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RAMSEY NUMBERS FOR A LARGE TREE VERSUS MULTIPLE COPIES OF COMPLETE GRAPHS OF DIFFERENT SIZES

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Abstract

For two graphs G and H, let $G \cup H$ be the union of vertex-disjoint copy of G and H. And the Ramsey number $R(G,H)$ is the minimum integer N such that any red-blue coloring of the edges of the complete graph K_N contains either a red copy of G or a blue copy of H. If G is connected and $v(G) \geq s(H)$, it is well known that $R(G,H) \geq (v(G)-1)(\chi(H)-1)+s(H)$, where $\chi(H)$ is the chromatic number of H and $s(H)$ is the size of the smallest color class taken over all proper vertex-colorings of H with $\chi(H)$ colors. Burr defined a connected graph G as *H-good* if the above inequality becomes equality. In this paper, for integers $t \geq 1$ and $m_1 \geq m_2 \geq \cdots \geq m_t$, we show that if n is sufficiently large, then any tree T_n is $\bigcup_{i=1}^{t} K_{m_i}$-good. In particular, we show that the condition of n being sufficiently large can be relaxed when T_n is a star.

Keywords: Ramsey number, tree, Ramsey goodness.

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1. Introduction

For two graphs G and H, the *Ramsey number* $R(G, H)$ is the minimum integer N such that any red-blue coloring of the edges of the complete graph K_N contains either a red copy of G or a blue copy of H. Although there have been many results on the Ramsey number of graphs [10, 13, 24], the exact value of $R(G, H)$ is known only if at least one of G and H belongs to one of a few families of graphs. An intriguing case is that H is fixed while G is in some sense "sparse". In this case, if G is connected, there is a universal lower bound.

As usual, we write $v(G)$ as the *number of vertices* of G, $\chi(G)$ as the *chromatic number* of G, and $s(G)$ as the *chromatic surplus* of G, that is, the size of the smallest color class taken over all proper vertex-colorings of G with $\chi(G)$ colors. For a connected graph G and a graph H with $v(G) \geq s(H)$, Burr [4] proved that

$$R(G, H) \geq (v(G) - 1)(\chi(H) - 1) + s(H). \tag{1.1}$$

Furthermore, we say that the connected graph G is *H-good* if the equality holds. There are many known cases of Ramsey-goodness, e.g. [1, 2, 6, 7, 12, 13, 18, 19, 21, 23, 26] and their references. Moreover, we refer the reader to the survey papers by Conlon, Fox and Sudakov [10], and Radziszowski [24].

In this paper, we consider a Ramsey-goodness problem related to trees. Let T_n be a *tree* on n vertices and K_n be a *complete graph* on n vertices. For two graphs G and H, let $G \cup H$ denote the *vertex-disjoint union* of G and H, and tG denote the union of t vertex-disjoint copies of G. In the 1970s, before the definition of Ramsey-goodness was given by Burr, a well-known result of Chvátal [8] showed that any tree is K_m-good for each $m \geq 2$, and an earlier result of Chvátal and Harary [9] showed that any tree is $2K_2$-good. Hu and Peng [16] extended this result and proved that T_n is $K_m \cup K_l$-good for integers $n \geq 3$ and $m \geq l \geq 2$. Furthermore, they [17] determined the exact value of $R(T_n, tK_2)$, which yields that T_n is not tK_2-good for small n. Luo and Peng [20] recently proved that T_n is tK_m-good for n sufficiently large. Moreover, they remarked that $n \geq 16m^3t^4$ is enough. Actually, Burr [5] proved that if t is at least double exponential in $k = \max\{v(G), v(H)\}$, then

$$R(G, tH) = tv(H) + R(\mathcal{D}(G), H) - 1, \tag{1.2}$$

where H is a connected graph and $\mathcal{D}(G)$ is a set of all graphs formed from G by removing a maximal independent set. Recently, Bucić and Sudakov [3] obtained an exponential improvement over the above result of Burr [5], i.e., they showed that if $t \geq 2^{O(k)}$, then (1.2) holds. In the case where $R(G, H)$ is not exponential in k, Sulser and Trujić [25] show that $t \geq O(k^{10}R(G, H)^2)$ is enough. However, for a general tree T_n, we can not apply (1.2) to get the exact value of $R(T_n, tH)$ for even large t.

