec{C}_m$ -factorization on  $s''$  of the  $K_{(\frac{m}{2};2)}^*$  for  $r'' + s'' = 2x - 2$ , and taking a  $\{(K_2^*)^{r'}, \vec{C}_m^{s'}\}$ -factorization of  $K_m^*$  give a  $\left\{ (K_2^*)^{\frac{m}{2}r''+r'}, \vec{C}_m^{\frac{m}{2}r''+s'} \right\}$ -factorization of  $K_{mx}^*$  where  $\frac{m}{2}r'' + r'$  is even.

Any even integer  $1 \leq r \leq mx - 1$  can be written as  $r = \frac{m}{2}r'' + r'$  for integers  $r' \in [0, m-1]$  and  $0 \leq r'' \leq 2x-2$ . Since  $r' \neq m-4$ , a solution to  $\text{HWP}^*(v; 2^r, m^s)$  exists for each even  $r \geq 2$  except possibly  $r = mx - 4 = v - 4$  and  $s \geq 1$  satisfying  $r + s = v - 1$ .

(b) Assume  $m \equiv 2 \pmod{4}$ . By Lemma 11, factorize  $K_n$  into  $\binom{n-1}{2}$   $C_n$ -factors for odd  $n$ , and get a  $C_n^*$ -factorization of  $K_n^*$  by Proposition 6. Also,  $K_m^*$  can be factorized as  $K_{\frac{m}{2}}^*[2] \oplus I_m^*$ . Since  $\frac{m}{2}$  is odd,  $K_m^*$  has a  $\left\{ (C_{\frac{m}{2}}^*[2])^{\frac{m-2}{4}}, I_m^* \right\}$ -factorization. By Lemma 19, each of  $C_{\frac{m}{2}}^*[2]$ -factors has a  $\{(K_2^*)^{r_0}, \vec{C}_m^{s_0}\}$ -factorization for  $r_0 \in \{0, 2, 4\}$  and  $r_0 + s_0 = 4$ . By Lemma 21,  $C_{\frac{m}{2}}^*[2] \oplus I_m^*$  has  $\{(K_2^*)^{r_1}, \vec{C}_m^{s_1}\}$ -factorization for  $r_1 \in \{0, 1, 3, 5\}$  and  $r_1 + s_1 = 5$ .

Those factorizations give a  $\{(K_2^*)^{r_2}, \vec{C}_m^{s_2}\}$ -factorization of  $K_m^*$  for  $r_2 = \frac{m-6}{4}$   $r_0 + r_1$  and  $s_2 = \frac{m-6}{4}s_0 + s_1$  with  $r_2 + s_2 = m - 1$ . Since we cannot get  $r_0 = 1$  or  $r_1 = 2$  from the above factorizations, it can be seen that  $r_2 = m - 4$  cannot

be obtained.

Placing a  $K_2^*$ -factorization on  $r'$  of the  $K_{(\frac{m}{2}:2)}^*$  factors for  $0 \leq r' \leq 2x-2$  where we choose  $r'$  is even, a  $\vec{C}_m$ -factorization on  $s'$  of the  $K_{(\frac{m}{2}:2)}^*$  with  $r' + s' = 2x-2$ , and taking a  $\{(K_2^*)^{r_2}, \vec{C}_m^{s_2}\}$ -factorization of  $K_m^*$  give a  $\{(K_2^*)^{\frac{m}{2}r' + r_2}, \vec{C}_m^{\frac{m}{2}s' + s_2}\}$ -factorization of  $K_{mx}^*$  where  $r = \frac{m}{2}r' + r_2$  and  $s = \frac{m}{2}s' + s_2$ . Also, we obtain the requested even integer  $r \in [1, mx-1]$ , from the sum of  $\frac{m}{2}r'$  and  $r_2$  for integers  $0 \leq r' \leq 2x-2$  and  $r_2 \in [0, m-1]$ . Since  $r_2 \neq m-4$ , a solution to  $\text{HWP}^*(v; 2^r, m^s)$  exists for even  $r \geq 2$  except possibly  $r = mx-4 = v-4$  and odd  $s \geq 1$  satisfying  $r + s = v-1$ .

If  $x$  is even, say  $x = 2t$ , factorize  $K_{mx}^*$  into a  $K_{2m}^*$ -factor and  $(2t-2)$   $K_{(m:2)}^*$ -factors.  $K_{(m:2)}^*$  has a  $K_2^*$ -factorization with  $m$   $K_2^*$ -factors and a  $\vec{C}_m$ -factorization with  $m$   $\vec{C}_m$ -factors by Lemmata 10 and 15, respectively. So, we must decompose  $K_{2m}^*$  into  $K_2^*$ -factors and  $\vec{C}_m$ -factors. As before,  $K_{2m}^*$  can be factorized as  $K_m^*[2] \oplus I_{2m}^*$ . So,  $K_{2m}^*$  has a  $\{(C_m^*[2])^{\frac{m-4}{2}}, I_{2m}^*, \Gamma_m^*\}$ -factorization. By Lemma 22, each of  $C_m^*[2]$ -factors has a  $\{(K_2^*)^{r_0}, \vec{C}_m^{s_0}\}$ -factorization for  $r_0 \in \{0, 2, 4\}$  and  $r_0 + s_0 = 4$ . By Corollary 23,  $C_m^*[2] \oplus I_{2m}^*$  has a  $\{(K_2^*)^{r_1}, \vec{C}_m^{s_1}\}$ -factorization for  $r_1 \in \{1, 3, 5\}$  and  $r_1 + s_1 = 5$ . By Lemma 25,  $\Gamma_m^*$  has a  $\{(K_2^*)^{r_2}, \vec{C}_m^{s_2}\}$ -factorization for  $m \equiv 2 \pmod{4}$  and  $r_2 \in \{1, 2, 3, 4, 6\}$  with  $r_2 + s_2 = 6$ . Using these factorizations, we obtain a solution to the problem for  $r = 2mt-4 = mx-4$  when  $m \equiv 2 \pmod{4}$  and even  $x$ . As a result,  $\text{HWP}^*(v; 2^r, m^s)$  has a solution for  $r = v-4$  and even  $\frac{v}{m}$  when  $m \equiv 2 \pmod{4}$ . ■

**Lemma 27.**  $C_4^*[2] \oplus I_8^*$  has a  $\{(K_2^*)^r, \vec{C}_4^s\}$ -factorization for  $r \in \{0, 1, 2, 3, 5\}$  with  $r + s = 5$ .

