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EXTENDING PARTIAL EDGE COLORINGS OF CARTESIAN PRODUCTS OF GRAPHS

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Abstract

We consider the problem of extending partial edge colorings of Cartesian products of graphs. More specifically, we suggest the following Evans-type conjecture. If G is a graph where every precoloring of at most k precolored edges can be extended to a proper $\chi'(G)$-edge coloring, then every precoloring of at most $k+1$ edges of $G\square K_2$ is extendable to a proper $(\chi'(G)+1)$-edge coloring of $G\square K_2$. In this paper we verify that this conjecture holds for trees, complete and complete bipartite graphs, as well as for graphs with small maximum degree. We also prove versions of the conjecture for general regular graphs where the precolored edges are required to be independent.

Keywords: precoloring extension, edge coloring, Cartesian product, list coloring.

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1. Introduction

An *edge precoloring* (or *partial edge coloring*) of a graph G is a proper edge coloring of some subset $E' \subseteq E(G)$; a *t-edge precoloring* (or just *t-precoloring*) is such a coloring with t colors. A t-precoloring φ is *extendable* if there is a proper t-edge coloring f such that $f(e) = \varphi(e)$ for any edge e that is colored under φ; f is called an *extension* of φ. In general, the problem of deciding whether a given precoloring is extendable is an $\mathcal{NP}$-complete problem, already for 3-regular bipartite graphs [12].

Edge precoloring extension problems seem to have been first considered in connection with the problem of completing partial Latin squares and the well-known Evans' conjecture that every $n \times n$ partial Latin square with at most $n-1$ non-empty cells is completable to a partial Latin square [11]. By a well-known correspondence, the problem of completing a partial Latin square is equivalent to asking if a partial edge coloring with $\Delta(G)$ colors of a balanced complete bipartite graph G is extendable to a proper $\Delta(G)$-edge coloring, where $\Delta(G)$ as usual denotes the maximum degree. Evans' conjecture was proved for large n by Häggkvist [15], and in full generality by Andersen and Hilton [2], and, independently, by Smetaniuk [18].

Another early reference on edge precoloring extension is [16], where the authors study the problem from the viewpoint of polyhedral combinatorics. More recently, the problem of extending a precoloring of a matching has been considered in [9]. In particular, it is conjectured that for every graph G, if φ is a precoloring of a matching M in G using $\Delta(G) + 1$ colors, and any two edges in M are at distance at least 2 from each other, then φ can be extended to a proper $(\Delta(G) + 1)$-edge coloring of G; here, by the *distance* between two edges e and e' we mean the number of edges in a shortest path between an endpoint of e and an endpoint of e'; a *distance-t matching* is a matching where any two edges are at distance at least t from each other. In [9], it is proved that this conjecture holds for e.g. bipartite multigraphs and subcubic multigraphs, and in [14] it is proved that a version of the conjecture with the distance increased to 9 holds for general graphs.

Quite recently, with motivation from results on completing partial Latin squares, questions on extending partial edge colorings of d-dimensional hypercubes Q_d were studied in [8]. Among other things, a characterization of partial colorings with at most d precolored edges that are extendable to proper d-edge colorings of Q_d is obtained, thereby establishing an analogue for hypercubes of the characterization by Andersen and Hilton [2] of $n \times n$ partial Latin squares with at most n non-empty cells that are completable to Latin squares. In particular, every partial coloring with at most $d-1$ colored edges is extendable to a d-edge coloring of Q_d. This line of investigation was continued in [6, 7] where similar questions are investigated for trees.

Denote by $G \square H$ the Cartesian product of the graphs G and H. Motivated by the result on hypercubes [8], which are iterated Cartesian products of K_2 with itself, in this paper we continue the investigation of edge precoloring extension of graphs with a particular focus on Evans-type questions for Cartesian products of graphs. We are particularly interested in the following conjecture, which would be a far-reaching generalization of a main result of [8].

Conjecture 1. *If G is a graph where every precoloring of at most k edges can be extended to a proper $\chi'(G)$-edge coloring, then every precoloring of at most $k+1$ edges of $G \square K_2$ is extendable to a proper $(\chi'(G)+1)$-edge coloring of $G \square K_2$.*

As we shall see, Conjecture 1 becomes false if we replace $(\chi'(G)+1)$ with $\chi'(G \square K_2)$ by the example of odd cycles.

It it straightforward that if every precoloring with $\Delta(G)$ colored edges of a connected Class 1 graph G is extendable to a $\Delta(G)$-edge coloring, then G is isomorphic to a star $K_{1,n}$. If G, on the other hand, is a connected Class 2 graph where any precoloring with at most $\Delta(G)+1$ colored edges is extendable to a $(\Delta(G)+1)$-edge coloring, then G is an odd cycle. Therefore, we shall generally only consider precolorings with at most $\chi'(G)-1$ colored edges.

In fact, it is easy to show that odd cycles and stars are the only connected graphs with the property that any partial $\chi'(G)$-edge coloring is extendable.

Proposition 2. *Every partial $\chi'(G)$-edge coloring of a connected graph G is extendable if and only if G is isomorphic to a star $K_{1,n}$ or and odd cycle.*

In this paper, we verify that Conjecture 1 holds for trees, complete and complete bipartite graphs. Moreover, we prove a version of Conjecture 1 for regular triangle-free graphs where the precolored edges are required to be independent; a version for graphs with triangles is proved as well. Finally, we prove it for graphs with small maximum degree, namely, graphs with maximum degree two and Class 1 graphs with maximum degree 3.

2. CARTESIAN PRODUCTS WITH TREES, COMPLETE GRAPHS AND COMPLETE BIPARTITE GRAPHS

In this section, we prove that Conjecture 1 holds for trees, complete bipartite graphs and complete graphs. In the following we shall say that an edge e is φ-*colored* if φ is a (partial) edge coloring and e is colored under φ. A color c *appears* at a vertex v if some edge incident with v is colored c; otherwise c is *missing* at v. Since we only consider edge colorings in this paper, we shall usually omit the word “edge” and just refer to a coloring of a graph.

Before we prove our results, let us briefly outline our general proof idea.

Proof outline. We consider a $\Delta(G)$-precoloring φ of $G \times K_2$ using colors $1, \dots, \Delta(G)$, where $G \times K_2$ consists of two copies G_1 and G_2 of G along with a matching M that joins vertices of G_1 and G_2. Henceforth, two edges are *corresponding* if their endpoints are joined by two edges of M. Similarly, two vertices are *corresponding* if they are joined by an edge of M. In the proofs we shall distinguish between the case when M contains precolored edges, and the case when it does not.

• When M has no precolored edges, then either all precolored edges are in G_1, or both G_1 and G_2 contains at least one precolored edge. In the latter case, we can extend the restrictions of φ to G_1 and G_2, respectively, using $\Delta(G)$ colors, and then color the edges of M by the color $\Delta(G) + 1$.

When all precolored edges are in G_1, then we remove the color from all edges colored by some fixed color appearing on at least one edge, say 1, and then take an extension of the obtained precoloring of G_1 using colors $2, \dots, \Delta(G)+1$. Next, we recolor the edges φ-colored 1 by the color 1, and thereafter color the edges of G_2 *correspondingly*, meaning that corresponding edges in G_1 and G_2 get the same color. Finally, we color the edges of M by the color in $\{1, \dots, \Delta(G) + 1\}$ missing at its endpoints.

• When M contains at least one precolored edge, then we generally aim to select corresponding independent edges of G_1 and G_2 (called *selected edges*), each of which is adjacent to as few precolored edges of G_1 and G_2 as possible, but adjacent to exactly one precolored edge of M. Next, we shall in most proofs color the selected edges by the color of the adjacent edges in M, and consider the resulting colorings of G_1 and G_2 (taken together with the restriction of φ to G_1 and G_2, respectively). If these colorings are extendable using colors $1, \dots, \Delta(G)$, then we take such extensions, recolor the selected edges by the color $\Delta(G) + 1$, and thereafter color the edges of M by its original color, or an arbitrary color not appearing at its endpoints, to obtain an extension of φ.

If the colorings of G_1 and G_2 are not extendable, then we employ some different techniques. The details are given in the proofs below.

Now we turn to the proofs of the main results in this section. Let us first consider trees. In [6], the following was proved.

Theorem 3 [6]**.** *Every partial coloring of at most $\Delta(T) - 1$ edges in a tree T is extendable to a proper $\Delta(T)$-coloring of T.*

Using this result we shall establish Conjecture 1 for the case of trees.

