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THE GENERALIZED TURÁN PROBLEM OF TWO INTERSECTING CLIQUES

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Abstract

For $s < r$, let $B_{r,s}$ be the graph consisting of two copies of K_r, which share exactly s vertices. Denote by $ex(n, K_r, B_{r,s})$ the maximum number of copies of K_r in a $B_{r,s}$-free graph on n vertices. About fifty years ago, Erdős and Sós determined $ex(n, K_3, B_{3,1})$. Recently, Gowers and Janzer showed that $ex(n, K_r, B_{r,r-1}) = n^{r-1-o(1)}$. It is a natural question to ask for $ex(n, K_r, B_{r,s})$ for general r and s. In this paper, we mainly consider the problem for $s = 1$. Utilizing Zykov's symmetrization, we determine the exact value of $ex(n, K_4, B_{4,1})$ for $n \geq 4$. For $r \geq 5$ and n sufficiently large, by the Füredi's structure theorem we show that $ex(n, K_r, B_{r,1}) = \mathcal{N}(K_{r-2}, T_{r-2}(n-2))$, where $\mathcal{N}(K_{r-2}, T_{r-2}(n-2))$ represents the number of copies of K_{r-2} in the $(r-2)$-partite Turán graph on $n-2$ vertices.

Keywords: generalized Turán number, Zykov's symmetrization, Füredi's structure theorem.

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1. Introduction

Let T be a graph and $\mathcal{F}$ be a family of graphs. We say that a graph G is $\mathcal{F}$-free if it does not contain any graph from $\mathcal{F}$ as a subgraph. Let $ex(n, T, \mathcal{F})$ denote the maximum possible number of copies of T in an $\mathcal{F}$-free graph on n vertices. The problem of determining $ex(n, T, \mathcal{F})$ is often called the generalized Turán problem. When $T = K_2$, it reduces to the classical Turán number $ex(n, \mathcal{F})$. For simplicity, we often write $ex(n, T, F)$ for $ex(n, T, \{F\})$.

Let T be a graph on t vertices. The s-blow-up of T is the graph obtained by replacing each vertex v of T by an independent set W_v of size s, and each edge uv of T by a complete bipartite graph between the corresponding two independent sets W_u and W_v. Alon and Shikhelman [1] showed that $ex(n, T, F) = \Theta(n^t)$ if and only if for any positive integer s, F is not a subgraph of the s-blow-up of T. Otherwise, there exists some $\epsilon(T, F) > 0$ such that $ex(n, T, F) \leq n^{t-\epsilon(T,F)}$.

For integers $s < r$, let $B_{r,s}$ be the graph consisting of two copies of K_r, which share exactly s vertices. Erdős and Sós in [3] determined the maximum number of hyperedges in a 3-uniform hypergraph without two hyperedges intersecting in exactly one vertex. From their result, it is easy to deduce the following theorem.

Theorem 1 (Erdős and Sós [3]). *For all n,*

$$ex(n, K_3, B_{3,1}) = \begin{cases} n, & n \equiv 0 \pmod 4; \\ n-1, & n \equiv 1 \pmod 4; \\ n-2, & n \equiv 2 \text{ or } 3 \pmod 4. \end{cases}$$

The celebrated Ruzsa-Szemerédi theorem [13] implies that $ex(n, K_3, B_{3,2}) = n^{2-o(1)}$. Recently, Gowers and Janzer [10] proposed a natural generalization of the Ruzsa-Szemerédi Theorem, and proved the following result.

Theorem 2 (Gowers and Janzer [10]). *For each $2 \leq s < r$,*

$$ex(n, K_r, \{B_{r,s}, B_{r,s+1}, \ldots, B_{r,r-1}\}) = n^{s-o(1)}.$$

For a graph G, let $V(G)$ and $E(G)$ be the vertex set and edge set of G, respectively. The *join* of two graphs G_1 and G_2, denoted by $G_1 \vee G_2$, is defined as $V(G_1 \vee G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 \vee G_2) = E(G_1) \cup E(G_2) \cup \{xy \colon x \in V(G_1), y \in V(G_2)\}$. The r-partite Turán graph on n vertices, denoted by $T_r(n)$, is a complete r-partite graph where the sizes of each part differ by at most one. Denote by $\mathcal{N}(T, G)$ the number of copies of T in G.

In this paper, by using Zykov's symmetrization [18] we determine $ex(n, K_4, B_{4,1})$ for $n \geq 4$.

Theorem 3. *For $4 \leq n \leq 6$, $ex(n, K_4, B_{4,1}) = \binom{n}{4}$. For $n = 7$, $ex(n, K_4, B_{4,1}) = \binom{6}{4}$. For $8 \leq n \leq 16$, $ex(n, K_4, B_{4,1}) = 4n - 15$. For $n \geq 17$,*

$$ex(n, K_4, B_{4,1}) = \left\lfloor \frac{(n-2)^2}{4} \right\rfloor,$$

and $K_2 \vee T_2(n-2)$ is the unique graph attaining the maximum number of copies of K_4.

Then, by using Füredi's structure theorem [7], we determine $ex(n, K_r, B_{r,1})$ for $r \geq 5$ and n sufficiently large.

Theorem 4. *For $r \geq 5$ and sufficiently large n,*

$$ex(n, K_r, B_{r,1}) = \mathcal{N}(K_{r-2}, T_{r-2}(n-2)),$$

and $K_2 \vee T_{r-2}(n-2)$ is the unique graph attaining the maximum number of copies of K_r.

Note that $B_{r,0}$ consists of two disjoint copies of K_r. We determine $ex(n, K_3, B_{3,0})$ for $n \geq 3$.

Theorem 5. *For $n \leq 5$, $ex(n, K_3, B_{3,0}) = \binom{n}{3}$. For $6 \leq n \leq 10$, $ex(n, K_3, B_{3,0}) = 3n - 8$. For $n \geq 11$, $ex(n, K_3, B_{3,0}) = \left\lfloor \frac{(n-1)^2}{4} \right\rfloor$.*

By applying Füredi's structure theorem, we determine $ex(n, K_r, B_{r,0})$ for $r \geq 4$ and n sufficiently large.

Theorem 6. *For $r \geq 4$ and sufficiently large n,*

$$ex(n, K_r, B_{r,0}) = \mathcal{N}(K_{r-1}, T_{r-1}(n-1)),$$

and $K_1 \vee T_{r-1}(n-1)$ is the unique graph attaining the maximum number of copies of K_r.

Let r, s be positive integers with $s < r$. An integer vector $(a_1, a_2, \ldots, a_t)$ is called a *partition* of r if $a_1 \geq a_2 \geq \cdots \geq a_t > 0$ and $\sum_{i=1}^{t} a_i = r$. Let $P = (a_1, a_2, \ldots, a_t)$ be a partition of r. If $\sum_{i \in I} a_i \neq s$ holds for every $I \subset \{1, 2, \ldots, t\}$, then we call P an *s-sum-free* partition of r. Denote by $\beta_{r,s}$ the maximum length of an s-sum-free partition of r.

Theorem 7. *For any $r > s \geq 2$, if $r \geq 2s + 1$,*

$$ex(n, K_r, B_{r,s}) = \Theta\big(n^{r-s-1}\big);$$

if $r \leq 2s$, then there exist positive reals c_1 and c_2 such that

$$c_1 n^{\beta_{r,s}} \leq ex(n, K_r, B_{r,s}) \leq c_2 n^s.$$

It seems hard to determine the exact value of $\beta_{r,s}$ for all r and s. The following proposition gives some bounds on $\beta_{r,s}$ and exact values of $\beta_{r,s}$ for $s \le 4$ and when r is even, s is odd.

Proposition 8. (i) *For* $6 \le s+1 \le r \le 2s$, $r-s \le \beta_{r,s} \le r/2$.

(ii)

$$\beta_{r,1} = \left\lfloor \frac{r}{2} \right\rfloor, \quad \beta_{r,2} = 1 + \left\lfloor \frac{r-1}{3} \right\rfloor.$$

$$\beta_{r,3} = \begin{cases} \max\left\{2 + \left\lfloor \frac{r-2}{4} \right\rfloor, r/2\right\}, & r \text{ is even;} \\ \max\left\{2 + \left\lfloor \frac{r-2}{4} \right\rfloor, 1 + \frac{r-5}{2}\right\}, & r \text{ is odd.} \end{cases}$$

$$\beta_{r,4} = \max\left\{3 + \left\lfloor \frac{r-3}{5} \right\rfloor, 1 + \left\lfloor \frac{r-2}{3} \right\rfloor\right\}.$$

(iii) *Suppose that* r *is even,* s *is odd and* $6 \le s+1 \le r \le 2s$, *then* $\beta_{r,s} = r/2$.

Utilizing the graph removal lemma, we establish an upper bound on $ex(n, K_4, B_{4,2})$.

Theorem 9. *For sufficiently large* n,

$$\frac{n^2 - 25}{12} \le ex(n, K_4, B_{4,2}) \le \frac{n^2}{9} + o(n^2).$$

We should mention that several papers considered related problems after the first version of this paper appeared on the arxiv. Gerbner and Patkós [9] determined $ex(n, K_k, B_{r,0})$ and $ex(n, K_k, B_{r,1})$ for all values of k, r if n is large enough. Zhang, Chen, Győri and Zhu [16] determined the exact value of $ex(n, K_r, (k+1)K_r)$ for all k, r if n is large enough, where $(k+1)K_r$ consists of $k+1$ disjoint copies of K_r. Some more related results can be found in [8, 11, 15, 17].

The rest of this paper is organized as follows. In Section 2, we prove Theorem 3 and Theorem 5. In Section 3, we prove Theorems 4 and 6. In Section 4, we prove Theorem 7. In Section 5, we prove Theorem 9.

2. The Values of $ex(n, K_4, B_{4,1})$ and $ex(n, K_3, B_{3,0})$

Zykov [18] introduced a useful tool to prove Turán's theorem, which is called Zykov's symmetrization. In this section, by using Zykov's symmetrization we first determine $ex(n, K_4, \{B_{4,1}, H_1, K_5\})$, where H_1 is a graph on seven vertices as shown in Figure 1. Then, we show that a $B_{4,1}$-free graph can be reduced to a $\{B_{4,1}, H_1, K_5\}$-free graph by deleting vertices and this happens without a loss of too many K_4's, which leads to a proof of Theorem 3.

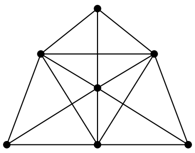

Figure 1. A graph H_1 on seven vertices.

For $S \subset V(G)$, let $G[S]$ denote the subgraph of G induced by S, and let $G - S$ denote the subgraph of G induced by $V(G) \setminus S$.

Lemma 10. *For $n \geq 2$,*

$$ex(n, K_4, \{B_{4,1}, H_1, K_5\}) = \left\lfloor \frac{(n-2)^2}{4} \right\rfloor,$$

and $K_2 \vee T_2(n-2)$ is the unique graph attaining the maximum number of K_4's.

Proof. Assume that G is a $\{B_{4,1}, H_1, K_5\}$-free graph with the maximum number of copies of K_4. We may further assume that each edge of G is contained in at least one copy of K_4, since otherwise we can delete it without decreasing the number of copies of K_4. For each $e \in E(G)$, let $\mathcal{K}_4(e)$ denote the set of copies of K_4 in G containing e. Let

$$E_1 = \big\{e \in E(G)\colon \text{there exist } K, K' \in \mathcal{K}_4(e) \text{ such that } E(K) \cap E(K') = \{e\}\big\}$$

and let G_1 be the subgraph of G induced by E_1.

Claim 11. *E_1 is a matching of G.*

Proof. Suppose to the contrary that there exists a path of length two in G_1, say vuw. Since $uv \in E_1$, there exist distinct vertices a_1, b_1, a_2, b_2 so that both $G[\{u, v, a_1, b_1\}]$ and $G[\{u, v, a_2, b_2\}]$ are copies of K_4. Since $uw \in E_1$, there exist distinct vertices c_1, d_1, c_2, d_2 so that both $G[\{u, w, c_1, d_1\}]$ and $G[\{u, w, c_2, d_2\}]$ are copies of K_4.

Case 1. $w \in \{a_1, b_1, a_2, b_2\}$ or $v \in \{c_1, d_1, c_2, d_2\}$. Since the two cases are symmetric, we only consider the case $w \in \{a_1, b_1, a_2, b_2\}$. By symmetry, we may assume that $a_1 = w$. Now $G[\{u, v, w, b_1\}]$ and $G[\{u, v, a_2, b_2\}]$ are both copies of K_4. Clearly, we have either $v \notin \{c_1, d_1\}$ or $v \notin \{c_2, d_2\}$. Without loss of generality, assume that $v \notin \{c_1, d_1\}$. If $\{c_1, d_1\} \cap \{a_2, b_2\} = \emptyset$, then $G[\{u, v, w, a_2, b_2, c_1, d_1\}]$ contains a copy of $B_{4,1}$, which contradicts the assumption that G is $B_{4,1}$-free. If $|\{c_1, d_1\} \cap \{a_2, b_2\}| = 1$, by symmetry we assume that $c_1 = a_2$, then $G[\{u, v, w, b_1, a_2, b_2, d_1\}]$ contains a copy of H_1, a contradiction. If $\{c_1, d_1\} = \{a_2, b_2\}$, then $G[\{u, v, w, a_2, b_2\}]$ is a copy of K_5, a contradiction.

