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ON $\mathcal{P}$ VERTEX-CONNECTIONS OF GRAPHS

Július Czap[1], Stanislav Jendrol'[2]

and

Peter Šugerek[1]

[1] *Department of Applied Mathematics and Business Informatics*
Faculty of Economics, Technical University of Košice
Košice, Slovakia

[2] *Institute of Mathematics, P.J. Šafárik University*
Košice, Slovakia

e-mail: julius.czap@tuke.sk
stanislav.jendrol@upjs.sk
peter.sugerek@tuke.sk

Abstract

A vertex colored graph G is $\mathcal{P}$ vertex-connected if every two vertices of G are connected by a path having property $\mathcal{P}$. The problem is to find the minimum integer k for which there exists a vertex k-coloring of G that makes it $\mathcal{P}$ vertex-connected. In this note we introduce some modifications of this problem and determine upper bounds for the corresponding graph parameters. We deal with four properties, namely with property to be zig-zag, to be dynamic, to be nonrepetitive, and to be conflict-free.

Keywords: edge coloring, vertex coloring, $\mathcal{P}$ connection, $\mathcal{P}$ vertex-connection.

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1. Introduction

In the last years, connection concepts such as rainbow connection, monochromatic connection, proper connection, conflict-free connection appeared in graph theory and obtained a lot of attention.

A path in an edge colored graph G is a rainbow path if its edges have different colors. An edge colored graph G is rainbow connected, if every two vertices are connected by at least one rainbow path in G. For a connected graph G, the rainbow connection number of G is defined as the smallest number of colors required to make it rainbow connected. The concept of rainbow connection in graphs was introduced by Chartrand *et al.* [7] in 2008. Since then, a lot of results on this concept have been obtained, see e.g. [17].

The opposite of the rainbow connection was introduced by Caro and Yuster [6] in 2011. A path in an edge colored graph G is a monochromatic path if all the edges of the path are colored with the same color. An edge colored graph G is monochromatically connected, if any two vertices of G are connected by a monochromatic path. For a connected graph G, the monochromatic connection number of G is defined as the largest number of colors used in an edge coloring that makes G monochromatically connected. For results on monochromatic connection of graphs we refer to a recent survey paper [19].

In 2012, Borozan *et al.* [2] introduced the concept of proper connection as a natural extension of the rainbow connection. A path in an edge colored graph G is a proper path if any two adjacent edges receive distinct colors. An edge colored graph G is properly connected, if every two vertices are connected by at least one proper path in G. For a connected graph G, the proper connection number of G is defined as the smallest number of colors required to make it properly connected. Readers who are interested in this topic are referred to [16].

Motivated by the above mentioned concepts and by conflict-free colorings of graphs, in 2018, Czap *et al.* [9] introduced the concept of conflict-free connection. A path in an edge colored graph G is called conflict-free, if there is a color used on exactly one of its edges. An edge colored graph G is said to be conflict-free connected if any two vertices are connected by at least one conflict-free path. The conflict-free connection number of a connected graph G is defined as the smallest number of colors in order to make it conflict-free connected.

As a natural counterpart of the concepts of rainbow connection, monochromatic connection, proper connection, and conflict-free connection, the concepts of rainbow vertex-connection [15], monochromatic vertex-connection [5], proper vertex-connection [8], and conflict-free vertex-connection [18] were introduced, respectively.

The above mentioned connection concepts can be seen as a particular case of the more general one, so called $\mathcal{P}$ connection, which was introduced by Brause *et al.* [3] in 2018 as follows.

Let $\mathbb{A} = \{a, b, c, \ldots\}$ be an alphabet, i.e., a set of colors, digits, symbols, etc., whose elements are called letters. A word W of length t over $\mathbb{A}$ is a sequence of letters, say $\ell_1\ell_2\cdots\ell_t$ where $\ell_i \in \mathbb{A}$ for $i \in \{1, 2, \ldots, t\}$. A property $\mathcal{P}$ is a set of words. If a word W belongs to $\mathcal{P}$, then we say that W has property $\mathcal{P}$.