In this paper, we determine the Ramsey number for a large tree versus multiple copies of complete graphs of different sizes and give the following main result.

Theorem 1. *Let $t \geq 1$, $s \geq 1$ and $m > m_1 \geq m_2 \geq \cdots \geq m_t \geq 1$ be integers. If n is sufficiently large, then*

$$R\left(T_n, sK_m \cup \left(\bigcup_{i=1}^{t} K_{m_i}\right)\right) = (n-1)(m-1) + s.$$

From this, we can obtain the following concise corollary.

Corollary 2. *Let $t \geq 1$ and $m_1 \geq m_2 \geq \cdots \geq m_t \geq 1$ be integers. If n is sufficiently large, then*

$$R\left(T_n, \bigcup_{i=1}^{t} K_{m_i}\right) = (n-1)(m_1 - 1) + p,$$

where p is the number of K_{m_1} in the union of $K_{m_i}, i \in [t]$.

Let G, H_1 and H_2 be graphs. Note that if $H_1 \subseteq H_2$, then $r(G, H_1) \leq r(G, H_2)$. The lower bound of Theorem 1 is from (1.1). Since $sK_m \cup \big(\bigcup_{i=1}^{t} K_{m_i}\big) \subseteq sK_m \cup tK_{m-1}$, we just need to prove the following theorem to finish the proof of Theorem 1.

Theorem 3. *Let $s \geq 1, t \geq 1$ and $m \geq 2$ be integers. If n is sufficiently large, then*

$$R(T_n, sK_m \cup tK_{m-1}) = (n-1)(m-1) + s.$$

We remark that $n \geq 16m^3(2s+t)^4$ is enough for Theorem 3. Moreover, the condition that n is sufficiently large in Theorem 3 can be relaxed to $n \geq s+t+1$ when $T_n = S_n$. Here, S_n denotes the *star* on n vertices.

Theorem 4. *Let $s \geq 1$, $t \geq 1$ and $m \geq 2$ be integers. If $n \geq s+t+1$, then*

$$R(S_n, sK_m \cup tK_{m-1}) = (n-1)(m-1) + s.$$

We will use the following notations and definitions throughout the paper. For a graph G, let $V(G)$ be the *vertex set* of G and $E(G)$ be the *edge set* of G. For $U \subseteq V(G)$, let $G - U$ denote the graph obtained from G by deleting U and all edges incident to U, and let $G[U]$ denote the induced graph of G on U. For $H \subseteq G$, let $G - H$ be the subgraph induced on $V(G) \setminus V(H)$. In a red-blue edge-colored graph, the *red/blue neighbors* of v is those vertices adjacent to v in red/blue, and the *red/blue degree* of v is the number of red/blue neighbors of v.

2. Preliminaries

2.1. t-tree

Let $n \geq t+1 \geq 2$ be integers. We call a graph G on n vertices a *t-tree* if there is an ordered t-set $A = \{v_1, \ldots, v_t\} \subseteq V(G)$ such that for each $j \in [t]$, v_j is adjacent to exactly one of $V(G) \setminus \{v_j, \ldots, v_t\}$. Clearly, T_n is an $(n-1)$-tree, and a matching tK_2 is a t-tree. Moreover, we denote the induced subgraph of G on $V(G) \setminus A$ as G', i.e., $G' = G[V(G) \setminus A]$. By the definition of t-tree, the number of components of $G - \{v_t\}$ is equal to the number of components of G. Moreover, for each $j \in [t-1]$, the number of components of $G - \{v_j, \ldots, v_t\}$ is equal to the number of components of $G - \{v_{j+1}, \ldots, v_t\}$. Consequently, G is connected if and only if G' is connected.

Theorem 5. *Let $t \geq 1$, $m \geq 2$ and $n \geq t+2$ be integers. Let G be a connected t-tree on n vertices. If G is tK_m-good, tK_{m-1}-good and $(t+1)K_{m-1}$-good, and G' is K_m-good, then G is $(K_m \cup tK_{m-1})$-good.*

