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DOUBLE DOMINATING SEQUENCES IN BIPARTITE AND CO-BIPARTITE GRAPHS

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Abstract

In a graph $G = (V, E)$, a vertex $u \in V$ dominates a vertex $v \in V$ if $v \in N_G[u]$. A sequence $S = (v_1, v_2, \ldots, v_k)$ of vertices of G is called a double dominating sequence of G if (i) for each i, the vertex v_i dominates at least one vertex $u \in V$ which is dominated at most once by the previous vertices of S and, (ii) all vertices of G have been dominated at least twice by the vertices of S. Grundy Double Domination problem asks to find a double dominating sequence of maximum length for a given graph G. In this paper, we prove that the decision version of the problem is NP-complete for bipartite and co-bipartite graphs. We look for the complexity status of the problem in the class of chain graphs which is a subclass of bipartite graphs. We use dynamic programming approach to solve this problem in chain graphs and propose an algorithm which outputs a Grundy double dominating sequence of a chain graph G in linear-time.

Keywords: double dominating sequences, bipartite graphs, chain graphs, NP-completeness, graph algorithms.

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1. Introduction

For a graph $G = (V, E)$, a set $D \subseteq V$ is called a *dominating set* of G, if for each vertex $x \in V$, $N_G[x] \cap D \neq \emptyset$. The Minimum Domination problem is to find a dominating set of a graph G having minimum cardinality. One of the

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fundamental problems in graph theory is the Minimum Domination problem and there is a huge amount of literature on this topic, see [7–10]. Further, Fink and Jacobson introduced the concept of double domination [4, 5]. For a graph G with no isolated vertices, a set $D \subseteq V$ is called a *double dominating set* of G, if for every vertex $x \in V$, $|N_G[x] \cap D| \geq 2$.

In 2014, Brešar *et al.* introduced the concept of dominating sequences. A motivation for introducing dominating sequences came from the well known domination game in which we get a vertex sequence as an outcome of a two-player game, played on a graph. For detailed description, one may refer [2].

Formally, a *dominating sequence* of G is a sequence S of vertices of G such that (i) each vertex of S dominates at least one vertex of G which was not dominated by any of the previous vertices of S, and (ii) every vertex of G is dominated by at least one vertex of S. The Grundy Domination problem is to find a longest dominating sequence of a given graph G. The Grundy Domination Decision (GDD) problem is the decision version of the Grundy Domination problem.

Recently, Haynes *et al.* proposed various kinds of vertex sequences, each of which is specified in terms of some conditions that must be satisfied by every subsequent vertex in the sequence [6]. Predictably, double domination in the sequence context is one of these variations. Before formally presenting the definitions related to this variant, we mention that for a sequence S, consisting of distinct vertices of a graph G, the corresponding set of vertices is denoted by $\widehat{S}$.

A sequence $S = (v_1, v_2, \ldots, v_n)$ is called a *double neighborhood sequence* of G if for each i, the vertex v_i dominates at least one vertex u of G which is dominated at most once by the vertices $v_1, v_2, \ldots, v_{i-1}$. If $\widehat{S}$ is a double dominating set of G, then we call S a *double dominating sequence* of G. A double dominating sequence of G with maximum length is called a *Grundy double dominating sequence* of G. The length of a Grundy double dominating sequence is the *Grundy double domination number* of G and is denoted by $\gamma_{gr}^{\times 2}(G)$. Given a graph G with no isolated vertices, the Grundy Double Domination (GD2) problem asks to find a Grundy double dominating sequence of G. The decision version of the Grundy Double Domination problem is as follows.

Decision Version: Grundy Double Domination Decision (GD2D) Problem
Input: A graph $G = (V, E)$ with no isolated vertices and $k \in \mathbb{Z}^+$.
Question: Is there a double dominating sequence of G of length at least k?

This concept was introduced in a slightly different manner by Haynes *et al.* in [6]. In their version, S_i denotes the subsequence $(v_1, v_2, \ldots, v_i)$ which consists of the first i vertices of S. If for each i, the vertex $v_i \in \widehat{S}$ dominates at least one vertex $x \in V(G) \setminus \widehat{S_{i-1}}$ which is dominated at most once by the vertices in $\widehat{S_{i-1}}$ and S is of maximal length, then S is called a *double dominating sequence* of G.

This definition does not obey the property that $\widehat{S}$ is a double dominating set of G. Brešar *et al.* introduced the former definition of double dominating sequences and argued that two invariants are equal in all graphs [3]. So, in this paper, we only consider the former version of double dominating sequences.

The Grundy double domination number of a tree T is exactly the number of vertices of T [6]. Recently, Brešar *et al.* proved that the GD2D problem is NP-complete for split graphs and can be solved efficiently for threshold graphs [3]. Here, we extend the literature of this variant by studying it for bipartite graphs.

The structure of the paper is as follows. In Section 2, we give some basic definitions and notations used throughout the paper. In Section 3, we prove that the GD2D problem is NP-complete even when restricted to bipartite and co-bipartite graphs. On the positive note, we present a linear-time algorithm for determining the Grundy double domination number of chain graphs in Section 4. Finally, we conclude the paper in Section 5.

2. Preliminaries

All graphs considered in this paper are simple, undirected and connected. Let $[n] = \{1, 2, \ldots, n\}$ for any positive integer n. Given a graph G, the *open neighborhood* of a vertex x is $N_G(x) = \{y \in V(G) : xy \in E(G)\}$, while the *closed neighborhood* of x is $N_G[x] = N_G(x) \cup \{x\}$. Two vertices $u, v \in V(G)$ are called *open twins* (*closed twins*) if $N_G(u) = N_G(v)$ ($N_G[u] = N_G[v]$). For a graph $G = (V, E)$, the subgraph induced on a set $U \subseteq V$, denoted by $G[U]$, is the subgraph of G whose vertex set is U and whose edge set consists of all edges in G that have both endpoints in U.

A complete graph on n vertices is denoted by K_n. An *independent set* of G is a set of vertices $A \subseteq V(G)$ such that no two vertices of A are adjacent in G. A *bipartite graph* $G = (X, Y, E)$ is a graph whose vertex set can be partitioned into two independent sets X and Y. The *complement* of G, denoted by $\overline{G}$, is the graph obtained by removing the edges of G and adding the edges that are not in G. A *co-bipartite graph* is a graph which is the complement of a bipartite graph. A bipartite graph $G = (X, Y, E)$ is a *chain graph* if there exists an ordering $\alpha = (x_1, x_2, \ldots, x_{n_1}, y_1, y_2, \ldots, y_{n_2})$ of vertices of G such that $N(x_1) \subseteq N(x_2) \subseteq \cdots \subseteq N(x_{n_1})$ and $N(y_1) \supseteq N(y_2) \supseteq \cdots \supseteq N(y_{n_2})$, where $X = \{x_1, x_2, \ldots, x_{n_1}\}$ and $Y = \{y_1, y_2, \ldots, y_{n_2}\}$. The ordering α is called a *chain ordering* of G and it can be found in linear-time [11].