Let G be a simple, connected, finite, and undirected graph with vertex set $V(G)$ and edge set $E(G)$. Let $P = v_1v_2\cdots v_t$ be a path in G with $v_i \in V(G)$ for

$i \in \{1, 2, \ldots, t\}$ and $v_i v_{i+1} \in E(G)$ for $i \in \{1, 2, \ldots, t-1\}$. Let $\mathcal{P}$ be a property over the alphabet $\mathbb{A}$. Considering an edge coloring $\varphi : E(G) \to \mathbb{A}$ (or a vertex coloring $\psi : V(G) \to \mathbb{A}$), we say that P has property $\mathcal{P}$ if the associated word $\varphi(v_1v_2)\varphi(v_2v_3)\cdots\varphi(v_{t-1}v_t)$ (or $\psi(v_1)\psi(v_2)\cdots\psi(v_t)$) has property $\mathcal{P}$. Now, let G be a connected graph, $\mathbb{A}$ be an alphabet with k letters, $\mathcal{P}$ be a property, and $\varphi : E(G) \to \mathbb{A}$ be an edge coloring ($\psi : V(G) \to \mathbb{A}$ be a vertex coloring) of G. The edge coloring φ makes G $\mathcal{P}$ connected (the vertex coloring ψ makes G $\mathcal{P}$ vertex-connected) if every two vertices of G are connected by a path having property $\mathcal{P}$. The problem is to find the minimum/maximum integer $k = |\mathbb{A}|$ for which there exists an edge coloring $\varphi : E(G) \to \mathbb{A}$ (a vertex coloring $\psi : V(G) \to \mathbb{A}$) that makes G $\mathcal{P}$ connected ($\mathcal{P}$ vertex-connected).

For instance, if $\mathcal{P}$ is the set of all words which does not contain identical letters, then we obtain a rainbow connection; if $\mathcal{P}$ is the set of all words in which consecutive letters are not identical, then we obtain a proper connection. Note that for a fixed graph G it suffices to consider only words of length at most $|V(G)|$.

As it is noted in [3], $\mathcal{P}$ connections of graphs play an important role for security and accessibility in communication networks.

In this note we deal with four classes of properties and we focus on vertex colorings.

2. Four Classes of Properties

Let $\mathbb{A} = \{1, 2, 3, \ldots\}$. We say that a property $\mathcal{P}$ is:

- E-property if every connected graph has a vertex coloring that makes it $\mathcal{P}$ vertex-connected,
- A-property if every connected graph has a vertex coloring such that all paths have property $\mathcal{P}$,
- E*-property if every connected graph has a proper vertex coloring that makes it $\mathcal{P}$ vertex-connected,
- A*-property if every connected graph has a proper vertex coloring such that all paths have property $\mathcal{P}$.

Let $\mathcal{E}$, $\mathcal{A}$, $\mathcal{E}^*$, $\mathcal{A}^*$ denote the set of all E-properties, A-properties, E^*-properties, A^*-properties, respectively. In general, it is not easy to decide to which set(s) a given property $\mathcal{P}$ belongs.

First, we describe the relations between these four sets. The following two properties will be very useful. Let $W = \ell_1 \cdots \ell_t$ be a word. We say that W is:

- monochromatic if $\ell_1 = \cdots = \ell_t$,

- zig-zag if one of the following holds
  - (i) $t = 1$,
  - (ii) $t = 2$ and $\ell_1 \neq \ell_2$,
  - (iii) $\ell_i > \max\{\ell_{i-1}, \ell_{i+1}\}$ or $\ell_i < \min\{\ell_{i-1}, \ell_{i+1}\}$ for any subword $\ell_{i-1}\ell_i\ell_{i+1}$ of W.

The property monochromatic, zig-zag consists of all monochromatic, zig-zag words, respectively.

Claim 1. *The property* zig-zag *is E-property.*

Proof. Let G be a connected graph and let S be its spanning tree. Clearly, S has a proper vertex coloring with two colors, say $1, 2$. Any such coloring induces a coloring of G which makes it zig-zag vertex-connected. ■

Claim 2. *The property* zig-zag *is E^*-property.*

Proof. Let G be a connected graph and let S be its spanning tree. S has a proper vertex coloring with two colors, say $1, 2$. Let a and b be the number of vertices colored with color 1 and 2, respectively. If we recolor the vertices colored by 1 with different colors from the set $\{1, 2, \ldots, a\}$ and the vertices colored by 2 with different colors from the set $\{a+1, a+2, \ldots, a+b\}$, then we obtain a proper vertex coloring of G which makes it zig-zag vertex-connected. Any two vertices are connected by a zig-zag path in G, since they are connected by such a path in S. ■

Claim 3. *The property* zig-zag *is not A-property.*

Proof. For instance, odd cycles have no vertex coloring such that all paths are zig-zag. ■

Claim 4. *Let $\mathcal{E}$, $\mathcal{A}$, $\mathcal{E}^*$, $\mathcal{A}^*$ denote the set of all E-properties, A-properties, E^*-properties, A^*-properties, respectively. Then*

(a) $\mathcal{E}^* \subset \mathcal{E}$;

(b) $\mathcal{A}^* \subset \mathcal{A}$;

(c) $\mathcal{A}^* \subset \mathcal{E}$;

(d) $\mathcal{A} \subset \mathcal{E}$;

(e) $\mathcal{A}^* \subset \mathcal{E}^*$;

(f) $\mathcal{A} \not\subseteq \mathcal{E}^*$ *and* $\mathcal{E}^* \not\subseteq \mathcal{A}$.