Theorem 4. *If T is a tree and φ a precoloring of $\Delta(T)$ edges in $T \square K_2$, then φ can be extended to a proper $(\Delta(T) + 1)$-coloring of $T \square K_2$.*

Proof. Let M be a perfect matching in $T\square K_2$ such that $T\square K_2 - M$ is isomorphic to two copies T_1 and T_2 of T.

If no precolored edges are in M, then we proceed as in the proof outline above, so we assume that M contains at least one precolored edge. By slight abuse of notation, we denote by $T_i + M$ the graph obtained from T_i by attaching every edge of M as a pendant edge of T_i.

If no precolored edges are in T_2, then since $T_1 + M$ is a forest with $\Delta(T)$ precolored edges, by Theorem 3 there is a proper $(\Delta(T)+1)$-coloring of $T_1 + M$ that agrees with φ. Hence, by coloring every edge of T_2 by the color of its corresponding edge in T_1, we obtain a proper $(\Delta(T)+1)$-coloring of $T\square K_2$.

Suppose now that both T_1 and T_2 contains at least one precolored edge. Set $G_i = T_i + M$ and consider the restriction of φ to G_1 and G_2, respectively. Note that both G_1 and G_2 are forests that each contains at most $\Delta(T) - 1$ precolored edges. We shall define extensions of these precolorings of G_1 and G_2, respectively, which agree on M. This yields an extension of φ.

Let V_M^1 be the set containing all vertices of degree $\Delta(T)+1$ in G_1 that are incident with some precolored edge from M and V_M^2 be the set containing all vertices of degree $\Delta+1$ in G_2 that are incident with some precolored edge from M. Note that $v_1 \in V_M^1$ if and only if the corresponding vertex $v_2 \in V_M^2$. We shall prove that there are matchings $M_1 \subseteq E(T_1)$ and $M_2 \subseteq E(T_2)$ such that M_1 covers V_M^1 and $e_2 \in M_2$ if and only if the corresponding edge $e_1 \in M_1$. Moreover we require that if $u_1v_1 \in M_1$ and the corresponding edge $u_2v_2 \in M_2$, then

(i) exactly one of u_1 and v_1 is in V_M^1, say u_1;

(ii) the components H_1 of $G_1 - u_1$ and H_2 of $G_2 - u_2$ containing v_1 and v_2, respectively, contain no precolored edges;

(iii) M_1 and M_2 do not contain any precolored edges.

Since all vertices of V_M^i have degree $\Delta(T)$ in T_i, $T_1 \cup T_2$ contains altogether at most $\Delta(T) - 1$ precolored edges, and all precolored edges of M are pairwise nonadjacent, we can indeed construct the required matchings M_1 and M_2 by for all precolored edges of M greedily selecting an adjacent edge of T_1 (similarly for T_2) so that (i)–(iii) holds.

Next, consider the edges of M that are not precolored but adjacent to some edge of M_1. We assign some fixed color c, arbitrarily chosen from $\{1, \dots, \Delta(T)\}$, to every uncolored edge of M that is adjacent to an edge of M_1 or M_2. Taken together with φ this yields a precoloring φ' of the graph $T\square K_2$.

We denote by M' the set containing all the remaining uncolored edges of M. Since all edges of M_i and M' are pairwise nonadjacent, the set $M_i'' = M_i \cup M'$ is a matching in G_i. We set $G_i'' = G_i - M_i''$. Then $E(G_1'') \cap M = E(G_2'') \cap M$.

By construction, each of the graphs G_1'' and G_2'' have maximum degree $\Delta(T)$, and since both G_1 and G_2 contain at most $\Delta(T) - 1$ precolored edges under φ, respectively, and (i)–(iii) holds, every connected component of G''_1 and G''_2 contains at most $\Delta(T)-1$ precolored edges under the coloring φ'. Hence, it follows from Theorem 3, that there is a proper $\Delta(T)$-coloring f_i of G''_i that agrees with the restriction of φ' to G''_i. Note that f_1 and f_2 agree on all edges of M that are in G''_1 (and G''_2).

It remains to color the edges of $M''_1 \cup M''_2$. We simply assign color $\Delta(T)+1$ to all edges of this set. Taken together with f_1 and f_2, this yields a proper $(\Delta(T)+1)$-edge coloring of $T\square K_2$ that is an extension of φ, because f_1 and f_2 agree on $M \cap E(G''_1)$. This completes the proof. ∎

Let us now consider complete bipartite graphs. Recall that the precoloring extension problem for the balanced complete bipartite graph $K_{n,n}$ corresponds to asking whether a partial Latin square can be completed to a Latin square. As mentioned above, motivated by Evans' conjecture [11], Andersen and Hilton [2] completely characterized partial Latin squares of order n with n nonempty cells that cannot be completed to a Latin square of order n. In the language of colorings they proved the following.

Theorem 5 [3]**.** *Let $n \geq 2$ be a positive integer. A precoloring φ of at most n edges of $K_{n,n}$ can be extended to a proper n-coloring of $K_{n,n}$ if and only if none of the following two conditions holds.*

(a) *For some uncolored edge uv there are n differently colored edges with end-vertices u or v.*

(b) *For some vertex v and some color c, the color c does not appear on any edge incident with v, but every uncolored edge incident with v is adjacent to an edge colored c.*

Using this result we now verify that Conjecture 1 holds for the Cartesian product $K_{n,n}\square K_2$.

Theorem 6. *Let $n \geq 2$ be a positive integer. If φ is a precoloring of n edges in $K_{n,n}\square K_2$, then φ can be extended to a proper $(n+1)$-coloring of $K_{n,n}\square K_2$.*

Proof. Let M be a perfect matching in $K_{n,n}\square K_2$ such that $K_{n,n}\square K_2 - M$ is isomorphic to two copies $K^1_{n,n}$ and $K^2_{n,n}$ of $K_{n,n}$. If no precolored edges are in M, then we proceed as in the proof outline above, so we assume that M contains at least one precolored edge.

Let us first assume that both $K^1_{n,n}$ and $K^2_{n,n}$ contains at least one precolored edge, Let V^i_M be the set containing all vertices in $K^i_{n,n}$ that are incident with some precolored edge from M. As in the preceding proof, $v_1 \in V^1_M$ if and only if the corresponding vertex $v_2 \in V^2_M$.

Proceeding along the lines in the proof of Theorem 4, we shall construct matchings $M_1 \subseteq E(K^1_{n,n})$ and $M_2 \subseteq E(K^2_{n,n})$ such that each vertex of V^1_M is incident with a unique edge of M_1, and $e_2 \in M_2$ if and only if the corresponding edge e_1 in $K^1_{n,n}$ is in M_1. Furthermore, we shall require that no edge e from $M_1 \cup M_2$ is adjacent to a precolored edge of $K^1_{n,n}$ or $K^2_{n,n}$ of the same color as the precolored edge of M that e is adjacent to, and that no edge of $M_1 \cup M_2$ is precolored. As above, since the vertex degree in $K^1_{n,n}$ and $K^2_{n,n}$ is n, $K^1_{n,n}$ and $K^2_{n,n}$ are bipartite, and $K_{n,n} \square K_2$ contains altogether n precolored edges, we can simply select the edges of M_1 (and M_2) greedily.

Now, from the restriction of φ to $K^i_{n,n}$, we define a new precoloring φ_i by coloring every edge of M_i by the color of the adjacent edge of M under φ. Since $K^i_{n,n}$ contains at most $n-1$ precolored edges under φ_i, there is an extension f_i of φ_i using colors $1, \dots, n$. Now, by recoloring all the edges in M_1 and M_2 by the color $n+1$ and coloring every uncolored edge of M by the color not appearing at its endpoints, we obtain an extension of φ.

Suppose now that no precolored edges are in $K^1_{n,n}$ or $K^2_{n,n}$, say $K^2_{n,n}$. Then $K^1_{n,n}$ contains at most $n-1$ precolored edges. Our proof of this case is similar to the proof of the preceding case. As above, let V^1_M be the set containing all vertices in $K^1_{n,n}$ that are incident with some precolored edge from M. Since vertices in $K^1_{n,n}$ have degree n, and $K^1_{n,n}$ is bipartite, there is a matching $M_1 \subseteq E(K^1_{n,n})$ covering V^1_M satisfying analogous conditions to the matchings constructed in the preceding case.