Recall that a relation on a set A is a subset of $A \times A$. We define a relation R on the vertex set of a chain graph $G = (X, Y, E)$ such that two vertices u and v of G are related if and only if they are open twins. It is easy to see that R is an equivalence relation so it provides a partition P of $V(G)$. Let $\{X_1, X_2, \ldots, X_k\}$

and $\{Y_1, Y_2, \dots, Y_k\}$ be the parts obtained from the relation R for the X and Y side, respectively. We write the partition P as $\{X_1, X_2, \dots, X_k, Y_1, Y_2, \dots, Y_k\}$. We keep the order of the sets in P so that it is satisfied that $N(X_1) \subset N(X_2) \subset \dots \subset N(X_k)$ and $N(Y_1) \supset N(Y_2) \supset \dots \supset N(Y_k)$. For each $i, j \in [k]$, it is easy to see that $N(X_i) = \bigcup_{r=1}^{i} Y_r$ and $N(Y_j) = \bigcup_{r=j}^{k} X_r$.

For two vertex sequences $S_1 = (v_1, \dots, v_n)$ and $S_2 = (u_1, \dots, u_m)$, in G, the *concatenation* of these two sequences is defined as the sequence $S_1 \oplus S_2 = (v_1, \dots, v_n, u_1, \dots, u_m)$. For an ordered set $A = \{u_1, u_2, \dots, u_k\}$ of vertices, (A) denotes the sequence of vertices $(u_1, u_2, \dots, u_k)$.

3. NP-Completeness

3.1. Bipartite graphs

Recall that the GD2D problem is NP-complete for general graphs [3]. In this subsection, we prove that the problem remains NP-complete for bipartite graphs.

Let $\mathcal{H} = (\mathcal{X}, \mathcal{E})$ be a hypergraph with no isolated vertices. An *edge cover* of $\mathcal{H}$ is a set of hyperedges from $\mathcal{E}$ that covers all vertices $\mathcal{X}$ of $\mathcal{H}$. A *legal hyperedge sequence* of $\mathcal{H}$ is a sequence of hyperedges $\mathcal{C} = (C_1, \dots, C_r)$ of $\mathcal{H}$ such that, for each i, $i \in [r]$, C_i covers a vertex not covered by C_j, for each $j < i$. In addition, if the set $\widehat{\mathcal{C}}$ is an edge cover of $\mathcal{H}$, then $\mathcal{C}$ is called an *edge covering sequence* of $\mathcal{H}$. The maximum length of an edge covering sequence of $\mathcal{H}$ is denoted by $\rho_{gr}(\mathcal{H})$. The Grundy Covering problem asks to find an edge covering sequence of $\mathcal{H}$ having size $\rho_{gr}(\mathcal{H})$. The Grundy Covering Decision (GCD) problem is the decision version of the Grundy Covering problem.

It is known that GCD problem is NP-hard in general graphs [1]. For $k \leq 2$, we can find an edge covering sequence of the hypergraph $\mathcal{H}$ of length at least k in polynomial time. So, the GCD problem is NP-complete for $k \geq 3$.

Theorem 1. *The* GD2D *problem is NP-complete for bipartite graphs.*

Proof. It is clear that the GD2D problem is in class NP. To show the NP-hardness, we give a polynomial reduction from the GCD problem in hypergraphs which is known to be NP-hard [1]. Given a hypergraph $\mathcal{H} = (\mathcal{X}, \mathcal{E})$ with $|\mathcal{X}| = n$ and $\mathcal{E} = \{\mathcal{E}_1, \mathcal{E}_2, \dots, \mathcal{E}_m\}$, $n, m \geq 2$, we construct an instance $G = (X^*, Y^*, E^*)$ of the GD2D problem, where G is a bipartite graph, as follows. $X^* = I \cup X'$ and $Y^* = \mathcal{E}'$, where $I = \{v_1, v_2, \dots, v_m\}$, $X' = \{x_1, x_2, \dots, x_n\}$ and $\mathcal{E}' = \{\alpha, e_1, e_2, \dots, e_m\}$. A vertex of X' corresponds to a vertex of $\mathcal{X}$ in the hypergraph $\mathcal{H}$ and the vertex e_i of $\mathcal{E}'$ corresponds to the hyperedge $\mathcal{E}_i$ of $\mathcal{H}$. Now, a vertex x of X' is adjacent to a vertex of $e_i \in \mathcal{E}'$ in G if and only if $x \in \mathcal{E}_i$ in $\mathcal{H}$. Each vertex of I is adjacent to each vertex of $\mathcal{E}'$ in G. Clearly, G is a bipartite graph. Figure 1 illustrates the construction of G when $\mathcal{H}$ is the hypergraph given by

$(X = \{x_1, x_2, x_3, x_4\}, \mathcal{E} = \{\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3, \mathcal{E}_4\})$, where $\mathcal{E}_1 = \{x_1, x_2, x_4\}$, $\mathcal{E}_2 = \{x_2, x_3\}$, $\mathcal{E}_3 = \{x_1, x_2\}$ and $\mathcal{E}_4 = \{x_2, x_3, x_4\}$.

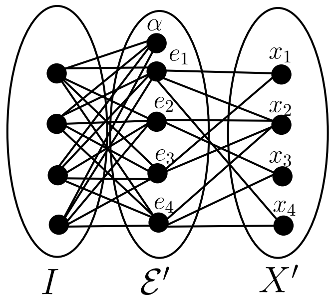

Figure 1. Construction of bipartite graph G from the hypergraph H.

Now, we show that $\rho_{gr}(\mathcal{H}) \geq k$ if and only if $\gamma_{gr}^{\times 2}(G) \geq n + m + k + 1$, for $k \geq 3$. First, let $(\mathcal{E}_{i_1}, \mathcal{E}_{i_2}, \ldots, \mathcal{E}_{i_{k'}})$ be an edge covering sequence of size at least k in $\mathcal{H}$. Then the sequence $(x_1, x_2, \ldots, x_n, v_1, v_2, \ldots, v_m, \alpha, e_{i_1}, e_{i_2}, \ldots, e_{i_{k'}})$ is a double dominating sequence of size at least $n + m + k + 1$ in G. So, we have $\gamma_{gr}^{\times 2}(G) \geq n + m + k + 1$.

For the converse part, we give a claim first.

Claim 1. *There exists a double dominating sequence of G of size at least $n + m + k + 1$ in which the first vertex from $\mathcal{E}'$ is the vertex α.*

Proof. Let S be a double dominating sequence of G of size at least $n+m+k+1$ and e_0 be the first vertex from $\mathcal{E}'$ appearing in S. If $e_0 = \alpha$, then there is nothing to prove. So, we assume that $e_0 \neq \alpha$. Now, we see that e_0 appears to dominate some vertices of both I and X'. If the vertex α appears in the sequence somewhere after e_0, we modify the sequence by putting the vertex α immediately before e_0 in the sequence. If the vertex α does not appear in the sequence, we modify the sequence by replacing the vertex e_0 by α. Note that the modified sequence, in both situations, is also a double dominating sequence of G of size at least $n + m + k + 1$ and now, the first vertex from $\mathcal{E}'$ appearing in the sequence is α.

Hence, the claim is true. □

Let S be a double dominating sequence of size at least $n + m + k + 1$ in any chosen bipartite graph G satisfying Claim 1. Note that $|\hat{S} \cap \mathcal{E}'| \geq k + 1$.

As $k \geq 3$, let e be the second vertex coming from $\mathcal{E}'$ in S. Now, let A denote the set of vertices appearing before the vertex α in S, B denote the set of vertices appearing after the vertex α and before the vertex e. Finally, let C denote the set of vertices appearing after the vertex e in S.

Claim 2. $|I \cap (A \cup B)| \geq 2$.