Case 2. $w \notin \{a_1, b_1, a_2, b_2\}$ and $v \notin \{c_1, d_1, c_2, d_2\}$. For $i, j \in \{1, 2\}$, we claim that $|\{a_i, b_i\} \cap \{c_j, d_j\}| = 1$. If $\{a_i, b_i\} \cap \{c_j, d_j\} = \emptyset$, then $G[\{u, v, w, a_i, b_i, c_j, d_j\}]$ contains $B_{4,1}$ as a subgraph, a contradiction. If $\{a_i, b_i\} = \{c_j, d_j\}$, then $G[\{u, v, w, a_i, b_i, c_i, d_i\}]$ contains $B_{4,1}$ as a subgraph, a contradiction. Hence $|\{a_i, b_i\} \cap \{c_j, d_j\}| = 1$. It follows that $\{a_1, b_1, a_2, b_2\} = \{c_1, d_1, c_2, d_2\}$. Then $G[\{u, v, w, a_1, b_1, a_2, b_2\}]$ contains H_1 as a subgraph, a contradiction. Thus, the claim holds. □

Let $G_2 = G - V(G_1)$. For two distinct vertices $u, v \in V(G)$ with $uv \notin E(G)$, define $C_{uv}(G)$ to be the graph obtained by deleting edges incident to u and adding edges in $\{uw \colon w \in N(v)\}$.

Claim 12. *For two distinct vertices $u, v \in V(G_2)$ with $uv \notin E(G)$, $C_{uv}(G)$ is a $\{B_{4,1}, H_1, K_5\}$-free graph.*

Proof. Let $\tilde{G} = C_{uv}(G)$. Since $uv \notin E(G)$, clearly we have $uv \notin E(\tilde{G})$. We first claim that $\tilde{G}$ is K_5-free. Otherwise, since G is K_5-free, there is a vertex set K containing u such that $\tilde{G}[K] \cong K_5$. Then $v \notin K$ since $uv \notin E(\tilde{G})$. It follows that $K \setminus \{u\} \cup \{v\}$ induces a copy of K_5 in G, a contradiction.

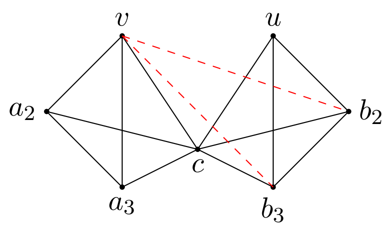

Figure 2. A copy of $B_{4,1}$ in $\tilde{G}$.

If $\tilde{G}$ contains a copy of $B_{4,1}$, let $S = \{a_1, a_2, a_3, b_1, b_2, b_3, c\}$ be a subset of $V(\tilde{G})$ such that both $\tilde{G}[\{a_1, a_2, a_3, c\}]$ and $\tilde{G}[\{b_1, b_2, b_3, c\}]$ are copies of K_4. If $u \notin S$, then $G[S]$ is a copy of $B_{4,1}$, a contradiction. If $u \in S$ but $v \notin S$, then $G[(S \setminus \{u\}) \cup \{v\}]$ is a copy of $B_{4,1}$, a contradiction. If $u, v \in S$, since $uv \notin E(\tilde{G})$, by symmetry we may assume that $a_1 = v$ and $b_1 = u$. Since u is a "clone" of v in $\tilde{G}$, we have $vb_2, vb_3 \in E(G)$ (as shown in Figure 2). Then both $G[\{v, c, a_2, a_3\}]$ and $G[\{v, c, b_2, b_3\}]$ are copies of K_4 in G. It follows that vc is an edge in E_1 in G, which contradicts the assumption that $v \in V(G) \setminus V(G_1)$. Thus $\tilde{G}$ is $B_{4,1}$-free.

If $\tilde{G}$ contains a copy of H_1, let $T = \{h, i, j, k, l, m, n\}$ be a subset of $V(\tilde{G})$ such that $\tilde{G}[\{h, i, j, k\}]$, $\tilde{G}[\{i, j, k, m\}]$, $\tilde{G}[\{i, k, l, m\}]$ and $\tilde{G}[\{j, k, m, n\}]$ are all copies of K_4 as shown in Figure 3. Similarly, we have $u, v \in T$. Since $uv \notin E(\tilde{G})$, by symmetry we have to consider three cases: (i) $h = u$, $n = v$; (ii) $h = u$, $m = v$ or (iii) $h = v$, $m = u$. If $h = u$ and $n = v$, then $vi \in E(G)$ since $ui \in E(\tilde{G})$. It follows that $\{i, j, k, m, v\}$ induces a copy of K_5 in G, which contradicts the assumption

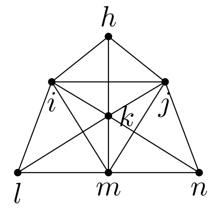

Figure 3. A copy of H_1 in $\tilde{G}$.

that G is K_5-free. If $h = u$ and $m = v$, then $kv \in E_1$ since both $G[\{k, v, i, l\}]$ and $G[\{k, v, j, n\}]$ are copies of K_4, which contradicts the fact that $v \in V(G_2)$. If $h = v$ and $m = u$, then $vl, vn \in E(G)$ since $ul, un \in E(\tilde{G})$. It follows that both $G[\{k, v, i, l\}]$ and $G[\{k, v, j, n\}]$ are copies of K_4, which contradicts the fact that $v \in V(G_2)$. Hence $\tilde{G}$ is H_1-free. □

By Zykov symmetrization, we prove the following claim.

Claim 13. *G_2 is a complete r-partite graph with $r \leq 4$.*

Proof. Recall that G is a $\{B_{4,1}, H_1, K_5\}$-free graph with the maximum number of copies of K_4 and each edge of G is contained in at least one copy of K_4. We define a binary relation R in $V(G_2)$ as follows: for any two vertices $x, y \in V(G_2)$, xRy if and only if $xy \notin E(G)$. We shall show that R is an equivalence relation. Since G is loop-free, it follows that R is reflexive. Since G is a undirected graph, it follows that R is symmetric.

Now we show that R is transitive. Suppose to the contrary that there exist $x, y, z \in V(G_2)$ such that $xy, yz \notin E(G_2)$ but $xz \in E(G_2)$. For $u, v \in V(G_2)$, let $k_4(u)$ be the number of copies of K_4 in G containing u, and $k_4(u, v)$ be the number of copies of K_4 in G containing u and v.

Case 1. $k_4(y) < k_4(x)$ or $k_4(y) < k_4(z)$. Since the two cases are symmetric, we only consider the case $k_4(y) < k_4(x)$. Let $\tilde{G} = C_{yx}(G)$. By Claim 12, $\tilde{G}$ is $\{B_{4,1}, H_1, K_5\}$-free since G is $\{B_{4,1}, H_1, K_5\}$-free. But now we have

$$\mathcal{N}(K_4, \tilde{G}) = \mathcal{N}(K_4, G) - k_4(y) + k_4(x) > \mathcal{N}(K_4, G),$$

which contradicts the assumption that G is a $\{B_{4,1}, H_1, K_5\}$-free graph with the maximum number of copies of K_4.

Case 2. $k_4(y) \geq k_4(x)$ and $k_4(y) \geq k_4(z)$. Let $G^* = C_{xy}(C_{zy}(G))$. By Claim 12, G^* is $\{B_{4,1}, H_1, K_5\}$-free. Since each edge in G is contained in at least one copies of K_4, it follows that

$$\begin{aligned}\mathcal{N}(K_4, G^*) &= \mathcal{N}(K_4, G) - (k_4(x) + k_4(z) - k_4(x, z)) + 2k_4(y) \\ &\geq \mathcal{N}(K_4, G) + k_4(x, z) > \mathcal{N}(K_4, G),\end{aligned}$$

which contradicts the assumption that G is a $\{B_{4,1}, H_1, K_5\}$-free graph with the maximum number of copies of K_4. Thus, we conclude that $xz \notin E(G)$ and R is transitive. Since R is an equivalence relation on $V(G_2)$ and G is K_5-free, it follows that G_2 is a complete r-partite graph with $r \leq 4$. □

Claim 14. *For any copy K of K_4 in G and any $uv \in E_1$, $|V(K) \cap \{u, v\}| \neq 1$.*

Proof. Suppose for contradiction that there exists $\{a, b, c, d, v\} \subset V(G)$ such that $G[\{a, b, c, d\}]$ is isomorphic to K_4 and bv is an edge in E_1, as shown in Figure 4.

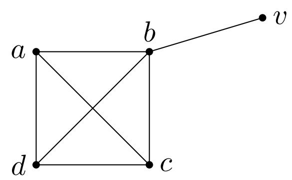

Figure 4. An edge in E_1 is attached to a copy of K_4.

Since $bv \in E_1$, there exist distinct vertices x_1, y_1, x_2, y_2 such that both $G[\{b, v, x_1, y_1\}]$ and $G[\{b, v, x_2, y_2\}]$ are copies of K_4 in G. Then either $|\{x_1, y_1\} \cap \{a, c, d\}| \leq 1$ or $|\{x_2, y_2\} \cap \{a, c, d\}| \leq 1$ holds since x_1, y_1, x_2, y_2 are distinct. By symmetry, we assume that $|\{x_1, y_1\} \cap \{a, c, d\}| \leq 1$. If $\{x_1, y_1\} \cap \{a, c, d\} = \emptyset$, then $G[\{b, v, x_1, y_1, a, c, d\}]$ contains a copy of $B_{4,1}$, a contradiction. If $|\{x_1, y_1\} \cap \{a, c, d\}| = 1$, without loss of generality, we assume that $x_1 = a$. Since both $G[\{a, b, x_2, v\}]$ and $G[\{a, b, c, d\}]$ are copies of K_4, it follows that $ab \in E_1$, which contradicts Claim 11. Thus, we conclude that $|V(K) \cap \{u, v\}| \neq 1$ for any copy K of K_4 in G and any $uv \in E_1$. □

Now let K be a copy of K_4 in G. Recall that E_1 is a matching in G and G_1 is the graph induced by E_1. If $|V(K) \cap V(G_1)| = 1$ or 3, then we will find an edge in E_1 attached to K, which contradicts Claim 14. Thus $|V(K) \cap V(G_1)| \in \{0, 2, 4\}$. Moreover, if $|V(K) \cap V(G_1)| = 2$, let $\{x, y\} = V(K) \cap V(G_1)$, then by Claim14 we have $xy \in E_1$. Recall that $\mathcal{K}_4(e)$ represents the set of copies of K_4 in G containing e for $e \in E(G)$. Define

$$\mathcal{K}_0(G) = \{K\colon K \text{ is a copy of } K_4 \text{ in } G \text{ and } V(K) \subset V(G_1)\};$$
$$\mathcal{K}_1(G) = \{K\colon K \text{ is a copy of } K_4 \text{ in } G \text{ and } V(K) \subset V(G_2)\};$$
$$\mathcal{K}_2(G) = \{K\colon K \in \mathcal{K}_4(e) \text{ for some } e \in E_1 \text{ and } |V(K) \cap V(G_1)| = 2\}.$$

Let $|V(G_1)| = n_1$, $|V(G_2)| = n - n_1 = n_2$. Since E_1 is a matching, it follows that n_1 is even. By Claim 14, for any $K \in \mathcal{K}_0(G)$ we have $E(K) \cap E_1$ is a matching of size 2. To derive an upper bound on $|\mathcal{K}_0(G)|$, we define a graph H

with $V(H) = E_1$ as follows. For any $e_1, e_2 \in E_1$, e_1e_2 is an edge of H if and only if there exists a copy of K_4 containing both e_1 and e_2. Since G is K_5-free, it is easy to see that H is triangle-free. Moreover, each copy of K_4 in G corresponds to an edge in H. Thus, by Mantel's Theorem [12] we have

$$|\mathcal{K}_0(G)| = e(H) \leq \left\lfloor \frac{|E_1|^2}{4} \right\rfloor = \left\lfloor \frac{n_1^2}{16} \right\rfloor.$$

We have shown that G_2 is a complete r-partite graph with $r \leq 4$ in Claim 13. If $r \leq 1$, then $\mathcal{K}_1(G) = \mathcal{K}_2(G) = \emptyset$. Thus, we have

$$\mathcal{N}(K_4, G) = |\mathcal{K}_0(G)| \leq \left\lfloor \frac{n_1^2}{16} \right\rfloor \leq \left\lfloor \frac{n^2}{16} \right\rfloor \leq \left\lfloor \frac{(n-2)^2}{4} \right\rfloor,$$

where the equalities hold if and only if $n = 4$ and G is isomorphic to K_4.

