Discussiones Mathematicae
Graph Theory 45 (2025) 595–614
https://doi.org/10.7151/dmgt.2547

THE DECYCLING NUMBER OF A GRAPH WITH LARGE GIRTH EMBEDDED IN A SURFACE

Ming-Yuan Ma and Han Ren

School of Mathematical Sciences
East China Normal University
Shanghai 200241, P.R. China

e-mail: 623mmy@sina.com
hren@math.ecnu.edu.cn

Abstract

It were conjectured that the decycling number of a bipartite planar graph of n vertices is at most $\frac{3n}{8}$, and that the decycling number of a planar graph of n vertices with girth at least five is at most $\frac{3n}{10}$. In this paper we show that the decycling number of a planar graph of n vertices with girth at least six (or eight) is at most $\frac{3n-6}{8}$ (or $\frac{3n-6}{10}$), which means that the first conjecture is true if the girth is at least six and the second conjecture holds if the girth is at least eight. If G is a connected graph 2-cell embedded in the orientable surface $S_\gamma(\gamma \geq 1)$, we prove that the decycling number of G is at most $\frac{3}{8}(n-2+2\gamma)$ (or $\frac{3}{10}(n-2+2\gamma)$) if the girth of G is at least $6+4\gamma$ (or $8+6\gamma$). Similarly, if G is 2-cell embedded in the non-orientable surface $N_{\bar{\gamma}}$, then the decycling number of G is at most $\frac{3}{8}(n-2+\bar{\gamma})$ (or $\frac{3}{10}(n-2+\bar{\gamma})$) if the girth of G is at least $6+2\bar{\gamma}$ (or $8+4\bar{\gamma}$).

Keywords: decycling number, girth, embedding.

2020 Mathematics Subject Classification: 05C10, 05C70.

1. Introduction

All graphs in this paper are simple and finite. Let $G = (V(G), E(G))$ be a graph in which S is a subset of $V(G)$. If the graph obtained from G by deleting all vertices in S is a forest, then S is called a *decycling set* of G. Sometimes a decycling set is said to be a feedback vertex set. The cardinality of a minimum decycling set of G is called the *decycling number* of G, which is denoted by $\nabla(G)$. Clearly, finding a decycling set S of G is equivalent to obtain a subset S' of $V(G)$ such that the subgraph of G induced by S' is a forest. Such a set as S' is called an *acyclic set* of G. The cardinality of a maximum acyclic set is referred to as a *forest number*, which is denoted by $a(G)$. Obviously, $\nabla(G) + a(G) = |V(G)|$.

The problem of finding a minimum decycling set of a graph is known to be NP-hard [16]. The decycling number of some classes of graphs, such as complete graphs and complete bipartite graphs, has been determined [6]. For a graph embedded in a surface, its decycling number has been explored [1, 13, 15, 19, 23, etc.]. It needs to point out that there is a following challenging conjecture for the decycling number of a planar graph.

Conjecture 1 [3, 12]**.** *If G is a planar graph of n vertices, then $\nabla(G) \leq \frac{n}{2}$.*

The conjecture is still open. However, there are a few of results on the decycling number of planar graphs. For instance, Hosono [15] showed that the decycling number of every outerplanar graph of n vertices is at most $\frac{n}{3}$. The authors [20] proved that if G is a planar graph with n edges such that the line graph $L(G)$ of G is also a planar graph, then $\nabla(L(G)) \leq \frac{n}{2}$.

Let G be a planar graph. If the girth of G is restricted, then the bound in Conjecture 1 can be improved. Alon *et al.* [5] showed that for every triangle-free cubic graph G of n vertices, $a(G) \geq \frac{5n}{8}$. Thus $\nabla(G) \leq \frac{3n}{8}$ if G is a bipartite cubic graph of order n. Akiyama and Watanabe [1], and Albertson and Haas [2] independently proposed the conjecture below.

Conjecture 2 [1, 2]**.** *If G is a bipartite planar graph of n vertices, then $\nabla(G) \leq \frac{3n}{8}$.*

Upon planar graphs with girth at least five, Kowalik *et al.* [17] proposed the following conjecture.

Conjecture 3 [17]**.** *If G is a planar graph of n vertices with girth at least five, then $\nabla(G) \leq \frac{3n}{10}$.*

Need to say that Conjectures 2 and 3 are still open. Since $\frac{n}{2}$ can be expressed as $\frac{3n}{6}$ and a bipartite graph has girth at least four, Conjectures 1, 2 and 3 seem to imply the following conjecture.

Conjecture 4. *Let $g \geq 3$ be an integer. If G is a planar graph of n vertices with girth at least g, then $\nabla(G) \leq \frac{3n}{2g}$.*

In this paper, we show the theorem below.

Theorem 5. *Let t and g be two integers with $t \geq 8$ and $t \equiv 0 \pmod 2$. Let G be a planar graph of n vertices with girth at least g. If $g \geq \frac{(t-8)^2}{2} + 6$, then $\nabla(G) \leq \frac{3n-6}{t}$.*

As the corollaries of Theorem 5, Conjecture 2 is true if the girth of G is at least six, and Conjecture 3 holds if the girth of G is at least eight. In addition, if G is a connected graph 2-cell embedded in a surface, we obtain a similar result to that in Theorem 5.

In 2016, Dross *et al.* [10] proposed the conjecture below.

Conjecture 6. *If G is a planar graph of m edges with girth at least g, then* $\nabla(G) \leq \frac{m}{g}$.

The conjecture is still open. Since $m \leq \frac{3}{2}(n-2)$ by Euler's formula if G has n vertices and $g \geq 6$, it follows that $\frac{m}{g} \leq \frac{3n-6}{2g}$ if $g \geq 6$. Consequently, for a planar graph with girth at least six, Conjecture 4 holds if $\nabla(G) \leq \frac{m}{g}$. In this paper we show the following result. Given an integer $g \geq 8$, if G is a planar graph of m edges with girth at least $2(g-4)^2+6$, then $\nabla(G) \leq \frac{m}{g}$.

The arrangement of the paper is as follows. In Section 2, we first study the relation between the number of vertices of degree two and the number of vertices of degree at least three in a planar graph. Then we discuss the decycling number of a planar graph with large girth. In Section 3, we explore the decycling number of a connected graph G which is 2-cell embedded in a surface, and we obtain a similar result to Theorem 5. At the end of this paper, we propose two conjectures on the decycling number of a connected graph 2-cell embedded in a surface as a generalization of Conjecture 4.

The remainder of this section is contributed for some terminologies on graphs or surfaces. The other undefined terminologies can be found in [7] or [21].

The degree of a vertex v in a graph G, denoted by $d_G(v)$, is the number of edges of G incident with v. The maximum degree and the minimum degree of G are denoted by $\Delta(G)$ and $\delta(G)$, respectively. Let X be a subset of $V(G)$. The subgraph of G induced by X is denoted by $G[X]$. The *girth* of a graph with at least one cycle is the length of a shortest cycle, and the girth of a forest is infinite. Let X and Y be two disjoint vertex subsets of $V(G)$. If one end of an edge e is in X and another end is in Y, then we say that e is *between* X and Y.

A *surface* is a compact connected 2-dimensional manifold without boundary. The orientable surface $S_\gamma(\gamma \geq 0)$ (or the non-orientable surface $N_{\bar{\gamma}}(\bar{\gamma} \geq 1)$) can be obtained from the sphere by attaching γ handles (or $\bar{\gamma}$ Möbius bands). The orientable surface S_1 and the non-orientable surface N_1 are usually said to be the torus and the projective plane, respectively. If a connected graph G is drawn in a surface Σ such that any edge does not pass through any vertex and any two edges do not cross each other, then we say that G is *embedded* in Σ. An embedding Π of G in Σ is called 2-*cell embedding* if any connected component of Σ-Π is homeomorphic to an open disc. In this paper a graph and its embedding are not distinguished if no confusion is caused.

At last, we give a proposition by Euler's formula, which is often used in the later proofs.

Proposition 7. *If G is 2-cell embedded in the surface S_γ (or the surface $N_{\bar{\gamma}}$) with girth at least g, then* $|E(G)| \le \frac{g}{g-2}(n-2)+\frac{2g}{g-2}\gamma$ $\left(\text{or } |E(G)| \le \frac{g}{g-2}(n-2)+\frac{g}{g-2}\bar{\gamma}\right)$.

2. The Decycling Number of a Planar Graph with Large Girth

This section starts with a lemma.