**Proof.** We represent  $C_4^*[2] \oplus I_8^*$  as the directed Cayley graph  $\vec{X}(\mathbb{Z}_8, S)$  with connection set  $S = \{\pm 1, \pm 3, 4\}$ .

For  $r = 0$ , we define a  $\vec{C}_4$ -factorization of  $C_4^*[2] \oplus I_8^*$ :

$$\mathcal{F}_1 = \{[(0, 1, 2, 3), (4, 5, 6, 7)], [(0, 3, 2, 1), (4, 7, 6, 5)], [(0, 5, 1, 4), (2, 7, 3, 6)], [(0, 4, 3, 7), (1, 5, 2, 6)], [(0, 7, 2, 5), (1, 6, 3, 4)]\}.$$

For  $r = 2$ , we define a  $\{(K_2^*)^2, \vec{C}_4^3\}$ -factorization of  $C_4^*[2] \oplus I_8^*$ :

$$\mathcal{F}_2 = \{[(0, 4)^*, (1, 5)^*, (2, 6)^*, (3, 7)^*], [(0, 7)^*, (1, 6)^*, (2, 5)^*, (3, 4)^*], [(0, 1, 2, 3), (4, 5, 6, 7)], [(0, 3, 6, 5), (1, 4, 7, 2)], [(0, 5, 4, 1), (2, 7, 6, 3)]\}.$$

The remaining cases are obtained from Corollary 23 for  $m = 4$ . ■

**Lemma 28.**  $K_{12}^*$  has a  $\{(K_2^*)^r, \vec{C}_4^s\}$ -factorization for  $r \in \{0, 1, 2, 3, 4, 5, 7, 9, 11\}$  with  $r + s = 11$ .

**Proof.** The cases  $r = 0$  and  $r = 11$  are obtained by Theorem 18 and Proposition 7, respectively. Since  $K_{12} - I$  has a  $C_4$ -factorization where  $I$  is a 1-factor of  $K_{12}$ , by Proposition 6,  $K_{12}^*$  can be factorized into five  $C_4^*$ -factors and one  $I^*$ -factor which is a  $K_2^*$ -factor of  $K_{12}^*$ . Also,  $C_4^*$  has a  $\vec{C}_4$ -factorization and  $K_2^*$ -factorization. So, we obtain a  $\{(K_2^*)^r, \vec{C}_4^s\}$ -factorization of  $K_{12}^*$  for  $r \in \{1, 3, 5, 7, 9\}$  with  $r + s = 11$ .

We represent  $K_{12}^*$  as the directed Cayley graphs  $\vec{X}(\mathbb{Z}_{12}, S)$  with connection set  $S = \{\pm 1, \pm 2, \pm 3, \pm 4, \pm 5, 6\}$ , and define the following factorizations of  $K_{12}^*$  for  $r = 2, 4$ , respectively.

$$\mathcal{F}_1 = \{[(0, 6)^*, (1, 7)^*, (2, 8)^*, (3, 9)^*, (4, 10)^*, (5, 11)^*], [(0, 10)^*, (4, 6)^*, (1, 5)^*, (7, 11)^*, (2, 9)^*, (3, 8)^*], [(0, 1, 2, 3), (4, 5, 6, 7), (8, 9, 10, 11)], [(0, 2, 1, 4), (3, 5, 7, 6), (8, 11, 10, 9)], [(0, 3, 1, 8), (2, 4, 11, 6), (5, 9, 7, 10)], [(0, 4, 2, 11), (1, 6, 8, 10), (3, 7, 9, 5)], [(0, 5, 8, 7), (1, 3, 4, 9), (2, 10, 6, 11)], [(0, 7, 5, 2), (1, 10, 8, 4), (3, 6, 9, 11)], [(0, 8, 6, 1), (2, 5, 10, 7), (3, 11, 9, 4)], [(0, 9, 6, 5), (1, 11, 4, 8), (2, 7, 3, 10)], [(0, 11, 1, 9), (2, 6, 10, 3), (4, 7, 8, 5)]\},$$

$$\mathcal{F}_2 = \{[(0, 6)^*, (1, 7)^*, (2, 8)^*, (3, 9)^*, (4, 10)^*, (5, 11)^*], [(0, 10)^*, (4, 6)^*, (1, 5)^*, (7, 11)^*, (2, 9)^*, (3, 8)^*], [(0, 8)^*, (2, 6)^*, (1, 10)^*, (4, 7)^*, (3, 11)^*, (5, 9)^*], [(0, 1)^*, (2, 3)^*, (4, 5)^*, (6, 7)^*, (8, 9)^*, (10, 11)^*], [(0, 2, 1, 3), (4, 8, 11, 9), (5, 7, 10, 6)], [(0, 3, 10, 5), (1, 8, 6, 11), (2, 4, 9, 7)], [(0, 4, 11, 2), (1, 6, 10, 9), (3, 5, 8, 7)], [(0, 5, 6, 9), (1, 2, 11, 4), (3, 7, 8, 10)], [(0, 7, 9, 11), (1, 4, 3, 6), (2, 10, 8, 5)], [(0, 9, 10, 7), (1, 11, 6, 8), (2, 5, 3, 4)], [(0, 11, 8, 4), (1, 9, 6, 3), (2, 7, 5, 10)]\}.$$

Therefore,  $K_{12}^*$  has a  $\{(K_2^*)^r, \vec{C}_4^s\}$ -factorization for  $r \in \{0, 1, 2, 3, 4, 5, 7, 9, 11\}$  with  $r + s = 11$ . ■

**Lemma 29.**  $K_{(4:3)}^*$  has a  $\{(K_2^*)^r, \vec{C}_4^s\}$ -factorization for  $r \in \{0, 1, 2, 4, 6, 8\}$  with  $r + s = 8$ .