If $r = 2$, then $\mathcal{K}_1(G) = \emptyset$. If $n_1 = 0$, then we have $\mathcal{N}(K_4, G) = 0$. Hence we may assume that $n_1 \geq 2$. We claim that each edge in $E(G_2)$ is contained in at most one copy of K_4 in $\mathcal{K}_2(G)$. Otherwise, by the definition of $\mathcal{K}_2(G)$, there exists an edge $e \in E(G_2)$ contained in two distinct copies of K_4, which contradicts the fact that $e \notin E_1$. Then

$$|\mathcal{K}_2(G)| \leq e(G_2) \leq \left\lfloor \frac{n_2^2}{4} \right\rfloor.$$

Thus, we have

$$\mathcal{N}(K_4, G) = |\mathcal{K}_0(G)| + |\mathcal{K}_2(G)| \leq \left\lfloor \frac{n_1^2}{16} \right\rfloor + \left\lfloor \frac{n_2^2}{4} \right\rfloor.$$

For even integer x with $2 \leq x \leq n$, let

$$f(x) = \left\lfloor \frac{x^2}{16} \right\rfloor + \left\lfloor \frac{(n-x)^2}{4} \right\rfloor.$$

Then

$$\begin{aligned} f(x-2) &= \left\lfloor \frac{(x-2)^2}{16} \right\rfloor + \left\lfloor \frac{(n-x+2)^2}{4} \right\rfloor \\ &\geq \left\lfloor \frac{x^2}{16} \right\rfloor - \frac{x-1}{4} - 1 + \left\lfloor \frac{(n-x)^2}{4} \right\rfloor + n - x + 1 \geq f(x) + n - \frac{5x-1}{4} \end{aligned}$$

and

$$f(x-2) \leq \left\lfloor \frac{x^2}{16} \right\rfloor - \frac{x-1}{4} + 1 + \left\lfloor \frac{(n-x)^2}{4} \right\rfloor + n - x + 1 \leq f(x) + n - \frac{5x-9}{4}.$$

Thus, $f(x-2) \geq f(x)$ for $x \leq \frac{4n+1}{5}$ and $f(x-2) \leq f(x)$ for $x \geq \frac{4n+9}{5}$. Therefore, for even n we have

$$\mathcal{N}(K_4, G) \leq \max\{f(2), f(n)\} = \max\left\{\left\lfloor \frac{(n-2)^2}{4} \right\rfloor, \left\lfloor \frac{n^2}{16} \right\rfloor\right\} \leq \left\lfloor \frac{(n-2)^2}{4} \right\rfloor,$$

where the equality holds if and only if G is isomorphic to $K_2 \vee T_2(n-2)$. For odd n we have

$$\mathcal{N}(K_4, G) \leq \max\{f(2), f(n-1)\} = \max\left\{\left\lfloor \frac{(n-2)^2}{4} \right\rfloor, \left\lfloor \frac{(n-1)^2}{16} \right\rfloor\right\} \leq \left\lfloor \frac{(n-2)^2}{4} \right\rfloor,$$

where the equality holds if and only if G is isomorphic to $K_2 \vee T_2(n-2)$.

If $r = 3$, there exists a triangle xyz in G_2. Since each edge in G is contained in at least one copy of K_4, by Claim 14 there exist $ab, cd \in E_1$ such that both $G[\{x, y, a, b\}]$ and $G[\{y, z, c, d\}]$ are copies of K_4 in G. Since E_1 is a matching, we have either $\{a, b\} = \{c, d\}$ or $\{a, b\} \cap \{c, d\} = \emptyset$. If $\{a, b\} = \{c, d\}$, then $G[\{x, y, z, a, b\}]$ is a copy of K_5, a contradiction. If $\{a, b\} \cap \{c, d\} = \emptyset$, then $G[\{x, y, z, a, b, c, d\}]$ contains $B_{4,1}$, a contradiction. Thus, we conclude that $r \neq 3$.

If $r = 4$, let V_1, V_2, V_3, V_4 be four vertex classes of G_2. Since G is $B_{4,1}$-free, at least two of $|V_i|$'s equal one. Without loss of generality, we assume that $|V_3| = |V_4| = 1$. Let $V_3 = \{u\}$ and $V_4 = \{v\}$. Since $uv \notin E_1$, it follows that one of $|V_1|$ and $|V_2|$ equal one. By symmetry let $|V_2| = 1$. Then, we have

$$|\mathcal{K}_1(G)| = |V_1| = n_2 - 3.$$

Moreover, we claim that $\mathcal{K}_2(G) = \emptyset$. Otherwise, assume that there exists $K \in \mathcal{K}_2(G)$ such that $V(K) \cap V(G_2) = \{x, y\}$. Since x, y also contained in some $K' \in \mathcal{K}_1(G)$, it follows that $E(K) \cap E(K') = \{xy\}$, which contradicts the fact that $xy \notin E_1$. Since $4 \leq n_2 \leq n$, we have

$$\begin{aligned}\mathcal{N}(K_4, G) = |\mathcal{K}_0(G)| + |\mathcal{K}_1(G)| &\leq \left\lfloor \frac{n_1^2}{16} \right\rfloor + n_2 - 3 \\ &\leq \max\left\{\left\lfloor \frac{(n-4)^2}{16} \right\rfloor + 1, n-3\right\} \leq \left\lfloor \frac{(n-2)^2}{4} \right\rfloor,\end{aligned}$$

in which the equality holds if and only if $n = 4$ and $G \cong K_4$ or $n = 5$ and $G \cong K_2 \vee T_2(3)$. Thus, the lemma holds. ■

Now we are in position to prove Theorem 3.

***Proof of Theorem* 3.** For $4 \leq n \leq 6$, K_n is $B_{4,1}$-free. Then $ex(n, K_4, B_{4,1}) = \binom{n}{4}$.

Now we assume that $n \geq 7$. Let G be a $B_{4,1}$-free graph on n vertices. We will show that G can be made $\{B_{4,1}, H_1, K_5\}$-free by deleting vertices, and such an operation will not lose too many copies of K_4.

Claim 15. *There exists a subset $V_1 \subset V(G)$ such that $G_1 = G - V_1$ is K_6-free and $\mathcal{N}(K_4, G_1) \geq \mathcal{N}(K_4, G) - 2.5|V_1|$.*

Proof. Assume that G contains K_6 as a subgraph. Since G is $B_{4,1}$-free, no K_4 can intersect the K_6 in $1, 2, 3$ vertices. By deleting the 6 vertices of K_6 from G, we lose $\binom{6}{4} = 15$ copies of K_4. Repeating this process, we arrive at a K_6-free graph G_1. Let V_1 be the set of deleted vertices. Clearly, $\mathcal{N}(K_4, G_1) \geq \mathcal{N}(K_4, G) - 2.5|V_1|$. □

Claim 16. *Let H_2 be a graph on six vertices as shown in Figure* 5. *There exists a subset $V_2 \subset V(G_1)$ such that $G_2 = G_1 - V_2$ is $\{H_1, H_2\}$-free and $\mathcal{N}(K_4, G_2) \geq \mathcal{N}(K_4, G_1) - 4|V_2|$.*

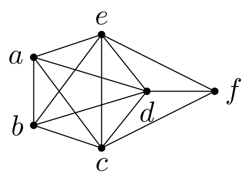

Figure 5. A graph H_2 on six vertices.

Proof. Assume that G_1 contains H_2 as a subgraph. Without loss of generality, we further assume that $A = \{a, b, c, d, e, f\}$ is a subset of $V(G_1)$ such that $G_1[A]$ contains H_2 (see Figure 5). We first claim that $V(K) \subset A$ for each copy K of K_4 containing f. Otherwise, if $|V(K) \cap A| = 1$, then K and $G_1[\{c, d, e, f\}]$ are both copies of K_4 that share exactly one vertex f, contradicting the fact that G_1 is $B_{4,1}$-free. If $|V(K) \cap A| = 2$, by symmetry we may assume that $V(K) \cap A = \{e, f\}$. Then K and $G_1[\{b, c, d, e\}]$ are both copies of K_4 that share exactly one vertex e, a contradiction. If $|V(K) \cap A| = 3$, by symmetry we assume that $V(K) \cap A = \{d, e, f\}$. Then K and $G_1[\{a, b, c, e\}]$ are both copies of K_4 that share exactly one vertex e, a contradiction. Thus, we conclude $V(K) \subset A$ for each copy K of K_4 containing f. Since G_1 is K_6-free, f has at most 4 neighbours within A. Now we delete f from G_1 to destroy a copy of H_2. By doing this, we lose at most $\binom{4}{3} = 4$ copies of K_4 since they are contained in A. We do it iteratively until the resulting graph is H_2-free. Let G_1' be the resulting graph and X_1 be the set of deleted vertices. Clearly, we have $\mathcal{N}(K_4, G_1') \geq \mathcal{N}(K_4, G_1) - 4|X_1|$.

Now G_1' is $\{B_{4,1}, H_2\}$-free. Assume that G_1' contains H_1 as a subgraph. Let $B = \{h, i, j, k, l, m, n\}$ be a subset of $V(G_1')$ such that $G_1'[B]$ contains H_1 (see Figure 3). It is easy to see that hm is not an edge in G_1'. Otherwise, $G_1'[\{h, i, j, k, m\}]$ is a copy of K_5 and $G_1'[\{h, i, j, k, m, l\}]$ contains a copy of H_2, a contradiction. Similarly, in and jl are not present in G_1'.

Now we claim that $V(K) \subset B \setminus \{m\}$ for each copy K of K_4 in G_1' containing h. Otherwise, we have one of the following cases.

- If $V(K) \cap B \subset \{h, l, n\}$, then K and $G_1'[\{h, i, j, k\}]$ form a copy of $B_{4,1}$;
- if $|V(K) \cap \{i, j, k\}| = 1$, then K and $G_1'[\{i, j, k, m\}]$ form a copy of $B_{4,1}$;
- if $V(K) \cap B = \{h, i, j\}$ or $\{h, i, k\}$, then K and $G_1'[\{j, k, m, n\}]$ form a copy of $B_{4,1}$;
- if $V(K) \cap B = \{h, j, k\}$, then K and $G_1'[\{i, k, l, m\}]$ form a copy of $B_{4,1}$.

Since G_1' is $B_{4,1}$-free, each of these cases leads to a contradiction.

By deleting h from G_1', we destroy a copy of H_1 and lose at most 4 copies of K_4. We do it iteratively until the resulting graph is H_1-free. Let G_2 be the resulting graph and X_2 be the set of deleted vertices. Clearly, we have $\mathcal{N}(K_4, G_2) \geq \mathcal{N}(K_4, G_1') - 4|X_2|$.

Let $V_2 = X_1 \cup X_2$. Clearly, G_2 is $\{H_1, H_2\}$-free and $\mathcal{N}(K_4, G_2) \geq \mathcal{N}(K_4, G_1) - 4|V_2|$. □

Claim 17. *There exists a subset $V_3 \subset V(G_2)$ such that $G_3 = G_2 - V_3$ is K_5-free and $\mathcal{N}(K_4, G_3) \geq \mathcal{N}(K_4, G_2) - 4|V_3|$.*

Proof. Since G_2 is $\{B_{4,1}, H_2\}$-free, it is easy to see that each pair of copies of K_5 in G_2 is vertex-disjoint. Let T be a copy of K_5 in G_2. We claim that $V(K) \subset V(T)$ for each copy K of K_4 in G_2 with $V(T) \cap V(K) \neq \emptyset$. Otherwise, if $|V(K) \cap V(T)| \leq 2$, then it is easy to find a copy of $B_{4,1}$ in G_2, a contradiction. If $|V(K) \cap V(T)| = 3$, then we will find a copy of H_2 in G_2, a contradiction. Thus, we conclude that $V(K) \subset V(T)$ for each copy K of K_4 in G_2 with $V(T) \cap V(K) \neq \emptyset$. By deleting a vertex $x \in V(T)$ from G_2, we lose 4 copies of K_4. Repeating this process, finally we arrive at a K_5-free graph G_3. Let V_3 be the set of deleted vertices. Clearly, we have G_3 is K_5-free and $\mathcal{N}(K_4, G_3) \geq \mathcal{N}(K_4, G_2) - 4|V_3|$. □

Let $x = |V_1|$ and $y = |V_2 \cup V_3|$. If $n - x = 4$, $\mathcal{N}(K_4, G) \leq 15\lfloor \frac{n}{6} \rfloor + 1$. And if $n - x \leq 3$, $\mathcal{N}(K_4, G) \leq 15\lfloor \frac{n}{6} \rfloor$.