Lemma 8. *Let G be a connected planar graph of n vertices with $\Delta(G)=\Delta\ge 3$ and $\delta(G)=2$. Let $n_{\ge 3}$ be the number of vertices of degree at least three in G, and let n_2 be the number of vertices of degree two in G. If the girth g of G is at least six, then*

$$n_{\ge 3} < \frac{4n}{g-2}.$$

Proof. Suppose on the contrary that $n_{\ge 3} \ge \frac{4n}{g-2}$. Then

$$\begin{aligned}|E(G)| &= \frac{1}{2}\sum\nolimits_{k=2}^{\Delta} kn_k = \frac{1}{2}\sum\nolimits_{k=3}^{\Delta} kn_k + n_2 \ge \frac{3}{2}n_{\ge 3} + n_2 \\ &= \frac{3}{2}n_{\ge 3} + (n-n_{\ge 3}) = n + \frac{1}{2}n_{\ge 3} \ge n + \frac{2n}{g-2} = \frac{gn}{g-2}.\end{aligned}$$

However, $|E(G)| \le \frac{g}{g-2}(n-2) < \frac{gn}{g-2}$ since G is a planar graph, a contradiction. ∎

For a graph G, let $\Gamma(G)$ be the sum of degrees of all vertices of G. If n_k is the number of vertices of degree k in G, then $\Gamma(G)=\Sigma_{k=1}^{\Delta(G)} kn_k$. Obviously, $\Gamma(G)=2m$ if G has m edges. If $\Delta(G)\le 2$ and the girth of G is large, then we have the result below.

Lemma 9. *Let $t\ge 8$ be an integer. Let G be a graph with $\Delta(G)\le 2$ and girth at least $\frac{(t-8)^2}{2}+6$. Then*

$$\nabla(G) \le \frac{\Gamma(G)}{t}.$$

Proof. If G does not have any cycle, then $\nabla(G)=0$. Obviously, $\nabla(G)\le\frac{\Gamma(G)}{t}$.

If G has at least one cycle, we can suppose that G is a connected graph. Otherwise, every component of G is argued in a similar way. Since $\Delta(G)\le 2$, G is exactly a cycle. Moreover, $\nabla(G)=1$ and $\Gamma(G)=2|V(G)|$. Since the girth of G is at least $\frac{(t-8)^2}{2}+6$, we have $\Gamma(G)\ge(t-8)^2+12$. Thus $\Gamma(G)-t\ge[(t-8)^2+12]-t$ $=(t-\frac{17}{2})^2+\frac{15}{4}>0$. In other words, $\Gamma(G)>t$. Hence $\nabla(G)<\frac{\Gamma(G)}{t}$. ∎

For a planar graph with maximum degree at least three, there is a similar result to that in Lemma 9.

Theorem 10. *Let g and t be two integers with $t \geq 8$ and $t \equiv 0 \pmod 2$. Let G be a planar graph of n vertices with $\Delta(G) \geq 3$, $\delta(G) = 2$, and girth at least g. If $g \geq \frac{(t-8)^2}{2} + 6$, then*

$$\nabla(G) \leq \frac{\Gamma(G)}{t}.$$

Proof. Suppose that the theorem is not true. Let G be a minimum counterexample with respect to the number of vertices.

We claim that G is a connected graph. Otherwise, suppose that G has k components $G_1, G_2, \ldots, G_k$, where $k \geq 2$. Clearly, the girth of any G_i is at least g. We observe that $\delta(G_i) = 2$ for $i = 1, 2, \ldots, k$. Otherwise, suppose that G_j is a component with $\delta(G_j) \geq 3$. Then $|E(G_j)| \geq \frac{3}{2}|V(G_j)|$. Since the girth of G_j is at least six, we have $|E(G_j)| \leq \frac{3}{2}(|V(G_j)| - 2)$, a contradiction. If $\Delta(G_i) \geq 3$ for an arbitrary i, then $\nabla(G_i) \leq \frac{\Gamma(G_i)}{t}$, since G is a minimum counterexample. In this case $\nabla(G) = \sum_{i=1}^{k} \nabla(G_i) \leq \sum_{i=1}^{k} \frac{\Gamma(G_i)}{t} = \frac{\Gamma(G)}{t}$, a contradiction. So there is some component, say G_k, with $\Delta(G_k) = 2$. Considering that $\Delta(G) \geq 3$, there is some component, say G_1, with $\Delta(G_1) \geq 3$. Since $\delta(G_k) = 2$, G_k is a cycle with at least g vertices. Then $\nabla(G_k) = 1$. Let G' be the union of $G_1, G_2, \ldots, G_{k-1}$. Then G' has fewer vertices than that of G with $\Delta(G') \geq 3$ and $\delta(G') = 2$. So $\nabla(G') \leq \frac{\Gamma(G')}{t}$. Since $\Gamma(G') = \Gamma(G) - \Gamma(G_k) \leq \Gamma(G) - 2g$, it follows that

$$\nabla(G) = \nabla(G') + 1 \leq \frac{\Gamma(G) - 2g}{t} + 1 = \frac{\Gamma(G)}{t} - \frac{2g - t}{t}.$$

Notice that $2g - t \geq (t-8)^2 - t + 12 = (t - \frac{17}{2})^2 + \frac{15}{4} > 0$. So $\nabla(G) < \frac{\Gamma(G)}{t}$, a contradiction.

Considering that $\Delta(G) \geq 3$, we have $n_{\geq 3} \geq 1$. Since $t \geq 8$, it follows that $g \geq 6$. By Lemma 8, $n_{\geq 3} < \frac{4n}{g-2}$. Therefore, $n_2 > \frac{(g-6)n}{g-2}$.

Let X be the set of all vertices of degree at least three in G, and let Y be the set of all vertices of degree two in G. We have two cases to consider.

Case 1. There is some vertex, say v, in X such that it is adjacent to at least $\frac{t}{2} - 3$ vertices in Y.

Suppose that $d_G(v) = a$, and suppose that $u_1, u_2, \ldots, u_l$ are all neighbors of v in Y. Then $a \geq 3$ and $l \geq \frac{t}{2} - 3$. We first delete the vertex v from G. Then the degree of any neighbor of v is decreased by one. Next, u_i is removed for $i = 1, 2, \ldots, l$. Let H_1 be the obtained graph. In the previous procedure, $a + l$ edges are deleted because the girth of G is larger than three. Since the deletion of an edge decreases two degrees, we have

$$\Gamma(H_1) = \Gamma(G) - 2a - 2l \leq \Gamma(G) - 6 - 2\left(\frac{t}{2} - 3\right) = \Gamma(G) - t.$$

If $\delta(H_1) \leq 1$, suppose that $w_1, w_2, \ldots, w_p$ are all vertices of degree at most one in H_1, where $p \geq 1$. Next, we remove those p vertices. If the minimum degree of the obtained graph is still at most one, then those vertices of degree at most one are deleted. The procedure is not stopped until a graph H_2 without any vertex of degree at most one is obtained. If H_2 has at most two vertices, then $\nabla(G) \leq 1$. Since $\delta(G) = 2$, G has at least one cycle. Thus $n \geq g$. Considering that $\Delta(G) \geq 3$, we have $\Gamma(G) \geq 2n + 1 \geq 2g + 1 \geq (t-8)^2 + 13$. Hence $\Gamma(G) - t \geq t^2 - 17t + 77 = (t - \frac{17}{2})^2 + \frac{19}{4} > 0$. In other words, $\frac{\Gamma(G)}{t} > 1$. So $\nabla(G) \leq \frac{\Gamma(G)}{t}$, a contradiction. If H_2 has at least three vertices, then $\delta(H_2) \geq 2$. If $\delta(H_2) \geq 3$, then $|E(H_2)| \geq \frac{3}{2}|V(H_2)|$, which violates the fact that $|E(H_2)| \leq \frac{3}{2}(|V(H_2)| - 2)$, since the girth of H_2 is at least six. So $\delta(H_2) = 2$. If $\Delta(H_2) = 2$, then $\nabla(H_2) \leq \frac{\Gamma(H_2)}{t}$ by Lemma 9. So $\nabla(G) \leq \nabla(H_2) + 1 \leq \frac{\Gamma(H_2)}{t} + 1 \leq \frac{\Gamma(H_1)}{t} + 1 \leq \frac{\Gamma(G)}{t}$, a contradiction. If $\Delta(H_2) \geq 3$, there are two cases to consider. If H_2 is a connected graph, then $\nabla(H_2) \leq \frac{\Gamma(H_2)}{t}$, because H_2 has fewer vertices than that of G. Therefore, $\nabla(G) \leq \nabla(H_2) + 1 \leq \frac{\Gamma(H_2)}{t} + 1 \leq \frac{\Gamma(H_1)}{t} + 1 \leq \frac{\Gamma(G)}{t}$, a contradiction. Otherwise, H_2 has s components, say $B_1, B_2, \ldots, B_s$, where $s \geq 2$. Proceeding a similar argument as $G_1, G_2, \ldots, G_k$, it follows that $\nabla(H_2) \leq \frac{\Gamma(H_2)}{t}$. Considering $\nabla(G) \leq \nabla(H_2) + 1$, we have $\nabla(G) \leq \frac{\Gamma(G)}{t}$, a contradiction.