**Proof.** The cases  $r = 0$  and  $r = 8$  are obtained by Theorem 17 and Lemma 10, respectively. By Theorem 8,  $K_{(4:3)}$  has a  $C_4$ -factorization and so,  $K_{(4:3)}^*$  has a  $C_4^*$ -factorization by Proposition 6. Since  $C_4^*$  has a  $K_2^*$ -factorization and a  $\vec{C}_4$ -factorization,  $K_{(4:3)}$  can be factorized into two  $K_2^*$ -factors and six  $\vec{C}_4$ -factors. Similarly, a  $\{(K_2^*)^r, \vec{C}_4^s\}$ -factorization of  $K_{(4:3)}$  is obtained for  $r \in \{4, 6\}$  with  $r + s = 8$ .

Finally, let  $V(K_{(4:3)}^*) = \bigcup_{i=0}^2 \{4i, 4i+1, 4i+2, 4i+3\}$  with the obvious vertex partition, and define the following factorization of  $K_{(4:3)}^*$  for  $r = 1$ .

$$\mathcal{F}_1 = \{[(0, 4, 2, 5), (1, 8, 3, 11), (6, 9, 7, 10)], [(0, 5, 1, 7), (2, 9, 4, 11), (3, 8, 6, 10)], [(0, 7, 1, 9), (2, 4, 3, 10), (5, 11, 6, 8)], [(0, 8, 1, 10), (2, 7, 3, 5), (4, 9, 6, 11)], [(0, 9, 2, 11), (1, 5, 3, 6), (4, 10, 7, 8)], [(0, 10, 4, 8), (1, 11, 5, 9), (2, 6, 3, 7)], [(0, 11, 3, 4), (1, 6, 2, 10), (5, 8, 7, 9)], [(0, 6)^*, (1, 4)^*, (2, 8)^*, (3, 9)^*, (5, 10)^*, (7, 11)^*]\}. \quad \blacksquare$$

In Theorem 26, we have given the necessary and sufficient conditions for the existence of a solution for  $\text{HWP}^*(v; 2^r, m^s)$  for even  $m \geq 6$ . The construction

in Theorem 26 is not valid when  $m = 4$ , therefore we also examine the case of  $m = 4$  in the following theorem.

**Theorem 30.** *Let  $r, s$  be nonnegative integers. Then  $\text{HWP}^*(v; 2^r, 4^s)$  has a solution if and only if  $r + s = v - 1$  except for  $s = 1$  or  $(r, v) = (0, 4)$ , and except possibly when at least one of the following conditions holds:*

- (i)  $r \geq 2$  even and  $v \equiv 4, 20 \pmod{24}$ ,
- (ii)  $s \in \{3, 5\}$  and  $v \equiv 12 \pmod{24}$ .

**Proof.** If you remove  $(v - 2)$  disjoint  $K_2^*$ -factors from  $K_v^*$ , then the remaining factor must be a  $K_2^*$ -factor in  $K_v^*$ . Thus, there is no  $\{(K_2^*)^{v-2}, \vec{C}_4^1\}$ -factorization of  $K_v^*$ . So, we may assume  $s \neq 1$ .

Since  $\text{HWP}^*(v; n^r, m^s)$  has a solution for  $r = 0$  except for  $(v, m) = (4, 4)$  by Theorem 1,  $\text{HWP}^*(4; 2^r, 4^s)$  has no solution for  $r = 0$ . As a result, we may assume that  $r \geq 1$ .

*Case 1.* ( $v \equiv 0 \pmod{8}$ ) Let  $v = 8k$  for a positive integer  $k$ . Note that,  $K_{8k}^*$  can be factorized as  $K_{4k}^*[2] \oplus I_{8k}^*$ . Also,  $K_{4k}^*[2]$  can be factorized into  $C_4^*[2]$ -factors and a  $K_2^*[2]$ -factor. The graph  $kC_4^*[2] \oplus I_{8k}^*$  can be considered as  $(C_4^*[2] \oplus I_8^*)$ -factor in  $K_{8k}^*$ . Therefore,  $K_{8k}^*$  has a  $\{(C_4^*[2])^{2k-1}, I_8^*, K_2^*[2]\}$ -factorization. Also,  $C_4^*[2]$  has a  $\{(K_2^*)^{r_0}, \vec{C}_4^{s_0}\}$ -factorization for  $r_0 \in \{0, 2, 4\}$  where  $r_0 + s_0 = 4$  by Lemma 22. Since  $K_2^*[2] = C_4^*$ ,  $K_2^*[2]$  has a  $\{(K_2^*)^{r_1}, \vec{C}_4^{s_1}\}$ -factorization for  $r_1 \in \{0, 2\}$  and  $r_1 + s_1 = 2$ . By Lemma 27,  $C_4^*[2] \oplus I_8^*$  has a  $\{(K_2^*)^{r_2}, \vec{C}_4^{s_2}\}$ -factorization for  $r_2 \in \{0, 1, 2, 3, 5\}$  where  $r_2 + s_2 = 5$ . These factorizations give a  $\{(K_2^*)^r, \vec{C}_4^s\}$ -factorization of  $K_{8k}^*$  for  $r \neq 8k - 2$  with  $r + s = 8k - 1$ .

