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ON INDEPENDENT COALITION IN GRAPHS AND INDEPENDENT COALITION GRAPHS

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Abstract

An independent coalition in a graph G consists of two disjoint, independent vertex sets V_1 and V_2, such that neither V_1 nor V_2 is a dominating set, but the union $V_1 \cup V_2$ is an independent dominating set of G. An independent coalition partition of G is a partition $\{V_1, \ldots, V_k\}$ of $V(G)$ such that for every $i \in [k]$, either the set V_i consists of a single dominating vertex of G, or V_i forms an independent coalition with some other part V_j. The independent coalition number $IC(G)$ of G is the maximum order of an independent coalition of G. The independent coalition graph $\mathrm{ICG}(G, \pi)$ of $\pi = \{V_1, \ldots, V_k\}$ (and of G) has the vertex set $\{V_1, \ldots, V_k\}$, vertices V_i and V_j being adjacent if V_i and V_j form an independent coalition in G. In this paper, a large family of graphs with $IC(G) = 0$ is described and graphs

G with $IC(G) \in \{n(G), n(G) - 1\}$ are characterized. Some properties of ICG(G, π) are presented. The independent coalition graphs of paths are characterized, and the independent coalition graphs of cycles described.

Keywords: dominating set, independent set, independent coalition, independent coalition number, independent coalition graph.

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1. Introduction

Let $G = (V(G), E(G))$ be a graph. A *coalition* in G consists of two disjoint sets V_1 and V_2 of vertices, such that neither V_1 nor V_2 is a dominating set, but the union $V_1 \cup V_2$ is a dominating set of G. A *coalition partition* of G, *c-partition* of G for short, is a partition $\{V_1, \ldots, V_k\}$ of $V(G)$ such that for every $i \in [k]$, either the set V_i consists of a single dominating vertex of G, or V_i forms a coalition with some other part V_j. Coalition partitions were introduced in 2020 in [8] and already extensively researched in [3, 4, 9–12]. Very recently, total coalition partitions of graphs have started to be explored in [1, 5] and connected coalition partitions of graphs in [2].

It is a generally accepted fact that the central concepts of graph domination are the domination itself, the total domination, and the connected domination, see the very comprehensive 2023 book [13] on the core concepts in domination which focuses precisely on these three topics. It therefore makes sense also to explore independent c-partitions in graphs which are defined just as c-partitions, except that, in addition, independence is required of the sets involved. More precisely, an *independent coalition* in G consists of two disjoint, independent vertex sets V_1 and V_2, such that neither V_1 nor V_2 is a dominating set, but the union $V_1 \cup V_2$ is an independent dominating set of G. An *independent c-partition* of G is a partition $\{V_1, \ldots, V_k\}$ of $V(G)$ such that for every $i \in [k]$, either the set V_i consists of a single dominating vertex of G or V_i forms an independent coalition with some other part V_j. The *independent coalition number*, $IC(G)$, of a graph G is the maximum order an independent c-partition in G. As it will be discussed later on, it is possible that a graph G does not admit an independent c-partition, in which case we set $IC(G) = 0$.

Independent coalitions were introduced/mentioned in [8], see also [11]. However, this concept was first explored in more detail in [15]. In this paper we continue the research in this direction and proceed as follows. In the rest of this section we define further concepts needed and introduce the relevant notation. In Section 2, we consider graphs with extremal independent coalition numbers. We first consider graphs G with $IC(G) = 0$ and conclude the section by graphs

G with $IC(G) \in \{n(G), n(G)-1\}$. We define and study the independent coalition graph of a graph in Section 3. In Section 4 we characterize the independent coalition graphs of paths, while in Section 5 we describe the independent coalition graphs of cycles.

Let G be a graph and $S \subseteq V(G)$. Then S is a *dominating set* if every vertex in $V(G) \setminus S$ has a neighbor in S, and S is an *independent set* if no two vertices from S are adjacent. By an *independent dominating set* we mean a set that is both dominating and independent. A vertex v of G which is adjacent to every other vertex is a *dominating vertex* of G. An *idomatic partition* of a graph is a partition of the vertices into independent dominating sets. Such partitions seem to be considered for the first time in 2000 in the paper [7] under the name *fall colorings*. Indeed, an idomatic partition is a proper coloring such that every vertex has every color in its open neighborhood. We say that a graph is *idomatic*, if its vertex set can be partitioned into independent dominating sets.