Proof. $\mathcal{E}^* \subseteq \mathcal{E}$, $\mathcal{A}^* \subseteq \mathcal{A}$, $\mathcal{A}^* \subseteq \mathcal{E}$, $\mathcal{A} \subseteq \mathcal{E}$, $\mathcal{A}^* \subseteq \mathcal{E}^*$ by the definition. $\mathcal{E}^* \neq \mathcal{E}$, $\mathcal{A}^* \neq \mathcal{A}$, $\mathcal{A}^* \neq \mathcal{E}$, since the property monochromatic belongs to $\mathcal{A}$ but does not belong to $\mathcal{E}^*$. $\mathcal{A} \neq \mathcal{E}$, $\mathcal{A}^* \neq \mathcal{E}^*$, because the property zig-zag belongs to $\mathcal{E}^*$ and does not belong to $\mathcal{A}$. Finally, the properties monochromatic and zig-zag show that $\mathcal{A} \nsubseteq \mathcal{E}^*$ and $\mathcal{E}^* \nsubseteq \mathcal{A}$. ∎

Let G be a connected graph. If $\mathcal{P}$ is an E-property (E^*-property), then the minimum integer $k = |\mathbb{A}|$ for which there exists a (proper) vertex coloring $\psi : V(G) \to \mathbb{A}$ that makes G $\mathcal{P}$ vertex-connected is denoted by $\pi_{\mathcal{P}}(G)$ ($\pi^*_{\mathcal{P}}(G)$). If $\mathcal{P}$ is an A-property (A^*-property), then the minimum $k = |\mathbb{A}|$, for which there exists a (proper) vertex coloring $\psi : V(G) \to \mathbb{A}$ such that all paths of G have property $\mathcal{P}$ is denoted by $\chi_{\mathcal{P}}(G)$ ($\chi^*_{\mathcal{P}}(G)$).

Evidently, $\pi^*_{\mathcal{P}}(G) \geq \chi(G)$ and $\chi^*_{\mathcal{P}}(G) \geq \chi(G)$, where $\chi(G)$ denotes the chromatic number of G.

In the case when $\mathcal{P}$ contains all monochromatic words it is meaningful to consider vertex colorings which make G $\mathcal{P}$ vertex-connected and maximize the number of used colors.

3. Flexible Properties

Let $\mathbb{A}$ be an alphabet. Let $\mathcal{P}$ be a property over $\mathbb{A}$. We say that $\mathcal{P}$ is flexible if it is closed in the following sense. If $a_1 a_2 \cdots a_k \in \mathcal{P}$ and $b_1 b_2 \cdots b_k$ is obtained from $a_1 a_2 \cdots a_k$ by replacing a_i with $b_i \in \mathbb{A}$ in such a way that $a_i \neq a_j$ implies $b_i \neq b_j$, then $b_1 b_2 \cdots b_k \in \mathcal{P}$.

Theorem 5. *Let G be a connected graph and let S be its spanning tree. If $\mathcal{P}$ is a flexible E-property, then*

$$\pi^*_{\mathcal{P}}(G) \leq \pi_{\mathcal{P}}(S) \cdot \chi(G).$$

Proof. Since $\mathcal{P}$ is an E-property, S has a vertex coloring ψ with $\pi_{\mathcal{P}}(S)$ colors that makes it $\mathcal{P}$ vertex-connected. Let ϕ be a proper vertex coloring of G with $\chi(G)$ colors. We associate the ordered pair $(\psi(v), \phi(v))$ to every vertex v of G. Now, we define a new vertex coloring ρ of G in the following way. Two vertices u, v of G receive the same color if and only if $(\psi(u), \phi(u)) = (\psi(v), \phi(v))$. The obtained coloring uses $\pi_{\mathcal{P}}(S) \cdot \chi(G)$ colors, moreover it is a proper coloring, since $\rho(u) = \rho(v)$ implies $\phi(u) = \phi(v)$.