Proof. Clearly, $|I \cap \hat{S}| \geq k \geq 3$. Since all neighbors of vertices of I have been dominated twice before the appearance of last vertex of I in the sequence, the last vertex from I in the sequence dominates itself and so, it cannot appear after e. Indeed, $I \cap C = \emptyset$. We also have, $|I \cap \hat{S}| = |I \cap (A \cup B)| + |I \cap C|$. Thus, $|I \cap (A \cup B)| \geq 2$ holds true. □

Claim 3. *There exists a double dominating sequence S_0 of G of size at least $n + m + k + 1$ satisfying Claim 1 such that $\widehat{S_0} \cap X' = X'$ and all vertices of X' appear before the vertex e in the sequence S_0.*

Proof. Here, we have two cases to consider.

Case 1. *There is a vertex $x \in X'$ which does not appear in the sequence S.* Let $x \in X'$ be a vertex such that it does not appear in the sequence S. This tells that there are two vertices coming from $\mathcal{E}'$, say e_i and e_j which appear in S and they dominate the vertex x first and second time respectively. Let v^* denote the vertex which appears just before the vertex e in the sequence S. Then, we see that the vertices e_i and e_j appear in the sequence S after v^*. Using Claim 2, we know that before the vertex e all vertices of $\mathcal{E}'$ are dominated twice. So, vertex e_j is appearing to dominate vertices of X' only.

Now, there are two possibilities. First, assume that e_i is appearing only to dominate the vertex x first time, then we modify S by adding x just before e and removing the vertex e_i from S. But, if e_i was appearing to dominate some vertices of $\{e_i\} \cup I \cup (X' \setminus \{x\})$ also, then we modify S by putting the vertex x just before e in the sequence. By this modification, we removed at most one vertex from the sequence and added a new vertex to S. Thus, S remains a double dominating sequence of size at least $n + m + k + 1$ in G with x appearing in S before the vertex e.

Case 2. *There is a vertex $x \in X' \cap \hat{S}$ which appears after e in S.* In this case, vertex x is appearing to dominate itself only. Since all vertices of $\mathcal{E}'$ are dominated twice before the vertex e, so we remove the vertex x from its place and put it just before e. Note that the size of S is not changed and so, S remains a double dominating sequence of size at least $n+m+k+1$ in G with x appearing in S before the vertex e. Therefore, the claim holds true. □

Claim 3 ensures that we can assume that $\widehat{S} \cap X' = X'$ and all vertices of X' appear before the vertex e in the sequence S. Combining all claims, we get that $|\hat{S} \cap (\mathcal{E}' \setminus \{\alpha\})| \geq k$ and these vertices of $(\mathcal{E}' \setminus \{\alpha\})$ are appearing only to dominate vertices of X' second time. So, these vertices of $\hat{S} \cap (\mathcal{E}' \setminus \{\alpha\})$ correspond to a legal hyperedge sequence of size at least k in the hypergraph $\mathcal{H}$. So, $\rho_{gr}(\mathcal{H}) \geq k$.

Therefore, the GD2D problem is NP-complete for bipartite graphs. ■

3.2. Co-bipartite graphs

In this subsection, we prove that the problem also remains NP-complete for co-bipartite graphs. For this, we give a polynomial reduction from the GDD problem in general graphs when $k \geq 4$, which is already known to be NP-complete [1]. Given a graph $G = (V, E)$ with $V = \{v_1, v_2, \ldots, v_n\}$ $(n \geq 2)$, we construct an instance $G' = (V', E')$ of the GD2D problem in the following way.

Define the vertex set V' as $V' = V_1 \cup V_2 \cup V_3$, where $V_r = \{v_i^r : i \in [n]\}$ for each r, $1 \leq r \leq 3$. Add the edges in G' in the following way. (i) Add the edges so that $G'[V_1]$ and $G'[V_2 \cup V_3]$ are complete subgraphs of G'. (ii) If $v_j \in N_G[v_i]$, then add an edge between v_i^1 and v_j^2. (iii) For each $i \in [n]$, add the edge $v_i^1 v_i^3$ in G'. Formally, define $E' = \{v_i^1 v_j^1, v_i^2 v_j^2, v_i^3 v_j^3 : 1 \leq i < j \leq n\} \cup \{v_i^2 v_j^3 : 1 \leq i \leq j \leq n\} \cup \{v_i^1 v_j^2 : v_j \in N_G[v_i]\} \cup \{v_i^1 v_i^3 : i \in [n]\}$. Clearly, G' is a co-bipartite graph. Figure 2 illustrates the construction of G' from a graph G.

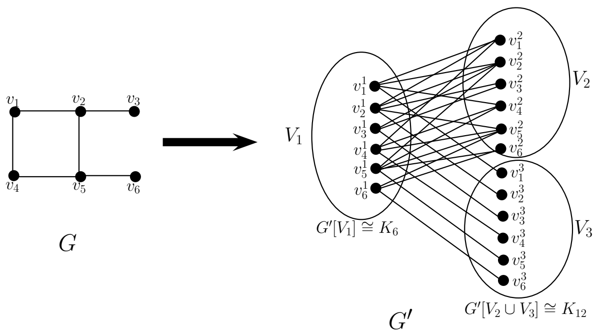

Figure 2. Construction of co-bipartite graph G' from the graph G.

To prove the NP-hardness of the GD2D problem in co-bipartite graphs, it is enough to prove the following theorem.

Theorem 2. *Let G' be the co-bipartite graph constructed from a graph $G = (V, E)$ with $V = \{v_1, v_2, \ldots, v_n\}$ $(n \geq 2)$ as explained above. Then, $\gamma_{gr}(G) \geq k$ if and only if $\gamma_{gr}^{\times 2}(G') \geq n + k$, for $k \geq 4$.*

Proof. First, let $S = (v_{i_1}, v_{i_2}, \ldots, v_{i_t})$ be a dominating sequence of G of size t, where $t \geq k$. Then the sequence $(v_1^3, v_2^3, \ldots, v_n^3, v_{i_1}^2, v_{i_2}^2, \ldots, v_{i_t}^2)$ is a double dominating sequence of G' of size at least $n+k$. So, we get that $\gamma_{gr}^{\times 2}(G') \geq n+k$.

Conversely, let S be a double dominating sequence of G' having size at least $n+k$. Now, we claim that there exists a double dominating sequence S^* of G' in which the following is true.

1. $V_1 \cap \widehat{S^*} = \emptyset$,
2. $V_3 \subseteq \widehat{S^*}$,

3. all vertices of V_3 appear at initial n places of the sequence S^*.

If S satisfies all the above conditions then there is nothing to prove. So, assume that $V_1 \cap \widehat{S} \neq \emptyset$. Then either $|V_1 \cap \widehat{S}| \geq 2$ or $|V_1 \cap \widehat{S}| \leq 1$.

