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LOWER BOUNDARY INDEPENDENT AND HEARING INDEPENDENT BROADCASTS IN GRAPHS

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Abstract

A *broadcast* on a connected graph G with vertex set $V(G)$ is a function $f : V(G) \to \{0, 1, \ldots, \operatorname{diam}(G)\}$ such that $f(v) \leq e(v)$ (the eccentricity of v) for all $v \in V(G)$. A vertex v is said to be *broadcasting* if $f(v) > 0$, with the set of all such vertices denoted V_f^+. A vertex u *hears* f from $v \in V_f^+$ if $d_G(u, v) \leq f(v)$. The broadcast f is *hearing independent* if no broadcasting vertex hears another. If, in the broadcast f, any vertex u that hears f from multiple broadcasting vertices satisfies $f(v) \leq d_G(u, v)$ for all $v \in V_f^+$, it is said to be *boundary independent.*

The *cost* of f is $\sigma(f) = \sum_{v \in V(G)} f(v)$. The minimum cost of a maximal boundary independent broadcast on G, called the *lower bn-independence number*, is denoted by $i_{bn}(G)$. The *lower h-independence number* $i_h(G)$ is defined analogously for hearing independent broadcasts. We prove that for an arbitrary connected graph G, either parameter equals the minimum of the corresponding parameter among that of the spanning trees of G. We use these results to prove that $i_{bn}(G) \leq i_h(G)$ for all graphs G. We also show that $i_h(G)/i_{bn}(G)$ is bounded.

Keywords: broadcast domination, broadcast independence, boundary independence, hearing independence.

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1. Introduction

There are several methods by which the concept of independent sets may be generalized to broadcast independence. If we require that no broadcasting vertex hears another, we obtain the definition of *cost independent* broadcasts introduced by Erwin in [5], which we refer to as *hearing independent*. The definition of boundary independent broadcasts, in which no broadcasts overlap on edges, was introduced by Neilson [16] and Mynhardt and Neilson [13] as an alternative to hearing independence. We further investigate the lower parameters $i_h(G)$ and $i_{bn}(G)$ on general graphs and trees.

We present broadcast definitions in Section 2, and preliminary results and observations in Section 3. In Section 4, we prove that for any connected graph G,

$$i_{bn}(G) = \min\{i_{bn}(T) : T \text{ is a spanning tree of G}\},$$

and

$$i_h(G) = \min\{i_h(T) : T \text{ is a spanning tree of G}\}.$$

We use these results to prove our main result in Theorem 9, namely that $i_{bn}(G) \leq i_h(G)$ for any graph G. In Section 5 we show that $i_h(G)/i_{bn}(G) \leq \frac{5}{4}$ for all graphs G. Open problems and directions for further research are discussed in Section 6. For terminology and general concepts in graphs theory not defined in this paper, see Chartrand, Lesniak, and Zhang [3].

2. Definitions and Background

Erwin [5] defined a *broadcast* on a connected graph G as a function $f : V(G) \to \{0, 1, \dots, \text{diam}(G)\}$ such that $f(v)$ is at most the eccentricity $e(v)$ for all vertices v. We say a vertex v is *broadcasting* if $f(v) \geq 1$, and that $f(v)$ is the *strength* of f from v. The *cost* or *weight* of f is $\sigma(f) = \sum_{v \in V(G)} f(v)$.

Given a broadcast f on G and a broadcasting vertex v, a vertex u *hears* f from v if $d_G(u, v) \leq f(v)$. We define the *f-neighbourhood* of v, denoted by $N_f[v]$, as the set of all vertices which hear f from v (including v itself).

The *f-private neighbourhood* of v, denoted by $PN_f(v)$, consists of those vertices that hear f only from v. The *f-boundary* of v is $B_f(v) = \{u \in N_f[v] \mid d(u, v) = f(v)\}$. The *$f$-private boundary* $PB_f(v)$ is defined analogously. In particular, $PB_f(v) = PN_f(v) \cap B_f(v)$. If $u \in N_f[v] \backslash B_f(v)$, v is said to *overdominate* u by k, where $k = f(v) - d_G(u, v)$. A vertex which does not broadcast or hear f from any broadcasting vertex is *undominated*.

We partition the set of broadcasting vertices V_f^+ into $V_f^1 = \{v \in V(G) \mid f(v) = 1\}$ and $V_f^{++} = \{v \in V(G) \mid f(v) > 1\}$. We denote the set of undominated vertices by U_f. A broadcast f is *dominating* if $U_f = \emptyset$. The broadcast domination number, $\gamma_b(G)$, is the minimum weight of such a broadcast. A broadcast f is a *radial broadcast* if $V_f^+ = \{c\}$, where c is a central vertex of G, and $f(c) = \operatorname{rad}(G)$. An overview of broadcast domination in graphs is given by Henning, MacGillivray, and Yang in [9].

We say an edge $e = uv$ *hears* f or *is covered by* $w \in V_f^+$ if $u, v \in N_f[w]$ and at least one endpoint does not lie on the f-boundary of w. If no such vertex w exists, then e is *uncovered*. The set of uncovered edges is denoted U_f^E.

An *independent set* on a graph G is a set of pairwise nonadjacent vertices. The minimum cardinality of a maximal independent set, called the *independent domination number* of G, is denoted by $i(G)$. A broadcast f is *hearing independent*, abbreviated *h-independent*, if $x \notin N_f[v]$ for any $x, v \in V_f^+$. A broadcast f is *boundary independent*, abbreviated *bn-independent*, if $N_f[v] \setminus B_f(v) \subseteq PN_f(v)$ for all $v \in V_f^+$. It follows directly from the definitions that any bn-independent broadcast is also h-independent, but the converse is false. A bn-independent broadcast f is illustrated in Figure 1, where the broadcasting vertices v_1 and v_2 are indicated in red. Here, $f(v_1) = 1$ and $f(v_2) = 2$.

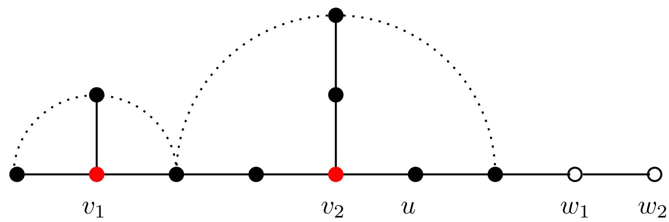

Figure 1. A bn-independent broadcast f on a tree. Vertices v_1 and v_2 broadcast at strengths 1 and 2, respectively. The vertex v_2 overdominates u by 1, whereas w_1 and w_2 are undominated.

Maximal independent broadcasts are those for which the broadcast strength cannot be increased at any vertex without violating the independence condition. If f and g are broadcasts on a graph G, we say that $g \leq f$ if $g(v) \leq f(v)$ for all $v \in V(G)$. If in addition $g(v) < f(v)$ for some v, we write $g < f$. A bn-independent broadcast f is *maximal bn-independent* if there exists no bn-independent broadcast g such that $g > f$. The bn-independent broadcast f illustrated in Figure 1 is not maximal bn-independent, because the broadcast g obtained from f by broadcasting from w_1 with a strength of 1 is also bn-independent, and $g > f$. In turn, g is not maximal h-independent, because the broadcast h obtained from f by broadcasting from w_1 with a strength of 2 is also h-independent, and $h > g$.

Mynhardt and Neilson [13] defined $i_{bn}(G)$ as the minimum weight of a maximal bn-independent broadcast on G, the *lower bn-independence number*. The minimum weight of a maximal h-independent broadcast, denoted by $i_h(G)$, is the *lower h-independence number*. We refer to a maximal h-independent broadcast of minimum cost as an *i_h-broadcast*, and to a maximal bn-independent broadcast of minimum cost as an *i_{bn}-broadcast*.

To illustrate the above-mentioned concepts, we depict a tree T and a broadcast f that is both bn-independent and h-independent in Figure 2. It can be seen (and will follow from Propositions 2 and 3, respectively) that f is maximal bn-independent but not maximal h-independent; broadcasting with a strength of 2 instead of 1 from v yields an h-independent broadcast g such that $g > f$. It can also be verified that f is an i_{bn}-broadcast while g is an i_h-broadcast; we leave the details to the reader. Hence, T is a tree with $i_{bn}(T) < i_h(T)$. As shown in [10, Proposition 2.2.1], the difference $i_h(G) - i_{bn}(G)$ can be arbitrary, even for trees.