We now suppose that $\delta(H_1) \geq 2$. Then we proceed a similar argument to that of H_2 in the previous paragraph, which yields a contradiction.

Case 2. Any vertex in X joins to at most $\frac{t}{2} - 4$ vertices in Y. Let η be the number of all edges between X and Y. Then $\eta \leq (\frac{t}{2} - 4)|X|$ in this case. Since $|X| = n_{\geq 3} < \frac{4n}{g-2}$, we have

$$\eta \leq \left(\frac{t}{2} - 4\right)|X| < \frac{4n}{g-2}\left(\frac{t}{2} - 4\right) = \frac{2t-16}{g-2}n. \tag{2.1}$$

We now consider the induced subgraph $G[Y]$ of G. Since any vertex in Y is of degree two in G, every component of $G[Y]$ is a path. If every path in $G[Y]$ has at most $\frac{t}{2} - 4$ vertices, then the number of paths in $G[Y]$ is at least $n_2/(\frac{t}{2} - 4)$. Since $n_2 > \frac{g-6}{g-2}n$ and $g \geq \frac{(t-8)^2}{2} + 6$, we have

$$n_2\Big/\left(\frac{t}{2} - 4\right) > \frac{g-6}{g-2}n\Big/\left(\frac{t}{2} - 4\right) \geq \frac{(t-8)^2}{2(g-2)}n\Big/\left(\frac{t}{2} - 4\right) = \frac{t-8}{g-2}n. \tag{2.2}$$

Since any end of a path in $G[Y]$ is adjacent to some vertex in X,

$$\eta > 2 \cdot \frac{t-8}{g-2}n = \frac{2t-16}{g-2}n,$$

which violates the formula (2.1). Thus there is some path, say P, in $G[Y]$ which has at least $\frac{t}{2} - 3$ vertices. Suppose that $P = y_1 y_2 \cdots y_s$, where $s \geq \frac{t}{2} - 3$. Let x_1

be the neighbor of y_1 in X, and let x_s be the neighbor of y_s in X, where x_1 may be x_s. Suppose that $d_G(x_1) = b$. Then $b \geq 3$. We now delete the vertex x_1, then the degree of y_1 is one in the obtained graph. Next, we remove $y_1, y_2, \ldots, y_s$ one by one. Let H_3 be the obtained graph. If $x_1 \neq x_s$, then

$$\Gamma(H_3) = \Gamma(G) - 2b - 2s \leq \Gamma(G) - 6 - 2\left(\frac{t}{2} - 3\right) \leq \Gamma(G) - t.$$

If x_1 is the same as x_s, then $x_1 y_1 \cdots y_s x_1$ is a cycle with at least g vertices in G. Thus $s + 1 \geq g$. So

$$\Gamma(H_3) = \Gamma(G) - 2b - 2s + 2 \leq \Gamma(G) - 6 - 2(g-1) + 2 = \Gamma(G) - 2g - 2. \tag{2.3}$$

Considering $g \geq \frac{(t-8)^2}{2} + 6$, we have $2g \geq (t-8)^2 + 12$. It is easy to check that $(t-8)^2 + 12 > t$ if $t \geq 8$. Thus $2g > t$ if $t \geq 8$. By the formula (2.3), we have

$$\Gamma(H_3) < \Gamma(G) - t - 2 < \Gamma(G) - t.$$

Next, we argue H_3 in a similar way to that for H_1 in Case 1. Then

$$\nabla(G) \leq \nabla(H_3) + 1 \leq \frac{\Gamma(H_3)}{t} + 1 \leq \frac{\Gamma(G)}{t},$$

which violates the assumption that G is a minimum counterexample. The proof is fulfilled. ■

Next, we shall prove Theorem 5. But we need to show the following lemma first.

Lemma 11. *Let G be a connected planar graph of n vertices. If the girth g of G is at least six, then $\Gamma(G) \leq 3n - 6$.*

Proof. Since the girth of G is at least six, we have

$$|E(G)| \leq \frac{6}{6-2}(n-2) = \frac{3}{2}(n-2).$$

Considering that $\Gamma(G) = 2|E(G)|$, we have $\Gamma(G) \leq 3n - 6$. ■

***Proof of Theorem* 5.** Since the deletion of any vertex of degree at most one does not affect the decycling number, we can suppose that $\delta(G) \geq 2$. If G is a connected graph, then the theorem follows from Lemmas 9, 11 and Theorem 10 directly. Otherwise, every component of G is argued in a similar way. Then the theorem holds. ■

In Theorem 5, if $t = 8$, then $g \geq 6$, and if $t = 10$, then $g \geq 8$. So we have the following results by Theorem 5.

Theorem 12. *Let G be a planar graph of n vertices. If the girth of G is at least six* (*or eight*), *then* $\nabla(G) \leq \frac{3n-6}{8}$ $\big($*or* $\nabla(G) \leq \frac{3n-6}{10}\big)$.

Obviously, Conjecture 2 (or Conjecture 3) is true if the girth of a planar graph is at least six (or eight) by Theorem 12.

Theorem 13. *Let $g \geq 8$ be an integer. Let G be a planar graph of m edges. If the girth of G is at least $2(g-4)^2+6$, then $\nabla(G) \leq \frac{m}{g}$.*

Proof. As in the proof of Theorem 5, we can suppose that $\delta(G) \geq 2$. Applying Lemma 9 and Theorem 10, we have $\nabla(G) \leq \frac{\Gamma(G)}{t}$ if the girth of G is at least $\frac{(t-8)^2}{2}+6$. Let $t = 2g$. Then $\frac{(t-8)^2}{2}+6 = 2(g-4)^2+6$. So $\nabla(G) \leq \frac{\Gamma(G)}{2g}$ if the girth of G is at least $2(g-4)^2+6$, where $t \equiv 0 \pmod 2$. Since $\Gamma(G) = 2m$, we have $\nabla(G) \leq \frac{m}{g}$. ■

Remark 14. Given an integer $g \geq 8$, if a planar graph of m edges has girth at least $2(g-4)^2+6$, then Conjecture 6 holds by Theorem 13.

3. The Decycling Number of a Graph Embedded in a Surface

Section 2 has discussed the decycling number of a planar graph with large girth. Notice that a graph is planar if and only if it can be embedded in the sphere. We now consider the decycling number of a connected graph with large girth which is 2-cell embedded in other surfaces. The section starts with a lemma.

Lemma 15. *Let G be a connected graph of n vertices with $\Delta(G) = \Delta \geq 3$, $\delta(G) = 2$, and the girth g at least six. Let $n_{\geq 3}$ be the number of vertices of degree at least three and n_2 the number of vertices of degree two in G. Then*

(1) $n_{\geq 3} < \frac{4n}{g-2} + \frac{4g}{g-2}\gamma$, *if G is 2-cell embedded in the orientable surface S_γ, where $\gamma \geq 1$, and*

(2) $n_{\geq 3} < \frac{4n}{g-2} + \frac{2g}{g-2}\bar{\gamma}$, *if G is 2-cell embedded in the non-orientable surface $N_{\bar{\gamma}}$.*

Proof. (1) If G is 2-cell embedded in the orientable surface S_γ, then $|E(G)| \leq \frac{g}{g-2}(n-2) + \frac{2g}{g-2}\gamma$ by Proposition 7. Thus $|E(G)| < \frac{g}{g-2}n + \frac{2g}{g-2}\gamma$. Suppose on the contrary that $n_{\geq 3} \geq \frac{4n}{g-2} + \frac{4g}{g-2}\gamma$. Then

$$\begin{aligned}|E(G)| &= \frac{1}{2}\sum\nolimits_{k=2}^{\Delta} kn_k = \frac{1}{2}\sum\nolimits_{k=3}^{\Delta} kn_k + n_2\\ &\geq \frac{3}{2}n_{\geq 3} + n_2 = \frac{3}{2}n_{\geq 3} + (n - n_{\geq 3}) = n + \frac{1}{2}n_{\geq 3}\\ &\geq n + \frac{2}{g-2}n + \frac{2g}{g-2}\gamma = \frac{g}{g-2}n + \frac{2g}{g-2}\gamma.\end{aligned}$$

Thus there is a contradiction.

