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MULTICOLOR RAMSEY NUMBERS AND STAR-CRITICAL RAMSEY NUMBERS INVOLVING FANS

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Abstract

For graphs G and H, the multicolor Ramsey number $r_{k+1}(G;H)$ is defined as the minimum integer N such that any edge-coloring of K_N by $k+1$ colors contains either a monochromatic G in the first k colors or a monochromatic H in the last color. We shall write two color Ramsey numbers as $r(G,H)$. For graphs F, G and H, let $F \to (G,H)$ signify that any red/blue edge coloring of F contains either a red G or a blue H. Define the star-critical Ramsey number $r^*(G,H)$ as $\max\{s \mid K_r \setminus K_{1,s} \to (G,H)\}$ where $r = R(G,H)$. A fan F_n is a graph that consists of n copies of K_3 sharing a common vertex, and a book $B_n^{(p)}$ is a graph that consists of n copies of K_{p+1} sharing a common K_p. In this note, we shall show the upper bounds for $r_{k+1}(K_{t,s};F_n)$, $r_{k+1}(K_{2,s};F_n)$, $r_{k+1}(C_{2t};F_n)$, some of which are sharp up to the sub-linear term asymptotically. We also obtain the value of $r^*(F_m, B_n^{(p)})$ as $n \to \infty$.

Keywords: multicolor Ramsey number, star-critical Ramsey number, fan, book.

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1. Introduction

For simple graph G, let $v(G) = |V(G)|$ and $e(G) = |E(G)|$, respectively. For graphs G and H, the multicolor Ramsey number $r_{k+1}(G;H)$ is defined as the

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minimum integer N such that any edge-coloring of K_N by $k+1$ colors contains either a monochromatic G in the first k colors or a monochromatic H in the last color. We shall write two color Ramsey number $r_2(G;H)$ as $r(G,H)$. Let $r(G,G)=r(G)$ be the diagonal Ramsey number. Then we call $r(G,H)$ the off-diagonal Ramsey number when $G\neq H$. For graphs F, G and H, let $F\to(G,H)$ signify that any red/blue edge coloring of graph F contains a red subgraph G or a blue subgraph H. Call graph F a Ramsey graph if $F\to(G,H)$ and $v(F)=r(G,H)$. Let $K_r\setminus F$ denote the graph obtained from K_r by deleting the edges of F from K_r, where F is viewed as a subgraph of K_r.

A new problem in Ramsey theory to consider is that the largest surplus graph F such that $K_r\setminus F$ remains a Ramsey graph. Thus we shall define F as a surplus subgraph of (G,H). If we consider F as a star, then we have the following definition.

Definition [26]**.** The definition of star-critical Ramsey number is

$$r^*(G,H)=\max\{s\mid K_r\setminus K_{1,s}\to(G,H)\},$$

where $r=r(G,H)$.

In this problem, one may ask how to add an extra vertex u and connect u with s vertices of K_{r-1} completely such that the resultant graph remains a Ramsey graph of (G,H). Let $K_{r-1}\sqcup K_{1,s}$ signify the graph that consists of a complete graph K_{r-1} and an extra vertex u obtained by connecting vertex u with random s vertices from $V(K_{r-1})$ completely. Hook and Isaak [14] introduced the definition of the star-critical Ramsey number first as $r_*(G,H)=\min\{s\mid K_{r-1}\sqcup K_{1,s}\to(G,H)\}$, where $r=r(G,H)$.

By above definitions, $r(G,H)=r^*(G,H)+r_*(G,H)+1$. The study of star-critical Ramsey numbers attracts much interests, see [17, 21, 28, 30].

For a vertex $u\in V(G)$ and set $U\subseteq V(G)$, let $N^R(u,U)$ and $N^B(u,U)$ denote the sets of neighbors of u in set U in graph R and B, respectively. Define $d^R(u,U)=|N^R(u,U)|$ and $d^B(u,U)=|N^B(u,U)|$. Let $\chi(G)$ denote the chromatic number of G. Let $\delta(G)$ and $\Delta(G)$ be the minimum degree and maximum degree of G, respectively. Let $V_1,V_2,\ldots,V_{\chi(G)}$ be the color classes of G with $|V_1|\le|V_2|\le\cdots\le|V_{\chi(G)}|$. Define $s(G)=|V_1|$, and

$$\tau(G)=\min\min_{\substack{v\in V_1\\|V_1|=s(G)}}\min_{2\le i\le\chi(G)}|N_G(v)\cap V_i|,$$

in which the first minimum takes over all proper vertex colorings of G.

For graphs G and H, where H is connected with $v(H)\ge s(G)$, if $r(G,H)=(\chi(G)-1)(v(H)-1)+s(G)$, then Burr [3] defined that H is G-good. Hao and Lin [13] gave a general upper bound for $r^*(G,H)$.