If G and H are graphs and k a positive integer, then $G \cup H$ denotes the disjoint union of G and H and kG the disjoint union of k copies of G. In addition, $G+H$ is the join of G and H, that is, the graph obtained from $G \cup H$ by adding all edges gh, where $g \in V(G)$ and $h \in V(H)$. Finally, the order of a graph G will be denoted by $n(G)$, and $[n]$ stands for the set $\{1, \ldots, n\}$.

2. GRAPHS G WITH $IC(G) \in \{0, n(G), n(G)-1\}$

In the seminal paper [8] a problem was posed whether every graph G admits an independent c-partition, that is, whether for every graph G we have $IC(G) > 0$. In [15], Samadzadeh and Mojdeh answered this question in negative by demonstrating that there exist graphs G with $IC(G) = 0$ as follows. Let X_n, $n \geq 4$, be the graph obtained from K_n with $V(K_n) = \{v_1, \ldots, v_n\}$ and two additional vertices v_{n+1}, v_{n+2} by adding the edges $v_n v_{n+1}$, $v_n v_{n+2}$, and $v_{n-1} v_{n+1}$. See Figure 1, where X_5 is drawn. Then it was proved in [15] that for $n \geq 4$, $IC(X_n) = 0$.

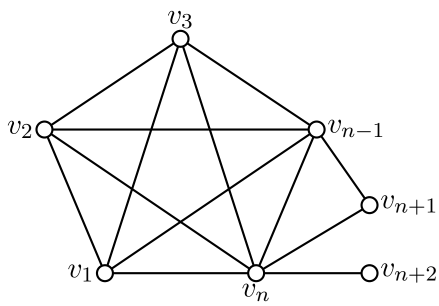

Figure 1. The graph X_5; so $n = 5$ in the figure.

In this section we present a significantly larger family of graphs which do not admit independent c-partitions. For it, the following fact that immediately follows by the definition of an independent c-partition will be useful.

Lemma 1. *Let x be a dominating vertex of a graph G. Then $IC(G) > 0$ if and only if $IC(G - x) > 0$.*

For a graph G, the graph $\widehat{G}$ is a graph defined as follows. Its vertex set is $V(\widehat{G}) = V(G) \cup \{x, y\}$ and the edge set is $E(\widehat{G}) = E(G) \cup \{xu : u \in V(G)\} \cup \{xy\}$. Now the main result of this section reads as follows.

Theorem 2. *Let G be a graph. Then $IC(\widehat{G}) > 0$ if and only if $G = H \cup sK_1$ for some $s \geq 0$ and some idomatic graph H.*

Proof. We use the notation from the definition of $\widehat{G}$, hence $V(\widehat{G}) = V(G) \cup \{x, y\}$, where x is a dominating vertex of $\widehat{G}$.

Assume first that $IC(\widehat{G}) > 0$. As x is a dominating vertex of $\widehat{G}$, Lemma 1 implies that $IC(G \cup K_1) > 0$. Set $G' = G \cup K_1$, where we may assume that $V(K_1) = \{y\}$. Let $\pi = \{V_1, \ldots, V_k\}$ be an independent c-partition of $V(G')$ and assume without loss of generality that $y \in V_1$. We distinguish two cases.

Case 1. $V_1 = \{y\}$.
Since V_j, $j \in \{2, \ldots, k\}$, must dominate G' together with some other set from π, the latter set must necessarily be V_1. Since $|V_1| = 1$, this in turn implies that V_j dominates G. It follows that $\{V_2, \ldots, V_k\}$ is an idomatic partition of G and consequently G is an idomatic graph. Clearly, we can write it as $G = G \cup 0K_1$.

Case 2. $|V_1| \geq 2$.
In this case $V_1 \cap V(G) \neq \emptyset$, which implies that $V_1 \cap V(G)$ is an independent set of G but it is not a dominating set of G, for otherwise V_1 would dominate G'. Moreover, $V_1 \cup V_j$ must be an independent dominating set of G' for every $j \in \{2, \ldots, k\}$. This in particular implies that each vertex of $V_1 \cap V(G)$ must be an isolated vertex of G and that each V_j, $j \geq 2$, is an independent dominating set of $H = G[V(G) \setminus V_1]$. If $|V_1 \cap V(G)| = s$, then we can conclude that $G = H \cup sK_1$, where H is an idomatic graph.