Hearing independence was further studied by Bessy and Rautenbach [1, 2] and by Dunbar, Erwin, Haynes, Hedetniemi and Hedetniemi [4]. The more recent study of bn-independent broadcasts was continued by Mynhardt and Neilson in [12, 14, 15] and by Marchessault and Mynhardt in [11].

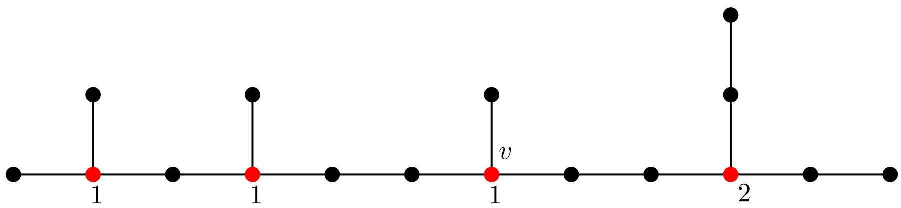

Figure 2. A tree with $i_{bn}(T) = 5$ and $i_h(T) = \mathrm{rad}(T) = 6$.

3. Preliminaries

Observe that if a broadcast f is h-independent or bn-independent but not dominating on a graph G, then f may be extended to a dominating broadcast g by successively broadcasting at strength 1 from an undominated vertex in $V(G)$ until no such vertices remain. We state this fact below for reference.

Observation 1. *If f is a maximal bn-independent or maximal h-independent broadcast, then f is dominating.*

Mynhardt and Neilson [13] extended Observation 1 to a necessary and sufficient condition for a bn-independent broadcast to be maximal bn-independent.

Proposition 2 [13]. *Let f be a bn-independent broadcast on a connected graph G. Then f is maximal bn-independent if and only if f is dominating, and either*

(i) $|V_f^+| = 1$, *or*

(ii) $B_f(v) \setminus PB_f(v) \neq \emptyset$ *for each* $v \in V_f^+$.

It is natural to consider the analogous result for maximal hearing independence.

Proposition 3. *Let f be an h-independent broadcast on a connected graph G. Then f is maximal hearing independent if and only if f is dominating, and either*

(i) $|V_f^+| = 1$, *or*

(ii) *for each $v \in V_f^+$ there exist $u \in B_f(v)$ and $w \in V_f^+ \setminus \{v\}$ such that $uw \in E(G)$, i.e., each broadcasting vertex has a vertex on its boundary that is adjacent to another vertex in V_f^+.*

To illustrate Proposition 3(ii), observe that in Figure 3, the dominating h-independent broadcast f cannot be increased at v, otherwise the vertex $w \in V_f^+$ adjacent to $u \in B_f(v)$ would hear f from v. Similarly, f cannot be increased at either vertex broadcasting at strength 1.

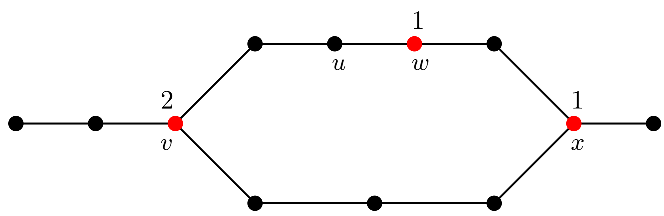

Figure 3. A dominating maximal h-independent broadcast f with $|V_f^+| = 3$.

Proof. Let f be a maximal h-independent broadcast on G. By Observation 1, f is dominating. Suppose there exists $v \in V_f^+$ such that $uw \notin E(G)$ for all $u \in B_f(v)$ and $w \in V_f^+ \setminus \{v\}$. Then either $|V_f^+| = 1$ (in which case $f(v) = e(v)$, where $e(v)$ denotes the eccentricity of v), or we may define a new broadcast f' where $f'(v) = f(v)+1$ and $f'(x) = f(x)$ for all $x \neq v$. Since f' is dominating and no broadcasting vertex hears another, f' is h-independent on G, contradicting the maximality of f.

Conversely, let f be a dominating h-independent broadcast such that (i) or (ii) hold. If $|V_f^+| = 1$ and $V_f^+ = \{v\}$, then $f(v) = e(v)$, otherwise f would not be dominating. Suppose, for a contradiction, that f satisfies (ii) and is not maximal h-independent. Then there exists $v \in V_f^+$ such that increasing the strength of the broadcast on v by 1 results in a new h-independent broadcast f'. By (ii), since $v \in V_f^+$, there exists $u \in B_f(u)$ adjacent to a broadcasting vertex $w \in V_f^+ \setminus \{v\}$. But then $w \in B_{f'}(v)$, a contradiction. Hence f is maximal h-independent. ■

If v and w are vertices in V_f^+ such that w is adjacent to a vertex in $B_f(v)$, we write $w \to v$ and say that w *provides a certificate that the broadcast cannot be increased at* v, or, in short, that w *certifies* v.

Observation 4. *By definition,* $w \to v$ *if and only if* $d(v, w) = f(v) + 1$. *Suppose* $w \to v$. *Since* $f(w) < d(v, w)$, $f(w) \le f(v)$. *Thus, if* $v \not\to w$, *then* $f(w) < f(v)$.

From Proposition 3, we derive conditions satisfied by a maximal h-independent broadcast that is also bn-independent.

Corollary 5. *Let* f *be a maximal h-independent broadcast on a connected graph* G. *If* f *is bn-independent, then either* $|V_f^+| = 1$ *or, for each* $v \in V_f^+$, *there exists* $u \in B_f(v)$ *adjacent to a vertex in* V_f^1. *Moreover,* f *is maximal bn-independent.*

Proof. Suppose f is a maximal h-independent broadcast on G such that f is bn-independent; that is, no edge of G hears more than one broadcasting vertex. The statement is obvious if $|V_f^+| = 1$, hence assume that $|V_f^+| \ge 2$.

By Proposition 3(ii), for every $v \in V_f^+$ there exists $w \in V_f^+$ such that $w \to v$, that is, v has a vertex u on its f-boundary that is adjacent to $w \in V_f^+$. If $f(w) \ge 2$, then w overdominates u, hence $N_f[v]$ and $N_f[w]$ intersect on an edge, which is a contradiction. Therefore $f(w) = 1$. Moreover, statement (ii) of Proposition 2 holds. Since f is dominating, the result follows. ∎

The following results of Marchessault and Mynhardt [11] will be used throughout this section. For a path P in a tree T, let $d(v, P)$ denote the minimum distance from a vertex $v \in V(T)$ to a vertex on P. The proposition is illustrated in Figure 4.

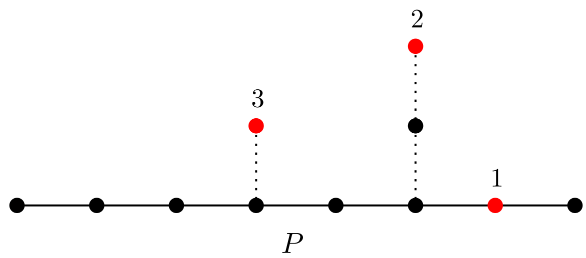

Figure 4. Vertices broadcasting to a path P of a tree. Observe that each broadcasting vertex v covers at most $2(f(v) - d(v, P))$ edges of P.

Proposition 6 [11]**.** *Let* P *be a path in a tree* T *and let* f *be a broadcast on* T. *Let* Touch(P) *denote the set of broadcasting vertices whose* f*-neighbourhoods intersect* P, *and let* Off(P) *denote the remaining broadcasting vertices, that is, those that do not broadcast to any vertex of* P. *Suppose*

$$\sigma_1 = \sum_{v \in \mathrm{Touch}(P)} d(v, P) \text{ and } \sigma_2 = \sum_{v \in \mathrm{Off}(P)} f(v).$$

Then

1. *f covers at most* $2\left(\sum_{v\in\mathrm{Touch}(P)} f(v)-\sigma_1\right)$ *edges of P, and*
2. *if f covers b edges of P, then* $\sigma(f)\ge\left\lceil\frac{b}{2}\right\rceil+\sigma_1+\sigma_2$.

