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EQUITABLE CHOOSABILITY OF PRISM GRAPHS

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Abstract

A graph G is equitably k-choosable if, for every k-uniform list assignment L, G is L-colorable and each color appears on at most $\lceil |V(G)|/k \rceil$ vertices. Equitable list-coloring was introduced by Kostochka, Pelsmajer, and West in 2003 [*A list analogue of equitable coloring*, J. Graph Theory 44 (2003) 166–177]. They conjectured that a connected graph G with $\Delta(G) \geq 3$ is equitably $\Delta(G)$-choosable, as long as G is not complete or $K_{d,d}$ for odd d. In this paper, we use a discharging argument to prove their conjecture for the infinite family of prism graphs.

Keywords: list coloring, equitable list coloring, prism graph, reducible configuration, discharging.

2020 Mathematics Subject Classification: 05C15.

1. Introduction

For terminology and notation not explicitly defined in this paper, see [2]. Let G be a graph. We say that G is *properly k-colorable* if there exists a mapping $c : V(G) \to [k]$ such that $c(u) \neq c(v)$ for every edge $uv \in E(G)$. In some graph coloring scenarios, we impose restrictions on which colors it is acceptable to use on each vertex. A *list assignment* L for G provides a list of acceptable colors, $L(v)$, to each vertex $v \in V(G)$. This list assignment is said to be *k-uniform* if $|L(v)| = k$ for every $v \in V(G)$. A *proper L-coloring* of G is a proper coloring in which each vertex is assigned a color from its list. If every k-uniform list assignment L of G admits a proper list coloring, we say that G is *k-choosable*. The choice number, $\mathrm{ch}(G)$, of a graph G is the smallest k for which G is k-choosable. The concept of list coloring was independently introduced by Vizing [7] and Erdős, Rubin, and Taylor [4].

A vertex coloring of G partitions $V(G)$ into sets called *color classes*. All vertices in a particular color class are assigned the same color, so if the coloring is proper, each color class is an independent set of vertices. Depending on the context, we may wish to ensure that the color classes are roughly uniform in size. A proper k-coloring is said to be *equitable* if the size of every color class is either $\lceil |V(G)|/k \rceil$ or $\lfloor |V(G)|/k \rfloor$. In [5], Kostochka, Pelsmajer, and West introduced a variation of equitable coloring for k-uniform list colorings. In particular, they say that a graph G is *equitably k-choosable* if every k-uniform list assignment of G admits a proper coloring in which each color class is $\lceil |V(G)|/k \rceil$*-bounded*. That is, they allow each color class to have any size up to $\lceil |V(G)|/k \rceil$.

In this paper, we will investigate the equitable choosability parameter of the infinite family of *prism graphs*, $\{\Pi_n : n \geq 3\}$. Note that the n-prism Π_n is the Cartesian product $C_n \square K_2$, whose drawing resembles a geometric prism with an n-sided polygon base. We will refer to the n edges which attach the two copies of C_n in Π_n as *rungs*. Note that the prism graphs are cubic, Hamiltonian, and planar (but not outerplanar). When n is even, the prism Π_n is bipartite. And when $n \geq 4$, the prism Π_n has girth 4.

With this in mind, there are two results from Erdős, Rubin, and Taylor that are very relevant to our research. In their seminal paper on list coloring, these authors were able to bound the choice number in the following way.

Theorem 1 [4]. *If a connected graph G is not K_n and not an odd cycle, then $ch(G) \leq \Delta(G)$.*

They were also able to completely characterize the family of 2-choosable graphs. They define the *core* of a graph G to be the induced subgraph H of G that is obtained by successively pruning degree-1 vertices until none remain. They also define $\Theta_{i,j,k}$ to be the graph consisting of two distinct vertices u and v attached by three internally-disjoint paths of respective lengths i, j, and k. Using these definitions, they articulate the following result.

Theorem 2 [4]. *A graph G is 2-choosable if and only if the core of G belongs to the set $\{K_1, C_{2m+2}, \Theta_{2,2,2m} : m \geq 1\}$.*

These results lead us to the following observation.

Observation 3. *$ch(\Pi_n) = 3$ for all $n \geq 3$.*

Additionally, there are several results known about equitable choosability that are relevant to us. Zhu and Bu proved that all outerplanar graphs with maximum degree 3 are equitably 3-choosable [9]. Dong and Zhang proved that a graph G with maximum average degree less than 3 is equitably k-colorable and equitably k-choosable for $k \geq \max\{\Delta(G), 4\}$ [3]. Wang and Lih proved the following theorem about graphs with maximum degree at most 3, and Pelsmajer proved an equivalent result in [6].

Theorem 4 [8]**.** *Every graph G with $\Delta(G) \leq 3$ is equitably k-choosable whenever $k > \Delta(G)$.*

In 1994, Chen, Lih, and Wu proposed the Equitable Δ-Coloring Conjecture.

Conjecture 5 (The Equitable Δ-Coloring Conjecture [1])**.** *Let G be a connected graph. If G is not a complete graph, or an odd cycle, or a complete bipartite graph $K_{2m+1,2m+1}$, then G is equitably $\Delta(G)$-colorable.*

They also proved the following special case of their conjecture.

Theorem 6 [1]**.** *A connected graph G with $\Delta(G) \leq 3$ is equitably $\Delta(G)$-colorable if it is different from K_m, C_{2m+1}, and $K_{2m+1,2m+1}$ for all $m \geq 1$.*

In 2003, when Kostochka, Pelsmajer, and West introduced their list analogue of equitable coloring, they included a list analogue of the Equitable Δ-Coloring Conjecture.