From the restriction of φ to $K^1_{n,n}$ we define a new precoloring φ_1 of $K^1_{n,n}$ by coloring every edge of M_1 by the color of its adjacent edge in M. Now, if φ_1 is extendable to a proper coloring of $K^1_{n,n}$ using colors $1, \dots, n$, then we obtain an extension of φ by recoloring all the edges in M_1 by color $n+1$, coloring $K^2_{n,n}$ correspondingly, and then coloring every uncolored edge of M by the unique color from $\{1, \dots, n+1\}$ not appearing at its endpoints. So assume that there is no such extension of φ_1. By Theorem 5, this means that the coloring φ_1 satisfies condition (a) or (b) of this theorem.

Suppose first that (a) holds. Then all colors $1, \dots, n$ appear on some edge under φ, so every color appears on precisely one edge. Since M contains at least one precolored edge, without loss of generality we may assume that one edge e_{M_1} in M_1 is colored with color 1. Now define a new coloring φ'_1 from φ_1 by removing color 1 from e_{M_1} of $K^1_{n,n}$. Since $K^1_{n,n}$ contains exactly $n-1$ φ'_1-colored edges, there is a proper coloring of $K^1_{n,n}$ using colors $2, \dots, n+1$ which is an extension of φ'_1. Now, by recoloring all the edges in M_1 of $K^1_{n,n}$, distinct from e_{M_1}, by color 1, coloring $K^2_{n,n}$ correspondingly, and then coloring every uncolored edge of M by the unique color not appearing on any edge incident with one of its endpoints, we obtain an extension of φ.

Suppose now that (b) but not (a) holds. Then there is some color c that appears on at least two edges under φ. If some color c' that is used once by φ appears on some edge in the matching M, then we may remove the color c' from the edge in M_1 that is colored c', and then proceed as in the preceding paragraph. If, on the other hand, all colors that are used once by φ do not appear on edges of M, then color c is the only color that appears on edges in M; so color c is the only color that appears on edges in M_1.

Now, since $K^1_{n,n}$ contains at most $n-1$ φ-colored edges, there is an extension of the restriction of φ to $K^1_{n,n}$ using colors $1, \dots, n$. Let M_c be the set of all edges of $K^1_{n,n}$ that are colored c and adjacent to a precolored edge of M. By recoloring all the edges of M_c by $n+1$, coloring $K^2_{n,n}$ correspondingly, and then coloring every uncolored edge of M by the unique color not appearing at its endpoints, we obtain an extension of φ. ■

Finally, let us consider complete graphs. Again our confirmation of Conjecture 1 is based on a result by Andersen and Hilton [3].

Theorem 7 [3]**.** *Let $n \geq 2$ be a positive integer.*

(i) *If φ is a precoloring of at most n edges of K_{2n}, then φ is extendable to a proper $(2n-1)$-coloring unless the precolored edges form a matching, where $n-1$ edges are colored by a fixed color c, and one edge is colored by some color $c' \neq c$.*

(ii) *If φ is a precoloring of at most $n+1$ edges of K_{2n-1}, then φ is extendable to a proper $(2n-1)$-coloring unless the precolored edges form a set of $n-2$ independent edges colored by a fixed color c, and a triangle, disjoint from the independent edges, the edges of which are colored by three different colors that are distinct from c.*

In particular, this implies that every partial coloring of at most $n-1$ edges is extendable to a proper $(2n-1)$-coloring of K_{2n}, and similarly, every partial coloring of at most n edges of K_{2n-1} is extendable to a proper $(2n-1)$-coloring.

The following establishes that Conjecture 1 holds for complete graphs.

Theorem 8. *Let $n \geq 2$ be a positive integer.*

(i) *If φ is a precoloring of at most n edges of $K_{2n} \square K_2$, then φ is extendable to a proper $2n$-coloring of $K_{2n} \square K_2$.*

(ii) *If φ is a precoloring of at most $n+1$ edges of $K_{2n-1} \square K_2$, then φ is extendable to a proper $2n$-coloring of $K_{2n-1} \square K_2$.*

We note that the number of colors used in part (ii) is best possible, since there are partial colorings of just two edges in $K_{2n-1} \square K_2$ that are not extendable to proper $(2n-1)$-colorings.

Before we prove the general case of Theorem 8, we separately consider the case of $K_5 \square K_2$. We first note the following lemma, which is easily proved using the fact that $K_{2n-1} \square K_2$ is Class 1.

Lemma 9. *If φ is a precoloring of $K_{2n-1}\square K_2$ with $n+1$ precolored edges, where at least n edges have the same color, then φ is extendable to a proper $2n$-coloring of $K_{2n-1}\square K_2$.*

In the following, we shall say that a matching *covers* a set S of edges, if every edge of S is adjacent to an edge of the matching.

Lemma 10. *If φ is a partial coloring of at most 4 edges of $K_5\square K_2$, then φ is extendable to a proper 6-coloring.*

Proof. Let M be a perfect matching in $K_5\square K_2$ such that $K_5\square K_2 - M$ is isomorphic to two copies K_5^1 and K_5^2 of K_5. If no precolored edge is in M, then we may proceed as in the proof outline, so we assume that at least one edge of M is precolored.

We shall consider many different cases. By Lemma 9, we may assume that at least two different colors are used in the precoloring φ.

Case 1. *All precolored edges are in $E(K_5^1) \cup M$.* Suppose first that exactly one color appears on the precolored edges of M, say color 1. Then we consider the precoloring of K_5^1 obtained from φ by removing the color 1 from any edge of K_5^1 that is precolored 1. By Theorem 7, this precoloring is extendable to a proper coloring of K_5^1 using colors $2, \dots, 6$. By recoloring any edge of K_5^1 that is φ-precolored 1 by the color 1, coloring K_5^2 correspondingly, and then coloring every uncolored edge of M by an appropriate color missing at its endpoints we obtain an extension of φ.

Suppose now that at least two colors appear on the precolored edges of M. We consider some different subcases.

Case 1.1. *Exactly two edges of M are precolored.* If there is at least one color c used on the precolored edges of K_5^1 that neither appears on an edge of M, nor is the edge precolored c adjacent to a precolored edge of M, then there is a matching $M_1 \subseteq E(K_5^1)$ of uncolored edges satisfying the following

- every edge of M_1 is adjacent to exactly one precolored edge of M,
- every precolored edge of M is adjacent to an edge of M_1,
- no edge of M_1 is adjacent to two precolored edges $e_1 \in E(K_5^1)$ and $e_2 \in M$ that have the same color under φ.

We call such a matching a *φ-good* matching. Moreover, it is easy to see that we can pick this matching such that if we color the edges of M_1 by the color of the precolored adjacent edges in M, then this coloring along with the restriction of φ to K_5^1 does not satisfy the condition in Theorem 7. Hence, the obtained 5-precoloring of K_5^1 is extendable. By recoloring every edge of M_1 by the color 6, coloring K_5^2 correspondingly, and then coloring the edges of M by an appropriate color not appearing at its endpoints, we obtain an extension of φ.

Suppose now that there is no φ-good matching, but there is a color c_1 that appears in K_5^1 but not in M. Since there is no φ-good matching, the edge e_1 colored c_1 is adjacent to an edge e of M that is colored $c_2 \neq c_1$, but not to the other precolored edge of M. Moreover, there is another edge e_1' in K_5^1 colored c_2. Now, it is easy to see that this implies that there is a proper edge coloring of $K_5 \square K_2$ with colors $\{1, \dots, 6\} \setminus \{c_2\}$ that agrees with all precolored edges that are not precolored c_2. Hence, φ is extendable.

Next, assume that the same two colors, say 1 and 2, are used both on the precolored edges of M and on K_5^1. Then there is at most one vertex of degree 4 in the graph obtained from K_5^1 by removing all precolored edges. We properly color the uncolored edges of K_5^1 using colors $3, 4, 5, 6$, color K_5^2 correspondingly, and then proceed as before.

Case 1.2. *Exactly three edges of M are precolored.* Let e_1 be the precolored edge of K_5^1 and assume first that only two colors, say 1 and 2, are used in the precoloring φ. By Lemma 9, we may assume that $\varphi(e_1) = 1$ and exactly one edge of M is colored 1. Then we can pick an uncolored edge e_1' of K_5^1, that is not adjacent to any edge φ-colored 1. We color $K_5^1 - \{e_1, e_1'\}$ properly by colors $3, 4, 5, 6$, color e_1 and e_1' by 1 and proceed as before.