Then,  $\text{HWP}^*(v; 2^r, 4^s)$  has a solution for  $r + s = v - 1$ ,  $s \neq 1$  and  $v \equiv 0 \pmod{8}$ .

*Case 2.* ( $v \equiv 4 \pmod{8}$ ) Let  $v = 8k + 4$  for a nonnegative integer  $k$ .

(a) Assume  $r$  is odd. Partition the vertices of  $K_{8k+4}^*$  into  $4k + 2$  sets of size 2, represent each set of size 2 vertices in  $K_{8k+4}^*$  with a single vertex and represent all double arcs between sets of size 2 as a single double arc, to get a  $K_{4k+2}^*$ . By Proposition 7,  $K_{4k+2}^*$  has a decomposition into  $4k + 1$   $K_2^*$ -factors. Construct a  $K_4^*$ -factor from one of the  $K_2^*$ -factors and a  $K_{(2;2)}^*$ -factor from each of the remaining  $4k$   $K_2^*$ -factors. Then, factorize  $K_{8k+4}^*$  into a  $K_4^*$ -factor and  $(4k)$   $K_{(2;2)}^*$ -factors.  $K_4^*$  has a decomposition into one  $K_2^*$  and two  $\vec{C}_4$ -factors or three  $K_2^*$ -factors, and  $K_{(2;2)}^*$  has a  $\{(K_2^*)^{r_0}, \vec{C}_4^{s_0}\}$ -factorization for  $r_0 \in \{0, 2\}$  satisfying  $r + s = 2$ . So,  $K_{8k+4}^*$  has a  $\{(K_2^*)^r, \vec{C}_4^s\}$ -factorization for odd  $r$ . Therefore,  $\text{HWP}^*(v; 2^r, 4^s)$  has a solution for odd  $r$  and  $v \equiv 4 \pmod{8}$ .

(b) Assume  $r$  is even, and also let  $k \equiv 1 \pmod{3}$ . Then, we have  $v = 24l + 12$  for some nonnegative integer  $l$ .

Representing each part of 4 vertices in  $K_{24l+12}^*$  with a single vertex and all double arcs between parts of size 4 as a single double arc, we have a  $K_{6l+3}^*$ . Since a Kirkman triple system exists for orders  $6l + 3$ , we have a  $C_3$ -factorization of  $K_{6l+3}^*$ . Then a  $C_3^*$ -factorization of  $K_{6l+3}^*$  is obtained by Proposition 6.

Construct a  $K_{12}^*$ -factor from one of the  $C_3^*$ -factors and  $K_{(4:3)}^*$ -factor from each of the remaining  $3l$   $C_3^*$ -factors. Then get a  $\{K_{12}^*, (K_{(4:3)}^*)^{3l}\}$ -factorization of  $K_{24l+12}^*$ . By Lemma 28,  $K_{12}^*$  has a  $\{(K_2^*)^{r_0}, \vec{C}_m^{s_0}\}$ -factorization for  $r_0 \in \{0, 1, 2, 3, 4, 5, 7, 9, 11\}$  with  $r_0 + s_0 = 11$ . Also,  $K_{(4:3)}^*$  has a  $\{(K_2^*)^{r_1}, \vec{C}_4^{s_1}\}$ -factorization by Lemma 29 for  $r_1 \in \{0, 1, 2, 4, 6, 8\}$  with  $r_1 + s_1 = 8$ . Those factorizations give a  $\{(K_2^*)^r, \vec{C}_m^s\}$ -factorization of  $K_{24l+12}^*$  where  $r = r_0 + ar_1$  and  $s = s_0 + bs_1$  satisfying  $r + s = 24l + 11 = v - 1$  with  $1 \leq r, s \leq v - 1$  and  $a + b = 3l$ . We obtain the requested even  $r \in [0, v - 1]$  except for  $r = v - 6$  and  $r = v - 4$ , from the sum of  $r_0$  and  $ar_1$ . Then  $\text{HWP}^*(v; 2^r, 4^s)$  has a solution for  $r + s = v - 1$ ,  $s \notin \{3, 5\}$  and  $v \equiv 12 \pmod{24}$ . ■

Proving Theorem 3 was accomplished by proving Theorems 26 and 30.

#### 4. SOLUTIONS TO $\text{HWP}^*(v; m^r, (2m)^s)$

In this section, we prove that for even  $m$ , a solution to  $\text{HWP}^*(v; m^r, (2m)^s)$  exists for  $r + s = v - 1$  and except possibly when  $s \in \{1, 3\}$ .

Firstly, factorize  $K_{2mx}^*$  into a  $K_{2m}^*$ -factor and  $(2x - 2)$   $K_{(m:2)}^*$ -factors.  $K_{(m:2)}^*$  has a  $\{\vec{C}_m^r, \vec{C}_{2m}^s\}$ -factorization for  $r \in \{0, m\}$  and  $r + s = m$ . Using Lemma 12 and Proposition 6, a  $\{(C_m^*[2])^{\frac{m-4}{2}}, I_{2m}^*, \Gamma_m^*\}$ -factorization of  $K_{2m}^*$  is also obtained. Therefore, in order to factorize  $K_{2mx}^*$  into  $\vec{C}_m$ -factors and  $\vec{C}_{2m}$ -factors,  $\Gamma_m^*$ ,  $C_m^*[2] \oplus I_{2m}^*$  and  $C_m^*[2]$  must be factorized into  $\vec{C}_m$ -factors and  $\vec{C}_{2m}$ -factors. The following lemmata examine the existence of a  $\{\vec{C}_m^r, \vec{C}_{2m}^s\}$ -factorization of these graphs for  $r + s \in \{4, 5, 6\}$ .

