Discussiones Mathematicae
Graph Theory 45 (2025) 615–636
https://doi.org/10.7151/dmgt.2549

THE DIRECTED UNIFORM HAMILTON-WATERLOO PROBLEM INVOLVING EVEN CYCLE SIZES

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Abstract

In this paper, factorizations of the complete symmetric digraph K_v^* into uniform factors consisting of directed even cycle factors are studied as a generalization of the undirected Hamilton-Waterloo Problem. It is shown, with a few possible exceptions, that K_v^* can be factorized into two nonisomorphic factors, where these factors are uniform factors of K_v^* involving K_2^* or directed m-cycles, and directed m-cycles or $2m$-cycles for even m.

Keywords: The Directed Hamilton-Waterloo Problem, 2-factorizations, directed cycle factorizations.

2020 Mathematics Subject Classification: 05C51, 05C70.

1. Introduction

In this paper, edges and arcs are denoted by using curly braces and parentheses, respectively. Throughout this paper, we denote by $K_{(x:y)}$ a complete equipartite graph having y parts of size x each. Also, for a simple graph G, we use G^* to denote the symmetric digraph with vertex set $V(G^*) = V(G)$ and arc set

$E(G^*) = \bigcup_{\{x,y\} \in E(G)} \{(x,y),(y,x)\}$. Hence, K_v^* and $K_{(x:y)}^*$ respectively denote the complete symmetric digraph of order v and the complete symmetric equipartite digraph with y parts of size x. We also use $(x,y)^*$ to denote the double arc which consists of (x,y) and (y,x).

A k-factor of a graph G is a k-regular spanning subgraph of G. A k-factorization of a graph G is a partition of the edge set of G into k-factors; in other words, it is a partition (decomposition) of the edge set of G into edge-disjoint k-factors. It is easy to see that a 2-factor consists of an Hamilton cycle, which is a cycle that visits each vertex exactly, or union of vertex-disjoint cycles. If a 2-factor consists only of cycles (directed cycles) of length m, it is called a C_m-factor ($\vec{C}_m$-factor). Furthermore, in the special case where $m = 2$, this factor becomes a K_2-factor. There are two well-studied 2-factorization problems. The Oberwolfach Problem asks for the existence of a decomposition of K_{2v+1} or $K_{2v} - I$ (i.e., K_{2v} with the edges of the 1-factor I removed) into copies of a given 2-factor F. The uniform version of the Oberwolfach Problem in which there is only one type of cycle in the factor F has been mostly solved, see [4, 5, 24, 31]. In the Hamilton-Waterloo Problem, there are two types of 2-factors. The uniform version of the Hamilton-Waterloo Problem asks for a 2-factorization of K_v (or for even v, 2-factorization of $K_v - I$) in which r of its 2-factors consist of only m-cycles and the remaining s of its 2-factors consist of only n-cycles, and we will denote it by $\mathrm{HWP}(v; m^r, n^s)$. Any of its solutions will be referred to as a $\{C_m^r, C_n^s\}$-factorization of K_v (or $K_v - I$ for even v).

Initially, small cases such as $(m,n) \in \{(4,6),(4,8), (4,16),(8,16),(3,5), (3,15),(5,15)\}$ are studied and solved with a few exceptions by Adams *et al.* [2], and later the cases where the cycle sizes are non-constant are investigated. The Hamilton-Waterloo Problem is nearly completely solved when both m and n are simultaneously either even or odd [10, 11, 15, 16]. When the parity of m and n is different, one of the cycle sizes is usually fixed. For instance, the cases $(m,n) \in \{(3,v),(3,3x),(4,n)\}$ have been studied, see [6, 20, 25, 29]. For more recent results on this problem, we refer the reader to [12, 13, 14]. Also, there exists an asymptotic solution (for sufficiently large v) [21] for the general form of the Oberwolfach and the Hamilton-Waterloo Problems. However, this asymptotic solution does not provide an explicit lower bound that guarantees the solvability of the problem. In [34], Traetta constructs solutions to the Oberwolfach Problem whenever F contains a cycle of length greater than an explicit lower bound.

The concept of factor and factorization can be applied to digraphs and one can consider the directed version of the Oberwolfach and Hamilton-Waterloo Problems. In the directed version of these problems, factorization of the complete symmetric digraph K_v^* into directed cycle factors is studied. The Directed Uniform Oberwolfach Problem is denoted by $\mathrm{OP}^*(m^k)$ where each 2-factor is composed of k directed m-cycles.

The following theorem summarizes the previous results on the Directed Oberwolfach Problem that will be used in this paper.

Theorem 1 [1, 3, 7, 9, 17, 18, 26, 33]**.** *Let m and k be nonnegative integers. Then,* $\mathrm{OP}^*(m^k)$ *has a solution if and only if* $(m,k) \notin \{(3,2),(4,1),(6,1)\}$.

The directed Oberwolfach Problem for complete symmetric equipartite digraphs and uniform-length cycles was solved by Francetić and Šajna in [19].

When it comes to the Directed Hamilton-Waterloo Problem, here K_v^* is decomposed into two types of directed 2-factors. If these factors consist of directed cycles of sizes m and n, respectively, the notation $\mathrm{HWP}^*(v;m^r,n^s)$ is used to denote the Directed Uniform Hamilton-Waterloo Problem.

In [35], the necessary conditions for the existence of a solution to the Directed Hamilton-Waterloo Problem are given.

Lemma 2 [35]**.** *If* $\mathrm{HWP}^*(v;m^r,n^s)$ *has a solution, then the following statements hold:*

(i) *if* $r > 0$, $v \equiv 0 \pmod{m}$,

(ii) *if* $s > 0$, $v \equiv 0 \pmod{n}$,

(iii) $r + s = v - 1$.

Additionally, the cases $(m,n) \in \{(3,5),(3,15),(5,15),(4,6),(4,8),(4,12),(4,16),(6,12),(8,16)\}$ are solved with a few possibly exceptions in [35].

In [30], factorizations of K_v into K_2-factors and C_m-factors are studied, and also new solutions to $\mathrm{HWP}(2m;\ m^r,(2m)^s)$ are given. Here, the problem of decomposing K_v^* into K_2^*-factors and $\vec{C}_m$-factors will be examined where $\vec{C}_m$ is the directed cycle of order m. Since K_2^* can be considered as $\vec{C}_2$, this problem can be included in the $\mathrm{HWP}^*(v;2^r,m^s)$. Afterwards, $\mathrm{HWP}^*(v;m^r,(2m)^s)$ will be studied.

In Section 2, we give some basic definitions and present some preliminary results that will be used in the next sections. In Section 3, we focus on finding solutions to $\mathrm{HWP}^*(v;2^r,m^s)$ for even m with $r+s=v-1$. Also a solution is denoted as a $\{(K_2^*)^r,\vec{C}_m^s\}$-factorization of K_v^*. In Section 4, we will concentrate on solving $\mathrm{HWP}^*(v;m^r,(2m)^s)$ for even m with $r+s=v-1$. Here are our main results.

Theorem 3. *Let r, s be nonnegative integers, and let $m \geq 4$ be even. Then* $\mathrm{HWP}^*(v;2^r,m^s)$ *has a solution if* $m|v$, $r+s=v-1$, $s \neq 1$, $(r,v) \neq (0,6)$, $(m,r,v) \neq (4,0,4)$, *and one of the following conditions holds:*

(i) $m > 4$, $s \neq 3$ *and* $m \equiv 0 \pmod{4}$,

(ii) $m > 4$, $m \equiv 2 \pmod{4}$, *and* $s \neq 3$ *when* $\frac{v}{m}$ *is odd,*

(iii) $m = 4$ *and* $v \equiv 0, 8, 16 \pmod{24}$,

(iv) $m = 4$, $v \equiv 12 \pmod{24}$ *and* $s \notin \{3, 5\}$,

(v) $m = 4$, $v \equiv 4, 20 \pmod{24}$ *and* r *is odd.*

Theorem 4. *Let* r, s *be nonnegative integers, and let* $m \geq 4$ *be even. Then* $\mathrm{HWP}^*(v; m^r, (2m)^s)$ *has a solution if and only if* $m|v$, $r + s = v - 1$ *and* $v \geq 4$ *except for* $(s, v, m) \in \{(0, 4, 4), (0, 6, 3), (0, 6, 6)\}$, *and except possibly when* $s \in \{1, 3\}$.

2. PRELIMINARY RESULTS

First, let us start with some definitions and notations that we will use throughout the paper.

Let G be a graph and $G_0, G_1, \ldots, G_{k-1}$ be k vertex disjoint copies of G with $v_i \in V(G_i)$ for each $v \in V(G)$. Let $G[k]$ denote the graph with vertex set $V(G[k]) = V(G_0) \cup V(G_1) \cup \cdots \cup V(G_{k-1})$ and edge set $E(G[k]) = \{\{u_i, v_j\} : \{u, v\} \in E(G) \text{ and } 0 \leq i, j \leq k - 1\}$. It is easy see that there is an $H[k]$-factorization of $G[k]$ if the graph G has an H-factorization. Note that $K_{(x:y)} \cong K_y[x]$.

If G_1 and G_2 are two edge-disjoint graphs with $V(G_1) = V(G_2)$, then we use $G_1 \oplus G_2$ to denote the graph on the same vertex set with $E(G_1 \oplus G_2) = E(G_1) \cup E(G_2)$. We will denote the vertex disjoint union of α copies of G by αG.