Suppose now that three colors appear in the precoloring φ. If the color of the precolored edge in K_5^1 does not appear on an edge of M, then there is some color c that appears on two edges in M, and a color c' that only appears on one edge of M. This implies that there is a matching $M_1 \subseteq E(K_5^1)$ of uncolored edges satisfying the following

- every edge of M_1 is adjacent to exactly one precolored edge of M,
- every precolored edge of M colored c is adjacent to an edge of M_1,
- the edge precolored c' is not adjacent to an edge of M_1.

Consider the precoloring obtained from the restriction of φ to K_5^1 by in addition coloring every edge of M_1 by the color c. It follows from Theorem 7 that there is an extension of this coloring to K_5^1 using colors $\{1, 2, 3, 4, 5, 6\} \setminus \{c'\}$. Now, we obtain an extension of φ by coloring the edges of M_1 by the color c', coloring K_5^2 correspondingly, and finally coloring the edges of M appropriately.

If, on the other hand, there is a color c which appears both in K_5^1 and in M, then we proceed as follows. We pick a proper coloring f of K_5^1 using colors $\{1, 2, 3, 4, 5, 6\} \setminus \{c\}$, so that for every vertex $v \in V(K_5^1)$, if there is a precolored edge of M incident with v, then the color of the edge of M does not appear at v under f. Next, we recolor the edges of K_5^1 φ-precolored c by the color c, and obtain an extension of φ as before.

The case when four colors appear on edges under φ can be dealt with using a similar argument as in the preceding paragraph.

Case 1.3. *Exactly four edges of M are precolored.* Suppose now that all four precolored edges are in M. If only two colors appear in φ, then by Lemma 9, we may assume that both colors appear on two edges. Thus there is a matching M_1 in K_5^1 covering all precolored edges of M and such that no edge in M_1 is adjacent to two edges precolored with different colors. Thus, φ is extendable, as before. If, on the other hand, three colors appear in φ, then there is a color c that appears on at least two edges. We can pick a proper coloring f of K_5^1 using colors $\{1, \dots, 6\} \setminus \{c\}$, such that no precolored edge of M is adjacent to an edge of the same color under f. Hence, φ is extendable. Note that we can use a similar argument if four colors appear on edges under φ.

Note that in all cases above, K_5^1 is first colored, and then K_5^2 is colored correspondingly. We shall use this property when we consider the next case.

Case 2. *Both K_5^1 and K_5^2 contains at least one precolored edge.* The condition implies that M contains one or two precolored edges. Assume first that M contains only one precolored edge, and so we may assume that K_5^1 contains two precolored edges, and K_5^2 one. Let e_1 and e_1' be the precolored edges of K_5^1, e_2 and e_2' the corresponding edges of K_5^2 respectively, and let e_2'' be the precolored edge of K_5^2, and e_1'' the corresponding edge of K_5^1. Furthermore, let e be the precolored edge of M.

Consider the restriction of φ to K_5^1. If we can assign the color $\varphi(e_2'')$ to e_1'' so that the resulting coloring φ_1 of K_5^1 is proper, then we may proceed as in Case 1 (since in that case K_5^2 is always colored correspondingly). Thus we may assume that either

(a) $e_1'' \in \{e_1, e_1'\}$, or
(b) e_1'' is adjacent to one of the edges in $\{e_1, e_1'\}$ and e_2'' has the same color as one adjacent edge in $\{e_1, e_1'\}$.

By Lemma 9, we may further assume that no color appears on three edges, and thus at most two edges are precolored by the same color. Then, unless e is adjacent to two precolored edges and there is another edge e_1 precolored $\varphi(e)$, that is disjoint from all these three edges, there is an uncolored edge $e_1^{(3)}$ in K_5^1 that is adjacent to e but not adjacent to a precolored edge of K_5^1 colored $\varphi(e)$, and, similarly for the corresponding edge $e_2^{(3)}$ of K_5^2. From the restriction of φ to K_5^1 and K_5^2 we obtain new 5-precolorings by coloring the edges $e_1^{(3)}$ and $e_2^{(3)}$ by the color $\varphi(e)$. Now by Theorem 7 these precolorings are extendable, and we may finish the argument by proceeeding as above.

Suppose now that e is adjacent to two precolored edges and there is one additional edge e_1 precolored $\varphi(e)$ in $E(K_5^1) \cup E(K_5^2)$. It is not hard to see that this implies that there are uncolored corresponding edges $e_1^{(4)} \in E(K_5^1)$ and $e_2^{(4)} \in E(K_5^2)$ that are not adjacent to any edges precolored $\varphi(e)$. We now construct new precolorings from the restrictions of φ to K_5^1 and K_5^2, respectively, by coloring $e_1^{(4)}$ and $e_2^{(4)}$ $\varphi(e)$, and also e_2 by the color $\varphi(e)$. These partial colorings are extendable using colors $1, 2, 3, 4, 5$. Moreover, by construction, the extensions of these precolorings satisfy that no edge colored c is adjacent to e. Hence, φ is extendable.

Let us now assume that M contains two precolored edges. So K_5^1 and K_5^2 both contains precisely one precolored edge, e_1 and e'_2, respectively. Denote the corresponding edges of K_5^2 and K_5^1 by e_2 and e'_1, respectively. As above, it follows that we may assume that either

(a) $e_1 = e'_1$ and $\varphi(e_1) \neq \varphi(e'_2)$, or

(b) e_1 and e_2 are adjacent and $\varphi(e_1) = \varphi(e'_2)$.

Suppose first that (a) holds. If both colors in $\{\varphi(e_1), \varphi(e'_2)\}$, say 1 and 2, appear on the precolored edges of M, then we pick corresponding uncolored edges $e''_1 \in E(K_5^1)$ and $e''_2 \in E(K_5^2)$ that are not adjacent to e_1 or e_2 and color e''_1 and e''_2 by 1 or 2 so that the resulting coloring of $K_5 \times K_2$ is proper. Thereafter, we color $K_5^1 - \{e'_1, e''_1\}$ and $K_5^2 - \{e'_2, e''_2\}$ properly using colors $3, 4, 5, 6$, and proceed as above.

If, on the other hand, at least one of the colors in $\{\varphi(e_1), \varphi(e_2)\}$ do not appear on the precolored edges of M, then there are matchings $M_1 \subseteq E(K_5^1)$ and $M_2 \subseteq E(K_5^2)$ of corresponding uncolored edges such that

- every precolored edge of M is adjacent to exactly one edge of M_i,
- every edge of M_i is adjacent to a precolored edge of M,
- no edge of M_i is adjacent to two precolored edges of the same color.

Now, consider the restriction of φ to K_5^1 and K_5^2, respectively. By, in addition, coloring the edges of M_1 and M_2 by the color of the adjacent precolored edge of M we obtain extendable 5-precolorings of K_5^1 and K_5^2, respectively. Given extensions of these precolorings, we may recolor the edges of M_1 and M_2 by color 6, and then color the edges of M appropriately to obtain an extension of φ as before.

Suppose now that (b) holds. Since $\varphi(e_1) = \varphi(e'_2)$, and we may by Lemma 9 assume that any color appears on at most two edges under φ, there are matchings M_1 and M_2 as described in the preceding paragraph. Thus, we proceed similarly, and this completes the proof of the lemma. ∎

***Proof of Theorem* 8.** We first prove part (i). Denote by M the matching of $K_{2n} \square K_2$ such that $K_{2n} \square K_2 - M$ is isomorphic to two copies K_{2n}^1 and K_{2n}^2 of K_{2n}. The cases when no precolored edges are in M can be handled as in the proof of Theorem 6, so we omit the details here.

In the case when M contains at least one precolored edge, then we may select a matching M_1 in K^1_{2n} as in the proof of Theorem 6 (and possibly also a matching M_2 of corresponding edges in K^2_{2n}). The only essential difference in the argument is that since K_{2n} contains triangles, we can only ensure that n such edges forming the matching M_1 can be selected greedily, although K_{2n} has vertex degree $2n-1$. Nevertheless, since $K_{2n}\square K_2$ contains at most n precolored edges, this suffices for our purposes. Apart from this difference, the argument is very similar to the one in the proof of Theorem 6, so we omit the details.

Let us now prove part (ii). Denote by M the matching of $K_{2n-1}\square K_2$ such that $K_{2n-1}\square K_2 - M$ is isomorphic to two copies K^1_{2n-1} and K^2_{2n-1} of K_{2n-1}.

The case of K_3 follows from the result on odd cycles proved in Section 4, and the case of K_5 is dealt with by the above lemma, so let us assume that $n \geq 4$.

We shall consider a number of different cases. In many of these cases we shall use strategies which are similar to the ones used in the proofs of Theorem 6 and/or Lemma 10, so we generally omit many details.