(2) If G is 2-cell embedded in the non-orientable surface $N_{\bar{\gamma}}$, then $|E(G)| \leq \frac{g}{g-2}(n-2) + \frac{g}{g-2}\bar{\gamma}$. If $n_{\geq 3} \geq \frac{4n}{g-2} + \frac{2g}{g-2}\bar{\gamma}$, then there is a contradiction by proceeding a similar argument as in the previous paragraph. Therefore, $n_{\geq 3} < \frac{4}{g-2}n + \frac{2g}{g-2}\bar{\gamma}$. ∎

Lemma 16 [8]**.** *Let $g \geq 4$ be an integer. Let G be a connected graph of n vertices with $\delta(G) = \delta \geq 2$ and girth at least g. Then*

$$n \geq 1 + \frac{(\delta-1)^{(g-1)/2} - 1}{\delta - 2}\delta, \text{ if } g \text{ is odd, or } n \geq \frac{2[(\delta-1)^{g/2} - 1]}{\delta - 2}, \text{ if } g \text{ is even.}$$

The following lemma is related to the number of the vertices and the girth of a connected graph which is 2-cell embedded in the orientable surface S_γ, which will be used in the proof of Theorem 18.

Lemma 17. *Let $t \geq 8$ be an integer. Let G be a connected graph of n vertices with girth at least g which is 2-cell embedded in the orientable surface S_γ where $\gamma \geq 1$. If $\delta(G) \geq 3$, then*

$$g < \left[\frac{(t-8)^2}{2} + 4\right]\gamma + \frac{(t-8)^2}{2} + 6.$$

Proof. Suppose on the contrary that $g \geq \left[\frac{(t-8)^2}{2} + 4\right]\gamma + \frac{(t-8)^2}{2} + 6$. Since $t \geq 8$ and $\gamma \geq 1$, we have $g \geq 10$. Since G is a connected graph with $\delta(G) \geq 3$, $n > 2^{g/2} - 1$ by Lemma 16. It is not hard to show that $2^x > 6x + 1$, where $x \geq 5$ is a variable. So $n \geq 3g$. Let $d = \frac{(t-8)^2}{2} + 4$. Then $d \geq 4$, since $t \geq 8$. Moreover, $n \geq 3g \geq 3\{\left[\frac{(t-8)^2}{2} + 4\right]\gamma + \frac{(t-8)^2}{2} + 6\} = 3d\gamma + \frac{3(t-8)^2}{2} + 18$. Thus

$$\gamma \leq \frac{1}{3d}\left[n - \frac{3(t-8)^2}{2} - 18\right] < \frac{1}{3d}(n-2). \tag{3.1}$$

Since $\delta(G) \geq 3$, we have $|E(G)| \geq \frac{3}{2}n$. On the other hand, $|E(G)| \leq \frac{g}{g-2}(n-2) + \frac{2g}{g-2}\gamma$ by Proposition 7. Hence

$$\frac{g}{g-2}(n-2) + \frac{2g}{g-2}\gamma \geq \frac{3}{2}n,$$

i.e.,

$$\frac{2g}{g-2}\gamma \geq \frac{3}{2}n - \frac{g}{g-2}(n-2).$$

So

$$\gamma \geq \left(\frac{1}{4} - \frac{3}{2g}\right)n + 1. \tag{3.2}$$

Considering that $g \geq \left[\frac{(t-8)^2}{2}+4\right]\gamma+\frac{(t-8)^2}{2}+6$, we have $g \geq d\gamma+6 \geq d+6$. Thus $\frac{1}{4}-\frac{3}{2g} \geq \frac{1}{4}-\frac{3}{2(d+6)}$. So

$$\gamma \geq \left[\frac{1}{4}-\frac{3}{2(d+6)}\right]n+1. \tag{3.3}$$

We now claim that

$$\frac{1}{3d}<\frac{1}{4}-\frac{3}{2(d+6)}.$$

Otherwise, $\frac{1}{3d} \geq \frac{1}{4}-\frac{3}{2(d+6)}$, then $3d^2-4d-24 \leq 0$. So $\frac{4-\sqrt{304}}{6} \leq d \leq \frac{4+\sqrt{304}}{6}$. However, $d \geq 4 > \frac{4+\sqrt{304}}{6}$, a contradiction. Thus $\gamma \geq \frac{n}{3d}+1$ by the formula (3.3), which violates the formula (3.1). Hence $g < \left[\frac{(t-8)^2}{2}+4\right]\gamma+\frac{(t-8)^2}{2}+6$. ■

Theorem 18. *Let t be an integer with $t \geq 8$ and $t \equiv 0 \pmod 2$. Let G be a connected graph of n vertices with girth at least g which is 2-cell embedded in the orientable surface S_γ. If $\Delta(G) \geq 3$, $\delta(G)=2$, and $g \geq \left[\frac{(t-8)^2}{2}+4\right]\gamma+\frac{(t-8)^2}{2}+6$, then*

$$\nabla(G) \leq \frac{\Gamma(G)}{t}.$$

Proof. We use the induction on γ. The base case is that $\gamma=0$. In this case the theorem holds by Theorem 10. Assume that $\nabla(G) \leq \frac{\Gamma(G)}{t}$ if $\gamma<k$, where $k>0$. We now consider the case that $\gamma=k$.

Suppose that the theorem is not true. Let G be a minimum counterexample with respect to the number of vertices. Let X be the set of all vertices of degree at least three and Y the set of all vertices of degree two in G. We have two cases to consider.

Case 1. There is some vertex, say w, in X such that it is adjacent to at least $\frac{t}{2}-3$ vertices in Y.

Suppose that $d_G(w)=c$, and suppose that $u_1,u_2,\ldots,u_j$ are all neighbors of w in Y. Then $c \geq 3$ and $j \geq \frac{t}{2}-3$. As in the proof of Theorem 10, let H_1 be the graph obtained from G by deleting $w,u_1,\ldots,u_j$. Then

$$\Gamma(H_1)=\Gamma(G)-2c-2j \leq \Gamma(G)-6-2\left(\frac{t}{2}-3\right)=\Gamma(G)-t.$$

If $\delta(H_1) \leq 1$, then we continuously remove the vertices of degree at most one. At last, we obtain a graph H_2. If H_2 has at most two vertices or $\Delta(H_2)=2$, then we proceed a similar argument to that in the proof of Theorem 10, which yields a contradiction. If $\delta(H_2) \geq 3$, then there is a contradiction by Lemma 17. So H_2 has at least three vertices with $\Delta(H_2) \geq 3$ and $\delta(H_2)=2$. There are two cases to consider. If H_2 is a connected graph, then H_2 can be 2-cell embedded in some orientable surface S_τ, where $\tau \leq k$. Notice the girth of H_2 is at least g. So $g \geq \left[\frac{(t-8)^2}{2} + 4\right]\tau + \frac{(t-8)^2}{2} + 6$. By the inductional assumption, $\nabla(H_2) \leq \frac{\Gamma(H_2)}{t}$. Hence $\nabla(G) \leq \nabla(H_2) + 1 \leq \frac{\Gamma(H_2)}{t} + 1 \leq \frac{\Gamma(H_1)}{t} + 1 \leq \frac{\Gamma(G)}{t}$, a contradiction. If H_2 is not a connected graph, let $F_1, F_2, \ldots, F_h$ be the all components of H_2, where $h \geq 2$. For $i = 1, 2, \ldots, h$, suppose that F_i is 2-cell embedded in some orientable surface S_{τ_i}, where $\tau_i \leq k$. Obviously, the girth of F_i is at least g. Since $g \geq \left[\frac{(t-8)^2}{2} + 4\right]k + \frac{(t-8)^2}{2} + 6 \geq \left[\frac{(t-8)^2}{2} + 4\right]\tau_i + \frac{(t-8)^2}{2} + 6$, we have $\delta(F_i) = 2$ by Lemma 17. For an arbitrary i, if $\Delta(F_i) \geq 3$, then $\nabla(F_i) \leq \frac{\Gamma(F_i)}{t}$ by the inductional assumption. If $\Delta(F_i) = 2$, then $\nabla(F_i) \leq \frac{\Gamma(F_i)}{t}$ by Lemma 9. Therefore, $\nabla(G) \leq \nabla(H_2) + 1 \leq \sum_{i=1}^{h} \frac{\Gamma(F_i)}{t} + 1 \leq \frac{\Gamma(G)-t}{t} + 1 = \frac{\Gamma(G)}{t}$, a contradiction.