Conversely, assume that $G = H \cup sK_1$, where H is an idomatic graph and $s \geq 0$. Let $\{V_1, \ldots, V_k\}$ be an idomatic partition of H and let S be the set of isolated vertices of G, where $S = \emptyset$ in case $s = 0$. Then we claim that $\{S \cup \{y\}, V_1, \ldots, V_k\}$ is an independent c-partition of G'. Indeed, $S \cup \{y\}$ is clearly an independent set of G' and as each V_i, $i \geq 1$, is an independent dominating set of H, we also have that $(S \cup \{y\}) \cup V_i$ is an independent dominating set of G'. By Lemma 1 we conclude that $IC(\widehat{G}) > 0$. ■

Let H_n, $n \geq 4$, be the graph obtained from K_{n-1} by attaching a pendant vertex to one of the vertices of K_{n-1}. Then $\widehat{H_n} \cong X_n$. Since the graph H_n is clearly not idomatic, Theorem 2 implies that $IC(X_n) = 0$.

Moreover, Theorem 2 yields a large variety of graphs which admit no independent coalition partition. If G is an arbitrary connected and not idomatic graph, then the theorem implies that $IC(\widehat{G}) = 0$. For instance, a cycle C_n is not idomatic if and only if n is odd and n is not congruent modulo 3, see [14]. Additional families of graphs that are not idomatic were constructed in [6], for instance graphs G with $\chi(G) > \delta(G) + 1$. For more information on the variety of domination partitions see [13, Chapter 12].

We now turn our attention to graphs with (almost) largest possible extremal independent coalition number, that is, to graphs G with $IC(G) \in \{n(G), n(G) -1\}$.

The following observation follows directly from definitions, but it is useful because it enables a direct, polynomial verification whether $IC(G) = n(G)$ holds for a given graph G.

Observation 3. *Let G be a graph without dominating vertices. Then $IC(G) = n(G)$ if and only if for any $v \in V(G)$ there exists $z \in V(G)$ with $z \notin N[v]$ such that $V(G) = N[v] \cup N[z]$.*

If G has k dominating vertices, then $IC(G) = k + IC(G')$, where G' is obtained from G by removing the k dominating vertices. Note that G' has no dominating vertices and hence Observation 3 can be applied to G. It is then straightforward to see that checking whether $IC(G) = n(G)$ holds for a graph G can be performed in $\mathcal{O}(n(G)^3)$ time.

To check whether $IC(G) = n(G) - 1$ holds, we can use the following proposition. Its proof is straightforward and hence not included.

Proposition 4. *Let G be a graph without dominating vertices. Then $IC(G) = n(G) - 1$ if and only if the following properties hold.*

- (i) *There exists a vertex $a \in V(G)$ such that for all vertices $b \neq a$ the set $\{a, b\}$ is not an independent dominating set.*
- (ii) *There exists two nonadjacent vertices $x, y \in V(G)$, such that there exists a vertex $w \in V(G)$, $w \neq x, y$, such that $\{w, x, y\}$ is an independent dominating set and for every vertex $u \in V(G)$ with $u \neq x, y$, the set $\{u, x, y\}$ is an independent dominating set or there exists a vertex $v \neq u$ such that $\{u, v\}$ is an independent dominating set.*

Note that the condition (i) of Proposition 4 rules out the possibility $IC(G) = n(G)$, and then (ii) checks whether $IC(G) = n(G) - 1$ holds. As we already mentioned, checking whether $IC(G) = n(G)$ holds (equivalently, condition (i)) can be done in $\mathcal{O}(n(G)^3)$ time. As for condition (ii), its testing for each pair of vertices x and y can be done in $\mathcal{O}(n(G)^3)$ time, leading to an $\mathcal{O}(n(G)^5)$ time algorithm for checking whether $IC(G) = n(G) - 1$ holds.