If $|V_1 \cap \widehat{S}| \geq 2$, let v_i^1 and v_j^1 be the first two vertices of V_1 which are appearing in S. Let A be the subset of $\widehat{S}$ which contains vertices of S appearing before v_i^1 in S. Let B be the subset of $\widehat{S}$ which contains vertices of S appearing after v_i^1 in S and before v_j^1. Finally, let C be the subset of $\widehat{S}$ which contains vertices of S appearing after v_j^1 in S. Now, we claim that $|(A \cup B) \cap (V_2 \cup V_3)| \geq 3$. To see this, if $|(A \cup B) \cap (V_2 \cup V_3)| \leq 2$ then $|(A \cup B) \cap (V_2 \cup V_3)| = 0$ or 1 or 2. In each of these three cases, we have, $|(A \cup B) \cap (V_2 \cup V_3)| + |C \cap (V_2 \cup V_3)| \leq 2$. So, $|\widehat{S} \cap (V_2 \cup V_3)| \leq 2$ which further implies that $|\widehat{S}| \leq n + 2$. This contradicts the assumption that $k \geq 4$. So, $|(A \cup B) \cap (V_2 \cup V_3)| \geq 3$. In this case, we see that all vertices of G' are dominated twice upto the appearance of vertex v_j^1. Thus, we get that $C = \emptyset$. Note that $|V_1 \cap \widehat{S}| = 2$. Since all vertices of $V_2 \cup V_3$ have been dominated twice before the appearance of v_j^1 and $v_i^1 \in V_1$ also appears before v_j^1, we get that v_j^1 appears to dominate some vertex $v_k^1 \in V_1$ second time. As, no vertex of G' appears after v_j^1, we have that $v_k^3 \notin \widehat{S}$. Now, we modify S by replacing the vertex v_j^1 by v_k^3 and get a new double dominating sequence of G' of same size. Hence, we can say that there exists a double dominating sequence of G' in which at most 1 vertex of V_1 appears. So, we assume that $|V_1 \cap \widehat{S}| \leq 1$.

First, we assume that $|V_1 \cap \widehat{S}| = 1$ and $V_1 \cap \widehat{S} = \{v_i^1\}$. Again, let A be the subset of $\widehat{S}$ which contains vertices of S appearing before v_i^1 in S and B be the subset of $\widehat{S}$ which contains vertices of S appearing after v_i^1 in S. Now, either $|A \cap (V_2 \cup V_3)| \leq 1$ or $|A \cap (V_2 \cup V_3)| \geq 2$. If $|A \cap (V_2 \cup V_3)| \leq 1$ then, after the first three vertices appearing in the sequence, all vertices of $V_2 \cup V_3$ are dominated twice and all vertices of V_1 are dominated at least once. Consequently, the subsequent vertices in the sequence must dominate some vertices of V_1 for the second time, and as a result, there cannot be more than n such vertices in the sequence. Hence, $|\widehat{S}| \leq n + 3$. This contradicts the assumption that $k \geq 4$. So, we have that $|A \cap (V_2 \cup V_3)| \geq 2$. Note that v_i^1 appears after at least two vertices of $V_2 \cup V_3$ in S, this implies that v_i^1 appears only to dominate some vertex of V_1 first or second time. There can be two cases now.

Case 1. *v_i^1 appears to dominate itself.* If there is a vertex $u \in V_2 \cup V_3$ which is a neighbor of v_i^1 in G' and $u \notin \widehat{S}$, then we modify S by replacing the vertex v_i^1 by u and get a new double dominating sequence of G' of same size in which no vertex of V_1 appears. So, assume that all neighbors of v_i^1 belonging to the set $V_2 \cup V_3$ appear in $\widehat{S}$. So, $v_i^3 \in \widehat{S}$.

Thus, we have that v_i^3 appears to dominate v_i^1 first or second time and all other neighbors of v_i^1 from the set $V_2 \cup V_3$ belong to the set B. In particular, the vertex v_i^2 is also in B and it appears after v_i^3 in S. Note that v_i^2 appears to dominate some vertex v_j^1 of V_1 second time. This implies that $v_j^3 \notin \widehat{S}$. Now, we modify S by replacing the vertex v_i^2 by v_j^3 and the vertex v_i^1 by v_i^2 to get a new double dominating sequence of G' of same size in which no vertex of V_1 appears.

Case 2. *v_i^1 is dominated twice before its appearance.* Here, suppose that v_i^1 appears to dominate some vertex v_j^1 of V_1, first or second time. If $v_j^2 \notin \widehat{S}$, then we can modify S by replacing the vertex v_i^1 by v_j^2 to get a new double dominating sequence of G' of same size in which no vertex of V_1 appears. Similarly if $v_j^3 \notin \widehat{S}$, we get a new double dominating sequence of G' of same size containing no vertex of V_1. So, assume that $v_j^2, v_j^3 \in \widehat{S}$. Note that at least one of the vertices v_j^2 and v_j^3 does not belong to the set A. This implies that the vertex v_j^2 appears to dominate some vertex v_k^1 of V_1 second time and $v_k^3 \notin \widehat{S}$. Now, we modify S by replacing the vertex v_j^2 by v_k^3 and the vertex v_i^1 by v_j^2 to get a new double dominating sequence of G' of same size in which no vertex of V_1 appears.

Hence, we can assume that S contains no vertex of V_1. Thus, condition (1) holds. Now, we need to show that $V_3 \subseteq \widehat{S}$. On the contrary, assume that this is not true. Let $v_i^3 \in V_3$ be a vertex which is not in $\widehat{S}$. This implies that the vertex v_i^1 is dominated both times by two vertices of V_2. Let $v_j^2 \in V_2$ be the vertex which dominates v_i^1 second time. Now, we modify S by replacing the vertex v_j^2 by v_i^3 to get a new double dominating sequence of G' of same size in which the vertex v_i^3 appears. By repeating this argument, we get that there is a double dominating sequence of G' in which all vertices of V_3 appears. So, we can assume that $V_3 \subseteq \widehat{S}$ and thus, condition (2) is also satisfied.

Now, it remains to show that all n vertices of V_3 appear at initial n places. For this, it is enough to show that $v_i^3 \in V_3$ dominates the vertex v_i^1 first time for each $i \in [n]$. So, let v_i^1 be a vertex of V_1 such that it is dominated first time by a vertex v_j^2 of V_2 and second time by v_i^3. Clearly v_i^3 appears after v_j^2 in S. Here, we see that $N_{G'}[v_i^3] \subseteq N_{G'}[v_j^2]$, so we can exchange the positions of these two vertices with each other and get a new double dominating sequence of G' of same size such that v_i^3 dominates v_i^1 first time. Hence, we get that all n vertices of V_3 appear at initial n places of S.

Therefore, we have that, at least k vertices of V_2 are appearing in S and all of them are appearing only to dominate vertices of V_1 second time. So, these vertices correspond to a dominating sequence of G of size at least k. Thus, $\gamma_{gr}(G) \geq k$. ■

Now, we can directly state the following theorem.

Theorem 3. *The* GD2D *problem is NP-complete for co-bipartite graphs.*

4. Algorithm for Chain Graphs

In this section, we present a linear-time algorithm to solve the GD2 problem in chain graphs. Let $G = (X, Y, E)$ denotes a chain graph and P is the partition of $V(G)$ obtained by the relation R. Recall that, $P = (X_1, X_2, \ldots, X_k, Y_1, Y_2, \ldots, Y_k)$. Let $|X| = n_1$ and $|Y| = n_2$. For $i \in [k]$, x^i denotes the vertex of X_i having minimum index in the chain ordering of G. Similarly, y^i denotes the vertex of Y_i having maximum index in the chain ordering of G. Below, we give a result which gives the Grundy double domination number of a complete bipartite graph.