In particular, if D is a diametrical path of a tree T and f covers every edge of D, then $\sigma(f)\ge\mathrm{rad}(T)$.

Recall that U_f^E denotes the set of edges uncovered by a broadcast f.

Proposition 7 [11]**.** *Let f be a bn-independent broadcast on a connected graph G such that $|V_f^+|\ge 2$. Then f is maximal bn-independent if and only if each component of $G-U_f^E$ contains at least two broadcasting vertices.*

Note that if each component of $G-U_f^E$ contains at least one broadcasting vertex, then f is dominating, since $G-U_f^E$ is a spanning subgraph of G. We next show that the first direction of Proposition 7 also holds for hearing independence.

Proposition 8. *Let f be a maximal h-independent broadcast on a connected graph G such that $|V_f^+|\ge 2$. Then each component of $G-U_f^E$ contains at least two broadcasting vertices.*

Proof. If some component C of $G-U_f^E$ contains only a single broadcasting vertex v, then all edges between $G-C$ and $B_f(v)$ are uncovered. But then increasing the broadcast strength of v by 1 results in a new h-independent broadcast of greater cost, a contradiction. ■

On the other hand, if f is an h-independent broadcast on a connected graph G such that $|V_f^+|\ge 2$ and all components of $G-U_f^E$ contain at least two broadcasting vertices, then f is not necessarily maximal h-independent as broadcasts may overlap on edges within components. Such a broadcast is illustrated in Figure 5.

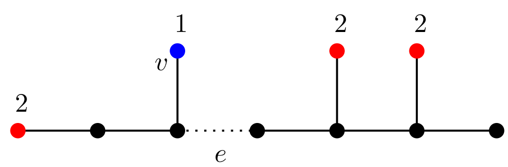

Figure 5. An h-independent broadcast f such that the removal of the f-uncovered edge e leaves two components, each of which contains two broadcasting vertices. As increasing the strength of the broadcast from v by 1 does not result in any broadcasting vertex hearing another, f is not maximal h-independent.

4. The Comparability of i_{bn} and i_h

Two graph parameters p and q are *incomparable* if there exist graphs G, G' for which $p(G) < q(G)$ and $p(G') > q(G')$. We write this as $p \diamond q$. In [13], Mynhardt and Neilson observed that $i \diamond i_{bn}$ and $i \diamond i_h$.

It is natural to ask whether $i_h(G)$ and $i_{bn}(G)$ are comparable or not. Suppose there exists a graph G with two or more vertices of high degree (i.e., many vertices could hear a broadcast of relatively small strength from these vertices) such that broadcasts from each of these vertices will dominate G only if some broadcasts may overlap on edges. Assuming a radial broadcast on G is not an i_{bn}-broadcast, it seems reasonable to imagine a case in which a maximal h-independent broadcast has lower cost than a maximal bn-independent broadcast. We proceed to show that this is impossible, solving an open problem posed in [11].

Theorem 9. *For any graph G, $i_{bn}(G) \leq i_h(G)$.*

Since the cost of a broadcast is equal to the sum of the costs of the broadcasts on each of its components, it suffices to consider connected graphs. We begin by proving special cases of broadcasts or graphs, including when G is a tree. The proof of Theorem 9 is presented in Section 4.3.

We first consider the case in which no vertices broadcast at strength greater than 1.

Proposition 10. *Let f be a broadcast on G such that $|V_f^{++}| = 0$. Then f is maximal bn-independent if and only if it is maximal h-independent.*

Proof. Suppose f is a maximal bn-independent or maximal h-independent (and hence dominating) broadcast on G such that $V_f^+ = V_f^1$. If $|V_f^+| = 1$, then f is both maximal bn-independent and maximal h-independent by part (i) of Propositions 2 and 3.

Otherwise, suppose $|V_f^1| \geq 2$ and let $v \in V_f^1$. Then $B_f(v) \setminus PB_f(v) \neq \emptyset$ if and only if v has a vertex on its boundary adjacent to another vertex broadcasting at strength 1, in other words, Proposition 2(ii) is equivalent to Proposition 3(ii). Therefore f is both maximal bn-independent and maximal h-independent. ■

4.1. Trees

Let $\ell(P)$ denote the length of the path P. The following result is a consequence of Proposition 6, and is stated here for clarity.

Corollary 11. *Let f be a broadcast on a tree T that covers all edges of T. Then $\sigma(f) \geq \mathrm{rad}(T)$.*

Proof. Let D be a diametrical path of T. By part 2 of Proposition 6 with $D = P$, $\sigma(f) \geq \left\lceil \frac{\ell(D)}{2} \right\rceil \geq \mathrm{rad}(T)$. ■

As in Proposition 6, given a path P in a tree, let $Touch(P)$ denote the set of broadcasting vertices whose f-neighbourhoods intersect P. Recall that for $v \in Touch(P)$, we use $d(v, P)$ to denote the minimum distance from v to a vertex on P.

Proposition 12. *Let P be a path of a tree T and let f be a broadcast on T. If f covers b edges of P, and k edges are covered more than once, then $\sigma(f) \geq \lceil \frac{b+k}{2} \rceil$.*

Proof. Consider $v \in Touch(P)$ and let u be the vertex on P for which the distance to v is smallest (possibly, $u = v$). Since T is a tree, there exists a unique $u - v$ path P_{uv} which, by choice of u, intersects P only on u. Thus, v covers at most $2(f(v) - d(u, v))$ edges of P. It follows that

$$b + k \leq \sum_{v \in Touch(P)} 2(f(v) - d(v, P)) \leq \sum_{v \in Touch(P)} 2f(v) \leq \sum_{v \in V_f^+} 2f(v),$$

hence $\sigma(f) \geq \frac{b+k}{2}$. As $\sigma(f)$ is an integer, we have that $\sigma(f) \geq \lceil \frac{b+k}{2} \rceil$. ■

We show next that Theorem 9 holds for trees.

Theorem 13. *For any tree T, $i_{bn}(T) \leq i_h(T)$.*

Proof. Suppose T is a tree such that $i_h(T) < i_{\text{bn}}(T)$ and let f be an i_h-broadcast on T. By Corollary 11, if f covers every edge of T, then $\sigma(f) \geq \text{rad}(T)$. Since $i_{bn}(T) \leq \text{rad}(T)$, the cost of f is strictly less than $\text{rad}(T)$, hence some edge of T is uncovered. In particular, $T - U_f^E$ contains at least two components.

Let $T_1, T_2, \ldots, T_k$ be the components of $T - U_f^E$ and let f_i denote the restriction of f to T_i. By Proposition 8, since f is maximal h-independent, each component T_i contains at least two broadcasting vertices. Hence, if f_i is bn-independent, then Proposition 7 implies that it is maximal bn-independent. Since $i_h(T) < i_{\text{bn}}(T)$, at least one restricted broadcast f_i is not bn-independent.

Assume without loss of generality that f_1 is not bn-independent on T_1. Then, since no edge of T_1 is uncovered, at least one edge hears more than one broadcasting vertex. If this edge lies along a diametrical path of T_1, then $\sigma(f_1) \geq \left\lceil \frac{\text{diam}(T_1)+1}{2} \right\rceil$ by Proposition 12. If no edge along the diametrical path is covered by multiple broadcasts, then some vertex off the diametrical path is broadcasting. By part 2 of Proposition 6, we again have that $\sigma(f_1) \geq \left\lceil \frac{\text{diam}(T_1)}{2} \right\rceil + 1 \geq \left\lceil \frac{\text{diam}(T_1)+1}{2} \right\rceil$.