Observe that, for any two vertices u and v of G there is a unique $u - v$ path in S. In S all paths have property $\mathcal{P}$ under the coloring ψ. Since $\mathcal{P}$ is flexible, all of these paths have property $\mathcal{P}$ under the coloring ρ of G ($\psi(u) \neq \psi(v)$ implies $\rho(u) \neq \rho(v)$). ∎

Theorem 6. *Let G be a connected graph, S be its spanning tree, and let $\mathcal{P}$ be a flexible E-property. If S admits a proper vertex coloring with $\pi_{\mathcal{P}}(S)$ colors that makes it $\mathcal{P}$ vertex-connected, then*

$$\pi^*_{\mathcal{P}}(G) \leq \pi_{\mathcal{P}}(S) \cdot \chi(G \setminus E(S)).$$

Proof. We can proceed as in the proof of Theorem 5, with such a modification that ϕ is a proper vertex coloring of $G \setminus E(S)$ with $\chi(G \setminus E(S))$ colors. ∎

4. Zig-Zag Vertex-Connection

As it was shown (Claim 2 and Claim 3), the property zig-zag is E^*-property but not A-property.

The smallest integer k, for which a connected graph G has a (proper) vertex coloring $\psi : V(G) \to \{1, 2, \ldots, k\}$ that makes G zig-zag vertex-connected is denoted by $\pi_{zz}(G)$ ($\pi^*_{zz}(G)$). Recall that a vertex colored graph G is zig-zag vertex-connected if any two distinct vertices are connected by a zig-zag path, i.e., by a path whose associated word is zig-zag.

From the proof of Claim 1 we immediately have

Theorem 7. *If G is a nontrivial connected graph, then $\pi_{zz}(G) = 2$.*

Since for any connected graph G it holds $\pi^*_{zz}(G) \geq \chi(G)$, we obtain

Corollary 8. *Let G be a nontrivial connected graph. Then $\pi^*_{zz}(G) = 2$ if and only if G is bipartite.*

Theorem 9. *If G is a connected graph and S is its spanning tree, then*

$$\chi(G) \leq \pi^*_{zz}(G) \leq 2 \cdot \chi(G \setminus E(S)).$$

Proof. Let ψ be a proper vertex coloring of S with colors 1 and 2. Let ϕ be a proper vertex coloring of $G \setminus E(S)$ with colors $1, 2, \ldots, \chi(G \setminus E(S))$. Now, we associate the ordered pair $(\psi(v), \phi(v))$ to every vertex v of G. Observe that if we assign the color k to vertices with associated pair $(1, k)$ and assign the color $k + \chi(G \setminus E(S))$ to vertices with associated pair $(2, k)$, then we obtain a proper vertex coloring. Moreover, this coloring makes G zig-zag vertex-connected, since S is zig-zag vertex-connected. ∎

Note that Theorem 9 does not follow from Theorem 6, since the property zig-zag is not flexible (for example, $3, 2, 4$ is a zig-zag word but if we replace 4 with 1, then the obtained word $3, 2, 1$ is not zig-zag).

Corollary 10. *If G is a connected graph, then*

$$\chi(G) \leq \pi_{zz}^*(G) \leq 2 \cdot \chi(G).$$

Theorem 9 can be strengthen as follows.

Theorem 11. *Let G be a connected graph and let S be its spanning tree with bipartition (V_1, V_2). Then*

$$\pi_{zz}^*(G) \leq \chi(G[V_1]) + \chi(G[V_2]),$$

where $G[V_i]$ is the induced subgraph of G for $i \in \{1, 2\}$.

Proof. Let ψ be a proper vertex coloring of $G[V_1]$ with colors $1, 2, \ldots, \chi(G[V_1])$ and let ϕ be a proper vertex coloring of $G[V_2]$ with colors $\chi(G[V_1])+1, \chi(G[V_1])+2, \ldots, \chi(G[V_1])+\chi(G[V_2])$. Clearly, these colorings induce a proper vertex coloring of G. Moreover, for any two vertices u and v of G there is a unique $u - v$ path in S, which is evidently a zig-zag path. ■

Problem 12. Is it true that $\pi_{zz}^*(G) = \chi(G)$ for any connected graph G?

If G is a graph with maximum degree Δ, then it has a proper vertex coloring with at most $\Delta+1$ colors (this bound is tight for odd cycles and complete graphs, moreover, Brooks proved that these are the only graphs for which we need $\Delta+1$ colors). In the following we prove that any connected graph admits a proper vertex coloring with at most $\Delta + 1$ colors such that any two vertices of G are connected by a zig-zag path.