The above definitions can be extended to digraphs. Let D be a digraph and $D_0, D_1, \ldots, D_{k-1}$ be k vertex disjoint copies of D with $v_i \in V(D_i)$ for each $v \in V(D)$. Then $D[k]$ has the vertex set $V(D[k]) = V(D_0) \cup V(D_1) \cup \cdots \cup V(D_{k-1})$ and arc set $E(D[k]) = \{(u_i, v_j) : (u, v) \in E(D) \text{ and } 0 \leq i, j \leq k - 1\}$.

Let G be a digraph and $R(G)$ denote the digraph on the same vertex set as G but the arcs are taken in opposite directions.

Let us define some special factors and cycles that will be used throughout this article. Let F_m be a 1-factor of K_m with edge set $E(F_m) = \{\{0, m/2\}, \{i, m-i\} : 1 \leq i \leq (m/2)-1\}$ and let $\mathcal{C} = (0, 1, 2, m-1, 3, m-2, \ldots, \frac{m}{2}-1, \frac{m}{2}+2, \frac{m}{2}, \frac{m}{2}+1)$ be an m-cycle, which are the same as in Wlacki's construction. Also, we define a factor F_m^* as a K_2^*-factor of K_m^* with $E(F_m^*) = \{(0, m/2)^*, (i, m-i)^* : 1 \leq i \leq (m/2) - 1\}$ and $\mathcal{C}^*$ is the symmetric version of the $\mathcal{C}$.

Using the above factors and cycles, we can define Γ_m and Γ_m^* as $\mathcal{C}[2] \oplus F_m[2]$ and $\mathcal{C}^*[2] \oplus F_m^*[2]$, respectively. We use these notations for the rest of the paper.

Let A be a finite additive group and let S be a subset of A, where S does not contain the identity of A. The Directed Cayley graph $\overrightarrow{X}(A; S)$ on A with connection set S is a digraph with $V(\overrightarrow{X}(A; S)) = A$ and $E(\overrightarrow{X}(A; S)) = \{(x, y) : x, y \in A, y - x \in S\}$.

Let m be an even integer and the vertex set of K_{2m}^* be $\mathbb{Z}_{2m}$. Let I_{2m}^* be a K_2^*-factor of K_{2m}^* with $E\left(I_{2m}^*\right) = \{(i, m+i)^* : 0 \leq i \leq m-1\}$ and define the bijective function $f : \mathbb{Z}_{2m} \to \mathbb{Z}_2 \times \mathbb{Z}_m$ with

$$f(i) = \begin{cases} (0,i) & \text{if } i < m, \\ (1,i) & \text{if } i \geq m. \end{cases}$$

Then $E\left(I_{2m}^*\right)$ can be restated as a set $\left\{\big((0,i),(1,i)\big)^* : 0 \leq i \leq m-1\right\}$ on $\mathbb{Z}_2 \times \mathbb{Z}_m$ using this bijective function.

We will represent $C_m^*[2]$ and $C_m^*[2] \oplus I_{2m}^*$ as the directed Cayley graphs $\overrightarrow{X}\big(\mathbb{Z}_2 \times \mathbb{Z}_m, S\big)$ and $\overrightarrow{X}\big(\mathbb{Z}_2 \times \mathbb{Z}_m,\, S \cup \{(1,0)\}\big)$ where $S = \{(0,1),(1,1),(0,-1),(1,-1)\}$.

Also, the arc set of F_m^* which is denoted by $E\left(F_m^*\right)$, can be expressed as $\left\{\big((0,0),(0,m/2)\big)^*, \big((0,i),(0,m-i)\big)^* : 1 \leq i \leq (m/2)-1\right\}$ using above bijective function. Thus, we can represent the vertex set and the edge set of Γ_m^* as $V(\Gamma_m^*) = \mathbb{Z}_2 \times \mathbb{Z}_m$ and $E(\Gamma_m^*) = \bigcup_{j=0}^{m-1} \left\{\big((i,j),(i,j+1)\big)^*, \big((i,j),(i+1,j+1)\big)^*\right\} \cup E\left(F_m^*\right)$ for $i = 0, 1$, respectively.

Häggkvist used $G[2]$ to build 2-factorizations that include even cycles [22].

Lemma 5 (Häggkvist Lemma)**.** *Let G be a path or a cycle with n edges and let H be a 2-regular graph on $2n$ vertices with all components even cycle. Then $G[2] \cong G' \oplus G''$ where $G' \cong G'' \cong H$. Therefore, $G[2]$ has an H-decomposition.*

The following proposition, which is useful for transferring the results of undirected graphs to digraph and symmetric digraph, states that if we have an H-factorization of the undirected graph G, then using this factorization an H^*-factorization of G^* can be obtained.

Proposition 6. *Let G be a graph and let H be a subgraph of G. If G has an H-factorization, then G^* has an H^*-factorization.*

It is known that K_{2x} has a 1-factorization [32]. Therefore, as a natural consequence of Proposition 6, the following proposition can be stated.

Proposition 7. *The complete symmetric digraph K_{2x}^* has a K_2^*-factorization for every integer $x \geq 1$.*

The following result of Liu on equipartite graph has been helpful in solving the Oberwolfach and Hamilton-Waterloo Problems. We will use this result to obtain a $\overrightarrow{C}_m$-factorization of $K_{(x:y)}^*$.

Theorem 8 [27]**.** *The complete equipartite graph $K_{(x:y)}$ has a C_m-factorization for $m \geq 3$ and $x \geq 2$ if and only if $m \mid xy$, $x(y-1)$ is even, m is even if $y = 2$ and $(x,y,m) \neq (2,3,3),(6,3,3),(2,6,3),(6,2,6)$.*

The necessary and sufficient condition for the existence of a 1-factorization of a complete equipartite graph $K_{(x:y)}$ is given by Hoffman and Rodger [23].

Theorem 9 [23]**.** *The complete equipartite graph $K_{(x:y)}$ has a 1-factorization if and only if xy is even.*

The following lemma is a straightforward consequence of Proposition 6 and Theorem 9.

Lemma 10. *The complete symmetric equipartite digraph $K^*_{(x:y)}$ has a K^*_2-factorization if and only if even xy.*

The following two well-known results of Walecki imply that K_m (a 1-factor removed graph of K_m when m is even) decomposes into Hamilton cycles. We will use these results and Proposition 6 to factorize K^*_m into symmetric Hamilton cycles in Section 3.

Lemma 11 [28]**.** *For all odd $m \geq 3$, K_m decomposes into $\left(\frac{m-1}{2}\right)$ Hamilton cycles.*

Lemma 12 [28]**.** *For all even $m \geq 4, K_m - F_m$ has an Hamilton cycle decomposition with prescribed cycles $\left\{\mathcal{C}, \sigma\left(\mathcal{C}\right), \sigma^2\left(\mathcal{C}\right), \ldots, \sigma^{\frac{m-4}{2}}\left(\mathcal{C}\right)\right\}$ for $\sigma = (0)(1, 2, 3 \ldots, m-2, m-1)$.*

Lemmata 13 and 14 show the existence of the $\{C^r_m, C^s_{2m}\}$-factorization of the $C_m[2]$ and $\left(\mathcal{C} \oplus F_m\right)[2]$ for $r+s=2$ and $r+s=3$, respectively. They will be used to find a $\left\{(K^*_2)^r, \vec{C}^s_m\right\}$-factorization of the $C^*_m[2]$ for $r \in \{0, 2, 4\}$, $r+s=4$ and a $\vec{C}_{2m}$-factorization of $\Gamma^*_m = \mathcal{C}^*[2] \oplus F^*_m[2]$.

Lemma 13 [30]**.** *Let m be an integer with $m \geq 3$. Then $C_m[2]$ has a $\{C^r_m, C^s_{2m}\}$-factorization for nonnegative integers r and s with $r+s=2$ except when m is odd and $r=2$, and except possibly when m is even and $r=1$.*

Lemma 14 [30]**.** *Let $m \geq 4$ be an even integer and $\Gamma_m = \mathcal{C}[2] \oplus F_m[2]$. Then Γ_m has a*

(i) *C_{2m}-factorization,*

(ii) *C_m-factorization when $m \equiv 0 \pmod 4$, and*

(iii) *$\left\{C^2_m, C^1_{2m}\right\}$-factorization when $m \equiv 2 \pmod 4$.*

Lemma 15 [18]**.** *Let $m \geq 4$ be an even integer and x be a positive integer. Then $K^*_{(\frac{mx}{2}:2)}$ has a $\vec{C}_m$-factorization.*

Theorem 16 [8]**.** *The complete symmetric equipartite digraph $K^*_{(x:y)}$ has a $\vec{C}_3$-factorization if and only if $3 \mid xy$ and $(x,y) \neq (1,6)$ with possible exceptions $(x,y) = (x,6)$, where $x \notin \{m : m$ is divisible by a prime less than $17\}$.*

Theorem 8 states that $K_{(x:y)}$ has a C_m-factorization with a few exceptions. This result will be used to show that $K^*_{(x:y)}$ has a $\overrightarrow{C}_m$-factorization. However, some of the exceptions in the undirected version do not exist in the symmetric version. It is shown that there is actually a solution for these exceptions in the symmetric version. Francetić and Šajna gave the following general result for the $\overrightarrow{C}_t$-factorization of $K^*_{(x:y)}$. The necessity part of this theorem is a consequence of Lemma 15, Proposition 6 and Theorems 8 and 16.