**Lemma 31.** *Let  $m \geq 4$  be an even integer. Then  $\Gamma_m^*$  has a  $\{\vec{C}_m^r, \vec{C}_{2m}^s\}$ -factorization for  $r \in \{0, 6\}$  and  $r + s = 6$ .*

**Proof.** *Case 1.* ( $r = 0$ ) By Lemma 14(i) and Proposition 6,  $\Gamma_m^*$  has a  $\vec{C}_{2m}$ -factorization.

*Case 2.* ( $r = 6$ ) By Lemma 14(ii) and Proposition 6,  $\Gamma_m^*$  has a  $\vec{C}_m$ -factorization for  $m \equiv 0 \pmod{4}$ .

When  $m \equiv 2 \pmod{4}$ , define the following  $m$ -cycles. Also, let  $\vec{C}_m^{(0)}$  and  $\vec{C}_m^{(1)}$  be the cycles  $\vec{C}_m^{(0)}$  and  $\vec{C}_m^{(2)}$  respectively, as stated in Lemma 25.



$$\vec{C}_m^{(2)} = (u_0, u_1, \dots, u_{m-1}) \text{ where } u_i = \begin{cases} (1, m-1-i) & \text{if } 0 \leq i \leq \frac{m}{2}, \\ (0, m-1-i) & \text{if } \frac{m}{2} + 1 \leq i \leq m-1. \end{cases}$$

$$\vec{C}_m^{(3)} = (y_0, y_1, \dots, y_{m-1}) \text{ where } y_0 = (0, 0), y_1 = (0, \frac{m}{2}), y_2 = (1, \frac{m}{2} + 1), y_3 = (1, \frac{m}{2} - 1) \text{ and}$$

$$y_i = \begin{cases} (1, \frac{m}{2} + (-1)^{i+1} \lfloor \frac{i}{2} \rfloor) & \text{if } i \equiv 0, 1 \pmod{4}, \\ (0, \frac{m}{2} + (-1)^i \lfloor \frac{i}{2} \rfloor) & \text{if } i \equiv 2, 3 \pmod{4}, \end{cases} \text{ for } 4 \leq i \leq m-1.$$

$$\vec{C}_m^{(4)} = (z_0, z_1, \dots, z_{m-1}) \text{ where}$$

$$z_i = \begin{cases} y_{m-i} + (1, 0) & \text{if } 1 \leq i \leq m-3, \\ y_{m-i} & \text{if } m-2 \leq i \leq m. \end{cases}$$

Using the above  $m$ -cycles, we obtain the following  $m$ -cycle factors.  $F_0 = \vec{C}_m^{(0)} \cup (\vec{C}_m^{(0)} + (1, 0))$ ,  $F_1 = \vec{C}_m^{(1)} \cup R(\vec{C}_m^{(1)} + (1, 0))$ ,  $F_2 = R(F_1)$ ,  $F_3 = \vec{C}_m^{(2)} \oplus (\vec{C}_m^{(2)} + (1, 0))$ ,  $F_4 = \vec{C}_m^{(3)} \cup (\vec{C}_m^{(3)} + (1, 0))$  and  $F_5 = \vec{C}_m^{(4)} \cup (\vec{C}_m^{(4)} + (1, 0))$ . Then  $\{F_0, F_1, F_2, F_3, F_4, F_5\}$  is a  $\vec{C}_m$ -factorization of  $\Gamma_m^*$ . So,  $\Gamma_m^*$  has a  $\vec{C}_m$ -factorization for even  $m \geq 4$ . ■

**Lemma 32.** *Let  $m \geq 4$  be an even integer. Then  $C_m^*[2] \oplus I_{2m}^*$  has a  $\{\vec{C}_m^r, \vec{C}_{2m}^s\}$ -factorization for  $r \in \{1, 3\}$  and  $r + s = 5$ .*

**Proof.** *Case 1.* ( $r = 1$ ) Let  $\vec{C}_m^{(0)} = (v_0, v_1, \dots, v_{m-1})$  be a directed  $m$ -cycle of  $C_m^*[2] \oplus I_{2m}^*$ , where  $v_i = (0, i)$  for  $0 \leq i \leq m-1$ , and it can be checked that  $F_1 = \vec{C}_m^{(0)} \cup (\vec{C}_m^{(0)} + (1, 0))$  is a directed  $m$ -cycle factor of  $C_m^*[2] \oplus I_{2m}^*$ . Also, let  $\vec{C}_{2m}^{(1)} = (u_0, u_1, \dots, u_{2m-1})$  be a directed  $2m$ -cycle of  $C_m^*[2] \oplus I_{2m}^*$ , where  $u_{2i} = (0, i)$ , and  $u_{2i+1} = (1, i)$  for  $0 \leq i \leq m-1$ . Similarly, it can be checked that  $F_2 = \vec{C}_{2m}^{(1)}$  and  $F_3 = \vec{C}_{2m}^{(1)} + (1, 0)$  are arc disjoint directed  $2m$ -cycle factors of  $C_m^*[2] \oplus I_{2m}^*$ .

Let  $\vec{C}_{2m}^{(2)} = (x_0, x_1, \dots, x_{2m-1})$  be a directed  $2m$ -cycle of  $C_m^*[2] \oplus I_{2m}^*$ , where  $x_0 = (0, 0)$ ,  $x_m = (1, 0)$ ,  $x_{i+1} = (0, m-1-i)$  for  $0 \leq i \leq m-2$  and  $x_{j+1+m} = (1, m-1-j)$  for  $0 \leq j \leq m-2$ .