Proof. By (1.1), it is sufficient to prove that $r(G, K_m \cup tK_{m-1}) \leq (n-1)(m-1)+1$. Let $N = (n-1)(m-1)+1$. Color $E(K_N)$ by red or blue arbitrarily. Let H be the resulting graph and $V = V(H)$. If H contains a blue copy of $K_m \cup tK_{m-1}$, then we are done. So we may assume that H contains no blue copy of $K_m \cup tK_{m-1}$. Note that $N \geq (n-1)(m-2)+t = R(G, tK_{m-1})$ since $n \geq t+2$ and G is tK_{m-1}-good. Thus, H contains either a red copy of G or a blue copy of tK_{m-1}. We only need to consider the latter and denote it by F. Recall that G' is K_m-good and note that

$$v(H-F) = (n-1)(m-1)+1-t(m-1) = (n-t-1)(m-1)+1 = R(G', K_m).$$

Consequently, $H-F$ contains either a red copy of G' or a blue copy of K_m. If $H-F$ contains a blue copy of K_m, then there is a blue copy of $K_m \cup tK_{m-1}$ containing F in H. A contradiction to our assumption. Thus, $H-F$ contains a red copy of G'.

Since G is a connected t-tree, let $A = V(G) \setminus V(G') \triangleq \{v_1, \ldots, v_t\}$ such that for each $j \in [t]$, v_j is adjacent to exactly one of $V(G') \cup \{v_1, \ldots, v_{j-1}\}$. Moreover, let v_j^* be the neighbor of v_j in $V(G') \cup \{v_1, \ldots, v_{j-1}\}$. In the following, we will extend the red copy of G' to a red copy of G by embedding v_j one by one.

Let $i \in [t-1] \cup \{0\}$ be an integer and assume that we have already found i vertices to embed $\{v_1, \ldots, v_i\}$, i.e., H contains a red copy of $G - \{v_{i+1}, \ldots, v_t\}$. Let $I = V(G') \cup \{v_1, \ldots, v_i\}$. We will prove that there exists a vertex to embed v_{i+1}, i.e., H contains a red copy of $G - \{v_{i+2}, \ldots, v_t\}$. Note that $|I \setminus \{v_{i+1}^*\}| = n-t+i-1 \geq i+1$ since $n \geq t+2$. Let $U \subseteq I \setminus \{v_{i+1}^*\}$ be a vertex set with $|U| = i+1$. Moreover, $v_{i+1}^* \notin V \setminus (I \setminus U)$ and

$$|V \setminus (I \setminus U)| = (n-1)(m-1)+1-(n-t+i)+i+1 = (n-1)(m-2)+t+1.$$

Since G is $(t+1)K_{m-1}$-good, $H[V\setminus(I\setminus U)]$ contains either a red copy of G or a blue copy of $(t+1)K_{m-1}$. We only need to consider the latter and denote the vertex set of these $t+1$ vertex-disjoint copies of K_{m-1} by $V_1, \ldots, V_{t+1}$, respectively. Note that for all $k \in [t+1]$, there exists at least one vertex $u_k \in V_k$ such that $u_k v^*_{i+1}$ is red. Otherwise, there is a blue copy of $K_m \cup tK_{m-1}$ containing $\{v^*_{i+1}\} \cup \left(\bigcup_{p=1}^{t+1} V_i\right)$ in H. A contradiction to our assumption. Furthermore, there exists $k_0 \in [t+1]$ such that $V_{k_0} \cap U = \emptyset$ since $|U| = i+1 < t+1$. Consequently, we can extend the red copy of $G - \{v_{i+1}, \ldots, v_t\}$ to a red copy of $G - \{v_{i+2}, \ldots, v_t\}$ by embedding v_{i+1} into u_{k_0}.

This process will stop once $i+1 = t+1$. When $i+1 = t+1$, there is a red copy of G in H, and we are done. ■

In order to obtain the conclusion related to T_n, we need the following theorem.

Theorem 6 (Luo and Peng [20]). *Let $t \geq 1$ and $m \geq 2$ be integers. There exists an n_0 such that if $n \geq n_0$, then*

$$R(T_n, tK_m) = (n-1)(m-1) + t.$$

Note that T_n is a t-tree for each $t \in [n-1]$. According to Theorem 6 and Theorem 5, we can directly obtain the following conclusion, which will be applied in the proof of Theorem 3.

Corollary 7. *Let $t \geq 1$ and $m \geq 2$ be integers. If n is sufficiently large, then*

$$R(T_n, K_m \cup tK_{m-1}) = (n-1)(m-1) + 1.$$

2.2. Some lemmas

In order to prove the main theorem, we need several results by Burr and Faudree [7], Erdős, Faudree, Rousseau and Shelp [11], and Hall [15].