Lemma 1 [13]**.** *For graph G with $\chi(G) \geq 2$ and connected graph H with $v(H) \geq s(G)$, if H is G-good, then*

$$r^*(G, H) \leq \max\{s(G) - 2, v(H) + s(G) - \delta(H) - \tau(G) - 1\}.$$

Call a graph $B_n^{(p)}$ to be a book that consists of n copies of K_{p+1} sharing a common K_p, and a graph F_n to be a fan that consists of n copies of K_3 sharing a common vertex. Book and fan graphs play important roles in graph Ramsey theory. It was shown by Rousseau and Sheehan [23] that $r(B_n) = 4n + 2$ for infinitely many n. Moreover, Conlon [5] obtained $r(B_n^{(m)}) \sim 2^m n$ as $n \to \infty$. Chen, Yu and Zhao [4] improved the bound of $r(F_n)$ as $\frac{9}{2}n - 5 \leq r(F_n) \leq \frac{11}{2}n + 6$. Recently, Dvořák and Metrebian [6] improved the upper bound as $r(F_n) \leq \frac{31}{6}n + 15$. For the results about the off-diagonal cases, see [29].

It is difficult to determine the exact values of Ramsey numbers involving fans. The case of multicolor Ramsey numbers involving fans is even worse. In this note, we focus on the multicolor Ramsey numbers and star-critical Ramsey numbers involving fans.

For positive functions $f(n)$ and $g(n)$, we write $f(n) = o(g(n))$ if $\frac{f(n)}{g(n)} \to 0$ as $n \to \infty$, and $f(n) = \Theta(g(n))$ if $c_1 g(n) \leq f(n) \leq c_2 g(n)$ for some constants $c_1, c_2 > 0$ and all large n.

Theorem 2. *Let $s \geq t \geq 3$ and $k \geq 1$ be fixed integers. If $\epsilon > 0$, then*

$$r_{k+1}(K_{t,s}; F_n) \leq 2n + (1 + \epsilon)k(s - t + 1)^{1/t}(2n)^{1-1/t}$$

for all sufficiently large n.

Then we consider a special case of $r_{k+1}(K_{t,s}; F_n)$.

Theorem 3. *Let $s \geq 2$ and $k \geq 1$ be fixed integers. Let $a = r_k(K_{2,s})$. If $\epsilon > 0$, then*

$$r_{k+1}(K_{2,s}; F_n) \leq 2n + \left\lceil k\sqrt{(s-1)2n} + \frac{k(ks - k + 1)}{2} + \epsilon \right\rceil + a$$

for all sufficiently large n.

Theorem 4. *Let $k, t \geq 1$ be fixed integers. If n is large, then*

$$r_{k+1}(C_{2t}; F_n) \leq 2n + (1 + o(1))c_t k n^{1/t}$$

where $c_t > 0$ is a constant depending on t only.

By the above results, we can get the following theorem.

Theorem 5. *Let $k, t, s \geq 1$ be fixed integers. Then there are infinitely many n such that $r_{k+1}(K_{2,s}; F_n) = 2n + (1 + o(1))k\sqrt{(s-1)2n}$ and $r_{k+1}(K_{3,3}; F_n) = 2n+(1+o(1))k(2n)^{2/3}$ as such $n \to \infty$, and $r_{k+1}(C_{2t}; F_n) = 2n+(1+o(1))c_t kn^{1/t}$ for all sufficiently large n where $c_t > 0$ is a constant depending on t only.*

Recently, Liu and Li [20] showed that for fixed integers m, p,

$$r\left(F_m, B_n^{(p)}\right) = 2(n + p - 1) + 1 \tag{1}$$

for large n. The star-critical Ramsey numbers involving large books also receive much attention, see [13, 18, 19]. It is determined in [19] that $r^*(K_2+G, B_n^{(p)}) \sim n$ for given graph G and fixed integer p as $n \to \infty$ where K_2+G is the join of graphs K_2 and G by connecting all the vertices of K_2 and all that of G completely. In this note, we also obtain the asymptotic value of $r^*(F_m, B_n^{(p)})$ as follows.

Theorem 6. *If m and p are fixed integers, then*

$$r^*(F_m, B_n^{(p)}) \sim n$$

as $n \to \infty$.

2. Proofs

For a graph G whose edges are colored by red and blue, let R and B denote the subgraphs of G induced by red and blue edges, respectively.