As before, the case when no precolored edge is in M can be dealt with as in the proof outline, so in the following we assume that at least one precolored edge is contained in M. The rest of the proof breaks into the following cases.

(1) Only one color appear on the precolored edges in M.

(2) At least two colors appear on the precolored edges in M, but at most one color appears on the edges in $E(K^1_{2n-1}) \cup E(K^2_{2n-1})$.

(3) At least two colors appear on the precolored edges of M, and at least two colors appear on the precolored edges in $E(K^1_{2n-1}) \cup E(K^2_{2n-1})$.

In many of the different subcases below we shall use matchings $M_1 \subseteq E(K^1_{2n-1})$ and $M_2 \subseteq E(K^2_{2n-1})$ of corresponding uncolored edges that covers all precolored edges of M, and in addition satisfy that

- no edge of M_i is adjacent to two precolored edges of M of different colors,
- no edge of M_i is adjacent to an edge of K^i_{2n-1} and an edge of M that are precolored by the same color.

We simply say that (M_1, M_2) is a φ-*good* pair of matchings, and that each of M_1, M_2 is a φ-*good* matching. When using such matchings, we usually consider the precolorings obtained from the restriction of φ to K^1_{2n-1} and K^2_{2n-1}, respectively, by coloring the edges of $M_1 \cup M_2$ by the color of the adjacent precolored edge of M. The resulting precolorings will be extendable to $(2n-1)$-colorings of K^1_{2n-1} and K^2_{2n-1}, respectively; indeed, it suffices to verify that none of these precolorings satisfy the condition in part (ii) of Theorem 7. Since such verifications are straightforward, we omit the exact details in the arguments below. Now, from these extensions, we obtain an extension of φ by recoloring the edges of $M_1 \cup M_2$ by the color $2n$, and then coloring the edges of M appropriately.

Case 1. *Only one color appears on the precolored edges of M.* Assume that color 1 appears on the edges of M. If all precolored edges are in $M \cup E(K^1_{2n-1})$, then consider the precoloring obtained from the restriction of φ to K^1_{2n-1} by removing the color 1 from all edges φ-colored 1. By Theorem 7, this precoloring is extendable to a proper coloring using colors $2, \dots, 2n$, and we obtain an extension of φ by recoloring the edges of K^1_{2n-1} that are φ-colored 1 by the color 1, coloring K^2_{2n-1} correspondingly and then coloring every edge of M by the color 1 or $2n$.

Suppose now that both K^1_{2n-1} and K^2_{2n-1} contains at least one precolored edge. By Lemma 9, we may assume that there are at least two edges in $E(K^1_{2n-1}) \cup E(K^2_{2n-1})$ precolored by a color distinct from 1. If there is a φ-good pair of matchings (M_1, M_2), then we consider the precolorings obtained from the restrictions of φ to K^1_{2n-1} and K^2_{2n-1}, respectively, by in addition coloring all edges of M_1 and M_2 by the color 1, and then proceed as outlined above.

Suppose now that there is no φ-good pair of matchings (M_1, M_2). Since at most $n-2$ edges of $E(K^1_{2n-1}) \cup E(K^2_{2n-1})$ are precolored 1, it follows that only one edge u_1u_2 of M is precolored (where $u_i \in V(K^i_{2n-1})$), there is a matching M' of $n-2$ edges in K^1_{2n-1} such that every edge of M' is either precolored 1, or the corresponding edge of K^2_{2n-1} is precolored 1. Moreover, u_1 is incident with two edges e_1 and e_2 that are independent from M' and satisfy that e_i or the corresponding edge of K^2_{2n-1} is precolored by a color distinct from 1. Now, from the restriction of φ to K^1_{2n-1} we define a new precoloring φ_1 by coloring all edges of M' by the color 1, and, in addition, coloring the unique edge of K^1_{2n-1} that is adjacent to both e_1 and e_2 by the color 1. We define an analogous precoloring φ_2 of K^2_{2n-1}.

By Theorem 7, both φ_1 and φ_2 are extendable to proper $(2n-1)$-colorings. Moreover, it is easy to see that neither u_1 nor u_2 is incident with an edge colored 1 in these extensions. Consequently, φ is extendable.

Case 2. *At least two colors appear on the precolored edges of M, but at most one color appears on the precolored edges of $E(K^1_{2n-1}) \cup E(K^2_{2n-1})$.*

We first consider the case when all precolored edges are in $M \cup E(K^1_{2n-1})$.

Suppose first that all precolored edges are contained in M. If every color appears on at most one edge in M, then φ is extendable, because we can choose a proper $(2n-1)$-coloring f of K^1_{2n-1} so that for every vertex $v \in V(K_{2n-1})$, if the edge of M incident with v is colored $i \in \{1, \dots, 2n-1\}$, then no edge incident with v is colored i under f. A similar argument applies if one color appears on at least two edges in M and all other colors appear on at most one edge.

If, on the other hand, there are two colors c_1 and c_2 that both appear on at least two precolored edges, then there is a φ-good matching M_1 in K^1_{2n-1}. Thus φ is extendable.

Suppose now that $E(K^1_{2n-1})$ contains at least one precolored edge, precolored c say. If all precolored edges of M are colored differently, except that several edges of M may be colored c, then there is a proper coloring f of K^1_{2n-1} using colors $\{1,\dots,2n\}\setminus\{c\}$, such that no precolored edge of M is adjacent to an edge of the same color under f. Hence, φ is extendable.

Assume instead that some color $c_1 \neq c$ appears on at least two edges of M and every other color appears on at most one edge of M. If at most one edge of K^1_{2n-1} is precolored c, then we proceed as before and pick a proper coloring f of K^1_{2n-1} using colors $\{1,\dots,2n\}\setminus\{c_1\}$ that agrees with the restriction of φ to K^1_{2n-1}, and where no precolored edge of M is adjacent to an edge of the same color under f. If instead color c is used on at least two edges of K^1_{2n-1}, then it is easy to see that there is a φ-good matching M_1. Hence, φ is extendable.

Finally, let us assume that there are two colors $c_1, c_2 \neq c$ that both appear on at least two edges of M. Again, this implies that there is a φ-good matching M_1, and so, φ is extendable.

Let us now consider the case when both K^1_{2n-1} and K^2_{2n-1} contains at least one edge precolored c. If all the colors on edges of M are distinct, except that several edges of M may be colored c, then φ is extendable as in the case when all precolored edges are in K^1_{2n-1}. Consequently, we assume that there is a color $c_1 \neq c$ that appears on at least two edges $u_1u_2, v_1v_2 \in M$, where $u_i, v_i \in V(K^i_{2n-1})$.

We may assume that at least one precolored edge of K^2_{2n-1} satisfies that the corresponding edge of K^1_{2n-1} is adjacent to a precolored edge of K^1_{2n-1}, since otherwise we can define a precoloring of K^1_{2n-1} by coloring every edge e that is precolored c, or satisfying that the corresponding edge of K^2_{2n-1} is precolored c, by the color c, and then proceed as in the case when only edges of K^1_{2n-1} and M are precolored. This assumption implies that there is a pair of φ-good matchings (M_1, M_2) (containing u_1v_1 and u_2v_2, respectively, if these edges are uncolored). As before, this implies that φ is extendable.

Case 3. *At least two colors appear on the precolored edges of M and at least two colors appear on the precolored edges of $E(K^1_{2n-1}) \cup E(K^2_{2n-1})$.*

Suppose first that all precolored edges lie in $M \cup E(K^1_{2n-1})$. If there is a φ-good matching M_1 in K^1_{2n-1}, then φ is extendable. Otherwise, if there is no φ-good matching, then at most two edges of M are precolored, because otherwise we could select three edges for M_1, each of which is adjacent to at least one precolored edge of K^1_{2n-1}, and thereafter select the rest of the edges of M_1 greedily.

Furthermore, since at least two colors appear on the edges in K^1_{2n-1}, if there is no φ-good matching, then one precolored edge of M, colored c_1 say, must be adjacent to an edge e' precolored $c_2 \neq c_1$, and there is a matching M' of $n-2$ edges in K^1_{2n-1}, disjoint from e', all edges of which are precolored c_1. It is straightforward to verify that this precoloring is extendable, e.g. by first taking an extension of the restriction of φ to the edges precolored by colors distinct from c_1 using colors $\{1,\dots,2n\}\setminus\{c_1\}$.

Let us now consider the case when both K^1_{2n-1} and K^2_{2n-1} contains at least one precolored edge. Again, the idea is to find a pair of φ-good matchings (M_1, M_2). If there are such matchings M_1 and M_2, then we can use them for finding an extension of φ as before.