Theorem 17 [19]. *Let x, y, and t be integers greater than 1, and let $g = \gcd(y,t)$. Assume one of the following conditions holds:*

(i) *$x(y-1)$ is even; or*

(ii) *$g \notin \{1,3\}$; or*

(iii) *$g = 1$, and $y \equiv 0 \pmod 4$ or $y \equiv 0 \pmod 6$; or*

(iv) *$g = 3$, and if $y = 6$, then x is divisible by a prime $p \leq 37$.*

Then the complete symmetric equipartite digraph $K^*_{(x:y)}$ has a $\overrightarrow{C}_t$-factorization if and only if $t \mid xy$ and t is even in case $y = 2$.

The following theorem presents a solution for the Directed Hamilton-Waterloo Problem for small even cycle factors. It will also help us in solving $\mathrm{HWP}^*(v; m^r, 2m^s)$ in Section 4, when $m = 4$.

Theorem 18 [35]. *For nonnegative integers r and s, $\mathrm{HWP}^*(v; m^r, n^s)$ has a solution for $(m,n) \in \{(4,6),(4,8),(4,12),(4,16),(6,12),(8,16)\}$ if and only if $r + s = v - 1$ and $\mathrm{lcm}(m,n)|v$.*

Using Lemmata 11 and 12, $K_{\frac{m}{2}}$ and $K_{\frac{m}{2}} - F_{\frac{m}{2}}$ factorize into $\left(\frac{m-2}{4}\right) C_{\frac{m}{2}}$ cycles and $\left(\frac{m-4}{4}\right) C_{\frac{m}{2}}$ cycles, respectively. Also, K_m is isomorphic to $K_{\frac{m}{2}}[2] \oplus I_m$. Hence, using Proposition 6, Lemmata 11 and 12, we will obtain a $\left\{\left(C^*_{\frac{m}{2}}[2]\right)^{\frac{m-6}{4}}, C^*_{\frac{m}{2}}[2] \oplus I^*_m\right\}$-factorization and a $\left\{\left(C^*_{\frac{m}{2}}[2]\right)^{\frac{m-12}{4}}, C^*_{\frac{m}{2}}[2] \oplus I^*_m, \Gamma^*_{\frac{m}{2}}\right\}$-factorization of K^*_m depending on whether $m \equiv 0$ or $2 \pmod 4$. Later, we will use these factorizations to obtain a $\{(K^*_2)^r, \overrightarrow{C}^s_m\}$-factorization of K^*_{mx}. Furthermore, we will need to have a $\{(K^*_2)^r, \overrightarrow{C}^s_{2m}\}$-factorization of $C^*_m[2]$ in order to factorize K^*_{mx} into K^*_2-factors and $\overrightarrow{C}_m$-factors.

Lemma 19. *Let $m \geq 4$ be an integer. Then $C^*_m[2]$ has a $\{(K^*_2)^r, \overrightarrow{C}^s_{2m}\}$-factorization for $r \in \{0,2,4\}$ and $r + s = 4$.*

Proof. First, note that $C_m[2]$ has a decomposition into two C_{2m}-factors by Häggkvist Lemma and each C_{2m}-factor has a decomposition into two 1-factors.

Case 1. ($r = 4$) Decompose $C_m[2]$ into four 1-factors by using C_{2m}-factors. Then a K^*_2-factorization of $C^*_m[2]$ is obtained by Proposition 6.

Case 2. ($r = 2$) Decompose $C_m[2]$ into one C_{2m} and two 1-factors. By Proposition 6, we get a $\{(K_2^*)^2, C_{2m}^*\}$-factorization of $C_m^*[2]$ and also C_{2m}^* has a $\vec{C}_{2m}$-factorization with two $\vec{C}_{2m}$-factors. So, we obtain a $\{(K_2^*)^2, \vec{C}_{2m}^2\}$-factorization of $C_m^*[2]$.

Case 3. ($r = 0$) Obtain a C_{2m}^*-factorization of $C_m^*[2]$ by Proposition 6. Since C_{2m}^* has a $\vec{C}_{2m}$-factorization with two $\vec{C}_{2m}$-factors, $C_m^*[2]$ has a $\vec{C}_{2m}$-factorization. ∎

Since I_{2m}^* and F_m^* are K_2^*-factors, the following result can be derived from Lemma 19.

Corollary 20. *Let $m \geq 4$ be an even integer. Then Γ_m^* has a $\{(K_2^*)^r, \vec{C}_{2m}^s\}$-factorization for $r \in \{0, 2, 4, 6\}$ with $r + s = 6$.*

Proof. $F_m^*[2]$ decomposes into two K_2^*-factors. Therefore, Γ_m^* has a $\{(K_2^*)^r, \vec{C}_{2m}^s\}$-factorization for $r \in \{2, 4, 6\}$ with $r + s = 6$ by Lemma 19. Also, Γ_m^* has a $\vec{C}_{2m}$-factorization by Lemma 14 and Proposition 6. ∎

The following lemma is quite useful in solving the Directed Hamilton-Waterloo Problem for $n = 2$ and even m when the values of r are even.

Lemma 21. *Let $m \geq 5$ be an integer. Then $C_m^*[2] \oplus I_{2m}^*$ has a $\{(K_2^*)^r, \vec{C}_{2m}^s\}$-factorization for $r \in \{0, 1, 3, 5\}$ and $r + s = 5$.*

Proof. The cases $r \in \{1, 3, 5\}$ can be directly obtained from Lemma 19.

When $r = 0$, we will examine the problem in two cases: m is odd or even.

Case 1. (odd $m \geq 5$) Define five directed $2m$-cycles in $C_m^*[2] \oplus I_{2m}^*$ as follows.
$\vec{C}_{2m}^{(0)} = (v_0, v_1, \ldots, v_{2m-1})$ where $v_i = \left(\lfloor \frac{i}{m} \rfloor, i\right)$,
$\vec{C}_{2m}^{(1)} = (u_0, u_1, \ldots, u_{2m-1})$ where,

$$u_{2i} = \begin{cases} (0, 2i) & \text{if } 0 \leq i \leq \frac{m-1}{2}, \\ (0, -2i-1) & \text{if } \frac{m+1}{2} \leq i \leq m-1, \end{cases}$$

and

$$u_{2i+1} = \begin{cases} (1, 2i+1) & \text{if } 0 \leq i \leq \frac{m-3}{2}, \\ (1, -2i-2) & \text{if } \frac{m-1}{2} \leq i \leq m-1, \end{cases}$$

$C_{2m}^{(2)} = (x_{0,}, x_1, \ldots, x_{2m-1})$ where

$$x_i = \begin{cases} \left(0, m - \lfloor \frac{i}{2} \rfloor\right) & \text{if } i \equiv 0, 3 \pmod 4, \\ \left(1, m - \lfloor \frac{i}{2} \rfloor\right) & \text{if } i \equiv 1, 2 \pmod 4, \end{cases} \quad \text{for } 0 \leq i \leq 2m-3,$$

and $x_{2m-2} = (1,1)$, $x_{2m-1} = (0,1)$. Also, $C_{2m}^{(3)} = (y_0, , y_1, \ldots, y_{2m-1})$ where

$$y_i = u_i + (1,2) \text{ for } 0 \le i \le m-3 \text{ and } m+2 \le i \le 2m-1,$$

$$y_{m-2} = (1,0),\ \ y_{m-1} = (0,1),\ \ y_m = (1,1),\ \ y_{m+1} = (0,0).$$

Finally, $\vec{C}_{2m}^{(4)} = (z_0, z_1, \ldots, z_{2m-1})$ where

$$z_i = x_i + (1,0) \text{ for } 3 \le i \le 2m-4,$$

$z_0 = (2,0)$, $z_1 = (0,m)$, $z_2 = (0,0)$, $z_{2m-1} = (1,1)$, $z_{2m-2} = (1,2)$, $z_{2m-3} = (0,1)$.

Then, $\{\vec{C}_{2m}^{(0)}, \vec{C}_{2m}^{(1)}, \vec{C}_{2m}^{(2)}, \vec{C}_{2m}^{(3)}, \vec{C}_{2m}^{(4)}\}$ is a $\vec{C}_{2m}$-factorization of $C_m^*[2] \oplus I_{2m}^*$.