If $\delta(H_1) \geq 2$, then there is a contradiction by proceeding a similar argument as H_2 in the previous paragraph.

Case 2. Any vertex in X joins to at most $\frac{t}{2} - 4$ vertices in Y. Let η be the number of all edges between X and Y. Then $\eta \leq (\frac{t}{2} - 4)|X|$ in this case. By Lemma 15, $|X| = n_{\geq 3} < \frac{4n}{g-2} + \frac{4g}{g-2}\gamma$. Considering that $\gamma = k \geq 1$, we have $g \geq 10$. Thus $\frac{2g}{g-2} < 3$. So

$$\eta \leq \left(\frac{t}{2} - 4\right)|X| < \left(\frac{4}{g-2}n + \frac{4g}{g-2}\gamma\right)\left(\frac{t}{2} - 4\right) = \frac{2t-16}{g-2}n + \frac{2g}{g-2}\gamma(t-8)$$

$$< \frac{2t-16}{g-2}n + 3\gamma(t-8). \tag{3.4}$$

We now consider the induced subgraph $G[Y]$ of G by Y. Since any vertex in Y has degree two in G, every component of $G[Y]$ is a path.

If every path in $G[Y]$ has at most $\frac{t}{2} - 4$ vertices, the number of paths in $G[Y]$ is at least $n_2/(\frac{t}{2} - 4)$. Considering that $\Delta(G) \geq 3$ and $\delta(G) = 2$, we claim that $\frac{n}{g-2} > \frac{3}{2}$. In fact, G has at least two cycles in this case. Let C_1 and C_2 be such two cycles. If C_1 and C_2 have at most one vertex in common, then $n \geq |V(C_1)| + |V(C_2)| - 1 \geq 2g - 1$. So $\frac{n}{g-2} > \frac{n}{g} \geq 2 - \frac{1}{g} > \frac{3}{2}$. Otherwise, C_1 and C_2 have at least two vertices in common. Let F be the union of C_1 and C_2. Let z_1 be a vertex in F whose degree is at least three. Let z_2 be the last vertex in F which has degree at least three when travelling C_1 starting from z_1. Then there are two internally disjoint paths P_1 and P_2 from z_1 to z_2 such that P_i is in C_i for $i = 1, 2$. Let P_3 be the path obtained from C_1 by deleting all edges in P_1 and isolated vertices. P_1, P_2, and P_3 are shown in Figure 1. Then $P_1 \cup P_2$ is a cycle, say C_3, and $P_2 \cup P_3$ is a cycle, say C_4. Let Q be the graph which is the union of P_1, P_2, P_3. Since the girth of G is at least g, we have $|E(Q)| \geq \frac{1}{2}[|E(C_1)| + |E(C_3)| + |E(C_4)|] \geq \frac{3}{2}g$. Considering that Q has two vertices of degree three, $n \geq |V(Q)| \geq \frac{3}{2}g - 2$. Thus $2n \geq 3g - 4 = 3(g-2) + 2$. So $\frac{n}{g-2} > \frac{3}{2}$.

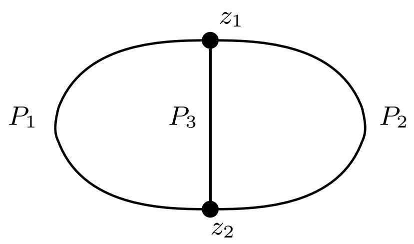


Figure 1. P_1, P_2 and P_3 in G.

Since $g \geq 6$, we have $\frac{4g}{g-2} \leq 6$. By Lemma 15, $n_2 = n - n_{\geq 3} > \frac{g-6}{g-2}n - \frac{4g}{g-2}\gamma$. Thus

$$n_2/\left(\frac{t}{2}-4\right) > \left(\frac{g-6}{g-2}n - \frac{4g}{g-2}\gamma\right)/\left(\frac{t}{2}-4\right) \geq \left(\frac{g-6}{g-2}n - 6\gamma\right)/\left(\frac{t}{2}-4\right).$$

Considering that $g \geq \left[\frac{(t-8)^2}{2}+4\right]\gamma + \left[\frac{(t-8)^2}{2}+6\right]$ and $\frac{n}{g-2} > \frac{3}{2}$, we have

$$\begin{aligned} n_2/\left(\frac{t}{2}-4\right) &> \left\{\frac{(t-8)^2}{2(g-2)}n + \frac{n}{g-2}\left[\frac{(t-8)^2}{2}\gamma + 4\gamma\right] - 6\gamma\right\}/\left(\frac{t}{2}-4\right) \\ &> \left\{\frac{(t-8)^2}{2(g-2)}n + \frac{3}{2}\left[\frac{(t-8)^2}{2}\gamma + 4\gamma\right] - 6\gamma\right\}/\left(\frac{t}{2}-4\right) \qquad (3.5) \\ &= \frac{t-8}{g-2}n + \frac{3}{2}\gamma(t-8). \end{aligned}$$

Since any end of a path in $G[Y]$ is adjacent to some vertex in X,

$$\eta \geq 2\left[\frac{t-8}{g-2}n + \frac{3}{2}\gamma(t-8)\right] = \frac{2t-16}{g-2}n + 3\gamma(t-8),$$

which violates the formula (3.4). So there is a path $P = y_1y_2\cdots y_q$, where $q \geq \frac{t}{2}-3$. Let x_1 be the neighbor of y_1 in X, and let x_q be the neighbor of y_q in X, where x_1 may be x_q. Suppose that $d_G(x_1) = b$. Then $b \geq 3$. Let H_3 be the graph obtained from G by deleting x_1, y_1, $\ldots$, y_q. Next, H_3 is argued as H_1 in Case 1 if $x_1 \neq x_s$. Otherwise, $x_1y_1\cdots y_qx_s$ is a cycle, which is argued in a similar way as in the proof of Theorem 10. Then $\Gamma(H_3) \leq \Gamma(G) - t$. Furthermore, $\nabla(G) \leq \nabla(H_3) + 1 \leq \frac{\Gamma(H_3)}{t} + 1 \leq \frac{\Gamma(G)}{t}$, which violates the assumption that G is a minimum counterexample. Thus the proof is completed. ∎

We now consider the case that a connected graph is 2-cell embedded in a non-orientable surface. We first give two lemmas.

Lemma 19. *Let t be an integer with $t \geq 8$. Let G be a connected graph of n vertices with girth at least g which is 2-cell embedded in the non-orientable surface $N_{\bar{\gamma}}$. If $\delta(G) \geq 3$, then*

$$g < \left[\frac{(t-8)^2}{2} + 2\right]\bar{\gamma} + \frac{(t-8)^2}{2} + 6.$$

Proof. Suppose on the contrary that $g \geq \left[\frac{(t-8)^2}{2} + 2\right]\bar{\gamma} + \frac{(t-8)^2}{2} + 6$. Since $t \geq 8$ and $\bar{\gamma} \geq 1$, we have $g \geq 8$. Since G is a connected graph with $\delta(G) \geq 3$, $n > 2^{g/2} - 1$ by Lemma 16. It is not hard to show that $2^x > 9x - 21$, where $x \geq 4$ is a variable. So $n \geq \frac{9}{2}g - 21$. Let $d = \frac{(t-8)^2}{2} + 2$. Then $d \geq 2$, since $t \geq 8$. Moreover, $n \geq \frac{9}{2}g - 21 \geq \frac{9}{2}\left\{\left[\frac{(t-8)^2}{2} + 2\right]\bar{\gamma} + \frac{(t-8)^2}{2} + 6\right\} - 21 = \frac{9}{2}d\bar{\gamma} + \frac{9(t-8)^2}{4} + 6$. Thus

$$\bar{\gamma} \leq \frac{2}{9d}[n - \frac{9(t-8)^2}{4} - 6] < \frac{2}{9d}(n-2). \tag{3.6}$$