The Turán number $ex(n, H)$ of H is defined as the maximum $e(G)$ of an H-free graph G of order n. A well known argument called *double counting* of Kövári, Sós and Turán [16] shows that

$$ex(N, K_{t,s}) \leq \frac{1}{2}\left[(s-1)^{1/t}N^{2-1/t} + (t-1)N\right] \tag{2}$$

for $s \geq t \geq 3$. Furthermore, Füredi [12] obtained

$$ex(N, K_{t,s}) \leq \frac{1}{2}\left[(s-t+1)^{1/t}N^{2-1/t} + tN + tN^{2-2/t}\right],$$

which improves (2) for $s \geq t \geq 3$.

Erdős, Füredi, Gould and Gunderson [10] showed that for $n \geq 1$ and $N \geq 50n^2$,

$$ex(N, F_n) \leq \left\lfloor \frac{N^2}{4} \right\rfloor + n^2 - cn,$$

where $c = 1$ for odd n and $c = 3/2$ for even n.

For even cycles C_{2t}, Bondy and Simonovits [2] proved that for any $t \geq 2$,

$$ex(N, C_{2t}) \leq c_t N^{1+1/t}$$

for large N, where $c_t > 0$.

Proof of Theorem 2. Let $\ell = (1+\epsilon)k(s-t+1)^{1/t}(2n)^{1-1/t}$ and $N = 2n+\ell$ where $\epsilon > 0$. Color the edges of K_N by colors $1, 2, \ldots, k+1$. Denote by G_i the spanning subgraph of order N induced by edges of K_N in color i. To prove $r_{k+1}(K_{t,s}; F_n) \leq N$, it suffices to show that $k \cdot ex(N; K_{t,s}) + ex(N; F_n) \leq \binom{N}{2}$. Otherwise if $\binom{N}{2} = \sum_{i=1}^{k+1} e(G_i) > k \cdot ex(N; K_{t,s}) + ex(N; F_n)$, then either $e(G_i) > ex(N; K_{t,s})$ for some $1 \leq i \leq k$, or $e(G_{k+1}) > ex(N; F_n)$. Thus we get a $K_{t,s}$ in color i or an F_n in color $k+1$. Then we have

$$\frac{N(N-1)}{2} > \frac{k}{2}\left[(s-t+1)^{1/t}N^{2-1/t} + tN + tN^{2-2/t}\right] + \left\lfloor \frac{N^2}{4} \right\rfloor + n^2 - cn,$$

where $c = 1$ for odd n and $c = 3/2$ for even n. Equivalently,

$$2N^2 - 2N > 2k(s-t+1)^{1/t}N^{2-1/t} + 2ktN + 2ktN^{2-2/t} + N^2 + 4(n^2 - cn).$$

So it holds

$$1 - \frac{2+2kt}{N} > \frac{2k(s-t+1)^{1/t}}{N^{1/t}} + \frac{2kt}{N^{2/t}} + \frac{4(n^2-cn)}{N^2}. \tag{3}$$

Note that $1 - \frac{2+2kt}{N} > 1 - \frac{2+2kt}{2n}$, $\frac{2k(s-t+1)^{1/t}}{N^{1/t}} < \frac{2k(s-t+1)^{1/t}}{(2n)^{1/t}}$. The third term on right side of (3) is

$$\begin{aligned}\frac{4(n^2-cn)}{N^2} &= \frac{4n^2-4cn}{(2n+\ell)^2} = \left(1-\frac{c}{n}\right)\left(1+\frac{\ell}{2n}\right)^{-2} \\ &= \left(1-\frac{c}{n}\right)\left(1-\frac{\ell}{n}+\Theta\left(\frac{1}{n^2}\right)\right) = 1 - \frac{c+\ell}{n} + \Theta\left(\frac{\ell}{n^2}\right).\end{aligned}$$

The second term on right side of (3) is $\frac{2kt}{N^{2/t}} = \Theta\left(\frac{1}{n^{2/t}}\right)$. So

$$1 - \frac{1+kt}{n} > \frac{2k(s-t+1)^{1/t}}{(2n)^{1/t}} + 1 - \frac{c+\ell}{n} + \Theta\left(\frac{\ell}{n^2}\right) + \Theta\left(\frac{1}{n^{2/t}}\right).$$

Since $\frac{\ell}{n^2} = \Theta\left(\frac{1}{n^{1+1/t}}\right) = o\left(\frac{1}{n^{2/t}}\right)$, we only need to show that

$$\frac{1+kt}{n} < -\frac{2k(s-t+1)^{1/t}}{(2n)^{1/t}} + \frac{c+\ell}{n} - \Theta\left(\frac{1}{n^{2/t}}\right).$$

Namely,

$$1 + kt < c + \ell - k(s - t + 1)^{1/t}(2n)^{1-1/t} - \Theta\left(n^{1-2/t}\right).$$

Then

$$\ell > k(s - t + 1)^{1/t}(2n)^{1-1/t} + \Theta\left(n^{1-2/t}\right) + 1 + kt - c.$$