3. Independent Coalition Graphs

Given an independent c-partition $\pi = \{V_1, \ldots, V_k\}$ of a graph G, we can associate it to a natural derived graph as follows. The *independent coalition graph* $\mathrm{ICG}(G, \pi)$ of π (and of G) has the vertex set $V(\mathrm{ICG}(G, \pi)) = \{V_1, \ldots, V_k\}$, and vertices V_i and V_j are adjacent if V_i and V_j form an independent coalition in G. In the following we present some general properties of the independent coalition graph.

Proposition 5. *Let G be a graph and π be an independent c-partition of G with $|\pi| = k$. Then the following holds.*

(i) $\Delta(\mathrm{ICG}(G, \pi)) \leq \Delta(G) + 1$.

(ii) *If $k \geq \delta(G) + 2$, then $\alpha(\mathrm{ICG}(G, \pi)) \geq k - \delta(G) - 1$.*

Proof. Let $\pi = \{V_1, \ldots, V_k\}$. Note first that $\Delta(\mathrm{ICG}(G, \pi)) = 0$ if and only if $G \cong K_k$. In this case (i) clearly holds, while in (ii) the condition $k \geq \delta(G) + 2 = k + 1$ is not fulfilled. Hence we may assume in the rest of the proof that $\Delta(\mathrm{ICG}(G, \pi)) > 0$.

(i) Let V_i be a vertex of $\mathrm{ICG}(G, \pi)$ with $\deg_{\mathrm{ICG}(G,\pi)}(V_i) = \Delta(\mathrm{ICG}(G, \pi)) \geq 1$. Then $|V_i| > 1$ and V_i is not a dominating set. Therefore, there exists a vertex $v \in V(G)$ with no neighbor in V_i. If V_i and V_j form an independent coalition, then $V_i \cup V_j$ is an independent dominating set, and therefore v has at least one neighbor in V_j. It follows that

$$\Delta(\mathrm{ICG}(G, \pi)) = \deg_{\mathrm{ICG}(G,\pi)}(V_i) \leq \deg_G(v) + 1 \leq \Delta(G) + 1,$$

which proves (i).

(ii) Assume now that $k \geq \delta(G) + 2$ and let v be a vertex of G with $\deg_G(v) = \delta(G)$. As the sets from π are independent, at most $\delta(G) + 1$ of them contain a vertex from $N_G[v]$. Consequently, at least $k - \delta(G) - 1 \geq 1$ sets of π do not contain a vertex from $N_G[v]$. It follows that no two of these $k - \delta(G) - 1$ sets form an independent coalition which in turn implies that $\alpha(\mathrm{ICG}(G, \pi)) \geq k - \delta(G) - 1$. ∎

Consider the graph G obtained from the complete graph K_n with a pendant edge. This graph G satisfies the equality in Proposition 5(i). Also the path graph P_n satisfies the equality of Proposition 5(ii).

Proposition 6. *Let G be a graph with $\delta(G) = 1$ and let π be an independent c-partition of G with $|\pi| = k \geq 3$. Then* $\mathrm{ICG}(G, \pi)$ *is a spanning subgraph of* $K_{2,k-2}$.

Proof. Let $\pi = \{V_1, \ldots, V_k\}$, let x be a vertex of degree 1 in G, and let y be its only neighbor. We may without loss of generality assume that $x \in V_1$ and $y \in V_2$. Then $\{V_1, V_2\} \notin E(\mathrm{ICG}(G, \pi))$. In addition, if $i, j \geq 3$, $i \neq j$, then $x \notin N_G[V_i \cup V_j]$ which in turn implies that $\{V_i, V_j\} \notin E(\mathrm{ICG}(G, \pi))$. Hence $\{V_3, \ldots, V_k\}$ is an independent set of $\mathrm{ICG}(G, \pi)$ which together with the fact that $\{V_1, V_2\} \notin E(\mathrm{ICG}(G, \pi))$ implies the result. ■

4. Independent Coalition Graphs of Paths

In this section we consider independent coalitions in paths. Their independent coalition numbers have already been determined as follows.