Let  $\vec{C}_{2m}^{(3)} = (y_0, y_1, \dots, y_{2m-1})$  be a directed  $2m$ -cycle of  $C_m^*[2] \oplus I_{2m}^*$ , where  $y_m = (1, 0)$ ,

$$y_i = \begin{cases} (0, m-i) & \text{if } i \text{ is even,} \\ (1, m-i) & \text{if } i \text{ is odd,} \end{cases} \text{ for } 0 \leq i \leq m-1,$$

and

$$y_i = \begin{cases} (1, 2m-i) & \text{if } i \text{ is even,} \\ (0, 2m-i) & \text{if } i \text{ is odd,} \end{cases} \text{ for } m+1 \leq i \leq 2m-1.$$

The factors  $F_4 = \vec{C}_{2m}^{(2)}$  and  $F_5 = \vec{C}_{2m}^{(3)}$  are arc disjoint directed  $2m$ -cycle factors of  $C_m^*[2] \oplus I_{2m}^*$ . Then  $\{F_1, F_2, F_3, F_4, F_5\}$  is a  $\{\vec{C}_m^1, \vec{C}_{2m}^4\}$ -factorization of  $C_m^*[2] \oplus I_{2m}^*$ .

*Case 2. ( $r = 3$ )* Let  $F_1, F_2$  and  $F_3$  be the same as in Case 1. Using the arcs of  $F_4 \cup F_5$ , we obtain two new  $\vec{C}_m$ -factors.

The factor  $F_4' = R(F_1)$  is a  $\vec{C}_m$ -factor of  $C_m^*[2] \oplus I_{2m}^*$ . Let  $\vec{C} = (y_0, y_1, \dots, y_{m-1})$  be a directed  $m$ -cycle of  $C_m^*[2] \oplus I_{2m}^*$ , where

$$y_i = \begin{cases} (0, i) & \text{if } i \text{ is even,} \\ (1, i) & \text{if } i \text{ is odd,} \end{cases} \quad \text{for } 0 \leq i \leq m-1.$$

It can be checked that  $F_5' = R(\vec{C}) \cup R(\vec{C} + (1, 0))$  is a directed  $m$ -cycle factor of  $C_m^*[2] \oplus I_{2m}^*$ .

So,  $\{F_1, F_2, F_3, F_4', F_5'\}$  is a  $\{\vec{C}_m^3, \vec{C}_{2m}^2\}$ -factorization of  $C_m^*[2] \oplus I_{2m}^*$ . ■

**Lemma 33.** *Let  $m \geq 4$  be an even integer. Then  $C_m^*[2]$  has a  $\{\vec{C}_m^r, \vec{C}_{2m}^s\}$ -factorization for  $r \in \{0, 2, 4\}$  and  $r + s = 4$ .*

**Proof.** The cases  $r \in \{0, 4\}$  are obtained by Lemmata 19 and 22. Let  $\vec{C}_{2m}^{(1)} = (u_0, u_1, \dots, u_{2m-1})$  be a directed  $2m$ -cycle of  $C_m^*[2]$ , where

$$u_i = \begin{cases} (0, i) & \text{if } 0 \leq i \leq m-1, \\ (1, i) & \text{if } m \leq i \leq 2m-1. \end{cases}$$

And it can be checked that  $F_1 = \vec{C}_{2m}^{(1)}$  is a  $\vec{C}_{2m}$ -factor of  $C_m^*[2]$ . Let  $\vec{C}_{2m}^{(2)} = (v_0, v_1, \dots, v_{2m-1})$  be a directed  $2m$ -cycle of  $C_m^*[2]$ , where

$$v_i = \begin{cases} u_i & \text{if } i \text{ is even,} \\ u_i + (1, 0) & \text{if } i \text{ is odd.} \end{cases}$$

The factor  $F_2 = \vec{C}_{2m}^{(2)}$  is a  $\vec{C}_{2m}$ -factor of  $C_m^*[2]$ . Let  $F_4'$  and  $F_5'$  be the same as in Lemma 32. Then  $\{F_1, F_2, F_4', F_5'\}$  is a  $\{\vec{C}_m^2, \vec{C}_{2m}^2\}$ -factorization of  $C_m^*[2]$ . ■

The proof of Theorem 4 can now be given.

**Theorem 4.** *Let  $r, s$  be nonnegative integers, and let  $m \geq 4$  be even. Then  $\text{HWP}^*(v; m^r, (2m)^s)$  has a solution if and only if  $m|v$ ,  $r + s = v - 1$  and  $v \geq 4$  except for  $(s, v, m) \in \{(0, 4, 4), (0, 6, 3), (0, 6, 6)\}$ , and except possibly when  $s \in \{1, 3\}$ .*

**Proof.** By Theorem 18,  $\text{HWP}^*(v; 4^r, 8^s)$  has a solution for  $r + s = v - 1$ , so we may assume that  $m \geq 6$ . Furthermore, by Theorem 1, a solution to the  $\text{HWP}^*(v; m^r, (2m)^s)$  exists when  $r = 0$  or  $s = 0$  and except for  $(s, v, m) \in \{(0, 4, 4), (0, 6, 3), (0, 6, 6)\}$ .