Proposition 4. *Let $G = (X, Y, E)$ be a complete bipartite graph. Then $\gamma_{gr}^{\times 2}(G) = \max\{|X|, |Y|\} + 1$.*

Proof. Without loss of generality, assume that $n_1 \geq n_2$ and $n_1 \geq 2$. We show that $\gamma_{gr}^{\times 2}(G) = n_1 + 1$. Clearly $(x_1, x_2, \ldots, x_{n_1}, y_1)$ is a double dominating sequence of G which implies that $\gamma_{gr}^{\times 2}(G) \geq n_1 + 1$.

Now, assume that S is a Grundy double dominating sequence of G such that $|\hat{S}| > n_1 + 1$. In this case, there can not be at least two vertices from both sets X and Y prior to the last vertex in the sequence. Thus, $|\hat{S}| \leq n_1 + 2$ and so, $|\hat{S}| = n_1 + 2$. Therefore, if the last vertex of the sequence is from X, then $|X \cap \hat{S}| = 2, |Y| = n_1$ and $Y \subseteq \hat{S}$. But so, just before the last vertex, every vertex of G is dominated twice, a contradiction. Similarly, if the last vertex of the sequence is from Y, then $|Y \cap \hat{S}| = 2$ and $X \subseteq \hat{S}$. But so, just before the last vertex, every vertex of G is dominated twice, a contradiction. Hence, $\gamma_{gr}^{\times 2}(G) = n_1 + 1$. ■

For technical reasons, we actually consider a slightly more generalized problem in chain graphs. Let $G = (X, Y, E)$ be a chain graph and $M \subseteq V(G)$. Vertices of M are called *marked vertices* of G. All remaining vertices of G are called *unmarked vertices*. We denote the set of unmarked vertices of G by V_0 and the subgraph of G induced on the set V_0 by G_0. The set of marked vertices satisfy all the conditions written in equation (1).

$$(1) \quad M \subseteq (X_k \cup Y_1),\ |M \cap X_k| \leq 1,\ |M \cap Y_1| \leq 1,\ |X_k \setminus M| \geq 1,\ |Y_1 \setminus M| \geq 1.$$

A sequence $S = (v_1, v_2, \ldots, v_k)$, where $v_i \in V_0$ for each $i \in [k]$, is called an *M-double neighborhood sequence* of (G, M) if for each i, the vertex v_i dominates at least one vertex u of G which is dominated at most once by its preceding vertices in the sequence S. In addition, if $\hat{S}$ is a double dominating set of G_0, then we call S an *M-double dominating sequence* of (G, M). Note that $\hat{S}$ may not be a double dominating set of G. An M-double dominating sequence with maximum length is called a *Grundy M-double dominating sequence* of (G, M). The length of a Grundy M-double dominating sequence of (G, M) is called the *Grundy M-double domination number* of (G, M) and is denoted by $\gamma_{grm}^{\times 2}(G, M)$. Given a chain graph G and $M \subseteq V(G)$ satisfying equation (1), the Grundy M-Double Domination (GMD2) problem asks to compute a Grundy M-double dominating sequence of (G, M).

Throughout this section, $\mathcal{G} = (G, M)$ denotes an instance of the GMD2 problem, where $G = (X, Y, E)$ is a chain graph and M is a subset of $V(G)$ satisfying equation (1). Let S be a Grundy M-double dominating sequence of $\mathcal{G}$. If $M = \emptyset$ then, S is also a Grundy double dominating sequence of G. So, the GD2 problem is a special case of the GMD2 problem.

Now, we state two important lemmas. The proofs of these lemmas are easy and hence are omitted.

Lemma 5. *Let $M \neq \emptyset$. Then, $\gamma_{grm}^{\times 2}(\mathcal{G}) \leq \gamma_{gr}^{\times 2}(G)$.*

Lemma 6. *For any Grundy M-double dominating sequence S of $\mathcal{G}$, we have that $X_k \cap \hat{S} \neq \emptyset$ and $Y_1 \cap \hat{S} \neq \emptyset$.*

We prove a lemma for complete bipartite graphs that forms the basis of our algorithm.

Lemma 7. $\gamma_{grm}^{\times 2}(\mathcal{G}) \in \{\max\{n_1, n_2\}, \max\{n_1, n_2\} + 1\}$*, for a complete bipartite graph G.*

Proof. There are four cases to consider.

Case 1. $M = \emptyset$. In this case, $\gamma_{grm}^{\times 2}(\mathcal{G}) = \gamma_{gr}^{\times 2}(G)$. Using Proposition 4, we have $\gamma_{grm}^{\times 2}(\mathcal{G}) = \max\{n_1, n_2\} + 1$.

Case 2. $M \cap X_k = \{x_{n_1}\}$ *and* $M \cap Y_1 = \emptyset$. Since $|M \cap X_k| = 1$, we have that $n_1 \geq 2$. We consider two subcases now.

Subcase 2.1. $n_1 = \max\{n_1, n_2\}$. Here, we have that $\gamma_{gr}^{\times 2}(G) = n_1 + 1$. Now, if $n_2 = 1$, $\gamma_{grm}^{\times 2}(\mathcal{G}) \leq |X| - 1 + |Y| = n_1$. As the sequence $(x_1, x_2, \ldots, x_{n_1 - 1}, y_1)$ is an M-double dominating sequence of $\mathcal{G}$ of length n_1. So, $\gamma_{grm}^{\times 2}(\mathcal{G}) = n_1 = \max\{n_1, n_2\}$. Otherwise, if $n_2 > 1$, the sequence $(x_1, x_2, \ldots, x_{n_1 - 1}, y_1, y_2)$ is an M-double dominating sequence of $\mathcal{G}$ of length $n_1 + 1$. Thus, we have that $\gamma_{grm}^{\times 2}(\mathcal{G}) = n_1 + 1 = \max\{n_1, n_2\} + 1$ using Lemma 5.

Subcase 2.2. $n_2 = \max\{n_1, n_2\}$. Here, we have that $\gamma_{gr}^{\times 2}(G) = n_2 + 1$. Since $n_1 \geq 2$, we have that $n_2 \geq 2$. The sequence $(y_1, y_2, \ldots, y_{n_2}, x_1)$ is an M-double dominating sequence of $\mathcal{G}$ of length $n_2 + 1$. Thus, we have $\gamma_{grm}^{\times 2}(\mathcal{G}) = n_2 + 1 = \max\{n_1, n_2\} + 1$ using Lemma 5.

Case 3. $M \cap Y_1 = \{y_1\}$ *and* $M \cap X_k = \emptyset$. This case is similar to Case 2.

Case 4. $M \cap X_k = \{x_{n_1}\}$ *and* $M \cap Y_1 = \{y_1\}$. Clearly, $n_1 \geq 2$ and $n_2 \geq 2$. We again consider two subcases.

Subcase 4.1. $n_1 = \max\{n_1, n_2\}$. Here, we have that $\gamma_{gr}^{\times 2}(G) = n_1 + 1$. If $n_2 \geq 3$, the sequence $(x_1, x_2, \dots, x_{n_1-1}, y_2, y_3)$ is an M-double dominating sequence of $\mathcal{G}$ of length $n_1 - 1 + 2 = n_1 + 1$. So, $\gamma_{grm}^{\times 2}(\mathcal{G}) = n_1 + 1 = \max\{n_1, n_2\} + 1$ using Lemma 5. But, if $n_2 = 2$, the sequence $(x_1, x_2, \dots, x_{n_1-1}, y_2)$ is an M-double dominating sequence of $\mathcal{G}$ of length $n_1 - 1 + 1 = n_1$. So, $\gamma_{grm}^{\times 2}(\mathcal{G}) = n_1 = \max\{n_1, n_2\}$ using the fact that $\gamma_{grm}^{\times 2}(G) \leq |X| - 1 + |Y| - 1 = n_1 - 1 + 2 - 1 = n_1$.