Theorem 7 [15]**.** *If $n \geq 1$, then*

$$IC(P_n) = \begin{cases} n; & n \leq 4, \\ 4; & n = 5, \\ 5; & n \in \{6, 7, 8, 9\}, \\ 6; & n \geq 10. \end{cases}$$

We say that a graph G is an *ICG* if G is isomorphic to the independent coalition graphs of some graph. In this section we complement Theorem 7 by determining which graphs are ICGs of paths.

Theorem 8. *A graph G is an* ICG *of some path if and only if*

$$G \in \{P_1, P_4, P_5, 2P_1, 2P_2, 2P_3, P_1 \cup P_2, P_2 \cup P_3\}.$$

Proof. Throughout the proof we will assume that $x_1, \ldots, x_n$ are consecutive vertices of P_n, $n \geq 1$. Further, we will represent an independent c-partition $\pi = \{V_1, \ldots, V_k\}$ of P_n by the vector $f(\pi) = (f_1(\pi), \ldots, f_n(\pi))$, where $x_i \in V_{f_i(\pi)}$. As an example consider the independent c-partition $\pi = \{\{x_1, x_5\}, \{x_2, x_4\}, \{x_3\}, \{x_6\}\}$ of P_6. Then π is represented by the vector $f(\pi) = (1, 2, 3, 2, 1, 4)$, where, for instance, $f_4(\pi) = 2$ means that $x_4 \in V_2$.

We first demonstrate that each of the graphs listed in the statement of the theorem is an ICG of some path. Considering P_1 and P_2 we obtain P_1 and $2P_1$ as ICG. For the remaining six graphs, here are instances of their realizations.

- P_4: the path P_7 with the independent c-partition $(1, 2, 3, 4, 3, 2, 1)$;
- P_5: the path P_{11} with the independent c-partition $(1, 2, 1, 5, 4, 3, 5, 4, 3, 2, 1)$;

- $2P_2$: the path P_6 with the independent c-partition $(1, 2, 3, 2, 1, 4)$;
- $2P_3$: the path P_{10} with the independent c-partition $(2, 1, 5, 6, 1, 2, 4, 3, 2, 1)$;
- $P_1 \cup P_2$: the path P_3 with the independent c-partition $(1, 2, 3)$;
- $P_2 \cup P_3$: the path P_9 with the independent c-partition $(1, 2, 3, 4, 2, 1, 5, 1, 2)$.

It remains to prove that no other graph but the above graphs is an ICG of some path. Since we have settled above all the cases for P_n, $n \le 3$, we may assume in the rest that $n \ge 4$.

We first recall that in [8, Lemma 1], it has been proved that $C(P_n) \le 6$. Since $IC(P_n) \le C(P_n)$, we have $IC(P_n) \le 6$. Consequently, Proposition 6 implies that independent coalition graphs of paths are spanning subgraphs of $K_{2,r}$ with $r \le 4$. In addition, since we have assumed that $n \ge 4$, no independent coalition graph of P_n contains isolated vertices. By inspection we find the following 20 non-isomorphic spanning subgraphs of $K_{2,r}$, where $r \le 4$ with minimum degree at least 1:

$$P_1, P_4, P_5, 2P_1, 2P_2, 2P_3, P_1 \cup P_2, P_2 \cup P_3, \tag{1}$$

$$P_3, K_{2,2}, K_{2,3}, K_{2,4}, K_{2,3} - e, K_{2,4} - e, K_{1,3} \cup P_2, F_1, F_2, F_3, F_4, F_5, \tag{2}$$

where the graphs F_i, $i \in [5]$, are depicted in Figure 2.

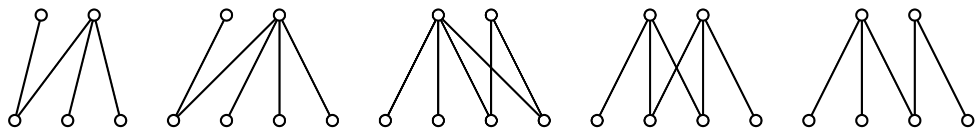

Figure 2. The graphs F_i, $i \in [5]$.

For the graphs from (1) we have established above that they are ICGs of paths, hence we need to prove that neither of the graphs from (2) is an ICG of some path.