Case 2. (even $m \ge 6$) Let $\vec{C}_{2m}^{(0)}$ be the same as in Case 1 and define the directed $2m$-cycles in $C_m^*[2] \oplus I_{2m}^*$ as follows.
$\vec{C}_{2m}^{(1)} = (x_0, x_1, \ldots, x_{2m-1})$ where $x_0 = (0,0)$ and

$$x_i = \begin{cases} \left(0, m - \lfloor \frac{i+2}{2} \rfloor\right) & \text{if } i \equiv 1,2 \pmod 4, \\ \left(1, m - \lfloor \frac{i+2}{2} \rfloor + 1\right) & \text{if } i \equiv 0,3 \pmod 4, \end{cases} \quad \text{for } 1 \leqslant i \leqslant 2m-8,$$

and $x_{2m-6+2i} = (0, 3-i)$ for $0 \le i \le 2$ and $x_{2m-7+2i} = (1, 3-i)$ for $0 \leqslant i \leqslant 3$. Also, $\vec{C}_{2m}^{(2)} = (u_0, u_1, \ldots, u_{2m-1})$ where $u_0 = (0,0)$, $u_1 = (1,0)$, $u_2 = (0, m-1)$ and

$$u_i = \begin{cases} \left(0, m - \lfloor \frac{i-1}{2} \rfloor - 1\right) & \text{if } i \equiv 0,1 \pmod 4, \\ \left(1, m - \lfloor \frac{i-1}{2} \rfloor\right) & \text{if } i \equiv 2,3 \pmod 4, \end{cases} \quad \text{for } 3 \le i \le 2m-9,$$

$$u_{2m-8+j} = \begin{cases} \left(0, 4 - \lfloor \frac{j}{2} \rfloor\right) \text{ if } j \equiv 0,2 \pmod 4, \\ \left(1, 4 - \lfloor \frac{j}{2} \rfloor\right) \text{ if } j \equiv 1,3 \pmod 4, \end{cases} \quad \text{for } 0 \leqslant j \leqslant 7, \text{ and when } m = 6,$$

$u_3 = (1,5)$ and we only use above piecewise function. $\vec{C}_{2m}^{(3)} = (y_0, y_1, \ldots, y_{2m-1})$ where $y_{2i+2} = (0, m-i)$ for $1 \le i \le m-4$, $y_{2i+1} = (1, m-i)$ for $1 \le i \le m-3$, $y_0 = (0,0)$, $y_1 = (1,1)$, $y_2 = (1,0)$, $y_{2m-4} = (1,2)$, $y_{2m-3} = (0,3)$, $y_{2m-2} = (0,2)$ and $y_{2m-1} = (0,1)$. $\vec{C}_{2m}^{(4)} = (z_0, z_1 \ldots, z_{2m-1})$ where $z_{9+2i} = (0, 4+i)$ for $1 \le i \le m-5$, $z_{10+2i} = (1, 4+i)$ for $0 \le i \le m-6$, $z_0 = (0,0)$, $z_1 = (1, m-1)$, $z_2 = (1,0)$, $z_3 = (0,1)$, $z_4 = (1,2)$, $z_5 = (1,1)$, $z_6 = (0,2)$, $z_7 = (1,3)$, $z_8 = (0,4)$, $z_9 = (0,3)$. Then $\left\{\vec{C}_{2m}^{(0)}, \vec{C}_{2m}^{(1)}, \vec{C}_{2m}^{(2)}, \vec{C}_{2m}^{(3)}, \vec{C}_{2m}^{(4)}\right\}$ is a $\vec{C}_{2m}$-factorization of $C_m^*[2] \oplus I_{2m}^*$. ■

By Lemma 13, we can decompose $C_m[2]$ into two C_m-factors for even m. So, we obtain the following lemma similar to Lemma 19. Also, the following corollaries are obtained as a result of this lemma.

Lemma 22. *Let $m \geq 4$ be an even integer. Then $C_m^*[2]$ has a $\{(K_2^*)^r, \vec{C}_m^s\}$-factorization for $r \in \{0,2,4\}$ with $r+s=4$.*

Corollary 23. *Let $m \geq 4$ be an even integer. Then $C_m^*[2] \oplus I_{2m}^*$ has a $\{(K_2^*)^r, \vec{C}_m^s\}$-factorization for $r \in \{1,3,5\}$ with $r+s=5$.*

Corollary 24. *Let $m \geq 4$ be an even integer. Then Γ_m^* has a $\{(K_2^*)^r, \vec{C}_m^s\}$-factorization for $r \in \{2,4,6\}$ with $r+s=6$.*

Recall that Γ_m^* is $\mathcal{C}^*[2] \oplus F_m^*[2]$.

Lemma 25. *Let $m \geq 4$ be an even integer. Γ_m^* has a $\{(K_2^*)^r, \vec{C}_m^s\}$-factorization for $m \equiv 2 \pmod 4$ and $r \in \{1,2,3,4,6\}$ with $r+s=6$.*

Proof. The cases $r \in \{2,4,6\}$ are obtained by Corollary 24.

For $r=1$, we define the following m-cycles.

$\vec{C}_m^{(0)} = (v_0, v_1, \ldots, v_{m-1})$ where $v_i = (0,i)$ for $0 \leqslant i \leqslant m-1$,

$$\vec{C}_m^{(1)} = (u_0, u_1, \ldots, u_{m-1}) \text{ where } u_i = \begin{cases} (0,i) & \text{if } i \text{ is even,} \\ (1,i) & \text{if } i \text{ is odd,} \end{cases}$$

$\vec{C}_m^{(2)} = (x_0, x_1, \ldots, x_{m-1})$ where $x_0 = (0,0)$ and for $1 \leq i \leq m-1$

$$x_i = \begin{cases} \left(\frac{1-(-1)^i}{2}, \frac{m}{2} - \left\lfloor \frac{i}{2} \right\rfloor\right) & \text{if } i \equiv 1,2 \pmod 4, \\ \left(\frac{1-(-1)^i}{2}, \frac{m}{2} + \left\lfloor \frac{i}{2} \right\rfloor\right) & \text{if } i = 0,3 \pmod 4, \end{cases}$$

$\vec{C}_m^{(3)} = (z_0, z_1, \ldots, z_{m-1})$ where $z_m = (1, m-1)$, $z_{m-1} = (0,0)$ and

$$z_i = \begin{cases} \left(0, \left\lfloor \frac{i}{2} \right\rfloor + 1\right) & \text{if } i \equiv 0 \pmod 4, \\ \left(1, \left\lceil \frac{i}{2} \right\rceil + 1\right) & \text{if } i \equiv 1 \pmod 4, \\ (0, m-i) & \text{if } i \equiv 2 \pmod 4, \\ (1, m-i) & \text{if } i \equiv 3 \pmod 4, \end{cases} \quad \text{for } 0 \leq i \leq \frac{m}{2}.$$

Let us choose the factor F_0 as isomorphic to $F_m^* \oplus (F_m^* + (1,0))$, then F_0 becomes a K_2^*-factor. Using the above m-cycles, we obtain five m-cycle factors: $F_1 = \vec{C}_m^{(0)} \cup (\vec{C}_m^{(0)} + (1,0))$, $F_2 = R(F_1)$, $F_3 = \vec{C}_m^{(1)} \cup R(\vec{C}_m^{(1)} + (1,0))$, $F_4 = \vec{C}_m^{(2)} \cup R(\vec{C}_m^{(2)} + (1,0))$, and $F_5 = \vec{C}_m^{(3)} \cup (\vec{C}_m^{(3)} + (1,0))$. Then $\{F_0, F_1, F_2, F_3, F_4, F_5\}$ is a $\{(K_2^*)^1, \vec{C}_m^5\}$-factorization of Γ_m^*.

For $r=3$, $F_1 \oplus F_2$ is a C_m^*-factor of Γ_m^* and has a factorization into two K_2^*-factors of Γ_m^* say $F_1^{'}$ and $F_2^{'}$. Then $\{F_0, F_1^{'}, F_2^{'}, F_3, F_4, F_5\}$ is a $\{(K_2^*)^3, \vec{C}_m^3\}$-factorization of Γ_m^*. ■

3. SOLUTIONS TO HWP$^*(v; 2^r, m^s)$

Now, we can give solutions to the Directed Hamilton-Waterloo Problem for K_2^* and $\vec{C}_m$ when even m.

Theorem 26. *Let r, s be nonnegative integers, and let $m \geq 6$ be even. Then* HWP$^*(v; 2^r, m^s)$ *has a solution if and only if $m|v$, $r+s = v-1$ and $v \geq 6$ except for $s = 1$ or $(r, v) = (0, 6)$, and except possibly when at least one of the following conditions holds:*

(i) $s = 3$ *and* $m \equiv 0 \pmod 4$,

(ii) $s = 3$, $m \equiv 2 \pmod 4$ *and* $\frac{v}{m}$ *is odd.*

Proof. Take $(v-2)$ disjoint K_2^*-factors of K_v^*, say $H_1^*, H_2^*, \ldots, H_{v-2}^*$. It is obvious that $K_v^* - (H_1^* \oplus H_2^* \oplus \cdots \oplus H_{v-2}^*)$ is a K_2^*-factor in K_v^*. Thus, there is no $\{(K_2^*)^{v-2}, \vec{C}_m^1\}$-factorization of K_v^*. Therefore, we may assume $s \neq 1$.

Since HWP$^*(v; n^r, m^s)$ has a solution for $r = 0$ except for $(v, m) = (6, 6)$ by Theorem 1, we may assume that $r \geq 1$.

Let $v = mx$ for a positive integer x. Partition the vertices of K_{mx}^* into $2x$ sets of size $\frac{m}{2}$, represent each part of $\frac{m}{2}$ vertices in K_{mx}^* with a single vertex and represent all double arcs between sets of size $\frac{m}{2}$ as a single double arc, to get a K_{2x}^*. By Proposition 7, K_{2x}^* has a decomposition into $(2x-1)$ K_2^*-factors. Then construct a K_m^*-factor of K_{mx}^* from one of the K_2^*-factors, and a $K_{(\frac{m}{2}:2)}^*$-factor of K_{mx}^* from each of the remaining $(2x-2)$ K_2^*-factors. Then K_{mx}^* can be factorized into a K_m^*-factor and $(2x-2)$ $K_{(\frac{m}{2}:2)}^*$-factors.

By Lemmata 10 and 15, $K_{(\frac{m}{2}:2)}^*$ decomposes into $\frac{m}{2}$ K_2^*-factors or $\frac{m}{2}$ $\vec{C}_m$-factors, respectively. As a result, we must decompose K_m^* into K_2^*-factors and $\vec{C}_m$-factors.