Since $\Theta\big(n^{1-2/t}\big) = o\big(n^{1-1/t}\big)$, we have that if $\epsilon > 0$ and $\ell = (1 + \epsilon)k(s - t + 1)^{1/t}(2n)^{1-1/t}$, then the claimed statement follows for large n. ∎

Lemma 7 [27]**.** *Let $s \geq 2$ be an integer. Then*

$$ex(N; K_{2,s}) \leq \frac{1}{2}\left(\sqrt{s-1}N^{3/2} + \frac{N}{2}\right).$$

Proof of Theorem 3. Let $\ell = \left\lceil k\sqrt{(s-1)2n} + \frac{k(ks-k+1)}{2} + \epsilon \right\rceil$ and $N = 2n + \ell + a$ where $\epsilon > 0$. Color the edges of K_N by colors $1, 2, \ldots, k + 1$. Denote by G_i the spanning subgraph of order N induced by edges of K_N in color i. Let $d^i(v)$ be the number of neighbors that are adjacent to v by color i for $1 \leq i \leq k + 1$.

We claim that if $(2n + \ell + a)\ell > 2k \cdot ex(N; K_{2,s})$, then $r_{k+1}(K_{2,s}; F_n) \leq N$. For any vertex $v \in V(K_N)$, we have $d^{k+1}(v) \leq 2n + a - 1$. Otherwise if $d^{k+1}(v) \geq 2n + a$, then $N^{k+1}(v)$ can only contain at most $n - 1$ copies of K_2 in color $k + 1$, otherwise we have an F_n in color $k+1$. So the graph induced by $N^{k+1}(v)$ contains a K_{a+2}, whose edges are colored by $i = 1, \ldots, k$. Then we get a $K_{2,s}$ in color i for some $i = 1, \ldots, k$ since $a = r_k(K_{2,s})$. Then

$$\sum_{i=1}^{k} d^i(v) \geq N - 1 - (2n + a - 1) = \ell.$$

Thus

$$\sum_{i=1}^{k} e(G_i) \geq \frac{(2n + \ell + a)\ell}{2} > k \cdot ex(N; K_{2,s})$$

such that $e(G_i) \geq ex(N; K_{2,s})$ for some $i = 1, \ldots, k$. Then we only need to prove $(2n + \ell + a)\ell > 2k \cdot ex(N; K_{2,s})$. Namely, $(2n + \ell + a)\ell > k\big(\sqrt{s-1}(2n + \ell + a)^{3/2} + \frac{2n+\ell+a}{2}\big)$. Equivalently,

$$k\sqrt{(s-1)2n} + \frac{k(ks-k+1)}{2} + \epsilon > k\sqrt{(s-1)(2n+\ell+a)} + \frac{k}{2}.$$

So

$$\frac{ks-k}{2} + \frac{\epsilon}{k} > \sqrt{(s-1)(2n+\ell+a)} - \sqrt{(s-1)2n},$$

which is true since

$$\sqrt{(s-1)(2n+\ell+a)}-\sqrt{(s-1)2n}=\frac{\sqrt{s-1}(\ell+a)}{\sqrt{2n+\ell+a}+\sqrt{2n}}\sim\frac{\sqrt{s-1}\ell}{2\sqrt{2n}}\to\frac{k(s-1)}{2}$$

as $n\to\infty$. ■

***Proof of Theorem* 4.** Let $\ell=(1+o(1))c_tkn^{1/t}$ and $N=2n+\ell$. Denote by G_i the spanning subgraph of order N induced by edges of K_N in color i. Similarly, it suffices to show that

$$k\cdot ex(N;C_{2t})+ex(N;F_n)\le\binom{N}{2}.$$

Namely,

$$N^2-2N\ge 4kc_tN^{1+1/t}+4\big(n^2-cn\big)$$

where $c=1$ for odd n and $c=3/2$ for even n, which is

$$1-\frac{2}{N}\ge\frac{4kc_t}{N^{1-1/t}}+\frac{4(n^2-cn)}{N^2}.$$

Since $1-\frac{2}{N}>1-\frac{1}{n}$, $\frac{4kc_t}{N^{1-1/t}}<\frac{4kc_t}{(2n)^{1-1/t}}$ and $\frac{4(n^2-cn)}{(2n+\ell)^2}=1-\frac{\ell+c}{n}+\Theta\big(\frac{\ell}{n^2}\big)$, we have

$$1-\frac{1}{n}\ge\frac{4kc_t}{(2n)^{1-1/t}}+1-\frac{\ell+c}{n}+\Theta\left(\frac{\ell}{n^2}\right).$$

Thus

$$\frac{\ell+c-1}{n}\ge\frac{4kc_t}{(2n)^{1-1/t}}+\Theta\left(\frac{\ell}{n^2}\right).$$