For $n - x \geq 5$, we have $n - x - y \geq 4$ since in Claim 16 and Claim 17 we only delete one vertex per operation. Note that G_3 is $\{B_{4,1}, H_1, K_5\}$-free. By Lemma 10 we have

$$\mathcal{N}(K_4, G_3) \leq \left\lfloor \frac{(n - x - y - 2)^2}{4} \right\rfloor.$$

By Claims 16 and 17, we have

$$\begin{aligned} \mathcal{N}(K_4, G) &\leq \left\lfloor \frac{(n - x - y - 2)^2}{4} \right\rfloor + 2.5x + 4y = \left\lfloor \frac{(n - x - y - 2)^2}{4} + 2.5x + 4y \right\rfloor \\ &\leq \left\lfloor \frac{(n - x - y - 2)^2}{4} + 4(x + y) \right\rfloor \end{aligned}$$

Let $z = x + y$. Since $f(z) = \frac{(n-z-2)^2}{4} + 4z$ is a convex function and $0 \le z \le n-4$, it follows that

$$\mathcal{N}(K_4, G) \le \max\left\{ \left\lfloor \frac{(n-2)^2}{4} \right\rfloor, 4n-15 \right\}.$$

For $n = 7$, $15\lfloor \frac{n}{6} \rfloor = \binom{6}{4} \ge \max\left\{ \left\lfloor \frac{(n-2)^2}{4} \right\rfloor, 4n-15 \right\}$ and $\binom{[6]}{4}$ is $B_{4,1}$-free. Then $ex(7, K_4, B_{4,1}) = \binom{6}{4} = 15$.

For $8 \le n \le 16$, $4n - 15 \ge \max\left\{ \left\lfloor \frac{(n-2)^2}{4} \right\rfloor, 15\lfloor \frac{n}{6} \rfloor + 1 \right\}$. $K_4 \vee K_{n-4}^c$ is a $B_{4,1}$-free graph with $4n-15$ copies of K_4, where K_{n-4}^c is an empty graph with $n-4$ vertices. Then $ex(n, K_4, B_{4,1}) = 4n-15$ for $8 \le n \le 16$.

For $n \ge 17$, $\left\lfloor \frac{(n-2)^2}{4} \right\rfloor > \max\left\{ 4n-15, 15\lfloor \frac{n}{6} \rfloor + 1 \right\}$. $K_2 \vee T_2(n-2)$ is a $B_{4,1}$-free graph with $\left\lfloor \frac{(n-2)^2}{4} \right\rfloor$ copies of K_4. Then $ex(n, K_4, B_{4,1}) = \left\lfloor \frac{(n-2)^2}{4} \right\rfloor$ for $n \ge 17$. Moreover, by Lemma 10, the equality holds if and only if G is isomorphic to $K_2 \vee T_2(n-2)$. Thus, the theorem holds. ■

By a similar argument, we can determine $ex(n, K_3, B_{3,0})$.

***Proof of Theorem* 5.** For $n \le 5$, K_n is $B_{3,0}$-free. Then $ex(n, K_3, B_{3,0}) = \binom{n}{3}$ for $3 \le n \le 5$.

Let G be a $B_{3,0}$-free graph on vertex set $[n]$. If G contains K_5 as a subgraph, let A be a subset of $V(G)$ such that $G[A]$ is a copy of K_5. Since G is $B_{3,0}$-free, every copy of K_3 is included in $G[A]$. Thus $\mathcal{N}(K_3, G) = \binom{5}{3} = 10 \le \min\left\{ 3n-8, \left\lfloor \frac{(n-1)^2}{4} \right\rfloor \right\}$ for $n \ge 6$.

Now we assume that G is K_5-free and $n \ge 6$.

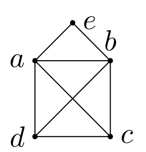

Figure 6. A graph H_3 on five vertices.

Claim 18. *There exists a subset $V' \subset V(G)$ such that $G' = G - V'$ is $\{B_{3,0}, K_4\}$-free and $\mathcal{N}(K_3, G') \ge \mathcal{N}(K_3, G) - 3|V'|$.*

Proof. Let H_3 be a graph on five vertices as shown in Figure 6. If G contains H_3 as subgraph, let $A = \{a, b, c, d, e\} \subset V(G)$ and $G[A]$ contains a copy of H_3. Since G is K_5-free, $V(K) \subset A$ for each copy K of K_3 containing e and e has at most 3 neighbours in $\{a, b, c, d\}$. So the number of copies of K_3 containing e is at most 3. Delete the vertex e from G and we lose at most 3 copies of K_3. We do it iteratively until the resulting graph $\tilde{G}$ is H_3-free.

If $\tilde{G}$ contains K_4 as subgraph, let $B = \{v_1, v_2, v_3, v_4\} \subset V(\tilde{G})$ and $\tilde{G}[B]$ is a copy of K_4. Since $\tilde{G}$ is $\{H_3, B_{3,0}\}$-free, $V(K) \subset B$ for each copy K of K_3 with $V(K) \cap V(B) \neq \emptyset$. Now we delete the vertex v_1 from $\tilde{G}$ and we lose 3 copies of K_3. Repeating this process, we arrive at a K_4-free graph G'.

Let V' be the set of vertices removed in the above two steps. Clearly, $\mathcal{N}(K_3, G') \geq \mathcal{N}(K_3, G) - 3|V'|$. □

Let $|V(G')| = n'$. Then $n' \geq 3$ by Claim 18.

Claim 19. *For $n' \geq 3$, $\mathcal{N}(K_3, G') \leq \left\lfloor \frac{(n'-1)^2}{4} \right\rfloor$.*

Proof. Let v be a vertex in G' with the maximal degree and $N \subset V(G')$ be the neighborhood of v. Since G' is K_4-free, $G'[N]$ is K_3-free.

If $|N| \leq 3$, $d(x) \leq 3$ for any $x \in V(G')$. For every $x \in V(G')$, the number of copies of K_3 containing x is at most 2. Thus $\mathcal{N}(K_3, G') \leq \left\lfloor \frac{2n'}{3} \right\rfloor \leq \left\lfloor \frac{(n'-1)^2}{4} \right\rfloor$ for $n' \geq 4$. For $n' = 3$, $\mathcal{N}(K_3, G') \leq 1 \leq \left\lfloor \frac{(n'-1)^2}{4} \right\rfloor$. So we assume that $|N| \geq 4$.

If there are three pairwise disjoint edges in $G'[N]$, every copy of K_3 in G' contains v. Thus $\mathcal{N}(K_3, G') = \left\lfloor \frac{|N|^2}{4} \right\rfloor \leq \left\lfloor \frac{(n-1)^2}{4} \right\rfloor$.

If the matching number of $G'[N]$ is 2, let v_1u_1 and v_2u_2 be two disjoint edges in $G'[N]$. Every edge in $G'[N]$ intersects $\{v_1, v_2, u_1, u_2\}$. Since $G'[N]$ is K_3-free, there are at most $|N| - 4$ edges in $\{e \in E(G') : |e \cap \{v_i, u_i\}| = 1, |e \cap (N \setminus \{v_1, v_2, u_1, u_2\})| = 1\}$, $i = 1, 2$. Moreover there are at most 4 edges in $G'[\{v_1, v_2, u_1, u_2\}]$. Thus the number of edges in $G'[N]$ is at most $2(|N|-4)+4 = 2(|N| - 2)$. For each copy K of K_3 in G' with $v \notin V(K)$, $N[K] \cap \{v_1, u_1\} \neq \emptyset$, $N[K] \cap \{v_2, u_2\} \neq \emptyset$ and K contains a vertex $u \in V(G') \setminus N \setminus \{v\}$. Since $G'[\{v, v_2, u_2\}]$ is a copy of K_3, u has at most one neighbor among v_1 and u_1. Analogously u has at most one neighbor among v_2 and u_2. Then for each $u \in V(G') \setminus N \setminus \{v\}$, there is at most one triangle containing u and the number of copies of K_3 that does not contain v is at most $n' - |N| - 1$. Thus,

$$\mathcal{N}(K_3, G') \leq 2(|N| - 2) + (n' - |N| - 1) = n' + |N| - 5 \leq \left\lfloor \frac{(n'-1)^2}{4} \right\rfloor.$$

If the matching number of $G'[N]$ is 1, $G'[N]$ is a star since $G'[N]$ is K_3-free. Let u be the center of $G'[N]$. Since $|N| \geq 4$, if K is a copy of K_3 that does not contain v, then $u \in V(K)$. Note that $d(v) \geq d(u)$. The neighborhood of u is $N \setminus \{u\} \cup \{v\}$ and there are no edges in $G'[N \setminus \{u\}]$. Then every copy of K_3 contains v. Thus $\mathcal{N}(K_3, G') \leq |N - 1| \leq n' - 2 \leq \left\lfloor \frac{(n'-1)^2}{4} \right\rfloor$. □

Let $|V'| = x$. Combining Claim 18 and Claim 19, we have

$$\mathcal{N}(K_3, G) \leq 3x + \left\lfloor \frac{(n-x-1)^2}{4} \right\rfloor = \left\lfloor 3x + \frac{(n-x-1)^2}{4} \right\rfloor.$$

Since $f(x) = 3x + \frac{(n-x-1)^2}{4}$ is a convex function and $0 \le x \le n-3$,

$$\mathcal{N}(K_3, G) \le \max\left\{\left\lfloor \frac{(n-1)^2}{4} \right\rfloor, 3n-8\right\}.$$

When $6 \le n \le 10$, $3n - 8 \ge \left\lfloor \frac{(n-1)^2}{4} \right\rfloor$. When $n \ge 11$, $3n-8 \le \left\lfloor \frac{(n-1)^2}{4} \right\rfloor$.

Moreover, $K_1 \vee T_{r-1}(n-1)$ is a $B_{3,0}$-free graph with $\left\lfloor \frac{(n-1)^2}{4} \right\rfloor$ copies of K_3, and $K_3 \vee K_{n-3}^c$ is a $B_{3,0}$-free graph with $3n-8$ copies of K_3, where K_{n-3}^c is an empty graph with $n-3$ vertices. Thus $ex(n, K_3, B_{3,0}) = 3n-8$ for $6 \le n \le 10$; and $ex(n, K_3, B_{3,0}) = \left\lfloor \frac{(n-1)^2}{4} \right\rfloor$ for $n \ge 11$. ∎

3. THE VALUES OF $ex(n, K_r, B_{r,1})$ AND $ex(n, K_r, B_{r,0})$

By using Füredi's structure theorem, Frankl and Füredi [4] determined the maximum number of hyperedges in an r-uniform hypergraph without two hyperedges sharing exactly s vertices for $r \ge 2s+2$. In this section, we determine $ex(n, K_r, B_{r,1})$ and $ex(n, K_r, B_{r,0})$ by following a similar approach.

First, we recall a result due to Frankl and Füredi in the intersection closed family (Lemma 5.5 in [4]). Let X be a finite set and 2^X be the family of all the subsets of X. We say that $\mathcal{I} \subset 2^X$ is *intersection closed* if for any $I, I' \in \mathcal{I}$, $I \cap I' \in \mathcal{I}$. We say $I \subset X$ is *covered* by $\mathcal{I}$ if there exists an $I' \in \mathcal{I}$ such that $I \subseteq I'$.

Theorem 20 (Frankl and Füredi [4]). *Let r and s be positive integers with $r \ge 2s+3$ and let F be an r-element set. Suppose that $\mathcal{I} \subset 2^F \setminus \{F\}$ is an intersection closed family such that $|I| \ne s$ for any $I \in \mathcal{I}$ and all the $(r-s-2)$-element subsets of F are covered by $\mathcal{I}$. Then there exists an $(s+1)$-element subset $A(F)$ of F such that*

$$\{I : A(F) \subset I \subsetneq F\} \subset \mathcal{I}.$$

We use $[n]$ to denote the set $\{1, \ldots, n\}$ and use $\binom{[n]}{r}$ to denote the collection of all r-element subsets of $[n]$. Let $\mathcal{F} \subset \binom{[n]}{r}$ be a hypergraph. We call $\mathcal{F}$ r-partite if there exists a partition $[n] = X_1 \cup \cdots \cup X_r$ such that $|F \cap X_i| = 1$ for all $F \in \mathcal{F}$ and $i \in \{1, 2, \ldots, r\}$.

We adopt the statement of Füredi's structure theorem given by Frankl and Tokushige in [5]. For clarity purpose, we recall some definitions from [5]. Let $\mathcal{F} \subset \binom{[n]}{r}$ be an r-partite hypergraph with partition $[n] = X_1 \cup \cdots \cup X_r$. For any $F \in \mathcal{F}$, define the *restriction* of $\mathcal{F}$ on F by

$$\mathcal{I}(F, \mathcal{F}) = \{F' \cap F : F' \in \mathcal{F} \setminus \{F\}\}.$$

A set of p hyperedges $F_1, \dots, F_p$ in $\mathcal{F}$ is called a *p-sunflower* if $F_i \cap F_j = C$ for every $1 \leq i < j \leq p$ and some set C. The set C is called *center* of the p-sunflower.

Füredi [7] proved the following fundamental result, which was conjectured by Frankl. It roughly says that every r-uniform hypergraph $\mathcal{F}$ contains a large r-partite subhypergraph $\mathcal{F}^*$ satisfying that $\mathcal{I}(F, \mathcal{F}^*)$ is isomorphic to $\mathcal{I}(F', \mathcal{F}^*)$ for any $F, F' \in \mathcal{F}^*$.