Case 1. (odd r) By Lemma 12, factorize K_m into an F_m-factor and $\left(\frac{m-2}{2}\right)$ C_m-factors. So, K_m^* can be factorized into an F_m^*-factor and $\left(\frac{m-2}{2}\right)$ C_m^*-factors by Proposition 6.

Since C_m^* can be decomposed into two K_2^*-factors or two $\vec{C}_m$-factors for even m, K_m^* has a $\{(K_2^*)^{2r_1+1}, \vec{C}_m^{2s_1}\}$-factorization where $r_1 + s_1 = \frac{m-2}{2}$.

Since K_{mx}^* has a $\left\{K_m^*, \left(K_{(\frac{m}{2}:2)}^*\right)^{(2x-2)}\right\}$-factorization, placing a K_2^*-factorization on r_0 of the $K_{(\frac{m}{2}:2)}^*$ factors for r_0 even and $0 \leq r_0 \leq 2x-2$, a $\vec{C}_m$-factorization on s_0 of the $K_{(\frac{m}{2}:2)}^*$ where $r_0 + s_0 = 2x-2$, and taking a $\{(K_2^*)^{2r_1+1}, \vec{C}_m^{2s_1}\}$-factorization of K_m^* give a $\{(K_2^*)^r, \vec{C}_m^s\}$-factorization of K_{mx}^* where $r = \frac{m}{2}r_0 + 2r_1+1$ and $s = \frac{m}{2}s_0+2s_1$ with $r+s = \frac{m}{2}(r_0+s_0)+2(r_1+s_1)+1 = mx-1 = v-1$.

Since any nonnegative odd integer $1 \leq r \leq mx-1$ can be written as $r = \frac{m}{2}r_0 + 2r_1 + 1$ for integers $0 \leq r_0 \leq 2x-2$ and $0 \leq r_1 \leq \frac{m-2}{2}$, a solution to

HWP$^*(v;2^r,m^s)$ exists for each odd $r\geq 1$ and $s\geq 1$ satisfying $r+s=mx-1=v-1$.

Case 2. (even r)

(a) Assume $m\equiv 0 \pmod 4$. Therefore, $\frac{m}{2}$ is even. Each $K^*_{(\frac{m}{2}:2)}$ decompose into $\frac{m}{2}$ K^*_2-factors or $\frac{m}{2}$ $\vec{C}_m$-factors. For this reason we need a $\{(K^*_2)^r,\vec{C}^s_m\}$-factorization of K^*_m for even r.

Also, $K^*_{\frac{m}{2}}$ can be factorized as $\bigoplus_{i=1}^{\frac{m-4}{4}} C^*_i\oplus F^*_{\frac{m}{2}}$ where each C^*_i is isomorphic to $C^*_{\frac{m}{2}}$. Then, $K^*_{\frac{m}{2}}[2]\cong\bigoplus_{i=1}^{\frac{m-4}{4}} C^*_i[2]\oplus F^*_{\frac{m}{2}}[2]$. Also, K^*_m is isomorphic to $K^*_{\frac{m}{2}}[2]\oplus I^*_m$. Therefore, K^*_m has a $\left\{\left(C^*_{\frac{m}{2}}[2]\right)^{\frac{m-12}{4}},C^*_{\frac{m}{2}}\oplus I^*_m,\Gamma^*_{\frac{m}{2}}\right\}$-factorization. By Lemma 19, each of $\frac{m-12}{4}$ $C^*_{\frac{m}{2}}[2]$-factors has a $\{(K^*_2)^{r_0},\vec{C}^{s_0}_m\}$-factorization for $r_0\in\{0,2,4\}$ and $r_0+s_0=4$. By Lemma 21, $C^*_{\frac{m}{2}}[2]\oplus I^*_m$ has a $\{(K^*_2)^{r_1},\vec{C}^{s_1}_m\}$-factorization for $r_1\in\{0,1,3,5\}$ and $r_1+s_1=5$. By Corollary 20, $\Gamma^*_{\frac{m}{2}}$ has a $\{(K^*_2)^{r_2},\vec{C}^{s_2}_m\}$-factorization for even m and $r_2\in\{0,2,4,6\}$ with $r_2+s_2=6$. Those factorizations give a $\{(K^*_2)^{r'},\vec{C}^{s'}_m\}$-factorization of K^*_m where $r'=\left(\frac{m-12}{4}\right)r_0+r_1+r_2$ and $s'=\left(\frac{m-12}{4}\right)s_0+s_1+s_2$ satisfying $r'+s'=\left(\frac{m-12}{4}\right)4+5+6=m-1$ with $0\leq r',s'\leq m-1$. If we choose $r_1=0$, we obtain a $\{(K^*_2)^{r'},\vec{C}^{s'}_m\}$-factorization of K^*_m for even r'. Since we cannot get $r_0=1$, $r_1=2$ or $r_2=3$ from the above factorizations, it can be seen that $r'=m-4$ cannot be obtained.

Placing a K^*_2-factorization on r'' of the $K^*_{(\frac{m}{2}:2)}$-factors for $0\leq r''\leq 2x-2$, a $\vec{C}_m$-factorization on s'' of the $K^*_{(\frac{m}{2}:2)}$ for $r''+s''=2x-2$, and taking a $\{(K^*_2)^{r'},\vec{C}^{s'}_m\}$-factorization of K^*_m give a $\left\{(K^*_2)^{\frac{m}{2}r''+r'},\vec{C}_m^{\frac{m}{2}r''+s'}\right\}$-factorization of K^*_{mx} where $\frac{m}{2}r''+r'$ is even.

Any even integer $1\leq r\leq mx-1$ can be written as $r=\frac{m}{2}r''+r'$ for integers $r'\in[0,m-1]$ and $0\leq r''\leq 2x-2$. Since $r'\neq m-4$, a solution to HWP$^*(v;2^r,m^s)$ exists for each even $r\geq 2$ except possibly $r=mx-4=v-4$ and $s\geq 1$ satisfying $r+s=v-1$.

(b) Assume $m\equiv 2 \pmod 4$. By Lemma 11, factorize K_n into $\left(\frac{n-1}{2}\right)$ C_n-factors for odd n, and get a C^*_n-factorization of K^*_n by Proposition 6. Also, K^*_m can be factorized as $K^*_{\frac{m}{2}}[2]\oplus I^*_m$. Since $\frac{m}{2}$ is odd, K^*_m has a $\left\{\left(C^*_{\frac{m}{2}}[2]\right)^{\frac{m-2}{4}},I^*_m\right\}$-factorization. By Lemma 19, each of $C^*_{\frac{m}{2}}[2]$-factors has a $\{(K^*_2)^{r_0},\vec{C}^{s_0}_m\}$-factorization for $r_0\in\{0,2,4\}$ and $r_0+s_0=4$. By Lemma 21, $C^*_{\frac{m}{2}}[2]\oplus I^*_m$ has $\{(K^*_2)^{r_1},\vec{C}^{s_1}_m\}$-factorization for $r_1\in\{0,1,3,5\}$ and $r_1+s_1=5$.

Those factorizations give a $\{(K^*_2)^{r_2},\vec{C}^{s_2}_m\}$-factorization of K^*_m for $r_2=\frac{m-6}{4}r_0+r_1$ and $s_2=\frac{m-6}{4}s_0+s_1$ with $r_2+s_2=m-1$. Since we cannot get $r_0=1$ or $r_1=2$ from the above factorizations, it can be seen that $r_2=m-4$ cannot be obtained.

Placing a K_2^*-factorization on r' of the $K^*_{(\frac{m}{2}:2)}$ factors for $0 \le r' \le 2x-2$ where we choose r' is even, a $\vec{C}_m$-factorization on s' of the $K^*_{(\frac{m}{2}:2)}$ with $r'+s' = 2x-2$, and taking a $\{(K_2^*)^{r_2}, \vec{C}_m^{s_2}\}$-factorization of K_m^* give a $\left\{(K_2^*)^{\frac{m}{2}r'+r_2}, \vec{C}_m^{\frac{m}{2}s'+s_2}\right\}$-factorization of K^*_{mx} where $r = \frac{m}{2}r' + r_2$ and $s = \frac{m}{2}s' + s_2$. Also, we obtain the requested even integer $r \in [1, mx-1]$, from the sum of $\frac{m}{2}r'$ and r_2 for integers $0 \le r' \le 2x-2$ and $r_2 \in [0, m-1]$. Since $r_2 \ne m-4$, a solution to HWP$^*(v; 2^r, m^s)$ exists for even $r \ge 2$ except possibly $r = mx-4 = v-4$ and odd $s \ge 1$ satisfying $r+s = v-1$.