Therefore it holds

$$\ell+c-1\ge 2kc_t(2n)^{1/t}+\Theta\left(\frac{\ell}{n}\right).$$

Note that $\Theta\big(\frac{\ell}{n}\big)=\Theta\big(\frac{1}{n^{1-1/t}}\big)=o\big(n^{1/t}\big)$. We shall get the desired upper bound, completing the proof. ■

Lemma 8 [27]**.** *Let $k,s\ge 1$ be fixed integers. Then there are infinitely many n such that*

$$r_{k+1}(K_{2,s};K_{1,2n})=2n+(1+o(1))k\sqrt{(s-1)2n}.$$

Lemma 9 [27]**.** *Let $k\ge 1$ be fixed integer. Then there are infinitely many n such that*

$$r_{k+1}(K_{3,3};K_{1,2n})=2n+(1+o(1))k(2n)^{2/3}.$$

Lemma 10 [27]**.** *Let H be a bipartite graph with $ex(N,H) \geq cN^{2-\eta}$ as $N \to \infty$, where c and η are positive constants. If there are extremal graphs G_N of order N for $ex(N,H)$ such that $\delta(G_N) \sim \Delta(G_N)$ as $N \to \infty$, then*

$$r_{k+1}(H; K_{1,n}) \geq n + (1-\epsilon)2kcn^{1-\eta}$$

for large n, where $\epsilon > 0$.

***Proof of Theorem* 5.** By Lemma 8 and Theorem 3, we have $r_{k+1}(K_{2,s}; F_n) = 2n+(1+o(1))k\sqrt{(s-1)2n}$ for infinitely many n as $n \to \infty$ since $r_{k+1}(K_{2,s}; F_n) \geq r_{k+1}(K_{2,s}; K_{1,2n})$.

By Lemma 9 and Theorem 2, we have $r_{k+1}(K_{3,3}; F_n) = 2n+(1+o(1))k(2n)^{2/3}$ for infinitely many n as $n \to \infty$ since $r_{k+1}(K_{3,3}; F_n) \geq r_{k+1}(K_{3,3}; K_{1,2n})$.

By Lemma 10 and Theorem 4, we have $r_{k+1}(C_{2t}; F_n) = 2n+(1+o(1))c_t kn^{1/t}$ for sufficiently large n since $r_{k+1}(C_{2t}; F_n) \geq r_{k+1}(C_{2t}; K_{1,2n})$. ∎

In terms of the proof of Theorem 6, the basic tool we mainly used is the stability theorem from Erdős and Simonovits. Intuitively, stability theorem describes that the structure of large graph G that has no subgraph G_1 is similar to that of $K_q(N/q)$ where $\chi(G_1) = q+1$ if $e(G) - e(K_q(N/q))$ is very small.

Lemma 11 [8, 9, 24]**.** *Let G be a given "forbidden" graph with $\chi(G) = k+1$. For each $\xi > 0$, there exist $\delta = \delta(\xi) > 0$ and $N_0 = N_0(\delta) > 0$ such that if H is a graph with $v(H) = N > N_0$ and $e(H) > \frac{k-1}{2k}N^2 - \delta N^2$ that contains no G, then there is a partition of $V(H)$ into classes $V_1, V_2, \ldots, V_k$ such that*

(i) *$N/k - \xi N < |V_i| < N/k + \xi N$ for each $i = 1, 2, \ldots, k$;*

(ii) *all but at most ξN^2 pairs (u,v) with $u \in V_i$ and $v \in V_j$ $(i \neq j)$ belong to $E(H)$;*

(iii) *at most ξN^2 pairs (u,v) with $u, v \in V_i$ belong to $E(H)$;*

(iv) *no vertex is adjacent to fewer vertices in some other class than the number of vertices to which it is adjacent in its own class.*

The following Regularity Lemma is due to Szemerédi. For graph G, let $X, Y \subset V(G)$ be nonempty subsets and $X \cap Y = \emptyset$. Let $e(X,Y)$ be the number of edges between sets X and Y. Denote by $d(X,Y) = \frac{e(X,Y)}{|X||Y|}$ the density of pair (X,Y). A pair (U,V) is called ϵ-regular if $|d(U,V) - d(X,Y)| < \epsilon$ whenever $U \subset X$, $V \subset Y$ such that $|U| \geq \epsilon|X|$ and $|V| \geq \epsilon|Y|$ for $\epsilon > 0$.