On the other hand, if there are no such matchings, then at most two edges of M are precolored; suppose e.g. that $e_1, e_2 \in M$ are precolored c_1 and c_2, respectively. Again it follows that one of e_1 and e_2, say e_1, is adjacent to an edge $e' \in E(K^1_{2n-1})$ (or $e' \in E(K^2_{2n-1})$) colored $c_3 \neq c_1$, and there is a matching M' in K^1_{2n-1} (K^2_{2n-1}) of $n-2$ edges, every edge of which is either colored c_1, or satisfies that the corresponding edge of K^2_{2n-1} (K^1_{2n-1}) is precolored c_1. Moreover, $M' \cup \{e', e_2\}$ is independent.

Thus we can color every edge of M' by the color c_1, retain the color of e' if $c_3 \neq c_2$, and also color the unique edge adjacent to both e' and e_2, but not e_1, by the color c_1 to obtain a precoloring of K^1_{2n-1} that is extendable using colorings $\{1, \dots, 2n\} \setminus \{c_2\}$. Moreover, every extension of this precoloring satisfies that no edge adjacent to e_1 is colored c_1. Hence, φ is extendable. This completes the proof of the theorem. ∎

3. Cartesian Products with General Graphs

We have not been able to confirm Conjecture 1 in the general case, but we can prove it for the case of regular triangle-free graphs when the precolored edges are independent.

Theorem 11. *If G is a triangle-free regular graph where every precoloring of at most $k < \Delta(G)$ independent edges are extendable to a $\chi'(G)$-coloring, then every precoloring of at most $k+1$ independent edges in $G \square K_2$ is extendable to a $(\chi'(G)+1)$-coloring.*

Proof. Without loss of generality, we assume that $k+1$ edges of $G \square K_2$ are precolored. We denote this precoloring by φ, by G_1 and G_2 the copies of G in $G \square K_2$, respectively, and by M the perfect matching between G_1 and G_2.

If no precolored edges are in M, then we proceed as in the proof outline above, so we assume that at least one precolored edge is contained in M, and consider two different cases.

Case 1. *All precolored edges are in $E(G_1) \cup M$.* We assume that at least one precolored edge is in M. Let E_M be the set of all precolored edges in M. As in the proof outline, we shall select a matching M_1 of $|E_M|$ uncolored edges in G_1 satisfying the following:

(i) every edge of E_M is adjacent to exactly one edge of M_1;

(ii) every edge of M_1 is adjacent to exactly one precolored edge of G.

Since G_1 is regular and triangle-free, at most one endpoint of a precolored edge of G_1 is adjacent to an endpoint of a precolored edge in M. Thus, since $G \square K_2$ contains at most $\Delta(G_1)$ precolored edges and all those precolored edges are independent, it is straightforward to verify that there is a set $M_1 \subseteq E(G_1)$ satisfying (i)–(ii); indeed, since every vertex of G_1 has degree $\Delta(G_1)$, we can simply select edges adjacent to the precolored edges of M greedily.

Now, by coloring all edges of M_1 by the color of the adjacent edge of M, and taking this coloring together with the restriction of φ to G_1, we obtain a precoloring φ_1 of G_1 with $k+1$ precolored independent edges. Without loss of generality we assume that some edge is colored 1 under φ_1. By removing the color 1 from every such edge φ_1-precolored 1, we obtain a precoloring that is extendable to a proper coloring of G_1 using colors $2, \dots, \chi'(G)+1$.

Next, for every edge of G_1 that is φ-precolored 1, we recolor this edge by 1. Similarly, for every φ_1-precolored edge e of M_1 such that $\varphi_1(e) \neq 1$, we recolor e by the color 1 and the adjacent edge of M by the color $\varphi_1(e)$. Finally, we recolor any edge of M that is φ-precolored 1 by the color 1. Since all φ_1-precolored edges are independent, the resulting partial coloring of G is proper. By coloring G_2 correspondingly and coloring all uncolored edges of M by a color missing at its endpoints, we obtain a proper coloring that is an extension of φ.

Case 2. *$E(G_1), E(G_2)$ and M contains at least one precolored edge each.* Let φ_1 and φ_2 be the restrictions of φ to G_1 and G_2, respectively. As in the preceding case we shall select a matching of $|E_M|$ uncolored edges $M_1 \subseteq E(G_1)$ and a matching $M_2 \subseteq E(G_2)$ of uncolored corresponding edges satisfying the following:

(i) every edge of E_M is adjacent to exactly one edge of M_i;

(ii) every edge of $M_1 \cup M_2$ is adjacent to exactly one precolored edge of G.

The existence of such sets M_1 and M_2 follows as in Case 2, since G_1 and G_2 are $\Delta(G)$-regular and $G \square K_2$ contains altogether at most $\Delta(G)$ precolored edges.

Now, consider the precolorings obtained from φ_1 and φ_2, respectively, by coloring every edge of M_1 and M_2 by the color of the adjacent precolored edge of M. Since G_1 and G_2 both contains at least one φ-precolored edge, the obtained precolorings φ'_1 and φ'_2, respectively, are by assumption extendable to proper $\chi'(G)$-colorings. Now, we recolor every edge of $M_1 \cup M_2$ by the color $\chi'(G)+1$, and then color every edge of M that is adjacent to an edge of $M_1 \cup M_2$ by the color of the adjacent edge of $M_1 \cup M_2$ (i.e., the color under φ if the edge of M is precolored). We color every edge of M that is not adjacent to an edge of $M_1 \cup M_2$ by the color $\chi'(G)+1$. Since M_i is a matching, the resulting coloring is proper, and thus also an extension of φ. ■

For graphs with triangles we have the following variant of Theorem 11.

Theorem 12. *If G is a regular graph where every precoloring of at most $k < \Delta(G)/2$ independent edges is extendable to a $\chi'(G)$-coloring, then every precoloring of at most $k+1$ independent edges in $G \square K_2$ is extendable to a $(\chi'(G)+1)$-coloring.*

The proof of this theorem is almost identical to the proof of the preceding one. The only essential difference is that when G is not triangle-free we have to assume that at most $\Delta(G)/2$ edges are precolored to be able to ensure that we can select independent edges in G_1 and G_2 that are adjacent to precolored edges of M and also not adjacent to any precolored edges in G_1 or G_2. We omit the details.

4. CARTESIAN PRODUCTS WITH SUBCUBIC GRAPHS

In this section, we consider Conjecture 1 for subcubic graphs, that is, graphs with maximum degree at most 3. First we prove that it holds for graphs with maximum degree two. The case of paths was considered above, so it suffices to consider cycles. We shall need some well-known auxiliary results on list coloring.

Lemma 13. *For every path P, if one edge $e \in E(P)$ has a list of size at least 1 and all other edges have lists of at least 2 colors, then P has a proper coloring using colors from the lists.*

Lemma 14. *If L is a list assignment for the edges of a cycle C, where every list has size at least two and not all edges have the same list, then C has a proper coloring using colors from the lists.*

Proposition 15. *Let $n \geq 2$ be a positive integer. If φ is a precoloring of two edges in $C_{2n} \square K_2$, then φ can be extended to a proper 3-coloring of $C_{2n} \square K_2$.*

Proof. Let M be matching of $C_{2n} \square K_2$, so that $C_{2n} \square K_2 - M$ consists of two copies C_{2n}^1 and C_{2n}^2 of C_{2n}. The cases when no precolored edge is contained in the matching M can be dealt with as above. If M contains exactly one precolored edge, then if the two precolored edges have distinct colors, then φ is trivially extendable. If, on the other hand, the two precolored have the same color, say 1, then we can properly color the uncolored edges in the copy of C_{2n} containing a precolored edge by colors 2 and 3; thus φ is extendable.

It remains to consider the case when no precolored edges are in C_{2n}^1 or C_{2n}^2. If the two precolored edges have the same color, then φ is trivially extendable. If they have different colors and have distance at least 2, then φ is extendable by Lemma 14. The case when the precolored edges are at distance 1 is straightforward. ■

Note that Proposition 15 does not hold for odd cycles. For instance, consider the Cartesian product $C_{2n+1}\Box K_2$ where two corresponding edges of the two copies of C_{2n+1} are colored by 1 and 2, respectively. If this precoloring is extendable to a proper 3-coloring of $C_{2n+1}\Box K_2$, then every edge in the matching M joining vertices of the copies of C_{2n+1} must be colored 3. Hence, the precoloring is not extendable.