If x is even, say $x = 2t$, factorize K^*_{mx} into a K^*_{2m}-factor and $(2t-2)$ $K^*_{(m:2)}$-factors. $K^*_{(m:2)}$ has a K_2^*-factorization with m K_2^*-factors and a $\vec{C}_m$-factorization with m $\vec{C}_m$-factors by Lemmata 10 and 15, respectively. So, we must decompose K^*_{2m} into K_2^*-factors and $\vec{C}_m$-factors. As before, K^*_{2m} can be factorized as $K_m^*[2] \oplus I^*_{2m}$. So, K^*_{2m} has a $\left\{\left(C_m^*[2]\right)^{\frac{m-4}{2}}, I^*_{2m}, \Gamma_m^*\right\}$-factorization. By Lemma 22, each of $C_m^*[2]$-factors has a $\{(K_2^*)^{r_0}, \vec{C}_m^{s_0}\}$-factorization for $r_0 \in \{0, 2, 4\}$ and $r_0 + s_0 = 4$. By Corollary 23, $C_m^*[2] \oplus I^*_{2m}$ has a $\{(K_2^*)^{r_1}, \vec{C}_m^{s_1}\}$-factorization for $r_1 \in \{1, 3, 5\}$ and $r_1 + s_1 = 5$. By Lemma 25, Γ_m^* has a $\{(K_2^*)^{r_2}, \vec{C}_m^{s_2}\}$-factorization for $m \equiv 2 \pmod 4$ and $r_2 \in \{1, 2, 3, 4, 6\}$ with $r_2 + s_2 = 6$. Using these factorizations, we obtain a solution to the problem for $r = 2mt-4 = mx-4$ when $m \equiv 2 \pmod 4$ and even x. As a result, HWP$^*(v; 2^r, m^s)$ has a solution for $r = v-4$ and even $\frac{v}{m}$ when $m \equiv 2 \pmod 4$. ∎

Lemma 27. *$C_4^*[2] \oplus I_8^*$ has a $\{(K_2^*)^r, \vec{C}_4^s\}$-factorization for $r \in \{0, 1, 2, 3, 5\}$ with $r+s = 5$.*

Proof. We represent $C_4^*[2] \oplus I_8^*$ as the directed Cayley graph $\vec{X}(\mathbb{Z}_8, S)$ with connection set $S = \{\pm 1, \pm 3, 4\}$.

For $r = 0$, we define a $\vec{C}_4$-factorization of $C_4^*[2] \oplus I_8^*$:

$\mathcal{F}_1 = \{[(0,1,2,3),(4,5,6,7)], [(0,3,2,1),(4,7,6,5)], [(0,5,1,4),(2,7,3,6)], [(0,4,3,7),(1,5,2,6)], [(0,7,2,5),(1,6,3,4)]\}$.

For $r = 2$, we define a $\{(K_2^*)^2, \vec{C}_4^3\}$-factorization of $C_4^*[2] \oplus I_8^*$:

$\mathcal{F}_2 = \{[(0,4)^*,(1,5)^*,(2,6)^*,(3,7)^*], [(0,7)^*, (1,6)^*, (2,5)^*, (3,4)^*], [(0,1,2,3),(4,5,6,7))], [(0,3,6,5),(1,4,7,2)], [(0,5,4,1),(2,7,6,3)]\}$.

The remaining cases are obtained from Corollary 23 for $m = 4$. ∎

Lemma 28. *K_{12}^* has a $\{(K_2^*)^r, \vec{C}_4^s\}$-factorization for $r \in \{0, 1, 2, 3, 4, 5, 7, 9, 11\}$ with $r+s = 11$.*

Proof. The cases $r = 0$ and $r = 11$ are obtained by Theorem 18 and Proposition 7, respectively. Since $K_{12}-I$ has a C_4-factorization where I is a 1-factor of K_{12}, by Proposition 6, K_{12}^* can be factorized into five C_4^*-factors and one I^*-factor which is a K_2^*-factor of K_{12}^*. Also, C_4^* has a $\vec{C}_4$-factorization and K_2^*-factorization. So, we obtain a $\{(K_2^*)^r, \vec{C}_4^s\}$-factorization of K_{12}^* for $r \in \{1,3,5,7,9\}$ with $r+ s = 11$.

We represent K_{12}^* as the directed Cayley graphs $\vec{X}(\mathbb{Z}_{12}, S)$ with connection set $S = \{\pm 1, \pm 2, \pm 3, \pm 4, \pm 5, 6\}$, and define the following factorizations of K_{12}^* for $r = 2, 4$, respectively.

$\mathcal{F}_1 = \{ [(0,6)^*,(1,7)^*, (2,8)^*,(3,9)^*, (4,10)^*,(5,11)^*], [(0,10)^*,(4,6)^*, (1,5)^*, (7,11)^*, (2,9)^*, (3,8)^*], [(0,1,2,3), (4,5,6,7), (8,9,10,11)], [(0,2,1,4), (3,5,7,6), (8,11,10,9)], [(0,3,1,8), (2,4,11,6), (5,9,7,10)], [(0,4,2,11), (1,6,8,10), (3,7, 9,5)], [(0,5,8,7), (1,3,4,9), (2,10,6,11)], [(0,7,5,2), (1,10,8,4), (3,6,9,11)], [(0,8,6,1), (2,5,10,7), (3,11,9,4)], [(0,9,6,5), (1,11,4,8), (2,7,3,10)], [(0,11, 1,9), (2,6,10,3), (4,7,8,5)]\}$,

$\mathcal{F}_2 = \{ [(0,6)^*, (1,7)^*, (2,8)^*, (3,9)^*, (4,10)^*, (5,11)^*], [(0,10)^*, (4,6)^*, (1,5)^*, (7,11)^*, (2,9)^*, (3,8)^*], [(0,8)^*, (2,6)^*, (1,10)^*, (4,7)^*, (3,11)^*, (5,9)^*], [(0,1)^*, (2,3)^*, (4,5)^*, (6,7)^*, (8,9)^*, (10,11)^*], [(0,2,1,3), (4,8,11,9), (5,7,10,6)], [(0,3, 10,5), (1,8,6,11), (2,4,9,7)], [(0,4,11,2), (1,6,10,9), (3,5,8,7)], [(0,5,6,9), (1,2, 11,4), (3,7,8,10)], [(0,7,9,11), (1,4,3,6), (2,10,8,5)], [(0,9,10,7), (1,11,6,8), (2,5,3,4)], [(0,11,8,4), (1,9,6,3), (2,7,5,10)]\}$.

Therefore, K_{12}^* has a $\{(K_2^*)^r, \vec{C}_4^s\}$-factorization for $r \in \{0,1,2,3,4,5,7,9,11\}$ with $r + s = 11$. ∎

Lemma 29. *$K_{(4:3)}^*$ has a $\{(K_2^*)^r, \vec{C}_4^s\}$-factorization for $r \in \{0,1,2,4,6,8\}$ with $r + s = 8$.*

Proof. The cases $r = 0$ and $r = 8$ are obtained by Theorem 17 and Lemma 10, respectively. By Theorem 8, $K_{(4:3)}$ has a C_4-factorization and so, $K_{(4:3)}^*$ has a C_4^*-factorization by Proposition 6. Since C_4^* has a K_2^*-factorization and a $\vec{C}_4$-factorization, $K_{(4:3)}$ can be factorized into two K_2^*-factors and six $\vec{C}_4$-factors. Similarly, a $\{(K_2^*)^r, \vec{C}_4^s\}$-factorization of $K_{(4:3)}$ is obtained for $r \in \{4,6\}$ with $r + s = 8$.

Finally, let $V(K_{(4:3)}^*) = \bigcup_{i=0}^{2}\{4i, 4i+1, 4i+2, 4i+3\}$ with the obvious vertex partition, and define the following factorization of $K_{(4:3)}^*$ for $r = 1$.

$\mathcal{F}_1 = \{[(0,4,2,5), (1,8,3,11), (6,9,7,10)], [(0,5,1,7), (2,9,4,11), (3,8,6,10)], [(0,7,1,9), (2,4,3,10), (5,11,6,8)], [(0,8,1,10), (2,7,3,5), (4,9,6,11)], [(0,9, 2,11), (1,5,3,6), (4,10,7,8)], [(0,10,4,8), (1,11,5,9), (2,6,3,7)], [(0,11,3,4), (1,6,2,10), (5,8,7,9)], [(0,6)^*, (1,4)^*, (2,8)^*, (3,9)^*, (5,10)^*, (7,11)^*]\}$. ∎

In Theorem 26, we have given the necessary and sufficient conditions for the existence of a solution for $\mathrm{HWP}^*(v; 2^r, m^s)$ for even $m \geq 6$. The construction in Theorem 26 is not valid when $m = 4$, therefore we also examine the case of $m = 4$ in the following theorem.

Theorem 30. *Let r, s be nonnegative integers. Then* $\mathrm{HWP}^*(v; 2^r, 4^s)$ *has a solution if and only if $r + s = v - 1$ except for $s = 1$ or $(r, v) = (0, 4)$, and except possibly when at least one of the following conditions holds:*

(i) $r \geq 2$ *even and* $v \equiv 4, 20 \pmod{24}$,

(ii) $s \in \{3, 5\}$ *and* $v \equiv 12 \pmod{24}$.

Proof. If you remove $(v - 2)$ disjoint K_2^*-factors from K_v^*, then the remaining factor must be a K_2^*-factor in K_v^*. Thus, there is no $\{(K_2^*)^{v-2}, \vec{C}_4^1\}$-factorization of K_v^*. So, we may assume $s \neq 1$.

Since $\mathrm{HWP}^*(v; n^r, m^s)$ has a solution for $r = 0$ except for $(v, m) = (4, 4)$ by Theorem 1, $\mathrm{HWP}^*(4; 2^r, 4^s)$ has no solution for $r = 0$. As a result, we may assume that $r \geq 1$.