Theorem 13. *If G is a connected graph with maximum degree Δ, then*

$$\pi_{zz}^*(G) \leq \Delta + 1.$$

Proof. We prove a slightly stronger assertion. We show that every connected graph G has a proper vertex coloring such that it uses at most $\Delta + 1$ colors and it makes at least one spanning tree of G zig-zag vertex-connected.

Suppose to the contrary that H is a counterexample with minimum number of vertices. Let v be a vertex of H such that $H - v$ is connected. The graph $H - v$ has fewer vertices than H, therefore it has a required coloring ψ. We show that the coloring ψ can be extended to a required one of H, which is a contradiction to H being a counterexample.

Let $N(v)$ be the set of all neighbors of v in H. Let T be a spanning tree of $H - v$ that is zig-zag vertex-connected.

If there is a vertex $u \in N(v)$ such that $\psi(u) = 1$, then it is sufficient to color the vertex v with a color which does not appear on its neighbors. In T any two vertices are connected by a unique path. Therefore, $\psi(u) = 1$ implies that

$\psi(y) > \psi(u)$ holds for every edge yu of T with $u \in N(v)$. Consequently, we can extend the zig-zag vertex-connected spanning tree T by adding the vertex v and the edge uv.

So we can assume that no vertex in $N(v)$ has color 1. If there is an edge xu of T with $u \in N(v)$ such that $\psi(x) < \psi(u)$, then it is sufficient to color the vertex v with color 1. In such a way we obtain a proper vertex coloring of H. Similarly as above, in T any two vertices are connected by a unique path, hence, $\psi(x) < \psi(u)$ implies that $\psi(y) < \psi(u)$ holds for every edge yu of T with $u \in N(v)$. So, we can extend the zig-zag vertex-connected spanning tree T by adding the vertex v and the edge uv. Finally, if for every edge xu of T with $u \in N(v)$ it holds $\psi(x) > \psi(u)$, then no vertex $u \in N(v)$ has color $\Delta + 1$. Observe if we color v with $\Delta + 1$, then we obtain a proper vertex coloring of H and T can be extended by adding the vertex v and an arbitrary edge uv with $u \in N(v)$. ■

5. Dynamic Vertex-Connection

Let $W = \ell_1 \cdots \ell_t$ be a word. We say that W is dynamic if one of the following holds.

(i) $t = 1$,

(ii) $t = 2$ and $\ell_1 \neq \ell_2$,

(iii) W contains at least three different letters.

The property dynamic consists of all dynamic words. Clearly, dynamic is A^*-property, since if we color the vertices of a connected n-vertex graph with n different colors, then every path is dynamic.

The smallest integer k, for which a connected graph G has a (proper) vertex coloring $\psi : V(G) \to \{1, 2, \ldots, k\}$ that makes G dynamic vertex-connected is denoted by $\pi_d(G)$ ($\pi_d^*(G)$). The minimum k for which there exists a (proper) vertex coloring $\psi : V(G) \to \{1, 2, \ldots, k\}$ such that all paths of G are dynamic is denoted by $\chi_d(G)$ ($\chi_d^*(G)$).

In the following the next two lemmas will be very useful.

Lemma 14 [18]**.** *If G is a 2-connected graph and w is a vertex of G, then for any two vertices u and v in G, there is a $u - v$ path containing w.*

Lemma 15 [9]**.** *If G is a 2-connected graph and e is an edge of G, then for any two vertices u and v in G, there is a $u - v$ path containing e.*

Theorem 16. *If G is a 2-connected graph, then*

$$\pi_d(G) = 3.$$

Proof. To obtain a required coloring of a 2-connected graph G we choose an edge $e = xy$ of G, then we color the endvertices of xy with different colors and color all the remaining vertices with a third color.

By Lemma 15, for any two vertices u and v in G, there is a $u - v$ path containing the edge xy.

Let u, v be two vertices in G. If $\{u, v\} \cap \{x, y\} = \emptyset$, then any $u - v$ path containing the edge xy is dynamic. If $\{u, v\} = \{x, y\}$, then any $u - v$ path is dynamic. Finally, without loss of generality assume that $u = x$ and $v \neq y$. In this case, 2-connectedness of G implies that G contains a $u - v$ path containing y (see Lemma 14), which is evidently dynamic. ■

Theorem 17. *Let G be a connected graph on at least three vertices and let m denote the maximum number of cut-edges incident with a vertex of G. Then*

$$\pi_d(G) = \max\{3, m + 1\}.$$

Proof. G has at least three vertices, so trivially $\pi_d(G) \geq 3$. Now assume that v is a vertex of G incident with m cut-edges. In this case $\pi_d(G) \geq m+1$, since any two of the neighbors of v are connected by a unique path (through v).