Subcase 4.2. $n_2 = \max\{n_1, n_2\}$. Similar to Subcase 4.1, we can prove that $\gamma_{grm}^{\times 2}(\mathcal{G})$ is either n_2 or $n_2 + 1$. ∎

Algorithm 1 computes a Grundy M-double dominating sequence of $\mathcal{G}$ based on the Lemma 7, when G is a complete bipartite graph.

Next, we state some lemmas for $\mathcal{G}$, when G is not a complete bipartite graph, that is, $k \geq 2$.

Lemma 8. *If there exists a Grundy M-double dominating sequence S^* of $\mathcal{G}$ such that $|X_k \cap \widehat{S^*}| \geq 3$, then exactly one of the following is true.*

1. $\gamma_{grm}^{\times 2}(\mathcal{G}) = |X| + k$.
2. $\gamma_{grm}^{\times 2}(\mathcal{G}) = |X| + k - 1$.

Proof. Let $X_k = \{a_{k_1}, a_{k_2}, \dots, a_{k_t}\}$. There can be two cases.

Case 1. $M \cap X_k = \emptyset$. In this case, no vertex of X_k is marked and $\big(x_1, x_2, \dots, x_{n_1}, y^k, y^{k-1}, \dots, y^1\big)$ is an M-double dominating sequence of $\mathcal{G}$ which implies that $\gamma_{grm}^{\times 2}(\mathcal{G}) \geq |X| + k$.

Since, $|X_k \cap \widehat{S^*}| \geq 3$, we get that $t \geq 3$. Note that we can assume that all vertices of $X_k \cap \widehat{S^*}$ appear in the same order as in the chain ordering. We also assume that all vertices of $(X_k \setminus \{a_{k_1}\}) \cap \widehat{S^*}$ appear together in S^*. Now, we show that there exists an M-double dominating sequence of $\mathcal{G}$ in which all vertices of X_k appear. To see this, suppose that there is a vertex a_{k_i}, where $i \geq 4$, such that $a_{k_i} \notin \widehat{S^*}$. This means that there are two vertices $y, y' \in Y \cap \widehat{S^*}$ which dominate a_{k_i} first and second time respectively. Note that y' appears after a_{k_3} in S^*. Here, we modify the sequence by replacing y' with the vertex a_{k_i} and get a new Grundy M-double dominating sequence of $\mathcal{G}$ in which the vertex a_{k_i} appears. By repeating this argument, we get a Grundy M-double dominating sequence of $\mathcal{G}$ in which all vertices of X_k appear. So, assume that $X_k \subseteq \widehat{S^*}$. We also assume that all vertices of $X_k \setminus \{a_{k_1}\}$ appear together in S^*.

Algorithm 1: $S = \text{GrundyM1}(G, M)$

Input: $\mathcal{G} = (G, M)$, where $G = (X, Y, E)$ is a complete bipartite graph and $M \subseteq V(G)$ satisfying equation (1), $X = \{x_1, \ldots, x_{n_1}\}$ and $Y = \{y_1, \ldots, y_{n_2}\}$.
Output: A Grundy M-double dominating sequence S of $\mathcal{G}$.

- **if** $M = \emptyset$ **then**
  - **if** $n_1 \geq n_2$ **then**
    - $S = (x_1, x_2, \ldots, x_{n_1}, y_1)$
  - **else**
    - $S = (y_1, y_2, \ldots, y_{n_2}, x_1)$
- **if** $M \cap X_k = \{x_{n_1}\}$ *and* $M \cap Y_1 = \emptyset$ **then**
  - **if** $n_1 \geq n_2$ **then**
    - **if** $n_2 = 1$ **then**
      - $S = (x_1, x_2, \ldots, x_{n_1-1}, y_1)$
    - **else**
      - $S = (x_1, x_2, \ldots, x_{n_1-1}, y_1, y_2)$
  - **else**
    - $S = (y_1, y_2, \ldots, y_{n_2}, x_1)$
- **if** $M \cap Y_1 = \{y_1\}$ *and* $M \cap X_k = \emptyset$ **then**
  - **if** $n_2 \geq n_1$ **then**
    - **if** $n_1 = 1$ **then**
      - $S = (y_2, y_3, \ldots, y_{n_2}, x_1)$
    - **else**
      - $S = (y_2, y_3, \ldots, y_{n_2}, x_1, x_2)$
  - **else**
    - $S = (x_1, x_2, \ldots, x_{n_2}, y_2)$
- **if** $M \cap X_k = \{x_{n_1}\}$ *and* $M \cap Y_1 = \{y_1\}$ **then**
  - **if** $n_1 \geq n_2$ **then**
    - **if** $n_2 \geq 3$ **then**
      - $S = (x_1, x_2, \ldots, x_{n_1-1}, y_2, y_3)$
    - **else**
      - $S = (x_1, x_2, \ldots, x_{n_1-1}, y_2)$
  - **else**
    - **if** $n_1 \geq 3$ **then**
      - $S = (y_2, y_3, \ldots, y_{n_2}, x_1, x_2)$
    - **else**
      - $S = (y_2, y_3, \ldots, y_{n_2}, x_1)$
- **return** S.

Next, we show that there exists a Grundy M-double dominating sequence of $\mathcal{G}$ in which all vertices of X appear. So, let x_0 be a vertex of X side which does not appear in S^*. This implies that there is a vertex $y_0 \in Y$ which appear in S^*, to dominate x_0 second time. Note that y_0 appears after all vertices of X_k in S^*. We modify S^* by replacing y_0 with the vertex x_0 and get a new Grundy M-double dominating sequence of $\mathcal{G}$ in which the vertex x_0 appears. By repeating this argument, we get a Grundy M-double dominating sequence S of $\mathcal{G}$ in which all vertices of X appear. Thus, $X \subseteq \hat{S}$. Since $N(X_k) = Y$ and $|X_k \cap \hat{S}| = |X_k| \geq 3$, we have that at most one vertex of Y appears before a_{k_3} in S. So, at most k vertices can appear in S from the Y side. Thus, $\gamma^{\times 2}_{grm}(\mathcal{G}) \leq |X| + k$ which further implies that $\gamma^{\times 2}_{grm}(\mathcal{G}) = |X| + k$.

Case 2. $M \cap X_k \neq \emptyset$. Let a_{k_t} be the marked vertex of X_k. Here, we see that $t \geq 4$. If $(X \setminus \{a_{k_t}\}) \nsubseteq \widehat{S^*}$, we can do similar modifications as done in Case 1 and get a new Grundy M-double dominating sequence S of $\mathcal{G}$ such that all vertices of $X \setminus \{a_{k_t}\}$ appear in S. Again, we see that at most k vertices can appear from Y side in S. Thus, $\gamma^{\times 2}_{grm}(\mathcal{G}) \leq |X| - 1 + k$ which further implies that $\gamma^{\times 2}_{grm}(\mathcal{G}) = |X| + k - 1$. ■

Similar to Lemma 8, we state another lemma for the Y side of G. Proof of Lemma 9 is similar to the one of Lemma 8.