Consider first P_3 and suppose that it is a an ICG of some path P_n with an independent c-partition $\pi = \{V_1, V_2, V_3\}$. We may assume without loss of generality that $f_1(\pi) = 1$ and $f_2(\pi) = 2$. Then $V_2 \cup V_3$ is an independent dominating set, hence $f_3(\pi) = 1$. Since also $V_1 \cup V_3$ is an independent dominating set, we have $f_4(\pi) = 2$. Continuing in this manner we conclude that $V_3 = \emptyset$, a contradiction.

Consider second $K_{2,3}$ and suppose that it is an ICG of some path P_n with an independent c-partition $\pi = \{V_1, \dots, V_5\}$. Assume without loss of generality that $\{1, 2\}$, $\{3, 4, 5\}$ is the bipartition of $K_{2,3}$, where $i \in V_i$ for $i \in [5]$. Let j be an arbitrary index such that $f_j(\pi) = 1$. Then $f_{j-1}(\pi) = 2$ (if $j - 1 \ge 1$)

and $f_{j+1}(\pi) = 2$ (if $j+1 \leq n$). In this way we see that $V_3 = V_4 = V_5 = \emptyset$, a contradiction. The same argument applies to $K_{2,2}$ and $K_{2,4}$.

Consider next $K_{2,3} - e$ and suppose that it is an ICG of some path P_n with an independent c-partition $\pi = \{V_1, \ldots, V_5\}$. Assume without loss of generality that $\{1, 2\}$, $\{3, 4, 5\}$ is the bipartition of $K_{2,3}$, where $\{2, 5\} \notin E(K_{2,3}) - e$, and $i \in V_i$ for $i \in [5]$. Let j be such that $f_j(\pi) = 1$. Then $f_{j-1}(\pi) = 2$ (if $j - 1 \geq 1$) and $f_{j+1}(\pi) = 2$ (if $j+1 \leq n$). Assume without loss of generality that $j+1 < n-1$ and that $\ell \geq j+2$ is the smallest index such that $f_\ell(\pi) \in \{3, 4, 5\}$. Then necessary $f_\ell(\pi) = 3$. Thus we have $f_{\ell-2}(\pi) = 1$, $f_{\ell-1}(\pi) = 2$, and $f_\ell(\pi) = 3$, which in turn implies that $f_{\ell+1}(\pi) \in \{4, 5\}$ (if $\ell + 1 \leq n$). But then at least one of $V_1 \cup V_4$ and $V_1 \cup V_5$ is not a dominating set because one of these sets does not dominate x_ℓ. This contradiction proves that $K_{2,3} - e$ is not an ICG of some path. A parallel argument can be used also for $K_{2,4} - e$ as well as for F_1, F_2, and F_3. (Note that in each of these graphs there exists a vertex from the smaller bipartition set adjacent to all the vertices from the other bipartition set.)

It remains to consider the graphs $K_{1,3} \cup P_2$, F_4, and F_5. As the arguments are similar, let us consider in detail only $K_{1,3} \cup P_2$. Suppose on the contrary that $K_{1,3} \cup P_2$ is an ICG of some path P_n with an independent c-partition $\pi = \{V_1, \ldots, V_6\}$. Assume without loss of generality that the vertices of $K_{1,3}$ are from $[4]$ with 1 being the vertex of degree 3, and that $V(P_2) = \{5, 6\}$, where $i \in V_i$ for $i \in [6]$. Consider a vertex x_j with $f_j(\pi) = 1$ and assume without loss of generality that $j < n - 1$. Then $f_{j+1}(\pi) \in \{5, 6\}$. If $f_{j+2}(\pi) = 1$, we repeat the pattern. Hence assume that $f_{j+2}(\pi) \in \{2, 3, 4\}$. But now no matter what the value of $f_{j+3}(\pi)$ (if $j+3 \leq n$) is, at least one of the sets $V_1 \cup V_2$, $V_1 \cup V_3$, and $V_1 \cup V_4$ is not a dominating set because at least one of them does not dominate x_{j+2}. This contradiction completes the argument for $K_{1,3} \cup P_2$. ■

5. INDEPENDENT COALITION GRAPHS OF CYCLES

Here we describe the independent coalition graphs of cycles. Similar to the result for paths, the number of ICGs of cycles is finite. This fact follows from the following known result that gives the independent coalition numbers of cycles.