A suspended path in a graph is a path all of whose internal vertices have degree two. An end-edge in a graph is an edge one of whose end vertices has degree one. A talon in a graph is a star consisting of end-edges. The following theorem tells us that any large tree contains one of the above structures.

Lemma 8 (Burr and Faudree [7]). *Any tree on n vertices contains either a suspended path on α vertices, or β independent end-edges, or a talon with $\left\lfloor \frac{n}{4\alpha\beta} \right\rfloor$ edges.*

The following two structure theorems also play an important role in the proof of the main result.

Lemma 9 (Erdős, Faudree, Rousseau and Shelp [11]). *Let a, b, c and d be positive integers such that $a \geq b(c-1)+d$. Consider a K_{a+b} on the vertex set $\{x_1, \ldots, x_a, y_1, \ldots, y_b\}$ whose edges are red-blue colored. Suppose that $x_1 x_2 \cdots x_a$ is a red path joining x_1 to x_a and no such paths joining x_1 to x_a exists with exactly $a+1$ vertices. Then either we have a blue K_c or there are d of the x_i that are joined in blue to all y_j.*

Lemma 10 (Hall's Theorem [15]). *Consider a complete bipartite graph $K_{a,b}, a \leq b$, whose parts are $X = \{x_1, x_2, \ldots, x_a\}$ and $Y = \{y_1, y_2, \ldots, y_b\}$, and whose edges are red-blue colored. Then either there is a red matching of size a, or for some $0 \leq c \leq a-1$, there is a blue $K_{c+1,b-c}$ with the $c+1$ vertices being in X.*

Also, we need the following theorem. Call a set $U \subseteq V(G)$ a (*vertex*) *cover* of G if every edge of G is incident with at least one vertex of U. For a graph G, let $\beta(G)$ be the minimum size of vertex cover taken over all vertex covers of G.

Theorem 11 (Hu and Peng [17]. $R(T_n, tK_2) = t + \max\{n, t + \beta(T_n)\} - 1$.

3. Proofs of Theorem 4 and Theorem 3

The lower bounds of Theorem 3 and Theorem 4 both follow from (1.1). Before we give the proof of the upper bound of Theorem 4, we need the following important result due to Hajnal and Szemerédi [14].

Theorem 12 (Hajnal and Szemerédi [14], Kierstead and Kostochka [22]). *Let n, l, a and b be positive integers such that $n = al + b$, where $0 \leq b < l$. Let G be a graph on n vertices with $\Delta(G) < l$. Then there exists a partition $A_1, \ldots, A_l$ of $V(G)$ such that A_i is an independent set for all $i \in [l]$, and $|A_j| = a+1$ for all $j \in [b]$ and $|A_j| = a$ for all $j \in [l] \setminus [b]$.*

***Proof of Theorem* 4.** Color $E(K_{(n-1)(m-1)+s})$ by red or blue arbitrarily, and let R and B be the graph induced by all red edges and all blue edges, respectively. We may assume that R contains no copy of S_n. Thus, $\Delta(R) < n-1$. By Theorem 12, there exists a partition $A_1, \ldots, A_{n-1}$ of $V(R)$ such that A_i is an independent set in R for all $i \in [n-1]$, and $|A_j| = m$ for all $j \in [s]$ and $|A_j| = m-1$ for all $j \in [n-1] \setminus [s]$. Note that $n-1-s \geq t$, thus $B[\bigcup_{i=s+1}^{n-1} A_i]$ contains a copy of tK_{m-1}. Consequently, B contains a copy of $sK_m \cup tK_{m-1}$, and we are done. ■

Now, we prove the upper bound of Theorem 3.

***Proof of Theorem* 3.** We use induction on m to prove the upper bound. It is trivial that the base case ($m = 2$) holds from Theorem 11. Assume that the result holds for $m-1$, and we will prove it for m. Now we use induction on s.

The base case ($s = 1$) is confirmed by Corollary 7. Assume that the assertion holds for $s - 1$, and we will prove that

$$R(T_n, sK_m \cup tK_{m-1}) \leq (n-1)(m-1) + s.$$

Let $N = (n-1)(m-1) + s$. Color $E(K_N)$ by red or blue arbitrarily and let H be the resulting graph. We will show that H contains either a red T_n or a blue $sK_m \cup tK_{m-1}$. By Lemma 8, we split the argument into three cases.