Theorem 21 (Füredi [7])**.** *For positive integers r and p, there exists a positive constant $c = c(r, p)$ such that every hypergraph $\mathcal{F} \subset \binom{[n]}{r}$ contains an r-partite subhypergraph $\mathcal{F}^*$ with partition $[n] = X_1 \cup \cdots \cup X_r$ satisfying* (i)–(iv).

(i) $|\mathcal{F}^*| \geq c|\mathcal{F}|$.

(ii) *For any $F_1, F_2 \in \mathcal{F}^*$, $\mathcal{I}(F_1, \mathcal{F}^*)$ is isomorphic to $\mathcal{I}(F_2, \mathcal{F}^*)$.*

(iii) *For $F \in \mathcal{F}^*$, $\mathcal{I}(F, \mathcal{F}^*)$ is intersection closed.*

(iv) *For $F \in \mathcal{F}^*$ and every $I \in \mathcal{I}(F, \mathcal{F}^*)$, I is the center of a p-sunflower in $\mathcal{F}^*$.*

We need the following two results. The first one is due to Deza, Erdős and Frankl [2].

Lemma 22 (Deza, Erdős and Frankl [2])**.** *Suppose that $\{E_1, \dots, E_{r+1}\}$ and $\{F_1, \dots, F_{r+1}\}$ are both $(r+1)$-sunflowers in r-uniform hypergraphs with centers C_1 and C_2, respectively. Then there exist i and j such that $E_i \cap F_j = C_1 \cap C_2$.*

The second one is due to Zykov [18]. He showed that the Turán graph maximizes the number of s-cliques in n-vertex K_{t+1}-free graphs for $s \leq t$.

Theorem 23 (Zykov [18])**.** *For $s \leq t$,*

$$ex(n, K_s, K_{t+1}) = \mathcal{N}(K_s, T_t(n)),$$

and $T_t(n)$ is the unique graph attaining the maximum number of copies of K_s.

Let $\mathcal{F} \subset \binom{[n]}{r}$ be a hypergraph and $x \in [n]$. Define

$$N_{\mathcal{F}}(x) = \left\{ T \in \binom{[n] \setminus \{x\}}{r-1} : T \cup \{x\} \in \mathcal{F} \right\}.$$

The degree of x in $\mathcal{F}$, denoted by $\deg_{\mathcal{F}}(x)$, is the cardinality of $N_{\mathcal{F}}(x)$.

Now we are ready to prove Theorem 4.

***Proof of Theorem* 4.** Let G be a $B_{r,1}$-free graph on $[n]$ with the maximum number of copies of K_r. Since $K_2 \vee T_{r-2}(n-2)$ is $B_{r,1}$-free, we may assume that $\mathcal{N}(K_r, G) \geq \mathcal{N}(K_{r-2}, T_{r-2}(n-2))$.

Let

$$\mathcal{F} = \left\{ F \in \binom{[n]}{r} : G[F] \text{ is a clique} \right\}.$$

Clearly, $|F_1 \cap F_2| \neq 1$ for any $F_1, F_2 \in \mathcal{F}$ since G is $B_{r,1}$-free. Now we apply Theorem 21 with $p = r+1$ to $\mathcal{F}$ and obtain $\mathcal{F}_1 = \mathcal{F}^*$ satisfying (i)–(iv). Then apply Theorem 21 to $\mathcal{F} - \mathcal{F}_1$ to obtain $\mathcal{F}_2 = (\mathcal{F} - \mathcal{F}_1)^*$, in the i-th step we obtain $\mathcal{F}_i = (\mathcal{F} - (\mathcal{F}_1 \cup \cdots \cup \mathcal{F}_{i-1}))^*$. We stop if there is an $F_0 \in \mathcal{F}_i$ and an $(r-3)$-element subset B_0 of F_0 such that B_0 is not covered by $\mathcal{I}(F_0, \mathcal{F}_i)$. Suppose that the procedure stops in the m-th step. By Theorem 21(ii), for every $F \in \mathcal{F}_m$ there is an $(r-3)$-element subset B of F such that B is not covered by $\mathcal{I}(F, \mathcal{F}_m)$.

Claim 24. *$|\mathcal{F} - (\mathcal{F}_1 \cup \cdots \cup \mathcal{F}_{m-1})| \leq c'\binom{n}{r-3}$ for some $c' > 0$.*

Proof. For any $F \in \mathcal{F}_m$, let B be an $(r-3)$-element subset of F that is not covered by $\mathcal{I}(F, \mathcal{F}_m)$. Then it follows that $B \not\subseteq E \cap F$ for any $E \in \mathcal{F}_m \setminus \{F\}$, that is, F is the only hyperedge in $\mathcal{F}_m$ that contains B. Thus $|\mathcal{F}_m| \leq \binom{n}{r-3}$. Now by Theorem 21(i),

$$|\mathcal{F} - (\mathcal{F}_1 \cup \cdots \cup \mathcal{F}_{m-1})| \leq c^{-1}|\mathcal{F}_m| \leq c'\binom{n}{r-3}.$$

□

Let $i \in \{1, 2, \ldots, m-1\}$ and $F \in \mathcal{F}_i$. By Theorem 21(iii), $\mathcal{I}(F, \mathcal{F}_i)$ is intersection closed. Since $|F_1 \cap F_2| \neq 1$ for any $F_1, F_2 \in \mathcal{F}_i$, $|I| \neq 1$ for each $I \in \mathcal{I}(F, \mathcal{F}_i)$. Now apply Theorem 20 with $s = 1$ to $\mathcal{I}(F, \mathcal{F}_i)$, we obtain a 2-element subset $A(F)$ of F such that

$$\{I : A(F) \subset I \subsetneq F\} \subset \mathcal{I}(F, \mathcal{F}_i).$$

Let $A_1, A_2, \ldots, A_h$ be the list of 2-element sets for which $A_j = A(F)$ for some $F \in \mathcal{F}_1 \cup \cdots \cup \mathcal{F}_{m-1}$. For $j = 1, \ldots, h$, let

$$\mathcal{H}_j = \{F \in \mathcal{F}_1 \cup \cdots \cup \mathcal{F}_{m-1} : A(F) = A_j\}$$

and

$$V(\mathcal{H}_j) = \bigcup_{F \in \mathcal{H}_j} F.$$

Claim 25. *$V(\mathcal{H}_1), \ldots, V(\mathcal{H}_h)$ are pairwise disjoint.*

Proof. Suppose for contradiction that $|V(\mathcal{H}_1) \cap V(\mathcal{H}_2)| \geq 1$. It follows that there exist $F_1 \in \mathcal{H}_1$ and $F_2 \in \mathcal{H}_2$ such that $|F_1 \cap F_2| \geq 1$. Then we can find two sets C_1 and C_2 satisfying $A_1 \subset C_1 \subsetneq F_1$, $A_2 \subset C_2 \subsetneq F_2$ and $|C_1 \cap C_2| = 1$ in the following way. If $|A_1 \cap A_2| = 1$, then let $C_1 = A_1$ and $C_2 = A_2$. If $A_1 \cap A_2 = \emptyset$, then let $C_1 = A_1 \cup \{x\}$ and $C_2 = A_2 \cup \{x\}$ for some $x \in F_1 \cap F_2$.

Since $F_1 \in \mathcal{F}_i$ for some $i \in \{1, \ldots, m-1\}$ and

$$C_1 \in \{I : A_1 \subset I \subsetneq F_1\} \subset \mathcal{I}(F_1, \mathcal{F}_i),$$

by Theorem 21(iv) C_1 is the center of an $(r+1)$-sunflower in $\mathcal{F}_i$. Therefore C_1 is the center of an $(r+1)$-sunflower in $\mathcal{F}$. Similarly, C_2 is also the center of an $(r+1)$-sunflower in $\mathcal{F}$. By Lemma 22, there exist $F_1', F_2' \in \mathcal{F}$ satisfying $|F_1' \cap F_2'| = |C_1 \cap C_2| = 1$, which contradicts the fact that $|F_1 \cap F_2| \neq 1$ for any $F_1, F_2 \in \mathcal{F}$. Thus the claim holds. $\square$

Assume that $A_i = \{u_i, v_i\}$ for $i = 1, \ldots, h$. Let G_i be the graph on the vertex set $V(\mathcal{H}_i)$ with the edge set

$$E(G_i) = \{uv \colon \{u, v\} \subset F \in \mathcal{H}_i\}.$$

Obviously, G_i is a subgraph of G and $vu_i, vv_i, u_iv_i \in E(G_i)$ for each $v \in V(\mathcal{H}_i) \setminus A_i$.

Claim 26. *$G_i - A_i$ is K_{r-1}-free for $i = 1, \ldots, h$.*

Proof. By symmetry, we only need to show that $G_1 - A_1$ is K_{r-1}-free. Suppose for contradiction that $\{a_1, a_2, \ldots, a_{r-1}\} \subset V(G_1) \setminus \{u_1, v_1\}$ induces a copy of K_{r-1} in $G_1 - A_1$. Since $u_1a_j \in E(G_1)$ for each $j = 1, \ldots, r-1$, $\{u_1, a_1, a_2, \ldots, a_{r-1}\}$ induces a copy of K_r in G. Note that $A_1 = \{u_1, v_1\}$ is the center of an $(r+1)$-sunflower in $\mathcal{F}$. Let $F_1, F_2, \ldots, F_{r+1}$ be such a sunflower with center A_1. Then there exists some F_j with $(F_j \setminus A_1) \cap \{a_1, a_2, \ldots, a_{r-1}\} = \emptyset$. It follows that $F_j \cap \{u_1, a_1, a_2, \ldots, a_{r-1}\} = \{u_1\}$. By the definition of $\mathcal{F}$, the subgraph of G induced by $F_j \cup \{u_1, a_1, a_2, \ldots, a_{r-1}\}$ contains $B_{r,1}$. This contradicts the fact that G is $B_{r,1}$-free and the claim follows. $\square$

Let $x_i = |V(\mathcal{H}_i)|$ for $i = 1, 2, \ldots, h$ and assume that $x_1 \geq x_2 \geq \cdots \geq x_h$. By Claim 25, $x_1 + \cdots + x_h \leq n$.

Claim 27. *$x_1 \geq n - c''$, for some constant $c'' > 0$.*

Proof. By Claim 26 and Theorem 23, the number of copies of K_{r-2} in $G_i - A_i$ is at most $\mathcal{N}(K_{r-2}, T_{r-2}(x_i - 2))$. It follows that

$$|\mathcal{H}_i| \leq \mathcal{N}(K_{r-2}, T_{r-2}(x_i - 2))$$

for each $i = 1, \ldots, h$. By Claims 24 and 25,

$$\begin{aligned} \mathcal{N}(K_r, G) &= |\mathcal{F} - (\mathcal{F}_1 \cup \cdots \cup \mathcal{F}_{m-1})| + |(\mathcal{F}_1 \cup \cdots \cup \mathcal{F}_{m-1})| \\ &= |\mathcal{F} - (\mathcal{F}_1 \cup \cdots \cup \mathcal{F}_{m-1})| + |\mathcal{H}_1| + \cdots + |\mathcal{H}_h| \qquad (1) \\ &\leq c' \binom{n}{r-3} + \sum_{i=1}^{h} \mathcal{N}(K_{r-2}, T_{r-2}(x_i - 2)). \end{aligned}$$

Since

$$\mathcal{N}(K_{r-2}, T_{r-2}(x_i-2)) \le \left(\frac{x_i-2}{r-2}\right)^{r-2},$$

we have

$$\begin{aligned}\mathcal{N}(K_r, G) &\le c'\binom{n}{r-3} + \sum_{i=1}^{h}\left(\frac{x_i-2}{r-2}\right)^{r-2}\\ &\le c'\binom{n}{r-3} + \sum_{i=1}^{h}(x_i-2)\cdot\frac{(x_1-2)^{r-3}}{(r-2)^{r-2}} \qquad (2)\\ &\le c'\binom{n}{r-3} + \frac{(x_1-2)^{r-3}(n-2)}{(r-2)^{r-2}}.\end{aligned}$$

By our assumption,

$$\mathcal{N}(K_r, G) \ge \mathcal{N}(K_{r-2}, T_{r-2}(n-2)) \ge \left(\frac{n-r}{r-2}\right)^{r-2}. \qquad (3)$$

Combining (2) and (3), we obtain that

$$1 \le c'\binom{n}{r-3}\left(\frac{r-2}{n-r}\right)^{r-2} + \frac{n-2}{n-r}\cdot\left(\frac{x_1-2}{n-r}\right)^{r-3}.$$

Since n is sufficiently large, we get $x_1 \ge (1-o(1))n$.