Since $\delta(G) \geq 3$, we have $|E(G)| \geq \frac{3}{2}n$. On the other hand, $|E(G)| \leq \frac{g}{g-2}(n-2) + \frac{g}{g-2}\bar{\gamma}$ by Proposition 7. Hence

$$\frac{g}{g-2}(n-2) + \frac{g}{g-2}\bar{\gamma} \geq \frac{3}{2}n,$$

i.e.,

$$\frac{g}{g-2}\bar{\gamma} \geq \frac{3}{2}n - \frac{g}{g-2}(n-2).$$

So

$$\bar{\gamma} \geq \left(\frac{1}{2} - \frac{3}{g}\right)n + 2. \tag{3.7}$$

Considering that $g \geq \left[\frac{(t-8)^2}{2} + 2\right]\bar{\gamma} + \frac{(t-8)^2}{2} + 6$, we have $g \geq d\bar{\gamma} + 6 \geq d + 6$. Thus $\frac{1}{2} - \frac{3}{g} \geq \frac{1}{2} - \frac{3}{d+6}$. So

$$\bar{\gamma} \geq \left[\frac{1}{2} - \frac{3}{d+6}\right]n + 2. \tag{3.8}$$

We now claim that

$$\frac{2}{9d} < \frac{1}{2} - \frac{3}{d+6}.$$

Otherwise, $\frac{2}{9d} \geq \frac{1}{2} - \frac{3}{d+6}$, then $9d^2 - 4d - 24 \leq 0$. So $\frac{2-\sqrt{220}}{9} \leq d \leq \frac{2+\sqrt{220}}{9}$. However, $d \geq 2 > \frac{2+\sqrt{220}}{9}$, a contradiction. Thus $\bar{\gamma} \geq \frac{2n}{9d} + 2$ by the formula (3.8), which violates the formula (3.6). Hence $g < \left[\frac{(t-8)^2}{2} + 2\right]\bar{\gamma} + \frac{(t-8)^2}{2} + 6$. ∎

Lemma 20. *Let t be an integer with $t \geq 8$ and $t \equiv 0 \pmod 2$. Let G be a connected graph of n vertices with girth at least g which is 2-cell embedded in the projective plane N_1. If $\Delta(G) \geq 3$, $\delta(G) = 2$, and $g \geq (t-8)^2 + 8$, then*

$$\nabla(G) \leq \frac{\Gamma(G)}{t}.$$

Proof. Suppose that the theorem is not true. Let G be a minimum counterexample with respect to the number of vertices. Let X be the set of all vertices of degree at least three and Y the set of all vertices of degree two in G. We consider two cases.

Case 1. There is some vertex, say u, in X such that it is adjacent to at least $\frac{t}{2} - 3$ vertices in Y.

Suppose that $d_G(u) = \alpha$, and suppose that $z_1, z_2, \ldots, z_\beta$ are all neighbors of u in Y. Then $\alpha \geq 3$ and $\beta \geq \frac{t}{2} - 3$. Let H_1 be the graph obtained from G by deleting $u, z_1, \ldots, z_\beta$. Proceeding a similar argument to that in the proof of Theorem 10, we have $\Gamma(H_1) \leq \Gamma(G) - t$.

If $\delta(H_1) \leq 1$, let H_2 be the graph obtained from H_1 by deleting the vertices of degree at most one continuously. If H_2 has at most two vertices or $\Delta(H_2) = 2$, then there is a contradiction by proceeding a similar argument to that in the proof of Theorem 10. If $\delta(H_2) \geq 3$, then there is also a contradiction by Lemma 19. So H_2 has at least two vertices with $\Delta(H_2) \geq 3$ and $\delta(H_2) = 2$. Now, we consider two cases. If H_2 is a connected graph, then H_2 can be 2-cell embedded in the sphere or the projective plane. If the former occurs, then $\nabla(H_2) \leq \frac{\Gamma(H_2)}{t}$ by Theorem 10. If the latter occurs, then $\nabla(H_2) \leq \frac{\Gamma(H_2)}{t}$, since G is a minimum counterexample. Thus $\nabla(G) \leq \nabla(H_2) + 1 \leq \frac{\Gamma(H_2)}{t} + 1 \leq \frac{\Gamma(H_1)}{t} + 1 \leq \frac{\Gamma(G)}{t}$, a contradiction. If H_2 is not a connected graph, let $Q_1, Q_2, \ldots, Q_\theta$ be the all components of H_2, where $\theta \geq 2$. For an arbitrary i, if Q_i can be embedded in the sphere, then $\nabla(Q_i) \leq \frac{\Gamma(Q_i)}{t}$. If Q_i is 2-cell embedded in the projective plane, then $\Delta(Q_i) \geq 3$. If $\delta(Q_i) \geq 3$, then the girth of Q_i is less than $(t-8)^2+8$ by Lemma 19. However, the girth of Q_i is at least g which is at least $(t-8)^2+8$, a contradiction. If $\delta(Q_i) = 2$, then $\nabla(Q_i) \leq \frac{\Gamma(Q_i)}{t}$, since G is a minimum counterexample. Therefore, $\nabla(G) \leq \nabla(H_2) + 1 \leq \sum_{i=1}^{\theta} \frac{\Gamma(Q_i)}{t} + 1 \leq \frac{\Gamma(G)-t}{t} + 1 = \frac{\Gamma(G)}{t}$, a contradiction.

We now suppose that $\delta(H_1) \geq 2$. Then there is a contradiction by proceeding a similar argument as H_2 in the previous paragraph.

Case 2. Any vertex in X joins to at most $\frac{t}{2} - 4$ vertices in Y. Let η be the number of all edges between X and Y. Then $\eta \leq \left(\frac{t}{2} - 4\right)|X|$ in this case. By Lemma 15, $|X| = n_{\geq 3} < \frac{4n}{g-2} + \frac{2g}{g-2}$. So

$$\eta \leq \left(\frac{t}{2} - 4\right)|X| < \left(\frac{4}{g-2}n + \frac{2g}{g-2}\right)\left(\frac{t}{2} - 4\right) = \frac{2t-16}{g-2}n + \frac{g}{g-2}(t-8).$$

Since $g \geq 8$, we have $\frac{2g}{g-2} < 3$. So

$$\eta < \frac{2t-16}{g-2}n + \frac{3}{2}(t-8). \tag{3.9}$$

We now consider the induced subgraph $G[Y]$ of G by Y. Since any vertex in Y has degree two in G, every component of $G[Y]$ is a path.

If every path in $G[Y]$ has at most $\frac{t}{2} - 4$ vertices, then the number of paths in $G[Y]$ is at least $n_2/\left(\frac{t}{2} - 4\right)$. Proceeding a similar argument to that in the proof Theorem 18, we claim that $\frac{n}{g-2} > \frac{3}{2}$.

Since $\frac{2g}{g-2} < 3$ and $n_2 = n - n_{\geq 3} > \frac{g-6}{g-2}n - \frac{2g}{g-2}$, we have

$$n_2/\left(\frac{t}{2} - 4\right) > \left(\frac{g-6}{g-2}n - \frac{2g}{g-2}\right)/\left(\frac{t}{2} - 4\right) > \left(\frac{g-6}{g-2}n - 3\right)/\left(\frac{t}{2} - 4\right).$$

Considering that $g \geq (t-8)^2 + 8$ and $\frac{n}{g-2} > \frac{3}{2}$, we have

$$\begin{aligned} n_2/\left(\frac{t}{2} - 4\right) &> \left\{\frac{1}{g-2}\left[(t-8)^2 + 2\right]n - 3\right\}/\left(\frac{t}{2} - 4\right) \\ &= \left\{\frac{(t-8)^2}{2(g-2)}n + \frac{n}{g-2}\left[\frac{(t-8)^2}{2} + 2\right] - 3\right\}/\left(\frac{t}{2} - 4\right) \\ &> \left\{\frac{(t-8)^2}{2(g-2)}n + \frac{3}{2}\left[\frac{(t-8)^2}{2} + 2\right] - 3\right\}/\left(\frac{t}{2} - 4\right) \qquad (3.10) \\ &= \frac{t-8}{g-2}n + \frac{3}{2}(t-8). \end{aligned}$$

Since any end of a path in $G[Y]$ is adjacent to some vertex in X, we have

$$\eta \geq 2\left[\frac{t-8}{g-2}n + \frac{3}{2}(t-8)\right] = \frac{2t-16}{g-2}n + 3(t-8),$$

which violates the formula (3.9). Next, proceed a similar argument as in the proof of Theorem 18. Then there is a contradiction. ■

We now consider the decycling number of a connected graph which is 2-cell embedded in a non-orientable surface.