Case 1. T_n has a suspended path on at least $[(s+t)m - s - 2t][(s+t)m - t - 1] + 2(s+t)$ vertices.

Let T' be the tree obtained from T_n by shortening the suspended path by $s+t$ vertices, i.e., $v(T') = n - s - t$. By the induction hypothesis, $R(T_n, sK_{m-1} \cup tK_{m-2}) = (n-1)(m-2) + s < N$. Thus, H contains either a red copy of T_n or a blue copy of $sK_{m-1} \cup tK_{m-2}$. We only need to consider the latter and denote it by A. Note that

$$v(H - A) = (n-1)(m-1) + s - s(m-1) - t(m-2) = (n-s-t-1)(m-1) + s + t.$$

By Theorem 6, $H - A$ contains either a red copy of T' or a blue copy of $(s+t)K_m$ (thus, a blue copy of $sK_m \cup tK_{m-1}$). We only need to consider the former. Let $b = s(m-1) + t(m-2), c = sm + t(m-1), d = s + t$ and $a = b(c-1) + d$. Then, T' has a suspended path on a vertices and denote them by $x_1, x_2, \ldots, x_a$. Furthermore, let $V(A) = \{y_1, y_2, \ldots, y_b\}$. By Lemma 9, in $H[V(T') \cup V(A)]$, one of the following holds.

(i) There is a blue K_c.

(ii) There are d vertices of $V(T')$ such that they are adjacent to all vertices of $V(A)$ in blue.

(iii) The red suspended path of T' can be lengthened by 1, keeping the same end vertices.

If (i) holds, then H contains a blue copy of K_c (thus, a blue copy of $sK_m \cup tK_{m-1}$), and we are done. If (ii) holds, then these $d = s + t$ vertices of $V(T')$ together with A form a blue copy of $sK_m \cup tK_{m-1}$ in H, and we are done. If (iii) holds, then let T'' be the tree obtained from T' by lengthening the suspended path of T' by 1, keeping the same end vertices. Note that

$$v(H - T'') = (n-1)(m-1) + s - (n-s-t+1) = R(T_n, sK_{m-1} \cup tK_{m-2}) + s + t - 2.$$

$H - T''$ contains either a red copy of T_n or a blue copy of $sK_{m-1} \cup tK_{m-2}$. We only need to consider the latter and denote it by A'. Note that T'' has a suspended path on $a + 1 \geq b(c-1) + d$ vertices. Therefore, as long as the red T'' has fewer than n vertices, we continue to apply Lemma 9 for the same argument until we get a red copy of T_n. We only need to verify that $v(H - T^*) = (n-1)(m-1) + s - (n-1)$

is greater than or equal to $R(T_n, sK_{m-1} \cup tK_{m-2})$, where T^* is the tree obtained from T' by lengthening the suspended path, keeping the same end vertices.

Case 2. T_n has at least $2s+t-2$ independent end-edges. Let T' be the tree obtained from T_n by removing $2s+t-2$ end vertices of degree one from independent end-edges of T_n. By the induction hypothesis, $R(T_n, (s-1)K_m \cup (s+t-1)K_{m-1}) = (n-1)(m-1)+s-1 < N$. Thus, H contains either a red copy of T_n or a blue copy of $(s-1)K_m \cup (s+t-1)K_{m-1}$. We only need to consider the latter and denote it by J. Note that $v(T') = n-2s-t+2$ and

$$v(H-J) = (n-1)(m-1)+s-(s-1)m-(s+t-1)(m-1) = (n-2s-t+1)(m-1)+1.$$

Since $T_{n'}$ is $K_{m'}$-good [8], $H-J$ contains either a red copy of T' or a blue copy of K_m. If $H-J$ contains a blue copy of K_m, then it together with J forms a blue copy of $sK_m \cup (s+t-1)K_{m-1}$ (thus, a blue copy of $sK_m \cup tK_{m-1}$), and we are done. Therefore, we only need to consider the former.

Let X be those vertices of T' that are neighbors of the removed vertices in T_n. Clearly, $|X| = 2s+t-2 \triangleq a$. Let $Y = V(H) \setminus V(T')$ and thus, $|Y| = (n-1)(m-2)+3s+t-3 \triangleq b$. Clearly, $a \le b$. By Lemma 10, one of the following holds.