Since $T - U_f^E$ has at least two components, for some $i \neq 1$ there exist $y \in V(T_i)$ and $x \in V(T_1)$ such that $xy \in E(T)$. If $\left\lceil \frac{\text{diam}(T_1)+1}{2} \right\rceil > \text{rad}(T_1)$, then $d(c, x) \leq \text{rad}(T_1) < \sigma(f_1)$ for any central vertex c of T_1. In the case where

$\left\lceil \frac{\operatorname{diam}(T_1)+1}{2} \right\rceil = \operatorname{rad}(T_1)$, such as illustrated in Figure 6, $\operatorname{diam}(T_1)$ is odd, and so we may choose a central vertex c such that $d(c,x) < \operatorname{rad}(T_1)$. Let g_1 be the broadcast on T defined by

$$g_1(v) = \begin{cases} f(v) & \text{if } v \in V(T) \setminus V(T_1), \\ 0 & \text{if } v \in V(T_1) \setminus \{c\}, \\ \sigma(f_1) & \text{if } v = c. \end{cases}$$

Observe that $V(T_1) \subseteq N_{g_1}[c]$ and, by choice of c, the vertex y hears g_1 from c.

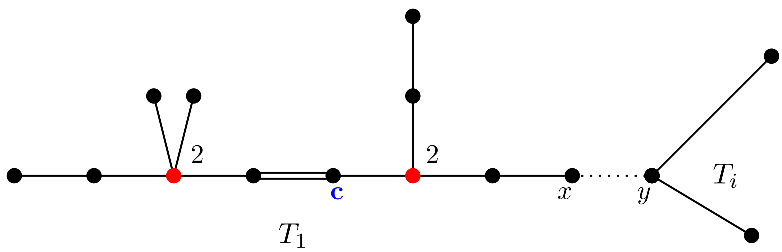

Figure 6. A component T_1 with $\sigma(f_1) = \operatorname{rad}(T_1) = 4$. A broadcast of strength 4 from c will cover the edge xy joining T_1 to T_i.

Let G_2 be the component of $T - U^E_{g_1}$ that contains T_1. As y hears g_1 from a vertex in T_1, G_2 must also contain the component of $T - U^E_f$ containing y. It follows that $|U^E_{g_1}| < |U^E_f|$ and that $T - U^E_{g_1}$ has fewer components than $T - U^E_f$.

Let h_2 be the restriction of g_1 to G_2. Note that h_2 covers all edges of G_2, and all other components of $T - U^E_{g_1}$ are trees T_i. In particular, each component of $T - U^E_{g_1}$ contains at least two vertices of $V^+_{g_1}$. Since $\sigma(g_1) = \sigma(f) < i_{\mathrm{bn}}(T)$, g_1 is not bn-independent, and so there exists at least one component of $T - U^E_{g_1}$ containing an edge that hears two broadcasting vertices. Assume without loss of generality that G_2 contains such an edge. Repeating the process, we again have that $\sigma(h_2) \geq \left\lceil \frac{\operatorname{diam}(G_2)+1}{2} \right\rceil$.

Because T is finite, we may apply this recursive procedure during which the number of uncovered edges strictly decreases until we eventually obtain a broadcast g_ℓ on T such that $U^E_{g_\ell} = \emptyset$. By Corollary 11, $\sigma(g_\ell) = \sigma(f) = \operatorname{rad}(T)$, a contradiction. ■

4.2. Spanning trees

Our aim in this subsection is to show that the lower bn-independent and lower h-independent domination numbers of a graph G are given by the minimum of these parameters, respectively, among all spanning trees of G. We begin by proving a result that is used frequently in the rest of this section.

Proposition 14. *Let f be a dominating broadcast on a connected graph G. If $G - U_f^E$ is connected, then $\sigma(f) \geq \mathrm{rad}(G)$. If, in addition, an edge of G hears a broadcast from more than one vertex, then for any vertex p of G, there exists a dominating broadcast g_p on G such that $\sigma(g_p) = \sigma(f)$, $|V_{g_p}^+| = 1$ and g_p overdominates p.*

Proof. Suppose f is a dominating broadcast of G such that $H_0 = G - U_f^E$ is connected. If $|V_f^+| = 1$, then $\sigma(f) \geq \mathrm{rad}(G)$, so assume that $|V_f^+| \geq 2$. Let p be an arbitrarily chosen vertex of G. Our goal is to define a sequence of equal-cost dominating broadcasts $f_0 = f, f_1, f_2, \ldots, f_k = g_p$ on G such that $|V_{f_1}^+| > |V_{f_2}^+| > \cdots > |V_{f_k}^+| = 1$. By careful construction of broadcasts, we will ensure that if an edge of G hears f from more than one broadcasting vertex, then some broadcast f_i in the sequence overdominates p. Furthermore, we will show that if p is overdominated in f_i, then p is overdominated in f_j for all $i \leq j \leq k$, such that the resulting radial broadcast f_k overdominates p.

Since f is dominating, there exists a vertex $w_0 \in V_f^+$ such that $p \in N_f[w_0]$. Since H_0 is connected and $|V_f^+| \geq 2$, there exists a vertex $u_0 \in V_f^+$ such that $N_f[u_0] \cap N_f[w_0] \neq \emptyset$. Then $d(u_0, w_0) \leq f(u_0) + f(w_0)$. Let $P_{u_0w_0}$ be a $u_0 - w_0$ geodesic in H_0. If an edge hears f from both u_0 and w_0, then $P_{u_0w_0}$ contains such an edge, in which case $d(u_0, w_0) < f(u_0) + f(w_0)$.

Let w_1 be the vertex on $P_{u_0w_0}$ at distance $f(w_0)$ from u_0 (and hence distance at most $f(u_0)$ from w_0). Observe that if $d(u_0, w_0) < f(u_0) + f(w_0)$, then w_1 is at distance at most $f(u_0) - 1$ from w_0. Define the broadcast f_1 by

$$f_1(v) = \begin{cases} f(v) & \text{if } v \in V(G) \setminus \{u_0, w_0, w_1\}, \\ 0 & \text{if } v \in \{u_0, w_0\}, \\ f(u_0) + f(w_0) & \text{if } v = w_1. \end{cases}$$

Clearly, $\sigma(f_1) = \sigma(f)$ and $|V_{f_1}^+| = |V_f^+| - 1$. Since f is dominating, to prove that f_1 is dominating, it suffices to show that each vertex in $N_f[u_0] \cup N_f[w_0]$ hears f_1. For any $v \in N_f[u_0]$,

$$d(v, w_1) \leq d(v, u_0) + d(u_0, w_1) \leq f(u_0) + f(w_0) = f_1(w_1),$$

hence v hears f_1 from w_1. Similarly, any vertex in $N_f[w_0]$ hears w_1. In particular, p hears w_1, and if an edge hears f from both u_0 and w_0, then w_1 overdominates p (because $d(u_0, w_0) < f(u_0) + f(w_0)$ and w_1 is at distance at most $f(u_0) - 1$ from w_0).

Let $H_1 = G - U_{f_1}^E$. Note that $U_{f_1}^E \subseteq U_f^E$ and that H_0 is a spanning subgraph of H_1. Thus H_1 is connected. Moreover, by the definition of f_1, regardless of whether an edge hears f from both u_0 and w_0,

- if w_0 overdominates p, then w_1 overdominates p, and

- if an edge e of H_0 hears f from two (or more) vertices, at least one of which belongs to $V_f^+ \setminus \{u_0, w_0\}$, then e hears f_1 from two vertices.

For $i \geq 1$, if $|V_{f_i}^+| \geq 2$, we repeat the above procedure. Since H_i is connected, there exists a vertex $u_i \in V_{f_1}^+$ such that $N_f[u_i] \cap N_f[w_i] \neq \emptyset$. As in the case of f_1, we define a dominating broadcast f_i by

$$f_i(v) = \begin{cases} f_{i-1}(v) & \text{if } v \in V(G) \setminus \{u_{i-1}, w_{i-1}, w_i\}, \\ 0 & \text{if } v \in \{u_{i-1}, w_{i-1}\}, \\ f_{i-1}(u_{i-1}) + f_{i-1}(w_{i-1}) & \text{if } v = w_i. \end{cases}$$

Let $H_i = G - U_{f_i}^E$. Then H_i is connected, $\sigma(f_i) = \sigma(f_{i-1})$, $|V_{f_i}^+| = |V_{f_{i-1}}^+| - 1$, the vertex p hears w_i, and if an edge hears f_{i-1} from both u_{i-1} and w_{i-1}, then, as in the case of w_1, w_i overdominates p. Moreover,

- if w_{i-1} overdominates p, then w_i overdominates p, and
- if an edge e of H_{i-1} hears f_{i-1} from two (or more) vertices, at least one of which belongs to $V_{f_{i-1}}^+ \setminus \{u_{i-1}, w_{i-1}\}$, then e hears f_i from two vertices.