Factorize  $K_{2mx}^*$  into a  $K_{2m}^*$ -factor and  $(2x - 2)$   $K_{(m:2)}^*$ -factors. By Theorem 17,  $K_{(m:2)}^*$  decomposes into  $m$   $\vec{C}_m$ -factors or  $m$   $\vec{C}_{2m}$ -factors. So,  $K_{2m}^*$  must be decomposed into  $\vec{C}_m$ -factors and  $\vec{C}_{2m}$ -factors. As before,  $K_{2m}^*$  can be factorized as  $K_m^*[2] \oplus I_{2m}^*$ . Consequently,  $K_{2m}^*$  has a  $\{(C_m^*[2])^{\frac{m-4}{2}}, I_{2m}^*, \Gamma_m^*\}$ -factorization. By Lemma 33, each of  $C_m^*[2]$ -factors has a  $\{\vec{C}_m^{r_0}, \vec{C}_{2m}^{s_0}\}$ -factorization for  $r_0 \in \{0, 2, 4\}$  and  $r_0 + s_0 = 4$ . By Lemmata 32 and 21,  $C_m^*[2] \oplus I_{2m}^*$  has a  $\{\vec{C}_m^{r_1}, \vec{C}_{2m}^{s_1}\}$ -factorization for  $r_1 \in \{0, 1, 3\}$  and  $r_1 + s_1 = 5$ . By Lemma 31,  $\Gamma_m^*$  has a  $\{\vec{C}_m^{r_2}, \vec{C}_{2m}^{s_2}\}$ -factorization for  $r_2 \in \{0, 6\}$  with  $r_2 + s_2 = 6$ . Those factorizations give a  $\{\vec{C}_m^r, \vec{C}_{2m}^s\}$ -factorization of  $K_{2m}^*$  where  $r = (\frac{m-6}{2})r_0 + r_1 + r_2$  and  $s = (\frac{m-6}{2})s_0 + s_1 + s_2$  satisfying  $r + s = (\frac{m-6}{2})4 + 5 + 6 = 2m - 1$  with  $0 \leq r, s \leq 2m - 1$  and  $s \notin \{1, 3\}$ .

Placing a  $\vec{C}_m$ -factorization on  $r'$  of the  $K_{(m:2)}^*$ -factors for  $0 \leq r' \leq 2x - 2$ , a  $\vec{C}_{2m}$ -factorization on  $s'$  of the  $K_{(m:2)}^*$  for  $r' + s' = 2x - 2$ , and taking a  $\{\vec{C}_m^r, \vec{C}_{2m}^s\}$ -factorization of  $K_{2m}^*$  give a  $\{\vec{C}_m^{mr'+r}, \vec{C}_{2m}^{ms'+s}\}$ -factorization of  $K_{2mx}^*$ . Then  $\text{HWP}^*(v; m^r, 2m^s)$  has a solution except possibly when  $s \in \{1, 3\}$ . ■

## REFERENCES

- [1] R.J.R. Abel, F.E. Bennett and G. Ge, *Resolvable perfect Mendelsohn designs with block size five*, Discrete Math. **247** (2002) 1–12.  
[https://doi.org/10.1016/S0012-365X\(01\)00157-1](https://doi.org/10.1016/S0012-365X(01)00157-1)
- [2] P. Adams, E.J. Billington, D.E. Bryant and S.I. El-Zanati, *On the Hamilton-Waterloo problem*, Graphs Combin. **18** (2002) 31–51.  
<https://doi.org/10.1007/s003730200001>
- [3] P. Adams and D. Bryant, *Resolvable directed cycle systems of all indices for cycle length 3 and 4*, unpublished.
- [4] B. Alspach and R. Häggkvist, *Some observations on the Oberwolfach problem*, J. Graph Theory **9** (1985) 177–187.  
<https://doi.org/10.1002/jgt.3190090114>
- [5] B. Alspach, P.J. Schellenberg, D.R. Stinson and D. Wagner, *The Oberwolfach problem and factors of uniform odd length cycles*, J. Combin. Theory Ser. A **52** (1989) 20–43.  
[https://doi.org/10.1016/0097-3165\(89\)90059-9](https://doi.org/10.1016/0097-3165(89)90059-9)
- [6] J. Asplund, D. Kamin, M. Keranen, A. Pastine and S. Özkan, *On the Hamilton-Waterloo problem with triangle factors and  $C_{3x}$ -factors*, Australas. J. Combin. **64** (2016) 458–474.

- [7] F.E. Bennett and X. Zhang, *Resolvable Mendelsohn designs with block size 4*, Aequationes Math. **40** (1990) 248–260.  
<https://doi.org/10.1007/BF02112298>
- [8] F.E. Bennett, R. Wei and L. Zhu., *Resolvable Mendelsohn triple systems with equal sized holes*, J. Combin. Des. **5** (1997) 329–340.  
[https://doi.org/10.1002/\(SICI\)1520-6610\(1997\)5:5<329::AID-JCD2>3.0.CO;2-H](https://doi.org/10.1002/(SICI)1520-6610(1997)5:5<329::AID-JCD2>3.0.CO;2-H)
- [9] J.C. Bermond, A. Germa and D. Sotteau, *Resolvable decomposition of  $K_n^*$* , J. Combin. Theory Ser. A **26** (1979) 179–185.  
[https://doi.org/10.1016/0097-3165\(79\)90067-0](https://doi.org/10.1016/0097-3165(79)90067-0)
- [10] D. Bryant and P. Danziger, *On bipartite 2-factorizations of  $K_n - I$  and the Oberwolfach problem*, J. Graph Theory **68** (2011) 22–37.  
<https://doi.org/10.1002/jgt.20538>
- [11] D. Bryant, P. Danziger and M. Dean, *On the Hamilton-Waterloo problem for bipartite 2-factors*, J. Combin. Des. **21** (2013) 60–80.  
<https://doi.org/10.1002/jcd.21312>
- [12] A.C. Burgess, P. Danziger, A. Pastine and T. Traetta, *Constructing uniform 2-factorizations via row-sum matrices: Solutions to the Hamilton-Waterloo problem*, J. Combin. Theory Ser. A **201** (2024) 105803.  
<https://doi.org/10.1016/j.jcta.2023.105803>
- [13] A.C. Burgess, P. Danziger and T. Traetta, *On the Hamilton-Waterloo problem with cycle lengths of distinct parities*, Discrete Math. **341** (2018) 1636–1644.  
<https://doi.org/10.1016/j.disc.2018.02.020>
- [14] A.C. Burgess, P. Danziger and T. Traetta, *The Hamilton-Waterloo problem with even cycle lengths*, Discrete Math. **342** (2019) 2213–2222.  
<https://doi.org/10.1016/j.disc.2019.04.013>
- [15] A.C. Burgess, P. Danziger and T. Traetta, *On the Hamilton-Waterloo problem with odd cycle lengths*, J. Combin. Des. **26** (2018) 51–83.  
<https://doi.org/10.1002/jcd.21586>
- [16] A.C. Burgess, P. Danziger and T. Traetta, *On the Hamilton-Waterloo problem with odd orders*, J. Combin. Des. **25** (2017) 258–287.  
<https://doi.org/10.1002/jcd.21552>
- [17] A.C. Burgess, N. Francetić and M. Šajna, *On the directed Oberwolfach problem with equal cycle lengths: the odd case*, Australas. J. Combin. **71** (2018) 272–292.
- [18] A.C. Burgess and M. Šajna, *On the directed Oberwolfach problem with equal cycle lengths*, Electron. J. Combin. **21** (2014) #P1.15.  
<https://doi.org/10.37236/2982>
- [19] N. Francetić and M. Šajna, *On the directed Oberwolfach problem for complete symmetric equipartite digraphs and uniform-length cycles*, J. Combin. Des. **31** (2023) 604–641.  
<https://doi.org/10.1002/jcd.21913>