Theorem 9 [15, Theorem 3.11]. *If $n \geq 3$, then*

$$IC(C_n) = \begin{cases} n; & n \leq 6, \\ 5; & n = 7, \\ 6; & n \geq 8. \end{cases}$$

Just as done for paths, we denote the consecutive vertices of C_n by $x_1, \ldots, x_n$ and represent an independent c-partition $\pi = \{V_1, \ldots, V_k\}$ of C_n by the vector

$f(\pi) = (f_1(\pi), \ldots, f_n(\pi))$, where $x_i \in V_{f_i(\pi)}$. As an example consider the independent c-partition $\pi = \{\{x_1, x_3\}, \{x_2, x_4\}, \{x_5\}, \{x_6\}, \{x_7\}\}$ of C_7. Then π is represented by the vector $f(\pi) = (1, 2, 1, 2, 3, 4, 5)$, where, for instance, $f_5(\pi) = 3$ means that $x_5 \in V_3$.

Clearly, the only cycle whose ICG contains an isolated vertex is C_3, more precisely, $\mathrm{ICG}(C_3, \pi) = 3K_1$, where π is the unique independent c-partition of C_3. For longer cycles we have the following.

Proposition 10. *Let π be an independent c-partition of C_n, $n \geq 4$. Then $\mathrm{ICG}(C_n, \pi)$ is a spanning subgraph of one of the graphs $(K_1 \cup K_2) + K_1$, $(K_1 \cup K_2) + 2K_1$, and $(K_1 \cup K_2) + 3K_1$.*

Proof. Let $\pi = \{V_1, \ldots, V_k\}$ be an independent c-partition of C_n. Since $n \geq 4$, $\mathrm{ICG}(C_n)$ has no isolated vertices. Moreover, $k = |\pi| \geq 4$. Indeed, if we would have $|\pi| = 3$, then there exist three consecutive vertices of C_n such that they respectively belong to the three parts of π. We may assume without loss of generality that $f_i(\pi) = i$ for $i \in [3]$. But then neither $V_2 \cup V_1$ nor $V_2 \cup V_3$ is an independent set, a contradiction.

We have thus seen that $k \geq 4$. On the other hand, $k \leq 6$ by Theorem 9. Just as above, there exist three consecutive vertices of C_n such that they respectively belong to the three parts of π and we may assume that $f_i(\pi) = i$ for $i \in [3]$. Then $V_2V_1 \notin E(\mathrm{ICG}(C_n, \pi))$ and $V_2V_1 \notin E(\mathrm{ICG}(C_n, \pi))$. It is possible however that $V_1V_3 \in E(\mathrm{ICG}(C_n, \pi))$. If $k = 4$, then $\mathrm{ICG}(C_n, \pi)$ is a spanning subgraph of $(K_1 \cup K_2) + K_1$. Assume $k = 5$. Then $V_4 \cup V_5$ is not a dominating set since the union does not dominate x_2. In this case $\mathrm{ICG}(C_n, \pi)$ is a spanning subgraph of $(K_1 \cup K_2) + 2K_1$. Assume finally that $k = 6$. Then by the same argument $V_j \cup V_{j'}$ is not a dominating set for any $j, j' \in \{4, 5, 6\}$, $j \neq j'$. Hence in this case $\mathrm{ICG}(C_n, \pi)$ is a spanning subgraph of $(K_1 \cup K_2) + 3K_1$. ∎

The variety of the ICGs of cycles thus appears larger than the ICGs of paths. Therefore, we will not make a precise analysis of which graphs from Proposition 10 are ICGs of cycles. Instead, we conclude with two ICGs of cycles which are not ICGs of paths.

- C_5: the cycle C_5 with the independent c-partition $(1, 2, 3, 4, 5)$;
- $3P_2$: the cycle C_6 with the independent c-partition $(1, 2, 3, 4, 5, 6)$.

Other realizations of the same graph can also exist. For instance, the graph $3P_2$ can be realized as the ICG of the cycle C_9 with the independent c-partition $(1, 3, 5, 1, 3, 5, 2, 4, 6)$ and as the ICG of the cycle C_{12} with the independent c-partition $(1, 3, 5, 1, 3, 5, 2, 4, 6, 2, 4, 6)$.

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