Since G is finite, there exists an integer k such that $V_k^+ = \{w_k\}$. Since f_k is dominating, $\sigma(f_k) = f_k(w_k) \geq \mathrm{rad}(G)$. Since $\sigma(f) = \sigma(f_k)$, $\sigma(f) \geq \mathrm{rad}(G)$, which proves the first part of the proposition. Furthermore, if an edge of G hears f from more than one broadcasting vertex, then the above procedure ensures that a $u_i - w_i$ geodesic $P_{u_i w_i}$, for some $i = 0, 1, \ldots, k-1$, contains such an edge, and that w_k overdominates p. This proves the second part of the proposition. ■

Let G be an arbitrary connected graph and $v \in V(G)$. A spanning tree T of G is *distance preserving from* v if $d_T(u, v) = d_G(u, v)$ for each $u \in V(G)$. As shown in [3, page 75], any connected graph G has a spanning subtree that is distance preserving from a chosen vertex v. We use this fact in the proof of Theorem 15, in which we show that the lower boundary independence number of an arbitrary connected graph G equals the minimum lower boundary independence number among those of its spanning trees.

Theorem 15. *For any connected graph* G,

$$i_{bn}(G) = \min\{i_{bn}(T) : T \textit{ is a spanning tree of } G\}.$$

Proof. Suppose there exists a spanning tree T of G such that $i_{bn}(T) < i_{bn}(G)$, and let f' be an i_{bn}-broadcast on T. Then $\sigma(f') < i_{bn}(G) \leq \mathrm{rad}(G)$. Let f be the corresponding broadcast on G. Since T is a spanning tree of G, f is a dominating broadcast on G. If $|V_{f'}^+| = 1$, then $\sigma(f') \geq \mathrm{rad}(T) \geq \mathrm{rad}(G) \geq i_{bn}(G)$, a contradiction. Hence we may assume that $|V_{f'}^+| = |V_f^+| \geq 2$.

Suppose f is bn-independent on G and consider any $v \in V_f^+ = V_{f'}^+$. Since f' is maximal bn-independent on T, Proposition 2(ii) implies that there exist a vertex $w \in V_{f'}^+$ and a vertex $x \in B_{f'}(v) \cap B_{f'}(w)$. Observe that $d_T(v,w) = f'(v) + f'(w) = f(v) + f(w)$. If $d_G(v,w) < d_T(v,w)$, then an edge incident with x hears f from both v and w, contradicting f being bn-independent. Therefore $x \in B_f(v) \cap B_f(w)$. Again by Proposition 2, f is maximal bn-independent. But then $i_{bn}(G) \leq \sigma(f) = \sigma(f') < i_{bn}(G)$, a contradiction. We deduce that f is not bn-independent on G.

Thus some edge of G is covered by two or more vertices in V_f^+. Moreover, since $\sigma(f) < i_{bn}(G) \leq \mathrm{rad}(G)$, Proposition 14 implies that $G - U_f^E$ is disconnected. Let H_1 be a component of $G - U_f^E$ containing an edge covered by two or more vertices in V_f^+ and let h_1 be the restriction of f to H_1. Since $G - U_f^E$ is disconnected, there exists a vertex x_1 of H_1 that is adjacent to a vertex y_1 of $G - H_1$. By Proposition 14, there exists a dominating broadcast g_1 on H_1 such that $\sigma(g_1) = \sigma(h_1)$, $V_{g_1}^+ = \{b_1\}$ for some $b_1 \in V(H_1)$, and b_1 covers the edge x_1y_1.

Define a new broadcast f_1 on G by

$$f_1(x) = \begin{cases} f(x) & \text{if } x \in V(G) \setminus V(H_1), \\ 0 & \text{if } x \in V(H_1) \setminus \{b_1\}, \\ \sigma(g_1) & \text{if } x = b_1. \end{cases}$$

Then f_1 is dominating, $\sigma(f_1) = \sigma(f)$, and $G - U_{f_1}^E$ has fewer components than $G - U_f^E$. Note that f_1 is not bn-independent on G, otherwise we obtain a contradiction as in the case of f. By Proposition 14, we may repeat the process until we obtain a dominating broadcast f_k such that $\sigma(f_k) = \sigma(f)$ and $G - U_{f_k}^E$ is connected. But then $\sigma(f_k) = \sigma(f') \geq \mathrm{rad}(G)$ (Proposition 14), a contradiction.

It remains to show that there exists a tree T spanning G such that $i_{bn}(T) = i_{bn}(G)$. Let f be an i_{bn}-broadcast on G and suppose $V_f^+ = \{v_1, \ldots, v_k\}$. For $i = 1, \ldots, k$, consider the subgraph G_i of G induced by $N_f[v_i]$. Let T_i be a spanning subtree of G_i that is distance preserving from v_i. Then the restriction of f to T_i, denoted by f_i, covers all edges of T_i. Let H be the subgraph of G induced by $\bigcup_{i=1}^k E(T_i)$.

Suppose H contains a cycle C. By construction, the edges of C are covered by a set of broadcasting vertices $V_C \subseteq V_f^+$ such that $|V_C| \geq 2$. Observe that each $v_i \in V_C$ covers an even number of edges on C. In particular, there exist $v_i, v_j \in V_C$ such that $B_f(v_i) \cap B_f(v_j)$ contains a vertex $b \in V(C)$. Let a be the vertex on C adjacent to b in T_i and let $H_1 = H - ab$. (See Figure 7.) Since $B_f(v_i) \cap C$ contains at least two vertices, there exists $b' \in V(B_f(v_i) \cap C) \setminus \{b\}$ lying on the boundary of another broadcasting vertex in H_1. The same holds for v_j. Thus, f is maximal bn-independent on H_1.

If H_1 contains a cycle, repeat the process, successively removing edges from cycles until the resulting graph H_r is acyclic. If H_r is connected, let $T = H_r$. Otherwise, since G is connected, we may add edges of $G - H_r$ to H_r joining components of H_r without creating cycles until we obtain a tree T spanning G.

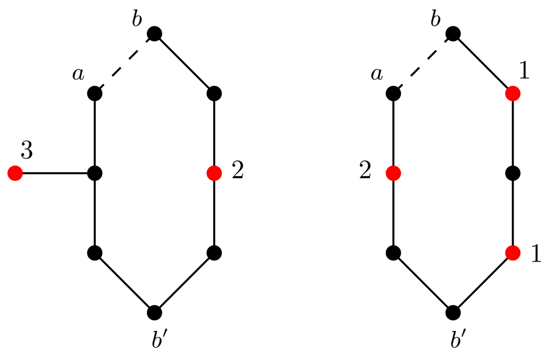

Figure 7. Two cycles whose edges are covered by multiple broadcasting vertices. We may always remove an edge ab from such a cycle without violating the maximal boundary independence condition.

Since f is maximal bn-independent on H_r, the construction ensures that f is a maximal bn-independent broadcast on T, as $B_f(v) \setminus PB_f(v) \neq \emptyset$ in H_r for each $v \in V_f^+$. Hence $i_{\mathrm{bn}}(T) \leq \sigma(f) = i_{\mathrm{bn}}(G)$. But we have already shown that $i_{\mathrm{bn}}(G) \leq i_{\mathrm{bn}}(T')$ for any spanning tree T' of G. Consequently, $i_{\mathrm{bn}}(G) = i_{\mathrm{bn}}(T) = \min\{i_{\mathrm{bn}}(T') : T'$ is a spanning tree of $G\}$. ■

To prove a result similar to Theorem 15 for h-independent broadcasts, we now define two graphs and a digraph associated with a graph G and a broadcast f on G.