Nevertheless, for odd cycles, we have the following analogue of Proposition 15.

Proposition 16. *Let $n \geq 1$ be a positive integer. If φ is a precoloring of three edges in $C_{2n+1}\Box K_2$, then φ can be extended to a proper 4-coloring of G.*

Proof. Let M be a perfect matching in $C_{2n+1}\Box K_2$ such that $C_{2n+1}\Box K_2 - M$ is isomorphic to two copies C^1_{2n+1} and C^2_{2n+1} of C_{2n+1}. Without loss of generality we assume that the precoloring of G uses colors $1, 2, 3$. We shall consider some different cases. Again, we omit the details in the case when M contains no precolored edges.

Case 1. *M contains exactly one precolored edge.* For any uncolored edge $e \in E(C_{2n+1}\Box K_2)$, we define a color list $L(e) \subseteq \{1, 2, 3, 4\}$ by setting

$$L(e) = \big\{1, 2, 3, 4\} \setminus \{\varphi(e') : e' \text{ is adjacent to } e\big\}.$$

If no precolored edges are in C^2_{2n+1}, then at most one uncolored edge of C^1_{2n+1} is adjacent to precolored edges of three distinct colors, which implies that at most one edge e has a list $L(e)$ of size 1 and all other edges of C^1_{2n+1} have lists of size at least 2. By Lemma 13, there is a proper coloring φ_1 of C^1_{2n+1}, which is an extension of the restriction of φ to C^1_{2n+1}. Hence, we obtain an extension of φ by coloring every edge of C^2_{2n+1} by the color of its corresponding edge in C^1_{2n+1}, and then coloring every edge of M with some color in $\{1, 2, 3, 4\}$ that is missing at its endpoints.

Suppose now that both C^1_{2n+1} and C^2_{2n+1} each contains exactly one precolored edge. Let $e_1 \in E(C^1_{2n+1})$, $e_2 \in M$, and $e_3 \in E(C^2_{2n+1})$ be the precolored edges of C^1_{2n+1}, M and C^2_{2n+1}, respectively.

First we treat the case when $n = 1$; it needs to be considered separately. If e_1 and e_2, and e_2 and e_3 are adjacent, then φ is extendable since $\varphi(e_2) \notin \{\varphi(e_1), \varphi(e_3)\}$; we can e.g. first properly color C^1_3 and C^2_3 using colors from $\{1, 2, 3, 4\} \setminus \{\varphi(e_2)\}$, and then color the remaining uncolored edges of M. If exactly two of the edges e_1, e_2, e_3 are pairwise adjacent, say e_1 and e_2, and $\varphi(e_3) = \varphi(e_2)$, then we can instead use colors $\varphi(e_1)$, $\varphi(e_2)$ and one additional color from $\{1, 2, 3, 4\}$ for coloring C^1_3 and C^2_3; if $\varphi(e_3) \neq \varphi(e_2)$, then we proceed similarly using colors $\{1, 2, 3, 4\} \setminus \{\varphi(e_2)\}$.

Suppose now that all the edges e_1, e_2, e_3 are pairwise nonadjacent. If at most two colors c_1 and c_2 appear on the precolored edges, then we may assume that

c_1 appears on at most one edge under φ. We color $C_3 \square K_2$ properly using the colors in $\{1,2,3,4\} \setminus \{c_1\}$, so that the edges precolored c_2 get the color c_2. Then we recolor the edge precolored c_1 by the color c_1. If, on the other hand, three colors appear on the precolored edges, then we properly color C_3^1 and C_3^2 using colors $\{1,2,3,4\} \setminus \{\varphi(e_2)\}$.

Now we treat the case when $n \geq 2$. We define a list assignment L for C_{2n+1}^1 by setting

$$L(e) = \{1,2,3\} \setminus \big\{\varphi(e') : e' \in E(C_{2n+1} \square K_2) \text{ is adjacent to } e\big\}.$$

Since at most one edge is adjacent to both e_1 and e_2, at most one uncolored edge e of C_{2n+1}^1 satisfies that $|L(e)| = 1$, and all other edges of C_{2n+1}^1 have lists of size at least two. By Lemma 13, there is a proper coloring φ_1 of C_{2n+1}^1 using colors $1,2,3$ which is an extension of the restriction of φ to C_{2n+1}^1. Arguing similarly, we can define a proper coloring φ_2 of C_{2n+1}^2 using colors $1,2,3$ which is an extension of the restriction of φ to C_{2n+1}^2. Finally, we obtain an extension of φ by coloring every uncolored edge of M with color 4.

Case 2. *M contains at least two precolored edges.* For the uncolored edges of $C_{2n+1} \square K_2$, we define a list assignment L by setting

$$L(e) = \{1,2,3,4\} \setminus \big\{\varphi(e') : e' \in E(C_{2n+1} \square K_2) \text{ is adjacent to } e\big\}.$$

If M contains exactly two precolored edges, then exactly one precolored edge is in C_{2n+1}^1 or C_{2n+1}^2, say C_{2n+1}^1. Then at most one edge of C_{2n+1}^1 is adjacent to precolored edges of three distinct colors, so at most one edge e satisfies that $|L(e)| = 1$, and all other edges of C_{2n+1}^1 have lists of size at least 2. Hence by Lemma 13, there is a proper coloring φ_1 of C_{2n+1}^1 using colors from the lists. By coloring C_{2n+1}^2 correspondingly, and then coloring the uncolored edges of M, we obtain an extension of φ. If, on the other hand, M contains all three precolored edges, then we can obtain an extension of φ by first coloring the uncolored edges of M, and then apply Lemma 14. ■

Note that even though any partial 3-coloring of C_{2n+1} is extendable, the upper bound of three precolored edges in Proposition 16 is best possible. For instance, consider a precoloring where two adjacent edges e_1 and e_2 of C_{2n+1}^1 are precolored 1 and 2, respectively, and the corresponding edges e_1' and e_2' of C_{2n+1}^2 are precolored 3 and 4, respectively (using the same notation as in the preceding proof).

Next, we shall verify that Conjecture 1 holds for Class 1 graphs of maximum degree 3.

Theorem 17. *If G is a Class 1 graph with $\Delta(G) = 3$ and every partial 3-coloring of at most $k < 3$ edges in G is extendable, then every partial 4-coloring of at most $k+1$ edges in $G \square K_2$ is extendable.*

Proof. The case when $k = 1$ is trivial, so let us assume that $k = 2$. Let G_1 and G_2 be copies of G in $G\square K_2$ and M the perfect matching joining vertices of G_1 with corresponding vertices of G_2.

Consider a partial 3-coloring φ of $G\square K_2$ with three precolored edges. Note that we may assume that G is connected, since otherwise we just consider every component of G separately. Next, we prove that G contains no triangle.

If G is a triangle, then we can apply Proposition 16, so we may assume that this is not the case. Thus if G contains a triangle $xyzx$, then at least one vertex, say x has degree 3. Then we can color an edge incident with x, which is not contained in the triangle, by the color 2, and yz by the color 1. The resulting partial coloring is not extendable to a proper 3-coloring, contradicting that any partial coloring with 2 precolored edges is extendable. Thus G is triangle-free.

In the remaining part of the proof of Theorem 17 we shall consider a large number of different cases. In many of these cases, we shall employ ideas that have been used above. Thus, in several places we just sketch the arguments, rather than giving all the details. As usual, we shall omit the details in the case when no precolored edges are contained in M, since we may proceed as in the proof outline above.

Case 1. *M contains exactly one precolored edge.* If the color of the precolored edge e of M only appears on e under φ, then the result is trivial. So assume that $e = u_1u_2$ is a precolored edge of M, that $\varphi(e) = 1$, where $u_i \in V(G_i)$, and that at least one other edge is precolored 1 under φ.

If the remaining two precolored edges are in G_1, then we consider the precoloring φ' of G_1 obtained from the restriction of φ to G_1 by removing color 1 from all edges precolored 1 in G_1. Then φ' is extendable to a proper 3-coloring using colors $2, 3, 4$. Next, we recolor the edges precolored 1 by the color 1, color G_2 correspondingly, and color every edge of M by a color missing at its endpoints to obtain an extension of φ.