Lemma 12 [25]**.** *For positive integer ℓ and real ϵ, there is a large integer $N = N(\ell, \epsilon)$ such that the vertex set $V(G)$ of any graph G has a partition $\bigcup_{i=1}^{q} V_i$ with $\ell \leq q \leq N$ if $n = v(G)$ is large enough, where $|V_0| < \epsilon n$, $|V_1| = |V_2| = \cdots = |V_q|$ and all but at most ϵq^2 pairs (V_i, V_j), $1 \leq i \neq j \leq q$, are ϵ-regular.*

Before proceeding to proof, we also need to introduce the following lemmas.

Lemma 13 [7]. *Let integers $t \geq 3, p \geq 2$. There exists a $c_{t,p} > 0$ such that if G is a K_t-free graph of order $n \geq R(K_t, K_p)$, then G contains at least $c_{t,p}n^p$ independent p-sets.*

Lemma 14 [15]. *Let $0 < \epsilon < \eta \leq 1$ and $(\eta - \epsilon)^{p-2} > \epsilon$. Suppose G is a graph and $V(G) = V \cup V_1 \cup \cdots \cup V_k$ is a partition with $|V| = |V_1| = \cdots = |V_k|$ such that each pair (V, V_i) is ϵ-regular and $e(V, V_i) \geq \eta |V||V_i|$, $1 \leq i \leq k$. Furthermore, G has at least*

$$k|V|(m - \epsilon p|V|^p)(\eta - \epsilon)^p$$

cliques ω with $|\omega| = p + 1$ and $|\omega \cap V| = p$, where m denotes the number of all cliques of size p of V.

Lemma 15 [15]. *For $0 < \epsilon < \eta < 1$ and an integer $s \geq 1$, let H be a graph obtained from a given graph F by replacing each vertex in $V(F)$ by s vertices and $E(F)$ with ϵ-regular pairs of density at least η. Let G be a subgraph of F with maximum degree $\Delta > 0$. If $\epsilon \leq (\eta - \epsilon)^{\Delta}/(\Delta + 2)$, then G is a subgraph of H.*

Lemma 16 [11]. *Suppose integer $k \geq 2$ and $\epsilon > 0$. Then there is $n_0 = n_0(k, \epsilon)$ that has the following property. If a graph G with $n = v(G) \geq n_0$ and*

$$e(G) \geq \left(\frac{k-2}{k-1} + \epsilon\right)\binom{n}{2},$$

then $K_k(\ell)$ is a subgraph of G for some $\ell \geq c\,\epsilon \log n$.

Lemma 17 [1]. *For integers $t \geq 1$ and $s \geq 2$, $r(tK_2, K_s) = 2(t-1) + s$.*

***Proof of Theorem* 6.** By Lemma 1, we only need to prove the lower bound. Take ζ to be a sufficiently small number and let $\xi \ll \zeta$. By (1), let $r = r\big(F_m, B_n^{(p)}\big) = 2(n + p - 1) + 1$. We may assume that there is neither a red F_m nor a blue $B_n^{(p)}$ in $G = K_r \setminus K_{1,M}$ where $M = (1 - \xi_0)n$ and $\xi_0 > 0$ is sufficiently small.

Claim 1. *The number of the red edges in graph K_N is at least $(1/4 - o(1))N^2$ in which $N = r - 1$.*

Proof. Since the proof is similar to [22] by Nikiforov and Rousseau, and the version in terms of $r\big(F_m, B_n^{(p)}\big)$ can also be found in [20], we only give a sketch of the proof. First, apply Lemma 12 on red subgraph of K_N, and we shall get a partition of $V(K_N) = V_0 \cup V_1 \cup \cdots \cup V_q$ such that almost all pairs (V_i, V_j) are ϵ-regular for $1 \leq i < j \leq q$. For any ϵ-regular pair (V_i, V_j), we say it is dense ϵ-regular if $d(V_i, V_j) > a$ for some positive constant a, and we can get the number of these dense pairs in blue subgraph of K_N is at least $(1/4 + o(1))q^2$. Otherwise by Lemmas 15 and 16, we can get a red F_m. By Lemma 13, there are $\Theta(N^p)$ blue cliques of size p in each set V_i for $1 \le i \le q$. By Lemma 14, almost all vertices in V_j can form the pages of a p-book for every dense ϵ-regular pair (V_i, V_j) in blue subgraph of K_N, and the base of each p-books is in one V_i. Then each ϵ-regular pair (V_i, V_j) with $d(V_i, V_j)$ not very close to 1 yields many additional pages to such books. Note that K_N contains no blue $B_n^{(p)}$. So there are at least $(1/4 - o(1))q^2$ ϵ-regular pairs (V_i, V_j) with $d(V_i, V_j)$ close to 1. Thus the number of the red edges is at least $(1/4 - o(1))N^2$. □

Then by Lemma 11, $V(K_N)$ namely $V(R)$ can be divided into two classes V_1, V_2 and for $i = 1, 2$,

(i) $N/2 - \xi N < |V_i| < N/2 + \xi N$;

(ii) all but at most ξN^2 pairs $\{u, v\}$ with $u \in V_i$, $v \in V_{3-i}$ are colored red;

(iii) at most ξN^2 pairs $\{u, v\}$ with $u, v \in V_i$ are colored red;

(iv) for any vertex $z \in V_i$, $d^R(z, V_i) \le d^R(z, V_{3-i})$.