Let n_1, n be two integers with $0 < n_1 < n$ and let H be an r-partite Turán graph on n vertices with vertex classes $V_1, V_2, \ldots, V_r$. Then there exist partitions $V_j = V_{j,1} \cup V_{j,2}$ for each $j = 1, 2, \ldots, r$ such that

$$\sum_{j=1}^{r}|V_{j,1}| = n_1$$

and both $H\left[\bigcup_{j=1}^{r} V_{j,1}\right]$ and $H\left[\bigcup_{j=1}^{r} V_{j,2}\right]$ are Turán graphs. There are $\mathcal{N}(K_r, T_r(n_1))$ copies of K_r in $H\left[\bigcup_{j=1}^{r} V_{j,1}\right]$, and $\mathcal{N}(K_r, T_r(n-n_1))$ copies of K_r in $H\left[\bigcup_{j=1}^{r} V_{j,2}\right]$. Moreover, the number of copies of K_r in H with $\left|V(K)\cap\left(\bigcup_{j=1}^{r} V_{j,1}\right)\right| = r-1$ and $\left|V(K)\cap\left(\bigcup_{j=1}^{r} V_{j,2}\right)\right| = 1$ is at most $\left\lfloor\frac{n-n_1}{r}\right\rfloor\cdot\mathcal{N}(K_{r-1}, T_r(n_1))$. Thus,

$$\begin{aligned}\mathcal{N}(K_r, T_r(n)) \;>\; &\mathcal{N}(K_r, T_r(n_1)) + \mathcal{N}(K_r, T_r(n-n_1))\\ &+\left\lfloor\frac{n-n_1}{r}\right\rfloor\cdot\mathcal{N}(K_{r-1}, T_r(n_1)). \qquad (4)\end{aligned}$$

Apply the inequality (4) inductively, we have

$$\sum_{i=2}^{h} \mathcal{N}(K_{r-2}, T_{r-2}(x_i - 2)) < \mathcal{N}(K_{r-2}, T_{r-2}(n - x_1)). \tag{5}$$

By (1) and (5), we see that

$$\mathcal{N}(K_r, G) \leq c'\binom{n}{r-3} + \mathcal{N}(K_{r-2}, T_{r-2}(x_1 - 2)) + \mathcal{N}(K_{r-2}, T_{r-2}(n - x_1)).$$

Apply the inequality (4) again, we obtain that

$$\begin{aligned} &\mathcal{N}(K_r, G) \\ &\leq c'\binom{n}{r-3} + \mathcal{N}(K_{r-2}, T_{r-2}(n-2)) - \left\lfloor \frac{n-x_1+2}{r} \right\rfloor \cdot \mathcal{N}(K_{r-3}, T_{r-2}(x_1-2)) \\ &\leq \mathcal{N}(K_{r-2}, T_{r-2}(n-2)) + c'\binom{n}{r-3} - \frac{n-x_1-r}{r} \cdot (r-2)\left(\frac{x_1-r}{r-2}\right)^{r-3}. \end{aligned} \tag{6}$$

It follows from (3) and (6) that

$$c'\binom{n}{r-3} \geq \frac{n-x_1-r}{r} \cdot (r-2)\left(\frac{x_1-r}{r-2}\right)^{r-3}.$$

Since $x_1 \geq (1-o(1))n$, we arrive at

$$c'\binom{n}{r-3} \geq \frac{n-x_1-r}{r} \cdot (r-2)\left(\frac{n-o(n)-r}{r-2}\right)^{r-3}.$$

It follows that $x_1 \geq n - c''$ for some $c'' > 0$. □

Let us define

$$\mathcal{K} = \left\{ F \in \mathcal{F} : \begin{array}{l} A_1 \subset F \text{ and for each } I \text{ with } A_1 \subset I \subsetneq F, \\ I \text{ is the center of an } (r+1)\text{-sunflower in } \mathcal{F} \end{array} \right\}.$$

Obviously, we have $\mathcal{H}_1 \subset \mathcal{K}$. Define

$$\mathcal{A} = \{F \in \mathcal{F} : A_1 \subset F, F \notin \mathcal{K}\} \text{ and } \mathcal{B} = \mathcal{F} - \mathcal{K} - \mathcal{A}.$$

Note that $V(\mathcal{K}) = \bigcup_{F\in\mathcal{K}} F$ and $V(\mathcal{B}) = \bigcup_{F\in\mathcal{B}} F$. We claim that $V(\mathcal{K}) \cap V(\mathcal{B}) = \emptyset$. Otherwise, there exist $F_1 \in \mathcal{K}$ and $F_2 \in \mathcal{B}$ with $|F_1 \cap F_2| \geq 1$. Note that $A_1 \subset F_1$ and $A_1 \not\subset F_2$. If $F_2 \cap A_1 = \emptyset$, let $C = A_1 \cup \{x\}$ with $x \in F_1 \cap F_2$. If $F_2 \cap A_1 \neq \emptyset$, then let $C = A_1$. It is easy to see that $|C \cap F_2| = 1$ in both of the two cases. Clearly, we have $A_1 \subset C \subsetneq F_1$. By the definition of $\mathcal{K}$, C is center of an $(r+1)$-sunflower in $\mathcal{F}$. Let $E_1, E_2, \ldots, E_{r+1}$ be such a sunflower. Since $|F_2 \setminus C| < r$, there exists some E_j such that $(E_j \setminus C) \cap (F_2 \setminus C) = \emptyset$. Then we have $|E_j \cap F_2| = |C \cap F_2| = 1$, a contradiction. Thus $V(\mathcal{K}) \cap V(\mathcal{B}) = \emptyset$.

By Claim 27, we have

$$|V(\mathcal{B})| \leq n - V(\mathcal{K}) \leq n - V(\mathcal{H}_1) \leq c''. \tag{7}$$

Let $\mathcal{C} = \{F \in \mathcal{A} \colon F \cap V(\mathcal{B}) = \emptyset\}$, $\mathcal{K}' = \mathcal{K} \cup \mathcal{C}$ and $\mathcal{A}' = \mathcal{A} \setminus \mathcal{C}$. Clearly, $V(\mathcal{K}') \cap V(\mathcal{B}) = \emptyset$, $F \cap V(\mathcal{K}') \supset A_1$ and $F \cap V(\mathcal{B}) \neq \emptyset$ for each $F \in \mathcal{A}'$.

Claim 28. $\mathcal{B} = \emptyset$.

Proof. Suppose for contradiction that there exists $B \in \mathcal{B}$. We first show that the degree of each vertex x in B is small. By (7), we have

$$\deg_{\mathcal{B}}(x) \leq \binom{|V(\mathcal{B})|}{r-1} \leq \binom{c''}{r-1}.$$

Note that $A_1 \subset F$ for any $F \in \mathcal{F} \setminus \mathcal{B}$ and $|F \cap F'| \neq 1$ for any $F, F' \in \mathcal{F}$. We have $A_1 \subset B'$ and $|B' \cap B| \geq 2$ for any $B' \in \mathcal{F} \setminus \mathcal{B}$ with $x \in B'$. Thus, the number of hyperedges containing x in $\mathcal{F} \setminus \mathcal{B}$ is at most $|B \setminus \{x\}| \cdot \binom{n}{r-4} = (r-1)\binom{n}{r-4}$. Therefore,

$$\deg_{\mathcal{F}}(x) \leq \deg_{\mathcal{B}}(x) + (r-1)\binom{n}{r-4} \leq \binom{c''}{r-1} + (r-1)\binom{n}{r-4}.$$

Let $u \in V(\mathcal{K}') \setminus A_1$ be the vertex with

$$\deg_{\mathcal{K}'}(u) = \max\left\{\deg_{\mathcal{K}'}(v) \colon v \in V(\mathcal{K}') \setminus A_1\right\}.$$

We show that $\deg_{\mathcal{K}'}(u) \geq c''' n^{r-3}$ for some constant $c''' > 0$. Since $F \cap V(\mathcal{B}) \neq \emptyset$ for each $F \in \mathcal{A}'$, we have

$$|\mathcal{A}'| + |\mathcal{B}| \leq \sum_{v \in V(\mathcal{B})} \deg_{\mathcal{F}}(v).$$

If $\deg_{\mathcal{K}'}(u) = o(n^{r-3})$, then

$$\begin{aligned}\mathcal{N}(K_r, G) = |\mathcal{K}'| + |\mathcal{A}'| + |\mathcal{B}| &\leq \frac{1}{r-2} \sum_{v \in V(\mathcal{K}') \setminus A_1} \deg_{\mathcal{K}'}(v) + \sum_{v \in V(\mathcal{B})} \deg_{\mathcal{F}}(v) \\ &\leq o(n^{r-2}) + c''\left((r-1)\binom{n}{r-4} + \binom{c''}{r-1}\right),\end{aligned}$$

which contradicts the assumption that $\mathcal{N}(K_r, G) \geq \mathcal{N}(K_{r-2}, T_{r-2}(n-2))$. Thus $\deg_{\mathcal{K}'}(u) \geq c''' n^{r-3}$ for some constant $c''' > 0$.

Since n is sufficiently large, for each $x \in B$ we have

$$\deg_{\mathcal{F}}(u) \geq \deg_{\mathcal{K}'}(u) \geq c''' n^{r-3} > \deg_{\mathcal{F}}(x).$$

We claim that there exists $x_0 \in B$ such that ux_0 is not an edge of G. Otherwise, if $ux \in E(G)$ for all $x \in B$, then $\{u\} \cup T$ induces a copy of K_r in G for any $T \in \binom{B}{r-1}$. Since $\deg_{\mathcal{K}'}(u) \geq c''' n^{r-3}$, there exists an hyperedge K in $\mathcal{K}'$ containing u. Recall that $V(\mathcal{K}') \cap V(\mathcal{B}) = \emptyset$. Then $\{u\} \cup T \cup K$ induces a copy of $B_{r,1}$ in G, a contradiction. Thus, there exists $x_0 \in B$ such that ux_0 is not an edge of G.

Now let G' be a graph obtained from G by deleting edges incident to x_0 and adding edges in $\{x_0 w \colon w \in N(u)\}$. We claim that G' is $B_{r,1}$-free. Otherwise, there exist two copies K, K' of K_r in G' with $V(K) \cap V(K') = \{y\}$ for some $y \in V(G')$. Since G is $B_{r,1}$-free, we may assume that $x_0 \in V(K)$. If $u \notin V(K')$, then $V(K) \cup V(K') \setminus \{x_0\} \cup \{u\}$ induces a copy of $B_{r,1}$ in G, a contradiction. If $u \in V(K')$, then $y \neq x_0$ since $x_0 y$ is not an edge in G'. Moreover, $V(K') \notin \mathcal{B}$ and $V(K) \setminus \{x_0\} \cup \{u\} \notin \mathcal{B}$ since $u \in V(\mathcal{K}')$. By the definition of $\mathcal{K}'$ and $\mathcal{A}'$, we see that both $V(K')$ and $V(K) \setminus \{x_0\} \cup \{u\}$ contains A_1. But now we have $V(K) \cap V(K') \supset A_1$ since $u, x_0 \notin A_1$, which contradicts our assumption that $V(K) \cap V(K') = \{y\}$. Thus G' is $B_{r,1}$-free.

Since $\deg_{\mathcal{F}}(u) > \deg_{\mathcal{F}}(x_0)$, we have

$$\mathcal{N}(K_r, G') = \mathcal{N}(K_r, G) - \deg_{\mathcal{F}}(x_0) + \deg_{\mathcal{F}}(u) > \mathcal{N}(K_r, G),$$

which contradicts the maximality of the number of copies of K_r in G. Thus, the claim follows. □

By Claim 28, A_1 is contained in every hyperedge of $\mathcal{F}$. Recall that $A_1 = \{u_1, v_1\}$. It follows that $xu_1, xv_1 \in E(G)$ for any $x \in V(G) \setminus A_1$. We claim that $G \setminus A_1$ is K_{r-1}-free. Otherwise, let $\{a_1, a_2, \ldots, a_{r-1}\} \subset V(G) \setminus A_1$ be a set that induces a copy of K_{r-1} in $G - A_1$. Since $u_1 a_j \in E(G)$ for each $j = 1, \ldots, r-1$, $\{u_1, a_1, a_2, \ldots, a_{r-1}\}$ induces a copy of K_r in G. Note that A_1 is the center of an $(r+1)$-sunflower in $\mathcal{F}$. Let $F_1, F_2, \ldots, F_{r+1}$ be such a sunflower with center A_1. Then there exists some F_j with $(F_j \setminus A_1) \cap \{a_1, a_2, \ldots, a_{r-1}\} = \emptyset$. It follows that $F_j \cap \{u_1, a_1, a_2, \ldots, a_{r-1}\} = \{u_1\}$. By the definition of $\mathcal{F}$, the subgraph of G induced by $F_j \cup \{u_1, a_1, a_2, \ldots, a_{r-1}\}$ contains $B_{r,1}$, a contradiction. Thus $G - A_1$ is K_{r-1}-free.

By Theorem 23, there are at most $\mathcal{N}(K_{r-2}, T_{r-2}(n-2))$ copies of K_{r-2} in $G - A_1$ and Turán graph $T_{r-2}(n-2)$ is the unique graph attaining the maximum number. Thus, the number of K_r in G is at most $\mathcal{N}(K_{r-2}, T_{r-2}(n-2))$ and $K_2 \vee T_{r-2}(n-2)$ is the unique graph attaining the maximum number of copies of K_r. ■

Now we prove Theorem 6 using Füredi's structure theorem.