Lemma 9. *If there exists a Grundy M-double dominating sequence S^* of $\mathcal{G}$ such that $|Y_1 \cap \widehat{S^*}| \geq 3$, then exactly one of the following is true.*

1. $\gamma^{\times 2}_{grm}(\mathcal{G}) = |Y| + k$.
2. $\gamma^{\times 2}_{grm}(\mathcal{G}) = |Y| + k - 1$.

Lemma 10. *Let $\mathcal{G}$ be an instance of the* GMD2 *problem such that there is no Grundy M-double dominating sequence S^* of $\mathcal{G}$ satisfying $|X_k \cap \widehat{S^*}| \geq 3$ or $|Y_1 \cap \widehat{S^*}| \geq 3$. Assume that S is a Grundy M-double dominating sequence of $\mathcal{G}$ such that $|X_k \cap \hat{S}| = 2$. Then either $|Y_1 \cap \hat{S}| = 1$ or there exists another Grundy M-double dominating sequence S' of $\mathcal{G}$ satisfying one of the following.*

(i) $|X_k \cap \widehat{S'}| = 2$ *and* $|Y_1 \cap \widehat{S'}| = 1$.

(ii) $|X_k \cap \widehat{S'}| = 1$ *and* $|Y_1 \cap \widehat{S'}| = 2$.

Proof. Since $\mathcal{G}$ has no Grundy M-double dominating sequence having at least 3 vertices from Y_1, we have that $|Y_1 \cap \hat{S}| \leq 2$. Moreover, Lemma 6 ensures that $|Y_1 \cap \hat{S}| = 1$ or $|Y_1 \cap \hat{S}| = 2$. Now, if $|Y_1 \cap \hat{S}| = 1$, there is nothing to prove. So, assume that $|Y_1 \cap \hat{S}| = 2$. Suppose that $X_k \cap \hat{S} = \{a, b\}$ and $Y_1 \cap \hat{S} = \{c, d\}$. Note that the sequence S ends with a vertex of the set $\{a, b, c, d\}$. There are two cases to consider.

Case 1. *S ends with c or d.* First, we assume that S ends with the vertex d. As all vertices of the set $\{a, b, c\}$ have been appeared before d, we get that d appears to dominate some vertex x^* of X second time. Note that $x^* \notin \hat{S}$. Now, if x^* is an unmarked vertex, we modify S by replacing the last vertex d by the vertex x^* and get a new Grundy M-double dominating sequence S' of $\mathcal{G}$ such that $|Y_1 \cap \widehat{S'}| = 1$. Otherwise, x^* is a marked vertex of G. This means that $x^* \in X_k$. As d is dominating x^* second time and $N(X_k) = Y$, we get that $|Y \cap \hat{S}| = 2$. In particular, $Y \cap \hat{S} = \{c, d\} = Y_1 \cap \hat{S}$. Since $k \geq 2$, there is a vertex $y_0 \in Y_2$ which is appearing nowhere in S. So, we modify S by replacing the last vertex d by the vertex y_0 and get a new Grundy M-double dominating sequence S' of $\mathcal{G}$ such that $|Y_1 \cap \widehat{S'}| = 1$. Hence, $\hat{S'}$ is the desired Grundy M-double dominating sequence of $\mathcal{G}$. Similar arguments can be given when S ends with the vertex c.

Case 2. *S ends with a or b.* In this case, we get a new Grundy M-double dominating sequence S' of $\mathcal{G}$ such that $|X_k \cap \widehat{S'}| = 1$ and $|Y_1 \cap \widehat{S'}| = 2$ by doing similar modifications as done in Case 1.

Therefore, the lemma holds. ∎

The proof of the next lemma is easy and hence is omitted.

Lemma 11. *Let $\mathcal{G}$ be an instance of the* GMD2 *problem such that there is no Grundy M-double dominating sequence S^* of $\mathcal{G}$ satisfying $|X_k \cap \widehat{S^*}| \geq 3$ or $|Y_1 \cap \widehat{S^*}| \geq 3$. Assume that S is a Grundy M-double dominating sequence of $\mathcal{G}$ such that $|X_k \cap \hat{S}| = 1$. Then $Y_k \subseteq \hat{S}$.*

Similar to Lemma 11, we state another lemma for G.

Lemma 12. *Let $\mathcal{G}$ be an instance of the* GMD2 *problem such that there is no Grundy M-double dominating sequence S^* of $\mathcal{G}$ satisfying $|X_k \cap \widehat{S^*}| \geq 3$ or $|Y_1 \cap \widehat{S^*}| \geq 3$. Assume that S is a Grundy M-double dominating sequence of $\mathcal{G}$ such that $|Y_1 \cap \hat{S}| = 1$. Then $X_1 \subseteq \hat{S}$.*

Using Lemmas 10, 11 and 12, we can directly state the following result.

Lemma 13. *Let $\mathcal{G}$ be an instance of the* GMD2 *problem such that there is no Grundy M-double dominating sequence S^* of $\mathcal{G}$ satisfying $|X_k \cap \widehat{S^*}| \geq 3$ or $|Y_1 \cap \widehat{S^*}| \geq 3$. Then one of the following is true.*

(i) *There exists a Grundy M-double dominating sequence S of $\mathcal{G}$ such that $|X_k \cap \widehat{S}| = 1$ and $Y_k \subseteq \hat{S}$.*

(ii) *There exists a Grundy M-double dominating sequence S of $\mathcal{G}$ such that $|Y_1 \cap \widehat{S}| = 1$ and $X_1 \subseteq \hat{S}$.*

Let $\mathcal{G}$ be an instance of the GMD2 problem such that there is no Grundy M-double dominating sequence S^* of $\mathcal{G}$ satisfying $|X_k \cap \widehat{S^*}| \geq 3$ or $|Y_1 \cap \widehat{S^*}| \geq 3$.

We call a Grundy M-double dominating sequence S of $\mathcal{G}$ as a *type* 1 *optimal sequence* of $\mathcal{G}$ if it satisfies that $|X_k \cap \hat{S}| = 1$ and $Y_k \subseteq \hat{S}$. Similarly, we call a Grundy M-double dominating sequence S of $\mathcal{G}$ a *type* 2 *optimal sequence* of $\mathcal{G}$ if it satisfies that $|Y_1 \cap \hat{S}| = 1$ and $X_1 \subseteq \hat{S}$.

Lemma 14. *Let $\mathcal{G}$ be an instance of the* GMD2 *problem. Then one of the following is true.*

(i) *There exists a type* 1 *optimal sequence of $\mathcal{G}$.*

(ii) *There exists a type* 2 *optimal sequence of $\mathcal{G}$.*

Proof. If a vertex of X is marked, we assume that it is the vertex x_{n_1} and if a vertex of Y is marked, we assume that it is the vertex y_1.

If $\mathcal{G}$ is an instance such that there is no Grundy M-double dominating sequence S^* of $\mathcal{G}$ satisfying $|X_k \cap \widehat{S^*}| \geq 3$ or $|Y_1 \cap \widehat{S^*}| \geq 3$ then the statement is true using Lemma 13.