Suppose now that both G_1 and G_2 contains precolored edges. If $d_{G_1}(u_1) = 3$, then $d_{G_2}(u_2) = 3$, and since $G - M$ contains at most two precolored edges and G is triangle-free, there are uncolored corresponding edges $e_1 \in E(G_1)$ and $e_2 \in E(G_2)$, adjacent to e, such that neither e_1, nor e_2, is adjacent to an edge of $G - M$ precolored 1. We define a new precoloring φ' from the restricton of φ to $G - M$ by coloring e_1 and e_2 by the color 1. The restrictions of φ' to G_1 and G_2, respectively, are extendable to proper 3-colorings of G_1 and G_2, respectively. Next we color e_1 and e_2 by the color 4, e by the color 1, and every uncolored edge of M by a color in $\{1, 2, 3, 4\}$ missing at its endpoints.

Now assume that $d_{G_1}(u_1) = 2$. If there is an uncolored edge $e_1 \in E(G_1)$ adjacent to e such that neither e_1, nor the corresponding edge $e_2 \in E(G_2)$, is adjacent to an edge of $G - M$ precolored 1, then we proceed as in the preceding paragraph. Otherwise, there are corresponding edges $e_1 \in E(G_1)$ and $e_2 \in E(G_2)$

that are adjacent to e and satisfies that e_1 is precolored or adjacent to an edge precolored 1, and e_2 is neither precolored, nor adjacent to a precolored edge of G_2; moreover, there are corresponding edges $e'_1 \in E(G_1)$ and $e'_2 \in E(G_2)$ adjacent to e that satisfy analogous conditions with the roles of G_1 and G_2 interchanged. Hence, from the restriction of φ to G_1 and G_2, respectively, we obtain extendable partial 3-colorings by coloring e'_1 and e_2 with colors from $\{2, 3\}$. Note that in these extensions no edge colored 1 is adjacent to e. Hence, φ is extendable. A similar argument applies when $d_{G_1}(u_1) = 1$.

Case 2. *M contains exactly two precolored edges.* We assume that G_1 contains the third precolored edge e_3. If all precolored edges of M have the same color, then we proceed as in Case 2 when $E(G_1) \cup M$ contains all precolored edges. Thus, we assume that two different colors 1 and 2 appear on the precolored edges of M; let u_1 and v_1 be the endpoints of these edges in G_1, respectively, where u_1 is incident with an edge of M precolored 1.

Suppose first that e_3 is colored by some color appearing on M, say 1. If there is an uncolored edge e' incident with u_1 that is neither incident with v_1, nor adjacent to e_3, then we color e' by the color 1 and take an extension of the coloring of e' and e_3 using colors $1, 3, 4$. We now proceed as before to obtain an extension of φ. Otherwise, if there is no such edge e', then

(a) $d_{G_1}(u_1) = 1$ and u_1 is adjacent to an endpoint of e_3, or

(b) $d_{G_1}(u_1) = 1$ and u_1 and v_1 are adjacent, or

(c) $d_{G_1}(u_1) = 2$ and u_1 is adjacent both to v_1 and an endpoint of e_3 (and these vertices are distinct).

If (a) holds, then we may simply take an extension of the restriction of φ to G_1 using colors $1, 3, 4$; if (b) or (c) holds, then we color u_1v_1 by color 3 and take an extension of the obtained coloring of G_1 using colors $1, 3, 4$. In all cases, it is straightforward that φ is extendable.

Suppose now that $\varphi(e_3) = 3$. If there is no uncolored edge incident with u_1 or v_1, then the result is trivial. Otherwise, if there is such an edge, then we proceed as in the preceding paragraph.

Case 3. *M contains exactly three precolored edges.* If all three precolored edges of M have the same color, then the result is trivial.

Suppose now that two colors appear on the precolored edges in M, say 1 and 2, and that color 2 appears on only one edge. Denote this edge by $e = u_1u_2$, where $u_1 \in V(G_1)$. If $d_{G_1}(u_1) \leq 2$, then properly color the edges incident with u_1 by 3 and 4. This precoloring of G_1 is extendable to a proper coloring of G_1 using colors $2, 3, 4$. We obtain an extension of φ by coloring G_2 correspondingly, coloring u_1u_2 by the color 2 and all other edges of M by the color 1. If $d_{G_1}(u_1) = 3$, then there is an edge u_1x of G_1 that is adjacent to only one precolored edge of M. We define a precoloring of G_1 by coloring u_1x by the color 2. This precoloring is extendable to a precoloring of G_1 by colors $2, 3, 4$. We now obtain an extension of φ by recoloring u_1x and proceeding as before.

Suppose now that three colors appear on the edges of M, i.e., that u_1u_2 is colored 1, v_1v_2 is colored 2, and w_1w_2 is colored 3, where $u_i, v_i, w_i \in V(G_i)$.

If there exist two distinct vertices $x, y \in V(G_1) \setminus \{u_1, v_1, w_1\}$ such that x and y can be matched to distinct vertices in $\{u_1, v_1, w_1\}$ by two independent edges, say u_1x and v_1y, then we color these edges by 1 and 2, respectively and take an extension of this precoloring of G_1 using colors $1, 2, 4$. Next, we recolor both u_1x and v_1y by the color 3, color u_1u_2 by the color 1, v_1v_2 by the color 2 and all other edges of M by the color 3; this yields an extension of φ. Otherwise, if no such edges exist, then $\{u_1, v_1, w_1\}$ has at most two neighbors outside $\{u_1, v_1, w_1\}$ in G_1. Moreover, since G_1 is connected, there must be at least one neighbor of $\{u_1, v_1, w_1\}$ in $V(G_1) \setminus \{u_1, v_1, w_1\}$ in G_1.

Suppose first that $\{u_1, v_1, w_1\}$ has exactly one neighbor $x \notin \{u_1, v_1, w_1\}$ in G_1. If all vertices in $\{u_1, v_1, w_1\}$ are adjacent to x, then we color xu_1 by 1 and xv_1 by color 4. By assumption, this coloring of G_1 is extendable to a proper coloring of G_1 using colors $1, 2, 4$. Since G is triangle-free, it follows that φ is extendable (by recoloring xu_1 by 3).

If only one vertex in $\{u_1, v_1, w_1\}$ is adjacent to x, say u_1, then either both v_1 and w_1 are adjacent to u_1, or both u_1 and w_1 are adjacent to v_1. In the first case we color u_1v_1 by 4 and u_1w_1 by 2; in the latter case we color v_1w_1 by 4 and u_1v_1 by 3. Both these partial colorings of G_1 are extendable to proper colorings of G_1 using colors $2, 3, 4$. Hence, φ is extendable.

Suppose now that two vertices in $\{u_1, v_1, w_1\}$ are adjacent to x, say u_1 and v_1. If w_1 is adjacent to both u_1 and v_1, then we color xv_1 by the color 1, and u_1w_1 by the color 2. This precoloring of G_1 is extendable (using colors $1, 2, 4$), and so, φ is extendable. If w_1 is only adjacent to one of u_1 and v_1, then we may proceed similarly.

Let us finally consider the case when $\{u_1, v_1, w_1\}$ has exactly two neighbors $x, y \notin \{u_1, v_1, w_1\}$ in G_1. Then x and y has only one common neighbor in $\{u_1, v_1, w_1\}$, say u_1. Thus v_1 and w_1 are only adjacent to vertices in $\{u_1, v_1, w_1\}$. Since G is triangle-free, we can properly color the edges incident with w_1 and v_1 by two colors from $\{2, 3, 4\}$ so that no vertex is incident with two edges of the same color. By assumption, there is an extension of the obtained coloring of G_1 using colors $2, 3, 4$. Hence, φ is extendable. This completes the proof of the theorem. ∎

Unfortunately, we are not able to prove a corresponding result for Class 2 graphs with maximum degree 3, since we cannot handle the presence of more precolored edges using our method. Nevertheless, we note that since $\Delta(L(G)) \leq 4$ if G is 3-regular, where $L(G)$ denotes the line graph of G, it follows from the characterization of non-degree-choosable graphs[1] proved in [5, 10] that one can decide in a polynomial time whether a given partial 4-coloring of a graph with maximum degree 3 is extendable to a proper 4-coloring. In particular, any partial coloring with at most three precolored edges of a subcubic Class 2 graph is extendable (while a precoloring of four edges is obviously not always possible to extend). Thus, subcubic Class 2 graphs constitutes another large family of graphs which admits an Evans-type result. So while there are well-known examples of subcubic Class 1 graphs that do not admit an Evans-type result (see e.g. [8]), such examples do not exist for Class 2 graphs.

Furthermore, let us note that the condition on degree here is best possible since there are 4-regular Class 2 graphs where not every partial coloring of at most 4 edges is extendable to a proper 5-coloring [3]; indeed K_5 is such a graph.

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