***Proof of Theorem* 6.** Let G be a $B_{r,0}$-free graph on vertex set $[n]$ and let

$$\mathcal{F} = \left\{F \in \binom{[n]}{r} : G[F] \text{ is a clique}\right\}.$$

Since G is $B_{r,0}$-free, $\mathcal{F}$ is an intersecting family. We apply Theorem 21 with $p = r + 1$ to $\mathcal{F}$ and obtain $\mathcal{F}^*$. Let $\mathcal{I} = \mathcal{I}(F, \mathcal{F}^*)$ for some fixed $F \in \mathcal{F}^*$. From Theorem 21(iv) and Lemma 22, we have $|I \cap I'| \geq 1$ for any $I, I' \in \mathcal{I}$. Let I_0 be a minimal set in $\mathcal{I}$. Since $\mathcal{I}$ is intersection closed, $I_0 \subset I$ for all $I \in \mathcal{I}$. Otherwise we have $I_0 \cap I \in \mathcal{I}$ and $|I \cap I_0| < |I_0|$, which contradicts the minimality of I_0. Now we distinguish two cases.

Case 1. $|I_0| = 1$. Let $I_0 = \{v\}$. By Theorem 21(iv), $\{v\}$ is center of an $(r + 1)$-sunflower in $\mathcal{F}^*$. Let $F_1, F_2, \ldots, F_{r+1}$ be hyperedges in such an $(r + 1)$-sunflower. If there is a hyperedge F in $\mathcal{F}$ with $v \notin F$, then it is easy to find some j such that $F_j \cap F = \emptyset$, which contradicts the fact that $\mathcal{F}$ is an intersecting family. Thus, v is contained in every hyperedge of $\mathcal{F}$. Let $G' = G[N(v)]$. Since each copy of K_r in G contains v, G' is K_r-free. By Theorem 23, we have

$$\mathcal{N}(K_r, G) \leq \mathcal{N}(K_{r-1}, G') \leq \mathcal{N}(K_{r-1}, T_{r-1}(n-1)),$$

and the equality holds if and only if $G \cong K_1 \vee T_{r-1}(n-1)$.

Case 2. $|I_0| \geq 2$. We claim that $F \setminus I_0$ is not covered by $\mathcal{I}$. Otherwise, assume that $F \setminus I_0 \subset I^*$ for some $I^* \in \mathcal{I}$. Since $I_0 \subset I$ for all $I \in \mathcal{I}$, we have $I_0 \subset I^*$. It follows that $I^* = F$, which contradicts the fact that $F \notin \mathcal{I}$. Hence $F \setminus I_0$ is not covered by $\mathcal{I}$. It follows that F is the only hyperedge in $\mathcal{F}^*$ containing $F \setminus I_0$. Theorem 21(ii) shows that $\mathcal{I}(F, \mathcal{F}^*)$ is isomorphic to $\mathcal{I}(F', \mathcal{F}^*)$ for any $F, F' \in \mathcal{F}^*$. For any $E \in \mathcal{F}^*$, there is an $(r - |I_0|)$-element subset T of E such that E is the only hyperedge in $\mathcal{F}^*$ containing T. Since $|I_0| \geq 2$, we have $|\mathcal{F}^*| \leq \binom{n}{r-2}$. By Theorem 21(i), for sufficiently large n, we have

$$\mathcal{N}(K_r, G) = |\mathcal{F}| \leq c^{-1}|\mathcal{F}^*| \leq c^{-1}\binom{n}{r-2} < \mathcal{N}(K_{r-1}, T_{r-1}(n-1)).$$

This completes the proof. ∎

4. BOUNDS ON $ex(n, K_r, B_{r,s})$ FOR GENERAL r AND s

Let $B_s^{(r)}$ be an r-uniform hypergraph consisting of two hyperedges that share exactly s vertices. Let $ex_r(n, B_s^{(r)})$ denote the maximum number of hyperedges in an r-uniform $B_s^{(r)}$-free hypergraph on n vertices. In [4], Frankl and Füredi proved the following theorem.

Theorem 29 (Frankl and Füredi [4]). *For $r \geq 2s+2$ and n sufficiently large,*

$$ex_r(n, B_s^{(r)}) = \binom{n-s-1}{r-s-1}.$$

For $r \leq 2s+1$, $ex_r(n, B_s^{(r)}) = O(n^s)$.

Now we prove Theorem 7 by using Theorem 29.

Proof of Theorem 7. Notice that $ex(n, K_r, B_{r,s}) \leq ex_r(n, B_s^{(r)})$, by Theorem 29 we have

$$ex(n, K_r, B_{r,s}) = O(n^{\max\{s, r-s-1\}}). \tag{8}$$

For $r \geq 2s+1$, it is easy to see that $K_{s+1} \vee T_{r-s-1}(n-s-1)$ is a $B_{r,s}$-free graph. Then

$$ex(n, K_r, B_{r,s}) \geq \mathcal{N}(K_{r-s-1}, T_{r-s-1}(n-s-1)).$$

By (8), we have $ex(n, K_r, B_{r,s}) = \Theta(n^{r-s-1})$.

For $r \leq 2s$, we present the following lower bound construction. Let $P = (a_1, a_2, \ldots, a_t)$ be an s-sum-free partition of r. Define a graph G_P on the vertex set $V(G) = X_1 \cup X_2 \cup \cdots \cup X_t$ with $X_i = \lfloor n/t \rfloor$ or $\lceil n/t \rceil$ for each $i = 1, 2, \ldots, t$. Let $G_P[X_i]$ be the union of $|X_i|/a_i$ vertex-disjoint copies of K_{a_i} for each $i = 1, 2, \ldots, t$ and $G_P[X_i, X_j]$ be a complete bipartite graph for $1 \leq i < j \leq t$.

We claim that G_P is $B_{r,s}$-free. Let K, K' be two copies of K_r in G_P. Since $G_P[X_i]$ is a union of vertex-disjoint copies of K_{a_i}, we have $|V(K) \cap X_i| \leq a_i$ and $|V(K') \cap X_i| \leq a_i$. It follows that $|V(K) \cap X_i| = a_i$ and $|V(K') \cap X_i| = a_i$ because of $a_1 + \cdots + a_t = r$. Since P is s-sum-free, we conclude that $|V(K) \cap V(K')| \neq s$. Thus, G_P is $B_{r,s}$-free. Moreover,

$$\mathcal{N}(K_r, G_P) = \prod_{i=1}^{t} \left\lfloor \frac{n}{ta_i} \right\rfloor \approx \left(t^t \prod_{i=1}^{t} a_i\right)^{-1} n^t.$$

Note that $\beta_{r,s}$ is defined to be the maximum length t in an s-sum-free partition of r. Thus, the construction gives that $ex(n, K_r, B_{r,s}) = \Omega(n^{\beta_{r,s}})$ for $r \leq 2s$. This completes the proof. ∎

Let $a_1, a_2, \ldots, a_k$ be a sequence of integers and let $m = \sum_{1 \leq i \leq k} a_k$. Let

$$\mathcal{S}(a_1, a_2, \ldots, a_k) = \left\{ \sum_{i \in I} a_i \colon \emptyset \neq I \subseteq [k] \right\}.$$

If $\mathcal{S}(a_1, a_2, \ldots, a_k) = [m]$, then we call $a_1, a_2, \ldots, a_k$ a *sum-complete* sequence.

Fact 1. *Let $a_1, a_2, \ldots, a_k$ be a sequence of integers with each $a_i \in \{1,2\}$. If at least one of a_i equals 1, then $a_1, a_2, \ldots, a_k$ is a sum-complete sequence.*

Proof. Suppose that $a_1, a_2, \ldots, a_k$ is not sum-complete. Then let h be the smallest integer such that $h \notin \mathcal{S}(a_1, a_2, \ldots, a_k)$. Clearly $h > 1$. Then $h-1 \in \mathcal{S}(a_1, a_2, \ldots, a_k)$. Let $h - 1 = \sum_{i \in I} a_i$. It follows that $a_i = 2$ for all $i \in [k] \setminus I$. Since $h - 1 < m$, there exists $j \in [k] \setminus I$ such that $a_j = 2$. Let $i_0 \in I$, $a_{i_0} = 1$, and let $I' = I \setminus \{i_0\} \cup \{j\}$. Then $h = \sum_{i \in I'} a_i$, a contradiction. ■

Fact 2. *Let $a_1, a_2, \ldots, a_k$ be a sum-complete sequence with $\sum_{1 \le i \le k} a_i = m$ and let $a_{k+1} \le m+1$. Then $a_1, a_2, \ldots, a_k, a_{k+1}$ is also sum-complete.*

Proof. Since $a_1, a_2, \ldots, a_k$ is sum-complete, then $\mathcal{S}(a_1, a_2, \ldots, a_k) = [m]$ and

$$\mathcal{S}(a_1, a_2, \ldots, a_k) + a_{k+1} = [a_{k+1} + 1, a_{k+1} + m].$$

Since $a_{k+1} \le m + 1$, we conclude that

$$\mathcal{S}(a_1, a_2, \ldots, a_k, a_{k+1}) = [m] \cup [a_{k+1} + 1, a_{k+1} + m] \cup \{a_{k+1}\} = [a_{k+1} + m]. \quad ■$$

Now we prove Proposition 8.

Proof of Proposition 8. (i) Since $r \le 2s$, $r - (s+1) \le s - 1$. The partition of r consisting of $r - (s+1)$ "1" and one "$s+1$" is s-sum-free. And there are $r - s$ integers in the partition $(1, 1, \ldots, 1, s+1)$. Then $\beta_{r,s} \ge r - s$.

Let $P = (a_1, a_2, \ldots, a_t)$ be an s-sum-free partition of r. If $a_i \ge 2$ for all $i = 1, 2, \ldots, t$, it is easy to see that $t \le r/2$.

Now we assume that $a_i = 1$ for some $i \in [t]$. Let $(a_i \colon i \in I)$ be a sum-complete subsequence of P with $|I|$ maximum. Clearly $|I| \ge 1$. Let $m = \sum_{i \in I} a_i$. We claim that $m \le r - s - 1$. Indeed, if $m \ge r - s$, then $r - s \in \mathcal{S}(a_i \colon i \in I)$ by definition of m, so $\sum_{i \in I'} a_i = r - s$ for some $I' \subset I$ and the complement has sum $r - (r - s) = s$, a contradiction. Thus $m \le r - s - 1$.

By Fact 2, $a_j \ge m + 2$ for all $j \in [t] \setminus I$. Note that $|I| \le m$. Thus,

$$r = \sum_{1 \le i \le t} a_i = \sum_{i \in I} a_i + \sum_{i \notin I} a_i \ge m + (t - |I|)(m+2) \ge m + (t - m)(m+2).$$

It follows that $t \le \frac{r-m}{m+2} + m =: f(m)$. It can be checked that $f(m) = m - 1 + \frac{r+2}{m+2}$ is a convex function. Since $1 \le m \le r - s - 1$, we conclude that

$$t \le \max\left\{\frac{r+2}{3}, r - (s+1) + \frac{s+1}{r-s+1}\right\}.$$

Since $r \ge 6$, $\frac{r+2}{3} \le \frac{r}{2}$. Let $g(r) = r - (s+1) + \frac{s+1}{r-s+1} - \frac{r}{2}$. Since $g(r)$ is convex and $g(s-1) = g(2s) = 0$, we have $g(r) = r - (s+1) + \frac{s+1}{r-s+1} - \frac{r}{2} \le 0$ for $s + 1 \le r \le 2s$. So we have $r - (s+1) + \frac{s+1}{r-s+1} \le \frac{r}{2}$. Thus, $\beta_{r,s} \le t \le \frac{r}{2}$.

(ii) For $s = 1$, "1" is not present in the 1-sum-free partition of r. Then $\beta_{r,1} \geq \lfloor \frac{r}{2} \rfloor$. $(2,2,\ldots,2,2)$ for r being even (or $(2,2,\ldots,2,3)$ for r being odd) is a 1-sum-free partition of r. Thus $\beta_{r,1} = \lfloor \frac{r}{2} \rfloor$.

For $s = 2$, the 2-sum-free partition of r contains at most one "1" and does not contain "2". Then $\beta_{s,2} \leq 1 + \lfloor \frac{r-1}{3} \rfloor$. Moreover, for $s = 3k$, $(3,3,\ldots,3)$ is a 2-sum-free partition of r. For $s = 3k+1$, $(1,3,3,\ldots,3)$ is a 2-sum-free partition of r. For $s = 3k+2$, $(1,3,3,\ldots,3,4)$ is a 2-sum-free partition of r. Thus $\beta_{s,2} = 1 + \lfloor \frac{r-1}{3} \rfloor$.