(i) There is a red matching of size $2s+t-2$ between X and Y.

(ii) For some $0 \le c \le 2s+t-3$, there is a blue copy of $K_{c+1,b-c}$ with $c+1$ vertices in X.

If (i) holds, then H contains a red copy of T_n, and we are done. If (ii) holds, then note that each vertex of X is adjacent to each K_{m-1} of $(s+t-1)K_{m-1} \subseteq J$ in at least one red edge. Otherwise, the vertex and J form a blue copy of $sK_m \cup (s+t-2)K_{m-1}$ and thus, H contains a blue copy of $sK_m \cup tK_{m-1}$ since $s \ge 2$. Consequently, the red degree of each vertex of X in J is at least $s+t-1$. Furthermore, $J \subseteq Y$ and thus, $b-c \le b-(s+t-1)$. Therefore, $c+1 \ge s+t$. On the other hand, $b-c \ge b-(2s+t-3) = (n-1)(m-2)+s$. Consequently, H contains a blue copy of $K_{s+t,(n-1)(m-2)+s}$ and denote it by C. In the part of C with size $(n-1)(m-2)+s$, by the induction hypothesis, there is either a red copy of T_n or a blue copy of $sK_{m-1} \cup tK_{m-2}$. We only need to consider the latter. Note that the blue copy of $sK_{m-1} \cup tK_{m-2}$ together with the part of C with size $s+t$ forms a blue copy of $sK_m \cup tK_{m-1}$ in H, and we are done.

Case 3. T_n has a talon with at least $c = \left\lfloor \frac{n}{4\alpha\beta} \right\rfloor$ edges, where $\alpha = [(s+t)m - s - 2t][(s+t)m-t-1] + 2(s+t)$ and $\beta = 2s+t-2$.

Denote the center of the talon by x. Let T' be the tree obtained from T_n by removing c end-vertices of the talon. By Theorem 4, H contains either a red copy of S_n or a blue copy of $sK_m \cup tK_{m-1}$. We only need to consider the former and denote the center of the red copy of S_n by y. Note that $v(H-\{y\}) = (n-1)(m-1)+s-1$, and by the induction hypothesis, $H-\{y\}$ contains either a red copy of T_n or a blue copy of $(s-1)K_m \cup tK_{m-1}$. We only need to consider the latter and denote it by D. If the red degree of each vertex in H is at least $n-c$, then we can embed a red copy of T' into H by putting x at y greedily since $v(T') = n-c$. Recall that y is the center of a red copy of S_n, and thus we can extend the red copy of T' to a red copy of T_n. Therefore, assume that there exists a vertex $z \in V(H)$ with blue degree at least $(n-1)(m-1)+s-1-(n-c)$ and let Z be the set of blue neighbors of z. Since n is sufficiently large,

$$|Z|-v(D) \geq (n-1)(m-1)+s-1-(n-c)-(s-1)m-t(m-1) \geq (n-1)(m-2)+1.$$

Since $T_{n'}$ is $K_{m'}$-good [8], $H[Z-V(D)]$ contains either a red copy of T_n or a blue copy of K_{m-1}. We only need to consider the latter. The blue K_{m-1} together with $\{z\}$ forms a blue K_m in $H-D$. Moreover, the blue copy of K_m together with D forms a blue copy of $sK_m \cup tK_{m-1}$ in H, and we are done.

All cases have been discussed, and the proof is complete. ■

By direct calculation, we have the following remark.

Remark 13. $n \geq 16m^3(2s+t)^4$ is enough for Theorem 3.

4. Remark

Theorem 1 illustrates that the Ramsey number for a large tree versus the disjoint union of complete graphs only depends on the order of the tree, the order of the maximum complete graph and the number of the maximum clique. What pair of graphs will satisfy this property?

Problem 14. Let s and t be positive integers. For which graphs $H, G, G_1, \ldots, G_t$, if $e(G) = \max\{e(G), e(G_1), \ldots, e(G_t)\}$, then $R\left(H, sG \cup \left(\bigcup_{i=1}^{t} G_i\right)\right)$ only depends on $v(H)$, $v(G)$ and s? If the size of graph does not make the property holds, then what parameter of graph will make the property holds? Maybe density?

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