- [20] H. Lei and H. Shen, *The Hamilton-Waterloo problem for Hamilton cycles and triangle-factors*, J. Combin. Des. **20** (2012) 305–316.  
<https://doi.org/10.1002/jcd.20311>
- [21] S. Glock, F. Joos, J. Kim, D. Kühn and D. Osthus, *Resolution of the Oberwolfach problem*, J. Eur. Math. Soc. JEMS **23** (2021) 2511–2547.  
<https://doi.org/10.4171/jems/1060>
- [22] R. Häggkvist, *A lemma on cycle decompositions*, North-Holland Math. Stud. **115** (1985) 227–232.  
[https://doi.org/10.1016/S0304-0208\(08\)73015-9](https://doi.org/10.1016/S0304-0208(08)73015-9)
- [23] D.G. Hoffman and C.A. Rodger, *The chromatic index of complete multipartite graphs*, J. Graph Theory **16** (1992) 159–163.  
<https://doi.org/10.1002/jgt.3190160207>
- [24] D.G. Hoffman and P.J. Schellenberg, *The existence of  $C_k$ -factorizations of  $K_{2n} - F$* , Discrete Math. **97** (1991) 243–250.  
[https://doi.org/10.1016/0012-365X\(91\)90440-D](https://doi.org/10.1016/0012-365X(91)90440-D)
- [25] M.S. Keranen and S. Özkan, *The Hamilton-Waterloo problem with 4-cycles and a single factor of  $n$ -cycles*, Graphs Combin. **29** (2013) 1827–1837.  
<https://doi.org/10.1007/s00373-012-1231-6>
- [26] A. Lacaze-Masmonteil, *Completing the solution of the directed Oberwolfach problem with cycles of equal length*, J. Combin. Des. **32** (2024) 5–30.  
<https://doi.org/10.1002/jcd.21918>
- [27] J. Liu, *The equipartite Oberwolfach problem with uniform tables*, J. Combin. Theory Ser. A **101** (2003) 20–34.  
[https://doi.org/10.1016/S0097-3165\(02\)00011-0](https://doi.org/10.1016/S0097-3165(02)00011-0)
- [28] E. Lucas, *Récréations Mathématiques* vol. 2 (Gauthier-Villars, Paris, 1892).
- [29] U. Odabaşı and S. Özkan, *The Hamilton-Waterloo problem with  $C_4$  and  $C_m$  factors*, Discrete Math. **339** (2016) 263–269.  
<https://doi.org/10.1016/j.disc.2015.08.013>
- [30] U. Odabaşı, *Factorizations of complete graphs into cycles and 1-factors*, Contrib. Discrete Math. **15** (2020) 80–89.  
<https://doi.org/10.11575/cdm.v15i1.62603>
- [31] D.K. Ray-Chaudhuri and R.M. Wilson, *Solution of Kirkman’s schoolgirl problem*, in: Proc. Symp. Pure Math., S. Motzkin (Ed(s)), Amer. Math. Soc. **19** (Providence, Rhode Island, 1971) 187–204.  
<https://doi.org/10.1090/pspum/019/9959>
- [32] M. Reiss, *Ueber eine Steinersche combinatorische Aufgabe welche im 45<sup>sten</sup> Bande dieses Journals, Seite 181, gestellt worden ist.*, in: Journal für die reine und angewandte Mathematik, Band **56** (1859) 326–344.  
<https://doi.org/10.1515/9783112368688-029>

- [33] T.W. Tillson, *A Hamiltonian decomposition of  $K_{2m}^*$ ,  $2m \geq 8$* , J. Combin. Theory Ser. B **29** (1980) 69–74.  
[https://doi.org/10.1016/0095-8956\(80\)90044-1](https://doi.org/10.1016/0095-8956(80)90044-1)
- [34] T. Traetta, *A constructive solution to the Oberwolfach problem with a large cycle*, Discrete Math. **347(5)** (2024) 113947.  
<https://doi.org/10.1016/j.disc.2024.113947>
- [35] F. Yetgin, U. Odabaşı and S. Özkan, *On the directed Hamilton-Waterloo problem with two cycle sizes*, Contrib. Discrete Math. (2023) accepted.

Received 20 June 2023

Revised 26 April 2024

Accepted 26 April 2024

Available online 20 June 2024