Case 1. ($v \equiv 0 \pmod 8$) Let $v = 8k$ for a positive integer k. Note that, K_{8k}^* can be factorized as $K_{4k}^*[2] \oplus I_{8k}^*$. Also, $K_{4k}^*[2]$ can be factorized into $C_4^*[2]$-factors and a $K_2^*[2]$-factor. The graph $kC_4^*[2] \oplus I_{8k}^*$ can be considered as $(C_4^*[2] \oplus I_8^*)$-factor in K_{8k}^*. Therefore, K_{8k}^* has a $\left\{\left(C_4^*[2]\right)^{2k-1}, I_8^*, K_2^*[2]\right\}$-factorization. Also, $C_4^*[2]$ has a $\left\{\left(K_2^*\right)^{r_0}, \vec{C}_4^{s_0}\right\}$-factorization for $r_0 \in \{0, 2, 4\}$ where $r_0 + s_0 = 4$ by Lemma 22. Since $K_2^*[2] = C_4^*$, $K_2^*[2]$ has a $\left\{\left(K_2^*\right)^{r_1}, \vec{C}_4^{s_1}\right\}$-factorization for $r_1 \in \{0, 2\}$ and $r_1 + s_1 = 2$. By Lemma 27, $C_4^*[2] \oplus I_8^*$ has a $\{(K_2^*)^{r_2}, \vec{C}_4^{s_2}\}$-factorization for $r_2 \in \{0, 1, 2, 3, 5\}$ where $r_2 + s_2 = 5$. These factorizations give a $\left\{\left(K_2^*\right)^r, \vec{C}_4^s\right\}$-factorization of K_{8k}^* for $r \neq 8k - 2$ with $r + s = 8k - 1$.

Then, $\mathrm{HWP}^*(v; 2^r, 4^s)$ has a solution for $r + s = v - 1$, $s \neq 1$ and $v \equiv 0 \pmod 8$.

Case 2. ($v \equiv 4 \pmod 8$) Let $v = 8k + 4$ for a nonnegative integer k.

(a) Assume r is odd. Partition the vertices of K_{8k+4}^* into $4k + 2$ sets of size 2, represent each set of size 2 vertices in K_{8k+4}^* with a single vertex and represent all double arcs between sets of size 2 as a single double arc, to get a K_{4k+2}^*. By Proposition 7, K_{4k+2}^* has a decomposition into $4k+1$ K_2^*-factors. Construct a K_4^*-factor from one of the K_2^*-factors and a $K_{(2:2)}^*$-factor from each of the remaining $4k$ K_2^*-factors. Then, factorize K_{8k+4}^* into a K_4^*-factor and $(4k)$ $K_{(2:2)}^*$-factors. K_4^* has a decomposition into one K_2^* and two $\vec{C}_4$-factors or three K_2^*-factors, and $K_{(2:2)}^*$ has a $\left\{\,(K_2^*)^{r_0}\,, \vec{C}_4^{s_0}\right\}$-factorization for $r_0 \in \{0, 2\}$ satisfying $r + s = 2$. So, K_{8k+4}^* has a $\left\{(K_2^*)^r, \vec{C}_4^s\right\}$-factorization for odd r. Therefore, $\mathrm{HWP}^*(v; 2^r, 4^s)$ has a solution for odd r and $v \equiv 4 \pmod 8$.

(b) Assume r is even, and also let $k \equiv 1 \pmod 3$. Then, we have $v = 24l + 12$ for some nonnegative integer l.

Representing each part of 4 vertices in K^*_{24l+12} with a single vertex and all double arcs between parts of size 4 as a single double arc, we have a K^*_{6l+3}. Since a Kirkman triple system exists for orders $6l+3$, we have a C_3-factorization of K_{6l+3}. Then a C_3^*-factorization of K^*_{6l+3} is obtained by Proposition 6.

Construct a K^*_{12}-factor from one of the C_3^*-factors and $K^*_{(4:3)}$-factor from each of the remaining $3l$ C_3^*-factors. Then get a $\{K^*_{12}, (K^*_{(4:3)})^{3l}\}$-factorization of K^*_{24l+12}. By Lemma 28, K^*_{12} has a $\{(K_2^*)^{r_0}, \overrightarrow{C}_m^{s_0}\}$-factorization for $r_0 \in \{0,1,2,3,4, 5,7,9,11\}$ with $r_0 + s_0 = 11$. Also, $K^*_{(4:3)}$ has a $\{(K_2^*)^{r_1}, \overrightarrow{C}_4^{s_1}\}$-factorization by Lemma 29 for $r_1 \in \{0,1,2,4,6,8\}$ with $r_1 + s_1 = 8$. Those factorizations give a $\{(K_2^*)^{r}, \overrightarrow{C}_m^{s}\}$-factorization of K^*_{24l+12} where $r = r_0 + ar_1$ and $s = s_0 + bs_1$ satisfying $r + s = 24l + 11 = v - 1$ with $1 \le r, s \le v - 1$ and $a + b = 3l$. We obtain the requested even $r \in [0, v-1]$ except for $r = v - 6$ and $r = v - 4$, from the sum of r_0 and ar_1. Then $\mathrm{HWP}^*(v; 2^r, 4^s)$ has a solution for $r + s = v - 1$, $s \notin \{3, 5\}$ and $v \equiv 12 \pmod{24}$. ■

Proving Theorem 3 was accomplished by proving Theorems 26 and 30.

4. SOLUTIONS TO $\mathrm{HWP}^*(v; m^r, (2m)^s)$

In this section, we prove that for even m, a solution to $\mathrm{HWP}^*(v; m^r, (2m)^s)$ exists for $r + s = v - 1$ and except possibly when $s \in \{1, 3\}$.

Firstly, factorize K^*_{2mx} into a K^*_{2m}-factor and $(2x-2)$ $K^*_{(m:2)}$-factors. $K^*_{(m:2)}$ has a $\{\overrightarrow{C}_m^r, \overrightarrow{C}_{2m}^s\}$-factorization for $r \in \{0, m\}$ and $r + s = m$. Using Lemma 12 and Proposition 6, a $\{(C_m^*[2])^{\frac{m-4}{2}}, I^*_{2m}, \Gamma^*_m\}$-factorization of K^*_{2m} is also obtained. Therefore, in order to factorize K^*_{2mx} into $\overrightarrow{C}_m$-factors and $\overrightarrow{C}_{2m}$-factors, Γ^*_m, $C_m^*[2] \oplus I^*_{2m}$ and $C_m^*[2]$ must be factorized into $\overrightarrow{C}_m$-factors and $\overrightarrow{C}_{2m}$-factors. The following lemmata examine the existence of a $\{\overrightarrow{C}_m^r, \overrightarrow{C}_{2m}^s\}$-factorization of these graphs for $r + s \in \{4, 5, 6\}$.

Lemma 31. *Let $m \ge 4$ be an even integer. Then Γ^*_m has a $\{\overrightarrow{C}_m^r, \overrightarrow{C}_{2m}^s\}$-factorization for $r \in \{0, 6\}$ and $r + s = 6$.*

Proof. *Case* 1. $(r = 0)$ By Lemma 14(i) and Proposition 6, Γ^*_m has a $\overrightarrow{C}_{2m}$-factorization.

Case 2. $(r = 6)$ By Lemma 14(ii) and Proposition 6, Γ^*_m has a $\overrightarrow{C}_m$-factorization for $m \equiv 0 \pmod 4$.

When $m \equiv 2 \pmod 4$, define the following m-cycles. Also, let $\overrightarrow{C}_m^{(0)}$ and $\overrightarrow{C}_m^{(1)}$ be the cycles $\overrightarrow{C}_m^{(0)}$ and $\overrightarrow{C}_m^{(2)}$ respectively, as stated in Lemma 25.

$\vec{C}_m^{(2)} = (u_0, u_1, \ldots, u_{m-1})$ where $u_i = \begin{cases} (1, m-1-i) & \text{if } 0 \le i \le \frac{m}{2}, \\ (0, m-1-i) & \text{if } \frac{m}{2}+1 \le i \le m-1. \end{cases}$

$\vec{C}_m^{(3)} = (y_0, y_1, \ldots, y_{m-1})$ where $y_0 = (0,0)$, $y_1 = \left(0, \frac{m}{2}\right)$, $y_2 = \left(1, \frac{m}{2}+1\right)$, $y_3 = \left(1, \frac{m}{2}-1\right)$ and

$$y_i = \begin{cases} \left(1, \frac{m}{2} + (-1)^{i+1} \left\lfloor \frac{i}{2} \right\rfloor\right) \text{ if } i \equiv 0,1 \pmod 4, \\ \left(0, \frac{m}{2} + (-1)^{i} \left\lfloor \frac{i}{2} \right\rfloor\right) \quad \text{ if } i \equiv 2,3 \pmod 4, \end{cases} \text{for } 4 \le i \le m-1.$$

$\vec{C}_m^{(4)} = (z_0, z_1, \ldots, z_{m-1})$ where

$$z_i = \begin{cases} y_{m-i} + (1,0) & \text{if } 1 \le i \le m-3, \\ y_{m-i} & \text{if } m-2 \le i \le m. \end{cases}$$

Using the above m-cycles, we obtain the following m-cycle factors. $F_0 = \vec{C}_m^{(0)} \cup \left(\vec{C}_m^{(0)} + (1,0)\right)$, $F_1 = \vec{C}_m^{(1)} \cup R(\vec{C}_m^{(1)} + (1,0))$, $F_2 = R(F_1)$, $F_3 = \vec{C}_m^{(2)} \oplus \left(\vec{C}_m^{(2)} + (1,0)\right)$, $F_4 = \vec{C}_m^{(3)} \cup \left(\vec{C}_m^{(3)} + (1,0)\right)$ and $F_5 = \vec{C}_m^{(4)} \cup \left(\vec{C}_m^{(4)} + (1,0)\right)$. Then $\{F_0, F_1, F_2, F_3, F_4, F_5\}$ is a $\vec{C}_m$-factorization of Γ_m^*. So, Γ_m^* has a $\vec{C}_m$-factorization for even $m \ge 4$. ■