Now we prove that $\pi_d(G) \leq \max\{3, m + 1\}$. We proceed by induction on the number of blocks of G. If G has only one block, then it is 2-connected, so the result follows from Theorem 16. Now assume that G has at least two blocks. Let H be a block of G incident with exactly one cut-vertex u of G. Let $G' = G - (H - u)$ be a graph obtained from G by removing all edges and vertices incident with H except for u. Let ψ be a coloring of G' with $\pi_d(G')$ colors that makes G' dynamic vertex-connected. We extend this coloring of G' to a coloring of G in the following way. There are two possibilities: either H is 2-connected or it is a complete graph on two vertices. In the former case we choose two adjacent vertices x, y of $H - u$, we color all the vertices of H except for x, y with color $\psi(u)$ and then we color x, y with two different colors distinct from $\psi(u)$. By Lemma 14 and Lemma 15, this coloring makes G dynamic vertex-connected. Now assume that H is a complete graph on two vertices. In this case we color the uncolored vertex of H with a color which appears neither on u nor on a neighbor w of u such that uw is a cut-edge in G'. ■

Corollary 18. *If T is a tree with maximum degree Δ, then*

$$\pi_d(T) = \Delta + 1.$$

Theorem 19. *If G is a 2-connected nonbipartite graph, then*

$$\pi_d^*(G) = \chi(G).$$

Proof. Let ψ be a proper vertex coloring of G with $\chi(G)$ colors. G is nonbipartite, therefore $\chi(G) \geq 3$. The coloring ψ uses $\chi(G)$ colors, hence for any two used colors a, b there is an edge in G whose endvertices are colored with a and b.

Let u, v be two vertices in G. If $\psi(u) = \psi(v)$, then we find an edge e with no endvertex of color $\psi(u)$. G is 2-connected, so there is a $u-v$ path in G containing the edge e; such a path exists by Lemma 15. If $\psi(u) \neq \psi(v)$, then we choose a third vertex w whose color is distinct from $\psi(u), \psi(v)$ and find a $u - v$ path containing w; such a path exists by Lemma 14. The obtained $u - v$ paths are dynamic, so $\pi_d^*(G) \leq \chi(G)$.

The inequality $\pi_d^*(G) \geq \chi(G)$ trivially holds. ■

It is easy to see that for the cycle on four vertices C_4 it holds $\pi_d^*(C_4) = 4$. For the other 2-connected bipartite graphs three colors are sufficient.

Theorem 20. *If $G \neq C_4$ is a 2-connected bipartite graph, then*

$$\pi_d^*(G) = 3.$$

Proof. Let $G \neq C_4$ be a 2-connected bipartite graph with partite sets V_1 and V_2, i.e., each edge of G has one endvertex in V_1 and one in V_2. Assume that $|V_1| \geq |V_2|$. Clearly, $|V_1| \geq 3$ and $|V_2| \geq 2$. First assume that G is not a complete bipartite graph. Then there are vertices $v_1 \in V_1$ and $v_2 \in V_2$ such that $v_1v_2 \notin E(G)$. Color the vertices from V_1 except for v_1 with color 1, the vertices from V_2 except for v_2 with 2, and the vertices v_1, v_2 with color 3. In such a way we obtain a proper vertex coloring of G. G is 2-connected, so it has no vertex of degree one. This implies that G contains an edge e_1 whose endvertices are colored with 2 and 3, an edge e_2 whose endvertices are colored with 1 and 3, and an edge e_3 whose endvertices are colored with 1 and 2.

Now, let u, v be two vertices of G. If they have different colors, then we choose a third vertex w colored with the third color. By Lemma 14, there is a $u - v$ path containing w. If they have the same color, say 1, then we choose an edge e_1 whose endvertices are colored with 2 and 3. By Lemma 15, there is a $u - v$ path containing e_1.

Finally, assume that G is a complete bipartite graph. Let x be a vertex from V_2. Color the vertices from V_1 with 1, the vertices from V_2 except for x with 2, and the vertex x with color 3. In the obtained proper vertex coloring there are only edges whose endvertices are colored with 1, 2 and edges whose endvertices are colored with 1 and 3.