Theorem 21. *Let t be an integer with $t \geq 8$ and $t \equiv 0 \pmod 2$. Let G be a connected graph of n vertices with girth at least g which is 2-cell embedded in the non-orientable surface $N_{\bar{\gamma}}$. If $\Delta(G) \geq 3$, $\delta(G) = 2$, and $g \geq [\frac{(t-8)^2}{2} + 2]\bar{\gamma} + \frac{(t-8)^2}{2} + 6$, then*

$$\nabla(G) \leq \frac{\Gamma(G)}{t}.$$

Proof. We use the induction on $\bar{\gamma}$. The base case is that $\bar{\gamma} = 1$. By Lemma 20, the theorem holds. Assume that $\nabla(G) \leq \frac{\Gamma(G)}{t}$ if $\bar{\gamma} < k$, where $k \geq 1$. We now consider the case that $\bar{\gamma} = k$. Suppose that the theorem does not hold. Let G be a minimum counterexample with respect to the number of vertices. Next we proceed a similar argument to that in the proof of Theorem 18. The difference is that the application of Lemma 17 is replaced with that of Lemma 19. Then the theorem is true. ■

Lemma 22. *Let G be a connected graph of n vertices with the girth g at least six.*

(1) *If G is 2-cell embeddable in the orientable surface S_γ, where $\gamma \geq 1$, then $\Gamma(G) \leq 3n - 6 + 6\gamma$.*

(2) *If G is 2-cell embeddable in the non-orientable surface $N_{\bar{\gamma}}$, then $\Gamma(G) \leq 3n - 6 + 3\bar{\gamma}$.*

Proof. (1) By Proposition 7, we have

$$|E(G)| \leq \frac{g}{g-2}(n-2) + \frac{2g}{g-2}\gamma.$$

Since $g \geq 6$, we have $\frac{g}{g-2} \leq \frac{3}{2}$. So

$$|E(G)| \leq \frac{3}{2}(n - 2 + 2\gamma).$$

Since $\Gamma(G) = 2|E(G)|$, we have $\Gamma(G) \leq 3n - 6 + 6\gamma$.

(2) Proceeding a similar argument as in (1), $\Gamma(G) \leq 3n - 6 + 3\bar{\gamma}$ if G is 2-cell embeddable in the non-orientable surface $N_{\bar{\gamma}}$. ■

The theorem below follows from Theorems 18, 21 and Lemma 22 directly.

Theorem 23. *Let t and g be two integers with $t \geq 8$, $t \equiv 0 \pmod 2$, and $g \geq 6$. Let G be a connected graph of n vertices with $\Delta(G) \geq 3$, $\delta(G) = 2$, and the girth at least g.*

(1) *If G is 2-cell embedded in the orientable surface S_γ, where $\gamma \geq 1$, and if $g \geq \left[\frac{(t-8)^2}{2} + 4\right]\gamma + \frac{(t-8)^2}{2} + 6$, then*

$$\nabla(G) \leq \frac{3}{t}(n - 2 + 2\gamma).$$

(2) *If G is 2-cell embedded in the non-orientable surface $N_{\bar{\gamma}}$, and if $g \geq \left[\frac{(t-8)^2}{2} + 2\right]\bar{\gamma} + \frac{(t-8)^2}{2} + 6$, then*

$$\nabla(G) \leq \frac{3}{t}(n - 2 + \bar{\gamma}).$$

If $t = 8$, then $\left[\frac{(t-8)^2}{2}+4\right]\gamma+\frac{(t-8)^2}{2}+6 = 6+4\gamma$, and $\left[\frac{(t-8)^2}{2}+2\right]\bar{\gamma}+\frac{(t-8)^2}{2}+6 = 6+2\bar{\gamma}$. Let G be a connected graph of n vertices with girth at least g which is 2-cell embedded in the orientable surface S_γ where $\gamma \geq 1$ (or the non-orientable surface $N_{\bar{\gamma}}$). By Lemma 17 (or Lemma 19), if $g \geq 6+4\gamma$ (or $g \geq 6+2\bar{\gamma}$), then $\delta(G) \leq 2$. If $\Delta(G) = 2$, then G is a cycle which cannot be 2-cell embedded in the orientable surface S_γ where $\gamma \geq 1$ and the non-orientable surface $N_{\bar{\gamma}}$. So $\Delta(G) \geq 3$. Considering that the deletion of any vertex of degree at most one does not affect the decycling number, we have the following result by Theorem 23.

Theorem 24. *Let G be a connected graph of n vertices with girth at least g.*

(1) *If G is 2-cell embedded in the orientable surface S_γ, where $\gamma \geq 1$, and if $g \geq 6+4\gamma$, then $\nabla(G) \leq \frac{3}{8}(n-2+2\gamma)$.*

(2) *If G is 2-cell embedded in the non-orientable surface $N_{\bar{\gamma}}$ and if $g \geq 6+2\bar{\gamma}$, then $\nabla(G) \leq \frac{3}{8}(n-2+\bar{\gamma})$.*

If $t = 10$, then $\left[\frac{(t-8)^2}{2}+4\right]\gamma+\frac{(t-8)^2}{2}+6 = 8+6\gamma$, and $\left[\frac{(t-8)^2}{2}+2\right]\bar{\gamma}+\frac{(t-8)^2}{2}+6 = 8+4\bar{\gamma}$. Similarly, we have the results below.

Theorem 25. *Let G be a connected graph of n vertices with girth at least g.*

(1) *If G is 2-cell embedded in the orientable surface S_γ, where $\gamma \geq 1$, and if $g \geq 8+6\gamma$, then $\nabla(G) \leq \frac{3}{10}(n-2+2\gamma)$.*

(2) *If G is 2-cell embedded in the non-orientable surface $N_{\bar{\gamma}}$ and if $g \geq 8+4\bar{\gamma}$, then $\nabla(G) \leq \frac{3}{10}(n-2+\bar{\gamma})$.*

4. Two Conjectures

Let G be a connected graph of n vertices with girth at least g. If G is 2-cell embedded in the orientable surface S_γ (or the non-orientable surface $N_{\bar{\gamma}}$), we have showed that $\nabla(G) \leq \frac{3}{8}(n-2+2\gamma)$ $\big($or $\nabla(G) \leq \frac{3}{8}(n-2+\bar{\gamma})\big)$ if $g \geq 6+4\gamma$ (or $g \geq 6+2\bar{\gamma}$), and that $\nabla(G) \leq \frac{3}{10}(n-2+2\gamma)$ $\big($or $\frac{3}{10}(n-2+\bar{\gamma})\big)$ if $g \geq 8+6\gamma$ (or $g \geq 8+4\bar{\gamma}$).

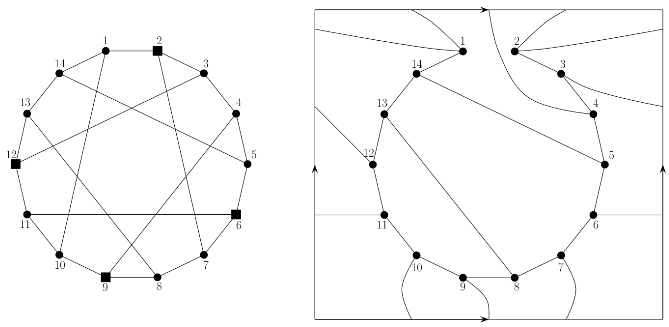
Notice that $\frac{3}{8}(n-2+2\gamma) < \frac{3}{8}(n+2\gamma)$ and that $\frac{3}{10}(n-2+2\gamma) < \frac{3}{10}(n+2\gamma)$. If G is 2-cell embedded in S_γ with $\gamma = 0$ (i.e., the sphere), Conjecture 4 tells us that $\nabla(G)$ may be no more than $\frac{3n}{2g}$. So we propose the following conjecture as a generalization of Conjecture 4.

Conjecture 26. *Let G be a connected graph of n vertices with girth at least g which is 2-cell embedded in the orientable surface S_γ. Then $\nabla(G) \leq \frac{3}{2g}(n+2\gamma)$.*

For the non-orientable surfaces, we have a similar conjecture.