So, assume that there is a Grundy M-double dominating sequence S^* of $\mathcal{G}$ such that $|X_k \cap \widehat{S^*}| \geq 3$ or $|Y_1 \cap \widehat{S^*}| \geq 3$. If $|X_k \cap \widehat{S^*}| \geq 3$ then, using Lemma 8, we get that $\gamma_{grm}^{\times 2}(\mathcal{G})$ is $|X| + k$ or $|X| + k - 1$. If $\gamma_{grm}^{\times 2}(\mathcal{G})$ is $|X| + k$, then $(x_1, x_2, \ldots, x_{n_1}, y^k, y^{k-1}, \ldots, y^1)$ is a type 2 optimal sequence of $\mathcal{G}$. Note that x_{n_1} is not a marked vertex of G in this case. Otherwise, if $\gamma_{grm}^{\times 2}(\mathcal{G})$ is $|X| + k - 1$, then $(x_1, x_2, \ldots, x_{n_1 - 1}, y^k, y^{k-1}, \ldots, y^1)$ is a type 2 optimal sequence of $\mathcal{G}$. Thus, if $|X_k \cap \widehat{S^*}| \geq 3$, there exists a type 2 optimal sequence of $\mathcal{G}$.

Similarly, if $|Y_1 \cap \widehat{S^*}| \geq 3$, then we get that there exists a type 1 optimal sequence of $\mathcal{G}$. This is ensured due to Lemma 9. ■

Finally, we state the lemma which completely characterizes the structure of an optimal solution for an instance of the GMD2 problem.

Lemma 15. *Let $\mathcal{G}$ be an instance of the* GMD2 *problem. Then one of the following is true.*

(i) *There exists a type* 1 *optimal sequence S of $\mathcal{G}$ in which the vertex of $X_k \cap \hat{S}$ appear in the last.*

(ii) *There exists a type* 2 *optimal sequence S of $\mathcal{G}$ in which the vertex of $Y_1 \cap \hat{S}$ appear in the last.*

Proof. Using Lemma 14, we have that, there exists a type 1 optimal sequence of $\mathcal{G}$ or there exists a type 2 optimal sequence of $\mathcal{G}$. If there exists a type 1 optimal sequence S of $\mathcal{G}$ then S contains one vertex of X_k and all vertices of Y_k. If the vertex of X_k does not appear as the last vertex of the sequence S, we relocate that vertex in the last and obtain a new sequence. Since $N(X_k) = Y_k$, the last vertex dominates the vertices of Y_k the second time and so, the new sequence is also a type 1 optimal sequence of $\mathcal{G}$. Hence, (i) is true. Using similar arguments, (ii) also holds true. ■

We use a dynamic programming approach to solve the GMD2 problem for an instance $\mathcal{G}$ in Algorihtm 2. Through Lemma 15, we characterized the structure of an optimal solution. Next, we define the optimal solution of the problem recursively in terms of the optimal solutions to subproblems. For GMD2 problem, we pick the subproblems as the problem of finding a Grundy M-double dominating sequence of $\mathcal{G}' = (G', M')$, where G' is a subgraph of G and $M' \subseteq V(G')$ satisfying equation (1).

Let S be a Grundy M-double dominating sequence of $\mathcal{G}$ which is a type 1 optimal sequence of $\mathcal{G}$ and the vertex of $X_k \cap \hat{S}$ appear in the last. We also assume that all vertices of Y_k appear together just before the vertex of X_k. Let G_1 denotes the subgraph of G induced on the set of vertices $(X \setminus X_k) \cup \{x_{t+1}\} \cup (Y \setminus Y_k)$, where $t = |X| - |X_k|$. Let $M_1 = \{x_{t+1}\} \cup (M \cap Y_1)$. Then the subsequence of S obtained by removing the last $|Y_k| + 1$ vertices of S is a Grundy M-double dominating sequence of (G_1, M_1).

Similarly, if S is a type 2 optimal sequence of $\mathcal{G}$ having the vertex of $Y_1 \cap \hat{S}$ in the last and G_2 denotes the subgraph of G induced on the set of vertices $(Y \setminus Y_1) \cup \{y_t\} \cup (X \setminus X_1)$, where $t = |M \cap Y_1| + 1$. Again, assume that all vertices of X_1 appear together just before the vertex of Y_1. Let $M_2 = \{y_t\} \cup (M \cap X_k)$. Then the subsequence of S obtained by removing the last $|X_1| + 1$ vertices of S is a Grundy M-double dominating sequence of (G_2, M_2).

Now, we give the algorithm to compute a Grundy M-double dominating sequence of $\mathcal{G}$.

Algorithm 2 computes a Grundy M-double dominating sequence of $\mathcal{G} = (G, M)$ by recursively appending some vertices at the end of the Grundy M-double dominating sequence of (G', M'), where G' is a subgraph of G. Note that this task can be performed in linear-time.

Based on the above discussion, we directly state the following theorem.

Theorem 16. *Algorithm* 2 *outputs a Grundy M-double dominating sequence of* $\mathcal{G} = (G, M)$ *in linear-time, where* G *is a chain graph.*

To solve the GD2 problem in a chain graph G, we compute a Grundy M-double dominating sequence of $(G, \emptyset)$ using Algorithm 2. So, we can state the following theorem.

Theorem 17. *A Grundy double dominating sequence of a chain graph* G *can be computed in linear-time.*

Algorithm 2: $S = \text{GrundyM}(G, M)$

```
Input: 𝒢 = (G, M), where G = (X, Y, E) is a chain graph and
       M ⊆ V(G) satisfying equation (1). X = {x_1, ..., x_{n_1}} and
       Y = {y_1, ..., y_{n_2}}.
Output: A Grundy M-double dominating sequence S of 𝒢.
if k = 1 then
    S = GrundyM1(G, M);
    return S;
else
    t = |X| − |X_k|, X'_{k−1} = X_{k−1} ∪ {x_{t+1}};
    if k ≥ 3 then
        X' = ∪_{i=1}^{k−2} X_i ∪ X'_{k−1};
    else
        X' = X'_{k−1};
    G^1_{k−1} = G[X' ∪ (Y \ Y_k)], M ∩ X_k = {x_{t+1}};
    S_1 = GrundyM(G^1_{k−1}, M) ⊕ (Y_k) ⊕ x_{t+1};
    t = |M ∩ Y_1| + 1, Y'_1 = Y_2 ∪ {y_t};
    if k ≥ 3 then
        Y' = ∪_{i=3}^{k} Y_i ∪ Y'_1;
    else
        Y' = Y'_1;
    G^2_{k−1} = G[(X \ X_1) ∪ Y'], M ∩ Y_1 = {y_t};
    S_2 = GrundyM(G^2_{k−1}, M) ⊕ (X_1) ⊕ y_t;
    if |Ŝ_1| ≥ |Ŝ_2| then
        return S_1;
        else
            return S_2;
```

5. Conclusion

We studied the GD2D problem in this paper. We proved that the problem is NP-complete for bipartite graphs and co-bipartite graphs. We also proved that the GD2D problem is efficiently solvable for chain graphs. We solved this problem in chain graphs using a dynamic programming approach. Since the class of chain graphs is a subclass of bipartite graphs, the gap between the efficient algorithms and NP-completeness in the subclasses of bipartite graphs has been narrowed a little. To find the status of the problem in the graph classes such as bipartite permutation graphs, convex bipartite graphs and chordal bipartite graphs can be the next research direction. These graph classes are subclasses of bipartite graphs and superclasses of chain graphs. Various types of vertex sequences were proposed for which computational complexities are still unknown in many graph classes [6]. These kind of vertex sequences are open for further research.

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