Let $V_i' = \{z \in V_i \mid d^R(z, V_{3-i}) \ge (1 - 2\sqrt{\xi})|V_{3-i}|\}$ for $i = 1, 2$.

Claim 2. *$|V_i'| \ge (1 - 3\sqrt{\xi})|V_i|$ for $i = 1, 2$.*

Proof. We shall only prove $|V_1'| \ge (1 - 3\sqrt{\xi})|V_1|$ by symmetry. Suppose to the contrary, $|V_1'| < (1 - 3\sqrt{\xi})|V_1|$. Then $|V_1 \setminus V_1'| \ge 3\sqrt{\xi}|V_1|$. For any vertex $z \in V_1 \setminus V_1'$, $d^R(z, V_2) < (1 - 2\sqrt{\xi})|V_2|$. So $d^B(z, V_2) \ge 2\sqrt{\xi}|V_2|$. Note that $|V_i| \ge (1/2 - \xi)N$. Therefore the number of blue edges between V_1 and V_2 is at least $d^B(z, V_2) \cdot |V_1 \setminus V_1'| > \xi N^2$, which contradicts to (ii). □

Select C_{i1} from $V_i \setminus V_i'$ such that $C_{i1} = \{z \in V_i \setminus V_i' \mid d^R(z, V_{3-i}') \ge \zeta |V_{3-i}'|\}$ and let C_{i2} be the set of the remaining vertices of $V_i \setminus V_i'$ for $i = 1, 2$.

Claim 3. *For $i = 1, 2$, we have $d^R(z, V_i') \le m - 1$ for any vertex $z \in C_{i1}$.*

Proof. By symmetry, suppose to the contrary that $d^R(z, V_1') \ge m$ for some vertex $z \in C_{11}$. Select m vertices in $N^R(z, V_1')$ and denote by $Z = \{z_1, z_2, \ldots, z_m\}$. Note that

$$d^R(z_i, V_2') \ge d^R(z_i, V_2) - |V_2 \setminus V_2'| \ge (1 - 2\sqrt{\xi})|V_2| - 3\sqrt{\xi}|V_2| = (1 - 5\sqrt{\xi})|V_2|$$

for $i = 1, 2, \ldots, m$. Thus we have

$$\left| \bigcap_{z_i \in W} N^R(z_i, V_2') \right| \ge m \left(1 - 5\sqrt{\xi}\right) |V_2| - (m-1)|V_2| = \left(1 - 5m\sqrt{\xi}\right) |V_2|.$$

Since $d^R(z, V_2') \ge \zeta |V_2'|$, it follows that

$$\begin{aligned} \left| \left(\bigcap_{z_i \in W} N^R(z_i, V_2') \right) \cap N^R(z, V_2') \right| &\ge \left(1 - 5m\sqrt{\xi}\right) |V_2| + \zeta |V_2'| - |V_2'| \\ &\ge \left(\frac{1}{2}\zeta - \frac{5}{2} m\sqrt{\xi} - 2\zeta\sqrt{\xi}\right) N \ge m \end{aligned}$$

for large n. Therefore R contains an F_m, a contradiction. □

Claim 4. $|C_{i2}| \leq c$ *with* $c = r(F_m, K_p) - 1$.

Proof. We shall only prove $|C_{12}| \leq c$ as symmetry. Suppose to the contrary, $|C_{12}| \geq c+1$. Note that for any vertex $z \in C_{12}$, $d^R(z, V_2') < \zeta|V_2'|$. Apply Claim 2, we can obtain that

$$\begin{aligned} d^R(z, V_1') &\leq d^R(z, V_1) \leq d^R(z, V_2) \leq d^R(z, V_2') + |V_2 \setminus V_2'| \\ &\leq \zeta|V_2'| + \frac{3\sqrt{\xi}}{1-3\sqrt{\xi}}|V_2'| \leq 2\zeta|V_2'| \end{aligned}$$

since $\xi \ll \zeta$. Therefore

$$\begin{aligned} d^B\big(z, V_1' \cup V_2'\big) &= |V_1'| + |V_2'| - \big(d^R(z, V_1') + d^R(z, V_2')\big) \\ &> (1-3\zeta)\big(|V_1'| + |V_2'|\big). \end{aligned}$$