Conjecture 27. *Let G be a connected graph of n vertices with girth at least g which is 2-cell embedded in the non-orientable surface $N_{\bar{\gamma}}$. Then $\nabla(G) \leq \frac{3}{2g}(n+\bar{\gamma})$.*

We observe that for the complete bipartite graph $K_{n,n}$ ($n \geq 3$), Conjectures 26 and 27 hold. It is known that the decycling number of $K_{n,n}$ is $n-1$ [6] and that $K_{n,n}$ can be 2-cell embedded in the orientable surface $S_{\left\lceil \frac{(n-2)^2}{4} \right\rceil}$ (or the non-orientable surface $N_{\left\lceil \frac{(n-2)^2}{2} \right\rceil}$) [24, 25]. Obviously, the girth of $K_{n,n}$ is four. If $n=3$, then $\frac{3}{4}\left\lceil \frac{(n-2)^2}{4} \right\rceil - \left(\frac{n}{4}-1\right) = \frac{3}{4}-\frac{3}{4}+1 > 0$. If $n \geq 4$, then $\frac{3}{4}\left\lceil \frac{(n-2)^2}{4} \right\rceil - \left(\frac{n}{4}-1\right) \geq \frac{3(n-2)^2}{16} - \frac{n}{4} + 1 \geq \frac{3(n-2)}{8} - \frac{n}{4} + 1 = \frac{n}{8} + \frac{1}{4} > 0$. So $\frac{3}{8}\left(2n + 2\left\lceil \frac{(n-2)^2}{4} \right\rceil\right) = \frac{3n}{4} + \frac{3}{4}\left\lceil \frac{(n-2)^2}{4} \right\rceil > \frac{3n}{4} + \frac{n}{4} - 1 = n-1$. Similarly, $\frac{3}{8}\left(2n + \left\lceil \frac{(n-2)^2}{2} \right\rceil\right) \geq \frac{3n}{4} + \frac{n}{4} - 1 = n-1$. Thus $\nabla(K_{n,n}) \leq \frac{3}{8}\left(2n + 2\left\lceil \frac{(n-2)^2}{4} \right\rceil\right)$ if $n \geq 3$. Similarly, $\nabla(K_{n,n}) \leq \frac{3}{8}\left(2n + \left\lceil \frac{(n-2)^2}{2} \right\rceil\right)$ if $n \geq 3$.



(1) (2)

Figure 2. (1) the graph H, (2) a 2-cell embedding of H in the torus.

We now give another example. The graph H shown in Figure 2(1) comes from [10]. It is easy to check that the vertices represented by black squares in Figure 2(1) form a decycling set of H. So $\nabla(H) \leq 4$. Notice that any subgraph of H obtained by deleting three vertices has 11 vertices and at least 12 edges which has at least one cycle. So we have $\nabla(H) > 3$. Thus $\nabla(H) = 4$. On the other hand, H can be 2-cell embedded in the torus (see Figure 2(2)) and the girth of H is six. So $\frac{3}{2g}(n+2\gamma) = 4$. Hence Conjecture 26 holds for the graph H.

Acknowledgements

The authors thank the referees for a careful reading of the manuscript and their helpful suggestions. The second author is supported by NSFC under the granted number 12271169 and Science and Technology Commission of Shanghai Municipality (No. 22DZ2229014).

References

[1] J. Akiyama and M. Watanabe, *Maximum induced forests of planar graphs*, Graphs Combin. **3** (1987) 201–202.
https://doi.org/10.1007/BF01788541

[2] M.O. Albertson and R.Haas, *A problem raised at the DIMACS Graph Coloring Week* (New Jersey, 1998).

[3] M.O Albertson and D.M. Berman, *The acyclic chromatic number*, in: Proc. Seventh Southeastern Conference on Combinatorics, Graph Theory, and Computing, Congr. Number. **17** (Vinnipeg, Man., 1976) 51–69.

[4] N. Alon, *Problems and results in extremal combinatorics*—I, Discrete Math. **273** (2003) 31–53.
https://doi.org/10.1016/S0012-365X(03)00227-9

[5] N. Alon, D. Mubayi and R. Thomas, *Large induced forests in sparse graphs*, J. Graph Theory **38** (2001) 113–123.
https://doi.org/10.1002/jgt.1028

[6] L.W. Beineke and R.C. Vandell, *Decycling graphs*, J. Graph Theory **25** (1997) 59–77.
https://doi.org/10.1002/(SICI)1097-0118(199705)25:1¡59::AID-JGT4¿3.0.CO;2-H

[7] J.A. Bondy and U.S.R. Murty, Graph Theory, Grad. Texts in Math. **244** (Springer, London, 2008).
https://doi.org/10.1007/978-1-84628-970-5

[8] B. Bollobás, Extremal Graph Theory (Academic Press, London-NewYork-San Francisco, 1978).

[9] O.V. Borodin, *A proof of Grünbaum's conjecture on the acyclic* 5*-colorability of planar graphs*, Dokl. Akad. Nauk SSSR **231** (1976) 18–20, in Russian.

[10] F. Dross, M. Montassier and A. Pinlou, *A lower bound on the order of the largest induced forest in planar graphs with high girth*, Discrete Appl. Math. **214** (2016) 99–107.
https://doi.org/10.1016/j.dam.2016.06.011

[11] F. Dross, M. Montassier and A. Pinlou, *Large induced forests in planar graphs with girth* 4, Discrete Appl. Math. **254** (2019) 96–106.
https://doi.org/10.1016/j.dam.2018.06.029

[12] P. Erdős, M. Saks and V.T. Sós, *Maximum induced trees in graphs*, J. Combin. Theory Ser. B **41** (1986) 61–79.
https://doi.org/10.1016/0095-8956(86)90028-6

[13] P. Festa, P.M. Pardalos and M.G.C. Reseude, *Feedback set problem*, in: Handbook of Combinatorial Optimization, D.-Z. Du and P.M. Pardalos (Ed(s)) (Springer, Boston, MA, 1999) 209–258.
https://doi.org/10.1007/978-1-4757-3023-4_4

[14] P.J. Heahood, *Map-colour theorem*, Quart. J. Pure and Appl. Math. **24** (1890) 332–338.

[15] K. Hosono, *Induced forests in trees and outerplanar graphs*, Proc. Fac. Sci. Tokai Univ. **25** (1990) 27–29.

[16] R.M. Karp, *Reducibility among combinatorial problems*, in: Complexity of Computer Computations, R.E. Miller, J.W. Thatcher and J.D. Bohlinger (Ed(s)) (Springer, Boston, MA, 1972) 85–103.
https://doi.org/10.1007/978-1-4684-2001-2_9

[17] L. Kowalik, B. Lužar and R. Škrekovski, *An improved bound on the large induced forests for triangle-free planar graphs*, Discrete Math. Theor. Comput. Sci. **12** (2010) 87–100.
https://doi.org/10.46298/dmtcs.487

[18] J.-P. Liu and C. Zhao, *A new bound on the feedback vertex sets in cubic graphs*, Discrete Math. **148** (1996) 119–131.
https://doi.org/10.1016/0012-365X(94)00268-N

[19] S.-D. Long and H. Ren, *The decycling number and maximum genus of cubic graphs*, J. Graph Theory **88** (2018) 375–384.
https://doi.org/10.1002/jgt.22218

[20] M.-Y. Ma and H. Ren, *The decycling number of a line graph*, (2024) a manuscript, submitted.

[21] B. Mohar and C. Thomassen, Graphs on Surfaces (Johns Hopkins University Press, 2001).
https://doi.org/10.56021/9780801866890

[22] D.A. Pike and Y. Zou, *Decycling Cartesian products of two cycles*, SIAM J. Discrete Math. **19** (2005) 651–663.
https://doi.org/10.1137/S089548010444016X

[23] H. Ren, C. Yang and T.-X. Zhao, *A new formula for the decycling number of regular graphs*, Discrete Math. **340** (2017) 3020–3031.
https://doi.org/10.1016/j.disc.2017.07.011

[24] G. Ringel, *Das Geschlecht des vollständigen paaren Graphen*, Abh. Math. Sem. Univ. Hamburg **28** (1965) 139–150.
https://doi.org/10.1007/BF02993245

[25] G. Ringel, *Der vollständige paare Graph auf nichtorientierbaren Flächen*, J. Reine Angew. Math. **220** (1965) 88–93.
https://doi.org/10.1515/crll.1965.220.88

Received 21 November 2023
Revised 17 April 2024
Accepted 17 April 2024
Available online 14 June 2024