Now, let u, v be two vertices of G. If both of them have color 1, then let w be an other vertex of color 1, and let y be a vertex of color 2. Then $uywxv$ is a dynamic path. In all other cases we proceed similarly as in the previous case. ■

Theorem 21. *Let G be a connected graph with at least one cut-vertex. Let m denote the maximum number of cut-edges incident with a vertex of G. Then*

$$\pi_d^*(G) = \begin{cases} \max\{3, m+1, \chi(G)\} & \textit{if } G \textit{ does not contain } C_4 \textit{ as a block,} \\ \max\{4, m+1, \chi(G)\} & \textit{otherwise.} \end{cases}$$

Proof. We can proceed similarly as in the proof of Theorem 17. ■

Let G^2 be the square of G, which is the graph obtained from G by adding the edges between pairs of vertices at distance two.

Theorem 22. *If G is a connected graph, then*

$$\chi_d(G) = \chi_d^*(G) = \chi(G^2).$$

Proof. Let ψ be a vertex coloring of G such that all paths of G are dynamic. Then ψ is a proper coloring, since every path on two vertices is dynamic. Now assume that u and v are at distance two in G. This implies that there is a third vertex w such that w is adjacent with both of them. Clearly, uwv form a path on three vertices in G. Every path of G is dynamic, so the vertices u, w, v are colored with different colors. Consequently, any two vertices at distance at most two have different colors. Therefore, $\chi_d^*(G) \geq \chi(G^2)$.

Now assume that ρ is a proper vertex coloring of G^2. If two vertices are adjacent in G, then they are adjacent in G^2 as well. If uwv is a path on three vertices in G, then the vertices u, w, v are mutually adjacent in G^2. Hence any proper vertex coloring of G^2 induces a proper vertex coloring of G such that any path on at least three vertices is colored with at least three colors, i.e., $\chi_d^*(G) \leq \chi(G^2)$. ■

6. Nonrepetitive Vertex-Connection

A word of the form $\ell_1\ell_2\cdots\ell_n\ell_1\ell_2\cdots\ell_n$ is called a repetition. A word is nonrepetitive if it does not contain a repetition as a block, i.e., a sequence of consecutive letters. The property nonrepetitive consists of all nonrepetitive words. Clearly, nonrepetitive is A^*-property, since if we color the vertices of a connected n-vertex graph with n different colors, then every path is nonrepetitive.

The smallest integer k, for which a connected graph G has a (proper) vertex coloring $\psi : V(G) \to \{1, 2, \ldots, k\}$ that makes G nonrepetitive vertex-connected is denoted by $\pi_{nr}(G)$ ($\pi_{nr}^*(G)$). The minimum k for which there exists a (proper) vertex coloring $\psi : V(G) \to \{1, 2, \ldots, k\}$ such that all paths of G are nonrepetitive is denoted by $\chi_{nr}(G)$ ($\chi_{nr}^*(G)$).

Determining $\pi_{nr}(G)$ is a nontrivial task even for paths. Indeed, the fact that $\pi_{nr}(P_n) = 3$ for all $n \geq 4$ follows from the famous result of Thue [21].

Brešar *et al.* [4] proved that every tree admits a proper vertex coloring with at most four colors that makes it nonrepetitive vertex-connected.

Theorem 23 [4]**.** *If T is a tree, then*

$$\pi_{nr}(T) = \pi^*_{nr}(T) = \chi_{nr}(T) = \chi^*_{nr}(T) \leq 4.$$

This result immediately implies.

Corollary 24. *If G is a connected graph, then $\pi_{nr}(G) \leq 4$.*

Now consider proper vertex colorings that make G nonrepetitive vertex-connected.

Theorem 25. *If G is a connected graph and S is its spanning tree, then*

$$\chi(G) \leq \pi^*_{nr}(G) \leq 4 \cdot \chi(G \setminus E(S)).$$

Proof. It follows from Theorem 6, Theorem 23, and from the fact that the property nonrepetitive is flexible. ∎

Corollary 26. *Every connected planar graph has a proper vertex coloring with at most* 16 *colors such that any two of its vertices are connected by a nonrepetitive path.*

Very recently was proven, by Dujmović *et al.* [12], that every planar graph G has a proper vertex coloring with at most 768 colors such that every path in G is nonrepetitive, i.e., $\chi^*_{nr}(G) \leq 768$. For several years, the problem whether planar graphs have bounded $\chi^*_{nr}(G)$ was widely recognized as the most important open problem in the field of nonrepetitive graph coloring.