Since $|C_{12}| \geq r(F_m, K_p)$ and R contains no F_m, there is a blue K_p in C_{12}. Then the number of common blue neighbors of p-clique in $V_1' \cup V_2'$ is at least

$$\begin{aligned} (1-3p\zeta)\left(|V_2'| + |V_2'|\right) &\geq (1-3p\zeta)\left(1-3\sqrt{\xi}\right)\left(|V_1| + |V_2|\right) \\ &\geq 2(1-4p\zeta)\left(n+p-1\right) \geq n \end{aligned}$$

for large n. Thus B contains a $B_n^{(p)}$, a contradiction. □

Since R contains no F_m, the graph induced by $N^R(u, V_i')$ contains no red mK_2 for each vertex u in V_{3-i}'. Note that

$$\left|N^R(u, V_i')\right| \geq \left|N^R(u, V_i)\right| - \left|V_i \setminus V_i'\right| \geq \left(1-5\sqrt{\xi}\right)|V_i|.$$

By Lemma 17, each set V_i' contains a blue clique X_i with

$$|X_i| \geq \left(1-5\sqrt{\xi}\right)|V_i| - (2m-2) \geq \left(1-6\sqrt{\xi}\right)|V_i|$$

for large n. Similar to the proof of Claim 3, we have $d_R(u, V_i') \leq m-1$ for each vertex $u \in V_i'$. Then we can put all the vertices of $V_i' \setminus X_i$ into C_{i1}, and redefine sets V_1 and V_2 as

$$V_1 = X_1 \cup C_{11} \cup C_{22};$$

$$V_2 = X_2 \cup C_{21} \cup C_{12}.$$

Then we can find a subset $X_i' \subseteq X_i$ and each vertex in $V_i \setminus X_i'$ is blue-adjacent to X_i completely with

$$\begin{aligned} |X_1'| &\geq |X_1| - m|C_{11}| - \zeta|C_{22}|\left|V_1'\right| \\ &\geq \left(1-6\sqrt{\xi}\right)|V_1| - m \cdot 6\sqrt{\xi}|V_1| - \zeta \cdot c|V_1| \\ &\geq \left(\frac{1}{2} - 2c\zeta\right)N \geq p \end{aligned}$$

for large n. Similarly, we can obtain $|X_2'| \geq p$. Note that $|V_1|+|V_2| = 2(n+p-1)$. If there exists a set V_i such that $|V_i| \geq N/2+1$, then B contains a $B_n^{(p)}$. Therefore, we conclude that $|V_1| = |V_2| = n+p-1$, and each set V_i contains a blue $B_{n-1}^{(p)}$ for $i = 1, 2$.

Now we consider a new vertex v. Since B contains no blue $B_n^{(p)}$, then $d^B(v, X_i') \leq p-1$. Thus there are at least

$$\begin{aligned} r-1-M-\sum_{i=1}^{2}|V_i \setminus X_i'| - 2d^B(v, X_i') &\geq N-(1-\xi_0)n-4c\zeta N-2p+2 \\ &\geq \left(1-9c\zeta-\sqrt{\xi}+\xi_0\right)n \geq \frac{1}{2}n \end{aligned}$$

red edges between v and $X_1' \cup X_2'$ for large n. Now we may assume that $d^R(v, X_2') \geq d^R(v, X_1')$ without loss of generality. By choosing $\xi_0 = \xi_0(\zeta)$ suitably, we can get that

$$\begin{aligned} d^R(v, X_1') &\geq r-1-M-|V_2|-|V_1 \setminus X_1'|-(p-1) \\ &\geq 2(n+p-1)-(1-\xi_0)n-(n+p-1)-4c\zeta(n+p-1)-(p-1) \\ &\geq (\xi_0-5c\zeta)n \geq m \end{aligned}$$

for large n. Note that $d^R(v, X_2') \geq \frac{1}{2} \cdot \frac{1}{2}n = \frac{1}{4}n$. Then select m vertices from $N^R(v, X_1')$ and denote them by $L = \{l_1, l_2, \ldots, l_m\}$. Note that $d^R(l_i, X_2') \geq |X_2'|-(p-1)$, otherwise there is a blue $B_n^{(p)}$. We can obtain that

$$\left|\bigcap_{l_i \in L} N^R(l_i, X_2')\right| \geq m\left(|X_2'|-(p-1)\right)-(m-1)|X_2'| = |X_2'|-m(p-1).$$

Moreover, we have

$$\begin{aligned} \left|\left(\bigcap_{l_i \in H} N_R(l_i, X_2')\right) \cap N_R(v, X_2')\right| &\geq |X_2'|-m(p-1)+\frac{1}{4}n-|X_2'| \\ &= \frac{1}{4}n-m(p-1) \geq m \end{aligned}$$

for large n. Hence we find a red F_m. ■

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