- The *neighbourhood graph* $\mathcal{N}_f(G)$ has V_f^+ as its vertex set, and two vertices $u, v \in V_f^+$ are adjacent in $\mathcal{N}_f(G)$ if and only if $N_f[u] \cap N_f[v] \neq \emptyset$.
- The *certification digraph* $\overrightarrow{\mathcal{C}}_f(G)$ has as its vertex set the set V_f^+, and (v, u) is an arc of $\overrightarrow{\mathcal{C}}_f(G)$ if and only if $v \to u$. Note that if (v, u) is an arc of $\overrightarrow{\mathcal{C}}_f(G)$, then (u, v) may or may not be an arc as well. We say that (v, u) is a *double arc* if (u, v) is also an arc, otherwise we say that (v, u) is a *single arc*. Note that if (v, u) is an arc of $\overrightarrow{\mathcal{C}}_f(G)$, then (v, u) is a double arc if and only if $f(v) = f(u)$.

As is standard terminology, the underlying graph of $\overrightarrow{\mathcal{C}}_f(G)$ (or of a subgraph $\overrightarrow{\mathcal{C}'}_f(G)$ of $\overrightarrow{\mathcal{C}}_f(G)$) is the graph obtained by replacing arcs of $\overrightarrow{\mathcal{C}}_f(G)$ (or $\overrightarrow{\mathcal{C}'}_f(G)$) by edges and identifying double edges; the underlying graph of $\overrightarrow{\mathcal{C}}_f(G)$ shown in Figure 8 is a triangle. Note that the underlying graph of $\overrightarrow{\mathcal{C}}_f(G)$ need not contain all edges of $\mathcal{N}_f(G)$. For example, let G be the graph depicted in Figure 3, and let g be the broadcast on G defined by $g(v) = 2 = g(x)$ and $g(y) = 0$ otherwise. Then $\mathcal{N}_g(G) = K_2$, but neither broadcasting vertex certifies the other, hence the underlying graph of $\overrightarrow{\mathcal{C}}_g(G)$ is $\overline{K_2}$.

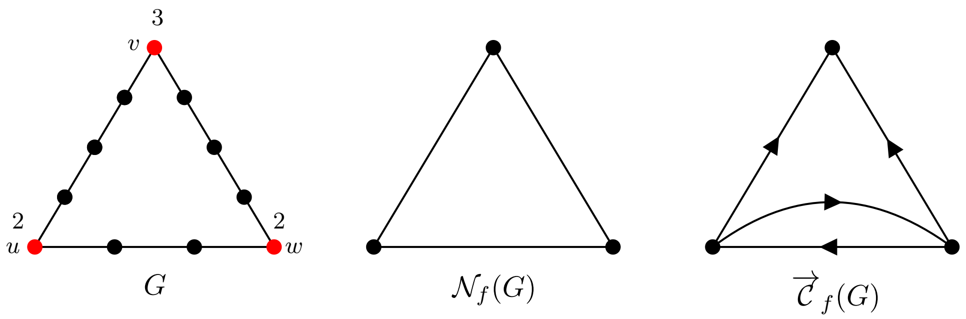

Figure 8. A maximal h-independent broadcast f on a graph G along with the corresponding neighbourhood graph and certification digraph. Observe that v is certified by u and w, and u and w certify each other.

Proposition 16. *If f is a maximal h-independent broadcast on G such that $|V_f^+| \geq 2$, then each vertex $u \in V_f^+$ is adjacent, in $\mathcal{N}_f(G)$, to a vertex $v \in V_f^+$ such that $f(v) \leq f(u)$.*

Proof. Let $u \in V_f^+$. By Proposition 3, since f is maximal h-independent, there exists $v \in V_f^+$ such that $v \to u$. By Observation 4, $f(v) \leq f(u)$. Moreover, u and v are adjacent in $\mathcal{N}_f(G)$. ∎

Proposition 17. *Let f be an h-independent broadcast on G such that $|V_f^+| \geq 2$. Then f is maximal h-independent if and only if f is dominating and each vertex of $\overrightarrow{\mathcal{C}}_f(G)$ has positive in-degree.*

Proof. If v has in-degree 0 for some $v \in \overrightarrow{\mathcal{C}}_f(G)$, by definition of $\overrightarrow{\mathcal{C}}_f(G)$, no vertex on the f-boundary of v is adjacent to a vertex in $V_f^+ \setminus \{v\}$. By part (ii) of Proposition 3, f is not maximal.

Conversely, if f is dominating and every vertex of $\overrightarrow{\mathcal{C}}_f(G)$ has positive in-degree, then f is maximal by part (ii) of Proposition 3. ∎

Proposition 18. *Let f be a maximal h-independent broadcast on G such that $|V_f^+| \geq 2$. Suppose C is a cycle in the underlying graph of $\overrightarrow{\mathcal{C}}_f(G)$. Then the subgraph of $\overrightarrow{\mathcal{C}}_f(G)$ with arcs corresponding to $E(C)$ contains a directed cycle of length at least 3 if and only if every edge of C corresponds to a double arc.*

Proof. Suppose the subgraph of $\overrightarrow{\mathcal{C}}_f(G)$ corresponding to C contains a directed cycle. Label the vertices of C as $v_1, v_2, \ldots, v_k$, where $k \geq 3$, such that $v_i \to v_{i+1}$ for all $1 \leq i \leq k-1$ and $v_k \to v_1$. By Observation 4, $f(v_i) \leq f(v_{i+1})$ for all $1 \leq i \leq k-1$ and $f(v_k) \leq f(v_1)$.

Without loss of generality, suppose to the contrary that $v_1 \to v_2$ but $v_2 \not\to v_1$. (See Figure 9.) Since (v_1, v_2) is a single arc if and only if $f(v_1) < f(v_2)$, we have that

$$f(v_2) > f(v_1) \geq f(v_k) \geq f(v_{k-1}) \geq \cdots \geq f(v_2),$$

which is impossible. The converse is obvious. ∎

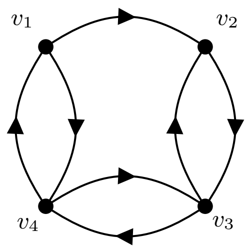

Figure 9. An example of the situation considered in Proposition 18. Since (v_1, v_2) is a single arc, we have that $f(v_1) < f(v_2)$, contradicting $f(v_2) = f(v_3) = f(v_4) = f(v_1)$ along the double arcs.

Corollary 19. *Let f be a maximal h-independent broadcast on G such that $|V_f^+| \geq 2$, and suppose C is a cycle in the underlying graph of $\overrightarrow{\mathcal{C}}_f(G)$. If C contains an edge e that does not corresponds to a double arc in $\overrightarrow{\mathcal{C}}_f(G)$, then C contains an edge e' corresponding to a single arc such that e and e' are oriented in opposite directions in the subgraph of $\overrightarrow{\mathcal{C}}_f(G)$ with arcs corresponding to the edges of C.*

Proof. Suppose not. Then the vertices of C may be labelled $v_1, v_2, \ldots, v_k$ such that e correspond to the single arc (v_k, v_1), and for all $1 \leq i \leq k-1$, v_i certifies v_{i+1}. But then the subgraph of $\overrightarrow{\mathcal{C}}_f(G)$ contains a directed cycle $v_1 \to v_2 \to \cdots \to v_k \to v_1$, contradicting Proposition 18. ∎

As mentioned before, any graph has a spanning subtree that is distance preserving from a chosen vertex v. In particular, by choosing v to be a central vertex of the graph G, we see that G has a spanning subtree T such that $\mathrm{rad}(T) = \mathrm{rad}(G)$. We use this fact in the proof of Theorem 20.