Lemma 32. *Let $m \ge 4$ be an even integer. Then $C_m^*[2] \oplus I_{2m}^*$ has a $\{\vec{C}_m^r, \vec{C}_{2m}^s\}$-factorization for $r \in \{1,3\}$ and $r + s = 5$.*

Proof. *Case* 1. $(r = 1)$ Let $\vec{C}_m^{(0)} = (v_0, v_1, \ldots, v_{m-1})$ be a directed m-cycle of $C_m^*[2] \oplus I_{2m}^*$, where $v_i = (0,i)$ for $0 \le i \le m-1$, and it can be checked that $F_1 = \vec{C}_m^{(0)} \cup \left(\vec{C}_m^{(0)} + (1,0)\right)$ is a directed m-cycle factor of $C_m^*[2] \oplus I_{2m}^*$. Also, let $\vec{C}_{2m}^{(1)} = (u_0, u_1, \ldots, u_{2m-1})$ be a directed $2m$-cycle of $C_m^*[2] \oplus I_{2m}^*$, where $u_{2i} = (0,i)$, and $u_{2i+1} = (1,i)$ for $0 \le i \le m-1$. Similarly, it can be checked that $F_2 = \vec{C}_{2m}^{(1)}$ and $F_3 = \vec{C}_{2m}^{(1)} + (1,0)$ are arc disjoint directed $2m$-cycle factors of $C_m^*[2] \oplus I_{2m}^*$.

Let $\vec{C}_{2m}^{(2)} = (x_0, x_1, \ldots, x_{2m-1})$ be a directed $2m$-cycle of $C_m^*[2] \oplus I_{2m}^*$, where $x_0 = (0,0)$, $x_m = (1,0)$, $x_{i+1} = (0, m-1-i)$ for $0 \le i \le m-2$ and $x_{j+1+m} = (1, m-1-j)$ for $0 \le j \le m-2$.

Let $\vec{C}_{2m}^{(3)} = (y_0, y_1, \ldots, y_{2m-1})$ be a directed $2m$-cycle of $C_m^*[2] \oplus I_{2m}^*$, where $y_m = (1,0)$,

$$y_i = \begin{cases} (0, m-i) & \text{if } i \text{ is even}, \\ (1, m-i) & \text{if } i \text{ is odd}, \end{cases} \quad \text{for } 0 \le i \le m-1,$$

and

$$y_i = \begin{cases} (1, 2m-i) & \text{if } i \text{ is even}, \\ (0, 2m-i) & \text{if } i \text{ is odd}, \end{cases} \quad \text{for } m+1 \le i \le 2m-1.$$

The factors $F_4 = \vec{C}_{2m}^{(2)}$ and $F_5 = \vec{C}_{2m}^{(3)}$ are arc disjoint directed $2m$-cycle factors of $C_m^*[2] \oplus I_{2m}^*$. Then $\{F_1, F_2, F_3, F_4, F_5\}$ is a $\{\vec{C}_m^1, \vec{C}_{2m}^4\}$-factorization of $C_m^*[2] \oplus I_{2m}^*$.

Case 2. $(r = 3)$ Let F_1, F_2 and F_3 be the same as in Case 1. Using the arcs of $F_4 \cup F_5$, we obtain two new $\vec{C}_m$-factors.

The factor $F_4' = R(F_1)$ is a $\vec{C}_m$-factor of $C_m^*[2] \oplus I_{2m}^*$. Let $\vec{C} = (y_0, y_1, \ldots, y_{m-1})$ be a directed m-cycle of $C_m^*[2] \oplus I_{2m}^*$, where

$$y_i = \begin{cases} (0, i) & \text{if } i \text{ is even}, \\ (1, i) & \text{if } i \text{ is odd}, \end{cases} \quad \text{for } 0 \leq i \leq m-1.$$

It can be checked that $F_5' = R(\vec{C}) \cup R(\vec{C} + (1, 0))$ is a directed m-cycle factor of $C_m^*[2] \oplus I_{2m}^*$.

So, $\{F_1, F_2, F_3, F_4', F_5'\}$ is a $\{\vec{C}_m^3, \vec{C}_{2m}^2\}$-factorization of $C_m^*[2] \oplus I_{2m}^*$. ■

Lemma 33. *Let $m \geq 4$ be an even integer. Then $C_m^*[2]$ has a $\{\vec{C}_m^r, \vec{C}_{2m}^s\}$-factorization for $r \in \{0, 2, 4\}$ and $r + s = 4$.*

Proof. The cases $r \in \{0, 4\}$ are obtained by Lemmata 19 and 22. Let $\vec{C}_{2m}^{(1)} = (u_0, u_1, \ldots, u_{2m-1})$ be a directed $2m$-cycle of $C_m^*[2]$, where

$$u_i = \begin{cases} (0, i) & \text{if } 0 \leq i \leq m-1, \\ (1, i) & \text{if } m \leq i \leq 2m-1. \end{cases}$$

And it can be checked that $F_1 = \vec{C}_{2m}^{(1)}$ is a $\vec{C}_{2m}$-factor of $C_m^*[2]$. Let $\vec{C}_{2m}^{(2)} = (v_0, v_1, \ldots, v_{2m-1})$ be a directed $2m$-cycle of $C_m^*[2]$, where

$$v_i = \begin{cases} u_i & \text{if } i \text{ is even}, \\ u_i + (1, 0) & \text{if } i \text{ is odd}. \end{cases}$$

The factor $F_2 = \vec{C}_{2m}^{(2)}$ is a $\vec{C}_{2m}$-factor of $C_m^*[2]$. Let F_4' and F_5' be the same as in Lemma 32. Then $\{F_1, F_2, F_4', F_5'\}$ is a $\{\vec{C}_m^2, \vec{C}_{2m}^2\}$-factorization of $C_m^*[2]$. ■

The proof of Theorem 4 can now be given.

Theorem 4. *Let r, s be nonnegative integers, and let $m \geq 4$ be even. Then* $\mathrm{HWP}^*(v; m^r, (2m)^s)$ *has a solution if and only if $m | v$, $r + s = v - 1$ and $v \geq 4$ except for $(s, v, m) \in \{(0, 4, 4), (0, 6, 3), (0, 6, 6)\}$, and except possibly when $s \in \{1, 3\}$.*

Proof. By Theorem 18, HWP$^*(v;4^r,8^s)$ has a solution for $r+s=v-1$, so we may assume that $m \geq 6$. Furthermore, by Theorem 1, a solution to the HWP$^*(v;m^r,(2m)^s)$ exists when $r=0$ or $s=0$ and except for $(s,v,m) \in \{(0,4,4),\ (0,6,3),(0,6,6)\}$.

Factorize K^*_{2mx} into a K^*_{2m}-factor and $(2x-2)$ $K^*_{(m:2)}$-factors. By Theorem 17, $K^*_{(m:2)}$ decomposes into m $\vec{C}_m$-factors or m $\vec{C}_{2m}$-factors. So, K^*_{2m} must be decomposed into $\vec{C}_m$-factors and $\vec{C}_{2m}$-factors. As before, K^*_{2m} can be factorized as $K^*_m[2] \oplus I^*_{2m}$. Consequently, K^*_{2m} has a $\{(C^*_m[2])^{\frac{m-4}{2}}, I^*_{2m}, \Gamma^*_m\}$-factorization. By Lemma 33, each of $C^*_m[2]$-factors has a $\{\vec{C}^{r_0}_m, \vec{C}^{s_0}_{2m}\}$-factorization for $r_0 \in \{0,2,4\}$ and $r_0+s_0=4$. By Lemmata 32 and 21, $C^*_m[2] \oplus I^*_{2m}$ has a $\{\vec{C}^{r_1}_m, \vec{C}^{s_1}_{2m}\}$-factorization for $r_1 \in \{0,1,3\}$ and $r_1+s_1=5$. By Lemma 31, Γ^*_m has a $\{\vec{C}^{r_2}_m, \vec{C}^{s_2}_{2m}\}$-factorization for $r_2 \in \{0,6\}$ with $r_2+s_2=6$. Those factorizations give a $\{\vec{C}^{r}_m, \vec{C}^{s}_{2m}\}$-factorization of K^*_{2m} where $r=\left(\frac{m-6}{2}\right)r_0+r_1+r_2$ and $s=\left(\frac{m-6}{2}\right)s_0+s_1+s_2$ satisfying $r+s=\left(\frac{m-6}{2}\right)4+5+6=2m-1$ with $0 \leq r,s \leq 2m-1$ and $s \notin \{1,3\}$.

Placing a $\vec{C}_m$-factorization on r' of the $K^*_{(m:2)}$-factors for $0 \leq r' \leq 2x-2$, a $\vec{C}_{2m}$-factorization on s' of the $K^*_{(m:2)}$ for $r'+s'=2x-2$, and taking a $\{\vec{C}^{r}_m, \vec{C}^{s}_{2m}\}$-factorization of K^*_{2m} give a $\{\vec{C}^{mr'+r}_m, \vec{C}^{ms'+s}_{2m}\}$-factorization of K^*_{2mx}. Then HWP$^*(v;m^r,2m^s)$ has a solution except possibly when $s \in \{1,3\}$. ∎

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Received 20 June 2023
Revised 26 April 2024
Accepted 26 April 2024
Available online 20 June 2024