Vertex coloring in which all paths are nonrepetitive was introduced by Alon *et al.* [1]. They obtained the following bounds for $\chi^*_{nr}(G)$.

Theorem 27 [1]**.** *If G is a graph with maximum degree Δ, then there are absolute constants c_1 and c_2 such that*

$$c_1 \cdot \frac{\Delta^2}{\log \Delta} \leq \chi^*_{nr}(G) \leq c_2 \cdot \Delta^2.$$

The precise upper bound shown was $2^{16}\Delta^2$. Several authors subsequently improved the constant 2^{16}. The best presently known upper bound on $\chi^*_{nr}(G)$ is the following one by Dujmović *et al.* [13] from 2016.

Theorem 28 [13]**.** *If G is a graph with maximum degree Δ, then*

$$\chi^*_{nr}(G) \leq (1 + o(1))\Delta^2.$$

Note that the topic of the nonrepetitive coloring is intensively studied. There is a very recent survey of Wood [22], which gives a unified and comprehensive presentation for the major results and proof methods as well as to highlight numerous open problems.

We offer the following.

Problem 29. Determine the precise value K such that $\pi^*_{nr}(G) \leq K$ for every connected graph G.

7. Conflict-Free Vertex-Connection

Let $W = \ell_1 \cdots \ell_t$ be a word. We say that W is conflict-free if at least one letter occurs exactly once in W. The property conflict-free consists of all conflict-free words. Clearly, conflict-free is A^*-property, since if we color the vertices of a connected n-vertex graph with n different colors, then every path is conflict-free.

The smallest integer k, for which a connected graph G has a (proper) vertex coloring $\psi : V(G) \to \{1, 2, \ldots, k\}$ that makes G conflict-free vertex-connected is denoted by $\pi_{cf}(G)$ ($\pi^*_{cf}(G)$). The minimum k for which there exists a (proper) vertex coloring $\psi : V(G) \to \{1, 2, \ldots, k\}$ such that all paths of G are conflict-free is denoted by $\chi_{cf}(G)$ ($\chi^*_{cf}(G)$).

Note that, $\chi_{cf}(G) = \chi^*_{cf}(G)$ for any connected graph G, since if a path on two vertices is conflict-free, then its vertices have different colors.

The concept of conflict-free vertex-connection was introduced very recently; even so, the study of $\pi_{cf}(G)$ has attracted a lot of attention, see e.g. [10, 11, 14, 18, 20].

For $\pi^*_{cf}(G)$ we have the following Vizing type theorem.

Theorem 30. *If G is a 2-connected graph or it has only one cut-vertex, then*

$$\chi(G) \leq \pi^*_{cf}(G) \leq \chi(G) + 1.$$

Proof. It suffices to show that $\pi^*_{cf}(G) \leq \chi(G) + 1$.

By Lemma 14, if w is a vertex of G, then for any two vertices u and v in G, there is a $u - v$ path containing w. Therefore, to get a required coloring of a 2-connected graph G we choose an arbitrary vertex x of G, then we properly color G with $\chi(G)$ colors and finally we recolor x with a new color $\chi(G) + 1$. If G has exactly one cut-vertex, say y, then our chosen vertex is y.

The upper bound is tight due to bipartite graphs. ■

If a graph G contains more cut-vertices, then the situation is much more complicated. Observe that for any tree T we have $\pi_{cf}(T) = \pi^*_{cf}(T) = \chi_{cf}(T) = \chi^*_{cf}(T)$. For paths, we have the following result.

Theorem 31 [18]**.** *If P_n is a path on n vertices, then*

$$\pi_{cf}(P_n) = \pi^*_{cf}(P_n) = \chi_{cf}(P_n) = \chi^*_{cf}(P_n) = \lceil \log_2(n+1) \rceil .$$

Recently, Ji *et al.* [14] proved the following.

Theorem 32 [14]**.** *If G is a connected graph with t cut-vertices, then*

$$\pi_{cf}(G) \leq \lceil \log_2(t+1) \rceil + 1.$$

It is easy to see that the property conflict-free is flexible. So the next result immediately follows from Theorems 6 and 32.

Corollary 33. *Let G be a connected graph and let S be its spanning tree with t cut-vertices. Then*

$$\pi^*_{cf}(G) \leq (\lceil \log_2(t+1) \rceil + 1) \cdot \chi(G \setminus E(S)).$$

We finish the paper with the following problem.

Problem 34. Determine the value $\chi_{cf}(G)$ for any graph G.

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