Theorem 20. *For any connected graph G,*

$$i_h(G) = \min\{i_h(T) : T \textit{ is a spanning tree of } G\}.$$

Proof. Suppose there exists a tree T spanning G such that $i_h(T) < i_h(G)$, and let f' be an i_h-broadcast on T. Then $\sigma(f') < i_h(G) \leq \mathrm{rad}(G)$. Let f be the corresponding broadcast on G. Since T is a spanning tree of G, f is a dominating broadcast on G. Suppose f is h-independent on G. Then f is maximal h-independent on G because f' is maximal h-independent on T – the same vertex that certifies $v \in V_{f'}^+$ in T also certifies $v \in V_f^+$ in G. But then $i_h(G) \leq \sigma(f) =$

$\sigma(f') = i_h(T) < i_h(G)$, a contradiction. We deduce that f is not h-independent on G. Since f is dominating but not h-independent on G, there exist vertices $u, v \in V_f^+$ such that $d_G(u,v) \leq f(v)$. Thus some edge of G is covered by two or more vertices in V_f^+. Moreover, since $\sigma(f) < i_h(G) \leq \mathrm{rad}(G)$, Proposition 14 implies that $G - U_f^E$ is disconnected. Proceed as in the proof of Theorem 15 to obtain a dominating broadcast f_k on G such that $\sigma(f_k) = \sigma(f') \geq \mathrm{rad}(G)$, a contradiction.

To show there exists a spanning tree T of G such that $i_h(T) = i_h(G)$, let f be an i_h-broadcast on G and let $G_0 = G - U_f^E$. If $|V_f^+| = 1$, then the minimality of f implies that $V_f^+ = \{c\}$, where c is a central vertex of G, and $f(c) = \mathrm{rad}(G)$. We may then choose T to be a spanning subtree of G that is distance preserving from c. Therefore we assume that $|V_f^+| \geq 2$. By Proposition 3(ii), for each $v \in V_f^+$ there exists $u \in V_f^+$ such that $u \to v$. Let $P_{u\to v}$ denote an arbitrarily chosen $u - v$ geodesic in G_0, which exists because every edge on a shortest $u - v$-path is covered by f.

We aim construct T such that it contains at least one geodesic $P_{u\to v}$ for every $v \in V_f^+$, so that the restriction of f to T, denoted f_T, is a dominating broadcast on T. Then Proposition 3(ii) will imply that f_T is a maximal h-independent broadcast on T such that $\sigma(f) = \sigma(f_T)$.

Consider the certification digraph $\overrightarrow{\mathcal{C}}_f(G_0)$ of G_0, and let H_0 be its underlying graph. Suppose C_0 is a cycle in H_0. The following two cases are illustrated in Figure 10.

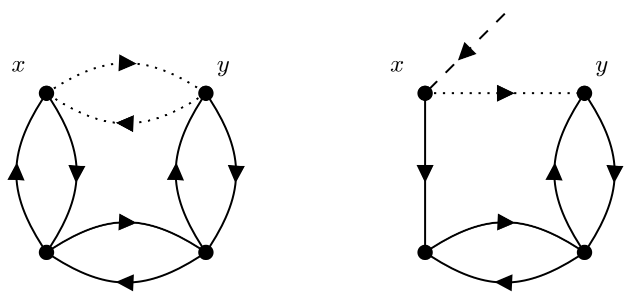

Figure 10. The two cases considered. In either case, both x and y have positive in-degree in $\overrightarrow{\mathcal{H}}_1$.

If C_0 corresponds to a directed cycle in $\overrightarrow{\mathcal{C}}_f(G_0)$, then each of its arcs corresponds to a double arc by Proposition 18. Let xy be an arbitrary edge of C_0 and let $\overrightarrow{\mathcal{H}}_1$ be the subgraph of $\overrightarrow{\mathcal{C}}_f(G_0)$ obtained by deleting the arcs (x,y) and (y,x).

If C_0 does not correspond to a directed cycle, then by Corollary 19, there exist two single arcs (x,y) and (x',y') oriented in opposite directions along the subgraph of $\overrightarrow{\mathcal{C}}_f(G_0)$ corresponding to C_0. We may select (x, y) and (x', y') such that either $y = y'$ or all edges between y and y' on C_0 correspond to double arcs, so that y and y' each has in-degree at least 2 in $\overrightarrow{\mathcal{C}}_f(G_0)$. Let $\overrightarrow{\mathcal{H}}_1$ be the subgraph obtained by deleting (x, y). By Proposition 17, every vertex of $\overrightarrow{\mathcal{C}}_f(G_0)$ has positive in-degree, hence x is certified by w for some $w \in V_f^+ \setminus \{x, y\}$ (which may or may not lie on C_0.)

Repeat the process: at each step i, if the underlying graph H_i of $\overrightarrow{\mathcal{H}}_i$ contains a cycle C_i, delete corresponding arcs as described for C_0. Eventually, we obtain a spanning subgraph $\overrightarrow{\mathcal{H}}_k$ of $\overrightarrow{\mathcal{C}}_f(G_0)$ such that its underlying graph H_k is acyclic.

Construct T as follows. For each arc (u, v) of $\overrightarrow{\mathcal{H}}_k$, let $P_{u\to v}$ be a $u-v$ geodesic in G, where only one such path is chosen if (u, v) is a double arc. Since H_k is acyclic, so is the spanning subgraph T_0 of G obtained by removing all edges of G not lying on one of the chosen paths.

By the construction of $\overrightarrow{\mathcal{H}}_k$, H_k and T_0, if a component of T_0 does not contain a vertex in V_f^+, then this component consists of an isolated vertex. Denote the restriction of f to T_0 by f_{T_0}. Suppose that T_0 contains at least two components dominated by f_{T_0}. If there exists an edge $a_0b_0 \in E(G)$ joining these two components, let $T_1 = T_0 \cup \{a_0b_0\}$ and let f_{T_1} denote the restriction of f to T_1. Repeat the process. At each step i, we construct a spanning forest T_i of G such that $T_i = T_{i-1} \cup \{a_{i-1}b_{i-1}\}$ for some edge $a_{i-1}b_{i-1}$ joining two components dominated by $f_{T_{i-1}}$.

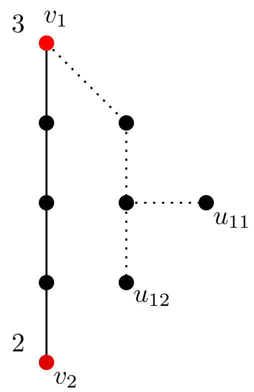

Figure 11. Isolated vertices are joined by geodesics to broadcasting vertices in T_k.

Eventually we obtain a spanning subgraph T_k of G consisting of a tree dominated by f_{T_k}, and a set of undominated isolated vertices S. If $S = \emptyset$, let $T = T_k$. Otherwise, let $v_1 \in V_f^+$ be a vertex of T_k that broadcasts to vertices $u_{11}, u_{12}, \ldots, u_{1s} \in S$. Join each u_{1i} to v_1 by a $u_{1i} - v_1$ geodesic P_{1i} to form the graph F_1 in which each u_{1i} hears v_1. See Figure 11. This can be done without creating cycles: suppose the subgraph induced by $P_{1i} \cup P_{1j}$ contains a cycle $C = (x_1, x_2, \ldots, x_r = y_s, y_{s-1}, \ldots, y_1 = x_1)$. Since P_{1i} and P_{1j} are geodesics, $r = s$, and the deletion of the edge $y_{s-1}y_s$, assuming that this is an edge of C farthest from v_1, does not result in some $u_{1\ell}$ becoming undominated by v_1. Let $S_1 = S \setminus V(F_1)$.

Repeat this procedure with S_1 and a vertex $v_2 \in V_f^+$ of T_k that broadcasts to vertices in S_1, etc., until no undominated vertices in S remain. Let T be the resulting spanning tree of G.

Let f_T be the restriction of f to T. Note that $|V_{f_T}^+| = |V_f^+|$ and $f_T(v) = f(v)$ for each $v \in V(G)$. Since f is h-independent and dominating, so is f_T. Thus, since each vertex of $\overrightarrow{\mathcal{H}}_k$ has positive in-degree, Proposition 17 implies that f_T is maximal h-independent. Therefore $i_h(T) \leq \sigma(f_T) = \sigma(f) = i_h(G)$. But we have already shown in the first part of the proof that $i_h(T) \geq i_h(G)$. Consequently, $i_h(G) = i_h(T) = \min\{i_h(T') : T' \text{ is a spanning tree of } G\}$. ∎

If H is a disconnected spanning subgraph of a connected graph G, then it is possible that $i_{bn}(H) < i_{bn}(G)$. The same is true for $i_h(H)$ and $i_h(G)$. For example, since $i_h(P_3) = i_{bn}(P_3) = 1$, $i_h(2P_3) = i_{bn}(2P_3) = 2$, but $i_h(P_6) = i_{bn}(P_6) = 3$. The situation is different for connected spanning subgraphs.