For $s = 3$, let $(a_1, a_2, \ldots, a_t)$ be a 3-sum-free partition of r. If $a_i = 1$ for some $i \in [t]$, "2" does not appear in the partition and there are at most two "1" in the partition. Then $t \leq 2 + \lfloor \frac{r-2}{4} \rfloor$ and $(1,1,4,4,\ldots,4,t)$ is a 3-sum-free partition of r where $t = 4,5,6,7$. If $a_i \neq 1$ for all $i \in [t]$, it is easy to see that $t \leq r/2$. And for r being even, $(2,2,\ldots,2)$ is a 3-sum-free partition of r with length $r/2$. When r is odd, there exists an integer a_i in the partition that is odd and $a_i \geq 5$. For r being odd, $t \leq 1 + \frac{r-5}{2}$. $(2,2,\ldots,2,5)$ is a 3-sum-free partition of r with length $1 + \frac{r-5}{2}$. Thus, $\beta_{r,3} = \max\left\{2 + \lfloor \frac{r-2}{4} \rfloor, r/2\right\}$ when r is even, and $\beta_{r,3} = \max\left\{2 + \lfloor \frac{r-2}{4} \rfloor, 1 + \frac{r-5}{2}\right\}$ when r is odd.

For $s = 4$ and $r \geq 4$, let $(a_1, a_2, \ldots, a_t)$ be a 4-sum-free partition of r. If $a_i = 1$ for some $i \in [t]$, $a_j \neq 3$ for all $j \in [t]$ and the sum of all "1" and "2" in the partition does not exceed 3. Then $t \leq 3 + \lfloor \frac{r-3}{5} \rfloor$. If $a_i \neq 1$ for all $i \in [t]$, there is at most one "2" in the partition and all other elements in the partition must be at least 3. Then we have $t \leq 1 + \lfloor \frac{r-2}{3} \rfloor$. $(1,1,1,5,5,\ldots,5,x)$ is a 4-sum-free partition of r with length $3 + \lfloor \frac{r-3}{5} \rfloor$, where $x \in \{5,6,7,8,9\}$. $(3,3,\ldots,3)$, $(2,3,3,\ldots,3,5)$ and $(2,3,3,\ldots,3)$ are 4-sum-free partition of r with length $1 + \lfloor \frac{r-2}{3} \rfloor$ for $r = 3k, 3k+1, 3k+2$. Thus $\beta_{r,4} = \max\left\{3 + \lfloor \frac{r-3}{5} \rfloor, 1 + \lfloor \frac{r-2}{3} \rfloor\right\}$.

(iii) From (i), $\beta_{r,s} \leq \frac{r}{2}$. If r is even and s is odd, $(2,2,\ldots,2)$ is an s-sum-free partition of r with length $r/2$. Thus we have $\beta_{r,s} = \frac{r}{2}$. ■

5. Bounds on $ex(n, K_4, B_{4,2})$

In this section, we derive an upper bound on $ex(n, K_4, B_{4,2})$ by utilizing the graph removal lemma.

Let $G = (V, E)$ be a graph. For any $E' \subset E(G)$, let $G[E']$ denote the subgraph of G induced by the edge set E', and let $G - E'$ denote the subgraph of G induced by $E(G) \setminus E'$. We use $v(G)$ to denote the number of vertices in a graph G.

Lemma 30 (Graph removal lemma [6])**.** *For any graph H and any $\epsilon > 0$, there exists $\delta > 0$ such that any graph on n vertices which contains at most $\delta n^{v(H)}$ copies of H may be made H-free by removing at most ϵn^2 edges.*

***Proof of Theorem* 9.** The lower bound in the theorem is due to the following construction. Suppose that $n = 6m+t$ with $t \leq 5$, let G^* be a graph on n vertices consisting of a set V of size $3m$, whose induced subgraph is a union of m disjoint copies of triangles, and an independent set U of size $3m+t$ as well as all the edges between V and U. Then, it is easy to see that G^* is $B_{4,2}$-free and

$$\mathcal{N}(K_4, G^*) = m(3m+t) = \frac{n^2 - t^2}{12} \geq \frac{n^2 - 25}{12}.$$

Thus, we are left with the proof of the upper bound.

Let G be a $B_{4,2}$-free graph on n vertices. We may further assume that each edge of G is contained in at least one copy of K_4.

Claim 31. *There is a subset $E' \subset E(G)$ with $|E'| = o(n^2)$ such that $G' = G - E'$ is K_5-free, and $\mathcal{N}(K_4, G) = \mathcal{N}(K_4, G') + o(n^2)$.*

Proof. For any edge e in G, there is at most one copy of K_5 containing e, since otherwise we shall find a copy of $B_{4,2}$. Thus, the number of K_5 in G is $O(n^2) = o(n^5)$. By the graph removal lemma, we can delete $o(n^2)$ edges to make G K_5-free. Let E' be the set of the deleted edges.

Note that the edge deletion is to remove the copy of K_5 in G, so the deleted edges are contained in some K_5 in G. Moreover, for any $e \in E'$, there is exactly one copy of K_5 in G containing e. We denote it by K. Then each copy of K_4 containing e is a subgraph of K, otherwise we shall find a copy of $B_{4,2}$. Thus, there are at most three copies K_4 in G containing e. Thus, edge deletion reduces at most $o(n^2)$ copies of K_4. $\square$

Let R be a subset of $E(G')$ consisting of all the edges contained in at least two copies of K_4 in G', and let $B = E(G') \setminus R$.

Claim 32. *There is a subset $T \subset B$ with $|T| = o(n^2)$ such that $G'[B \setminus T]$ is K_4-free, and $\mathcal{N}(K_4, G') = \mathcal{N}(K_4, G' - T) + o(n^2)$.*

Proof. By the definition of the set B, each edge in B is contained in at most one copy of K_4 in G'. Thus, the number of copies of K_4 in $G'[B]$ is at most $O(n^2) = o(n^4)$. By the graph removal lemma, we can delete $o(n^2)$ edges to make $G'[B]$ K_4-free. Moreover, for any deleted edge e, since $e \in B$ it follows that e is contained in exactly one copy of K_4 in G'. By deleting the edges, at most $o(n^2)$ copies of K_4 are removed. $\square$

Let $G^* = G' - T$, $B^* = B \setminus T$. Then the edge set of G^* consists of R and B^*, and $G^*[B^*]$ is K_4-free. In Claim 32, the edge deletion is to remove the copy of K_4 in $G'[B]$, and each deleted edge is contained in exactly one copy of K_4 in $G'[B]$. Then each edge in R is still contained in at least two copies of K_4 in G^* and every edge in B^* is contained in at most one copy of K_4 in G^*. We say a copy of K_4 in G^* is *right-colored* if three of its edges form a triangle in $G^*[R]$ and the other three edges form a star in $G^*[B^*]$.

Claim 33. *All the copies of K_4 in G^* are right-colored.*

Proof. Suppose that $S = \{v_1, v_2, v_3, v_4\}$ induces a copy of K_4 in G^*. Clearly, at least one edge in $G^*[S]$ is contained in R. Without loss of generality, assume that v_1v_2 be such an edge. Since v_1v_2 is contained in at least two copies of K_4 in G^*, assume that $G^*[\{v_1, v_2, v_s, v_t\}]$ be another copy of K_4 containing v_1v_2. If $\{v_s, v_t\} \cap \{v_3, v_4\} = \emptyset$, then we find a copy of $B_{4,2}$ in G^*, a contradiction. Thus, we have $|\{v_s, v_t\} \cap \{v_3, v_4\}| = 1$. Assume that $v_s = v_3$, then both v_1v_3 and v_2v_3 are contained in at least two copies of K_4. It follows that v_1v_3 and v_2v_3 are edges in R. Thus, there are three edges in $G^*[S]$ belonging to R that form a triangle in G^*.

Next we show that v_1v_4, v_2v_4 and v_3v_4 are all edges in B^*. If not, assume that $v_3v_4 \in R$. Then, all the copies of K_4 containing v_1v_2 should also contain v_3 or v_4, otherwise we shall find a copy of $B_{4,2}$. Without loss of generality, assume that all the copies of K_4 containing v_1v_2 contain v_3 as well. Let $G^*[\{v_1, v_2, v_3, v_4\}]$ and $G^*[\{v_1, v_2, v_3, v_5\}]$ be two such copies of K_4. Similarly, all the copies of K_4 containing v_3v_4 should also contain v_1 or v_2. Without loss of generality, assume that $G^*[\{v_3, v_4, v_1, v_2\}]$ and $G^*[\{v_3, v_4, v_1, v_6\}]$ be two such copies of K_4. Clearly, we have $v_5 \neq v_6$ for G^* is K_5-free. However, at this time both $G^*[\{v_1, v_3, v_4, v_6\}]$ and $G^*[\{v_1, v_3, v_2, v_5\}]$ form a copy of K_4, which implies $G^*[\{v_1, v_2, v_3, v_4, v_5, v_6\}]$ contains a copy of $B_{4,2}$, a contradiction. Thus, $v_3v_4 \in B^*$.

Similarly, we can deduce that v_1v_4 and v_2v_4 are edges in B^*. Therefore, $G^*[S]$ is right-colored and the claim holds. □

Since $G^*[B^*]$ is K_4-free, by Turán theorem [14] there are at most $\frac{n^2}{3}$ edges in $G^*[B^*]$. Moreover, since all the copies of K_4 in G^* are right-colored, it follows that each copy of K_4 in G^* contains three edges in B^*. Thus, we have

$$\mathcal{N}(K_4, G^*) \leq \frac{|B^*|}{3} \leq \frac{n^2}{9}.$$

From Claims 31 and 32, it follows that

$$\mathcal{N}(K_4, G) = \mathcal{N}(K_4, G^*) + o(n^2) \leq \frac{n^2}{9} + o(n^2),$$

which completes the proof. ■

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REFERENCES

[1] N. Alon and C. Shikhelman, *Many T copies in H-free graphs*, J. Combin. Theory Ser. B **121** (2016) 146–172.
https://doi.org/10.1016/j.jctb.2016.03.004

[2] M. Deza, P. Erdős and P. Frankl, *Intersection properties of systems of finite sets*, Proc. Lond. Math. Soc. (3) **36** (1978) 369–384.
https://doi.org/10.1112/plms/s3-36.2.369

[3] P. Erdős, *Problems and results in graph theory and combinatorial analysis*, in: Proc. 5th British Combin. Conf. (1975) 169–192.

[4] P. Frankl and Z. Füredi, *Forbidding just one intersection*, J. Combin. Theory Ser. A **39** (1985) 160–176.
https://doi.org/10.1016/0097-3165(85)90035-4

[5] P. Frankl and N. Tokushige, Extremal Problems for Finite Sets, Stud. Math. Libr. **86** (American Mathematical Society, Providence, 2018).
https://doi.org/10.1090/stml/086

[6] J. Fox, *A new proof of the graph removal lemma*, Ann. of Math. (2) **174** (2011) 561–579.
https://doi.org/10.4007/annals.2011.174.1.17

[7] Z. Füredi, *On finite set-systems whose every intersection is a kernel of a star*, Discrete Math. **47** (1983) 129–132.
https://doi.org/10.1016/0012-365X(83)90081-X

[8] D. Gerbner, *Generalized Turán results for disjoint cliques*, Discrete Math. **347(7)** (2024) 114024.
https://doi.org/10.1016/j.disc.2024.114024

[9] D. Gerbner and B. Patkós, *Generalized Turán results for intersecting cliques*, Discrete Math. **347(1)** (2024) 113710.
https://doi.org/10.1016/j.disc.2023.113710

[10] W.T. Gowers and B. Janzer, *Generalizations of the Ruzsa-Szemerédi and rainbow Turán problems for cliques*, Combin. Probab. Comput. **30** (2021) 591–608.
https://doi.org/10.1017/S0963548320000589

[11] Z. Lv, E. Győri, Z. He, N. Salia, C. Tompkins, K. Varga and X. Zhu, *Generalized Turán numbers for the edge blow-up of a graph*, Discrete Math. **347(1)** (2024) 113682.
https://doi.org/10.1016/j.disc.2023.113682

[12] M. Mantel, *Problem* 28, Wiskundige Opgaven **10** (1907) 60–61.

[13] I.Z. Ruzsa and E. Szemerédi, *Triple systems with no six points carrying three triangles*, in: Combinatorics, Proc. Fifth Hungarian Colloq. Keszthely (1976), Colloq. Math. Soc. J. Bolyai **18** (North Holland, Amsterdam-New York, 1978) 939–945.

[14] P. Turán, *On an extremal problem in graph theory*, Mat. Fiz. Lapok **48** (1941) 436–452.

[15] X. Yuan and W. Yang, *On generalized Turán number of two disjoint cliques*, Graphs Combin. **38(4)** (2022) #116.
https://doi.org/10.1007/s00373-022-02518-7

[16] F. Zhang, Y. Chen, E. Győri and X. Zhu, *Maximum cliques in a graph without disjoint given subgraph*, Discrete Math. **347(4)** (2024) 113863.
https://doi.org/10.1016/j.disc.2023.113863

[17] X. Zhu, Y. Chen, D. Gerbner, E. Győri and H.H. Karim, *The maximum number of triangles in F_k-free graphs*, European J. Combin. **114** (2023) 103793.
https://doi.org/10.1016/j.ejc.2023.103793

[18] A.A. Zykov, *On some properties of linear complexes*, Mat. Sb. (N.S.) **24(66)** (1949) 163–188.

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