Corollary 21. *If H is a connected spanning subgraph of a graph G, then $i_{bn}(H) \geq i_{bn}(G)$ and $i_h(H) \geq i_h(G)$.*

Proof. If H is a connected spanning subgraph of G, then $\min\{i_h(T) : T \text{ is a spanning tree of } H\} \subseteq \min\{i_h(T) : T \text{ is a spanning tree of } G\}$, and the result follows from Theorem 20. Similarly, by Theorem 15, $i_{bn}(H) \geq i_{bn}(G)$. ∎

4.3. Proof of Theorem 9

The proof of Theorem 9, the main result of the paper, now follows by combining Theorems 13, 15 and 20.

Theorem 9. *For any graph G, $i_{bn}(G) \leq i_h(G)$.*

Proof. By Theorem 20, $i_h(G) = \min\{i_h(T) : T \text{ is a spanning tree of } G\}$. Let T_0 be a spanning tree of G such that $i_h(T_0) = i_h(G)$. By Theorem 13, $i_h(T_0) \geq i_{bn}(T_0)$. Hence

$$\begin{aligned} i_h(G) &= \min\{i_h(T) : T \text{ is a spanning tree of } G\} = i_h(T_0) \geq i_{bn}(T_0) \\ &\geq \min\{i_{bn}(T) : T \text{ is a spanning tree of } G\} = i_{bn}(G), \end{aligned}$$

where the last identity follows from Theorem 15. ∎

5. The Ratio $i_h(G)/i_{bn}(G)$

In [11], Marchessault and Mynhardt asked whether the difference $i_h(G) - i_{bn}(G)$ may be arbitrarily large. A construction of such an infinite family of graphs is presented in [10, Proposition 2.2.1]. Marchessault and Mynhardt also posed the problem of bounding the ratio $i_h(G)/i_{bn}(G)$ for general graphs, which we consider in this section.

Recall that $i \diamond i_{bn}$ and $i \diamond i_h$. The ratios $\frac{i(G)}{i_{bn}(G)}$ and $\frac{i(G)}{i_h(G)}$, in general, may be arbitrarily large: for example, $i(K_{n,n}) = n$ whereas $i_{bn}(K_{n,n}) = i_h(K_{n,n}) = 2$ for all $n \geq 2$. In [11], Marchessault and Mynhardt found that $i_{bn}(G) \leq \left\lceil \frac{4i(G)}{3} \right\rceil$ and asked whether the ratio $\frac{i_h(G)}{i_{bn}(G)}$ may be similarly bounded.

In the previous section, we found that i_h and i_{bn} are comparable. In particular, since $i_{bn}(G) \leq i_h(G)$ for all G, $\frac{i_{bn}(G)}{i_h(G)} \leq 1$. We now prove that $\frac{i_h(G)}{i_{bn}(G)} \leq \frac{5}{4}$ for all graphs G.

Proposition 22 [11]**.** *If T' is a subtree of a tree T, then $i_{bn}(T') \leq i_{bn}(T)$.*

Recall that P_n denotes the path on n vertices. Since a tree T with diameter d contains the path P_{d+1} as a subtree, we may bound $i_{bn}(T)$ below by the value $i_{bn}(P_{d+1})$, which was determined exactly by Neilson in [16].

Proposition 23 [16]**.** *For any $n \neq 3$, $i_{bn}(P_n) = \left\lceil \frac{2n}{5} \right\rceil$.*

The exception is P_3, which admits a maximal bn-independent broadcast of cost 1.

Theorem 24. *For any graph G, $1 \leq i_h(G)/i_{bn}(G) \leq 5/4$.*

Proof. Since $i_{bn}(G)$ is equal to the sum of the costs of i_{bn}-broadcasts on all components of G, it suffices to consider graph with one component, so assume G is connected. By Theorem 15, there exists a tree T spanning G such that $i_{bn}(T) = i_{bn}(G)$.

Let $d = \operatorname{diam}(T)$ and let $D \cong P_{d+1}$ be a diametrical path of T. It follows from Propositions 22 and 23 that $i_{bn}(T) \geq \left\lceil \frac{2(d+1)}{5} \right\rceil \geq \frac{2(d+1)}{5}$. Since T spans G, $\operatorname{rad}(G) \leq \operatorname{rad}(T)$. Finally, since $i_h(G) \leq \operatorname{rad}(G)$ for any connected graph G, we have that

$$\frac{i_h(G)}{i_{bn}(G)} = \frac{i_h(G)}{i_{bn}(T)} \leq \frac{5 \cdot \operatorname{rad}(G)}{2(d+1)} \leq \frac{5 \cdot \operatorname{rad}(G)}{4 \cdot \operatorname{rad}(T)} \leq \frac{5 \cdot \operatorname{rad}(T)}{4 \cdot \operatorname{rad}(T)} = \frac{5}{4}.$$

The lower bound follows from Theorem 9. ■

6. Open problems

It is unknown whether there exists a graph G for which $\frac{i_h(G)}{i_{bn}(G)} = \frac{5}{4}$. As there exist graphs with $i_{bn}(G) = 5$ and $i_h(G) = 6$ (see Figure 2), a sharp upper bound for $\frac{i_h(G)}{i_{bn}(G)}$ must lie between $\frac{6}{5}$ and $\frac{5}{4}$.

Problem 1. Improve the bound $\frac{i_h(G)}{i_{bn}(G)} \leq \frac{5}{4}$, or show it is best possible for an infinite family of graphs.

Recall that $\gamma_b(G)$ denotes the minimum cost of a dominating broadcast on G. Neilson showed in [16] that $\gamma_b(G) \leq i_{bn}(G)$ for all graphs G. Since $i_{bn}(G) \leq i_h(G) \leq \mathrm{rad}(G)$, we have that $i_{bn}(G) = i_h(G)$ for all G such that $\gamma_b(G) = \mathrm{rad}(G)$, known as *radial* graphs. Radial trees were characterized by Herke and Mynhardt in [8].

Let f be an i_{bn}-broadcast on a graph G. If $|V_f^+| = 1$, or if $V_f^+ = V_f^1$, then $i_h(G) = i_{bn}(G)$ and f is an i_h-broadcast on G. In particular, $i_{bn}(G) = i_h(G)$ for paths and cycles. Equality also holds for graphs G such that $\mathrm{rad}(G) \leq 5$, if f is an i_{bn}-broadcast with $\sigma(f) \leq 4$, then $G - U_f^E$ consists of either a single component (in which case $\sigma(f) = \mathrm{rad}(G)$), or two components, each with two broadcasts of strength 1. It would be of interest to further classify graphs for which these parameters are equal.

Question 1. *For which graphs G is $i_{bn}(G) = i_h(G)$?*

Trees with exactly one branch vertex are known as *generalized spiders*. The generalized spider $S = S(n_1, n_2, \ldots, n_k)$ consists of a branch vertex b of degree k, and $k \geq 3$ paths or 'legs' $L_1, \ldots, L_k$, each with one endpoint at b, such that $\ell(L_i) = n_i$ for all $1 \leq i \leq k$. It was shown in [10] that generalized spiders satisfy the equality in Question 6.2.

Theorem 25 [10]**.** *Let $S = S(n_1, n_2, \ldots, n_k)$ be a generalized spider. Then $i_{bn}(S) = i_h(S)$.*

However, a closed formula to determine the exact values of these parameters remains unknown.

Problem 2. Determine $i_{bn}(S)$ for all generalized spiders S.

The problem of determining $\gamma(G)$ for a given graph G is known to be NP-complete [6]. In [7], however, Heggernes and Lokshtanov showed that the minimum broadcast domination problem is solvable in polynomial time for all graphs.

Problem 3. Study the complexity of determining $i_{bn}(G)$ and $i_h(G)$ for trees or other graph classes.

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