Conjecture 7 [5]**.** *If G is a connected graph with maximum degree at least 3, then G is equitably $\Delta(G)$-choosable, unless G is a complete graph or is $K_{d,d}$ for some odd d.*

Note that there are many more publications on equitable choosability. We have chosen to highlight only those that are most relevant to our main result, which verifies that Conjecture 7 holds for the infinite family of prism graphs.

Theorem 8. *Π_n, $n \geq 3$, is equitably 3-choosable.*

For a graph to be equitably k-choosable, it must be k-choosable. According to Observation 3, the prisms are 3-choosable, and they are *not* equitably 1- or 2-choosable because they are neither 1- nor 2-choosable. Also, by Theorem 4, the prisms *are* equitably k-choosable for $k \geq 4$. So the question that remains is whether or not the prisms are equitably 3-choosable. In the remaining sections of this paper, we will answer this question in the affirmative.

Our proof of Theorem 8 is organized as follows. In Section 2, we show that Π_n is equitably 3-choosable for $n \in \{3, 4, 5\}$. In Section 3, we complete the proof that all prisms are equitably 3-choosable. This argument is broken into two subsections. In Section 3.1 we prove that in a minimum list coloring of Π_n, $n \geq 6$, the two largest color classes must differ in size by at least 2 vertices. Finally, in Section 3.2 we use a discharging argument to prove that a minimum list coloring of Π_n, $n \geq 6$, must be equitable.

2. Small Prisms

In this section we prove that Π_3, Π_4, and Π_5 are equitably 3-choosable. The proofs for Π_4 and Π_5 were done by the third author in their 2021 undergraduate thesis and are recreated below. For the proof of Lemma 9, recall that the *independence number*, $\alpha(G)$, of a graph G is the cardinality of a maximum set of pairwise nonadjacent vertices of G.

Lemma 9. Π_3 *and* Π_5 *are equitably* 3*-choosable.*

Proof. First, recall that Π_3 and Π_5 are both 3-choosable by Observation 3, so every 3-list-assignment of these graphs admits a proper coloring.

For Π_3 to be equitably 3-choosable, every 3-list-assignment of Π_3 must admit a proper coloring in which the cardinality of a largest color class is at most $\lceil 6/3 \rceil = 2$. Since each color class in a proper coloring is an independent set, and the independence number of Π_3 is $\alpha(\Pi_3) = 2$, it must be that every color class contains at most 2 vertices. Thus, Π_3 is equitably 3-choosable.

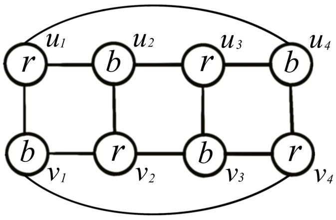
Similarly, for Π_5 to be equitably 3-choosable, the cardinality of a largest color class must be at most $\lceil 10/3 \rceil = 4$. The independence number of Π_5 is $\alpha(\Pi_5) = 4$, so every color class contains at most 4 vertices, and Π_5 is equitably 3-choosable. ■

To prove that Π_4 is equitably 3-choosable, it is not sufficient to simply calculate the independence number. Instead, we perform a case analysis to show that every coloring which is not $\lceil |V(\Pi_4)|/3 \rceil$-bounded can be transformed into a coloring which is.

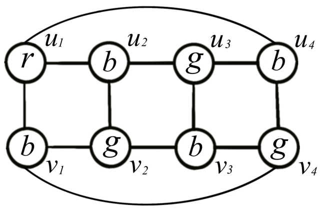
Lemma 10. Π_4 *is equitably* 3*-choosable.*

Proof. First, recall that $\mathrm{ch}(\Pi_4) = 3$ by Observation 3, so every 3-list-assignment of this graph admits a proper coloring. Let L be an arbitrary 3-list-assignment of Π_4 and let $c : V \to \mathcal{C}$ be a proper L-coloring of Π_4, where $\mathcal{C}$ is a set of at least three colors. Without a loss of generality, suppose a largest color class of c corresponds to the color b. We will denote this color class Blue.

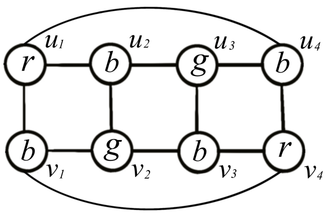
If $|\text{Blue}| \le \lceil 8/3 \rceil = 3$, then c is 3-bounded and we have found the coloring we need. So suppose instead that $|\text{Blue}| > 3$. The independence number of Π_4 is $\alpha(\Pi_4) = 4$, so it must be that $|\text{Blue}| = 4$. Thus, due to the symmetries of Π_4, the coloring c must look like one of the configurations in Figure 1. Note that these cases are distinguished by the different possible colorings of u_1, v_2, u_3, and v_4, which must all have colors other than b.



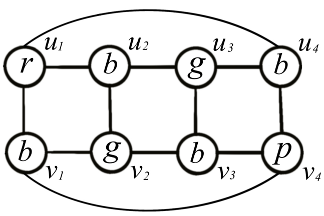
Case 1



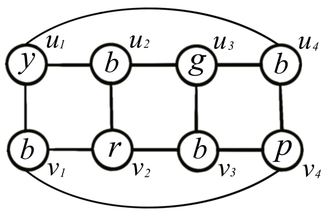
Case 2



Case 3



Case 4



Case 5

Figure 1. Potential colorings of Π_4.

In Case 1 there are two color classes of cardinality 4. However, lists $L(u_1)$ and $L(v_3)$ must each contain a color which is neither r nor b. (This color may or may not be the same in the two lists.) Define a new coloring c' in which $c'(u_1), c'(v_3) \notin \{b, r\}$ and $c' = c$ for all other vertices. This new coloring c' is a proper L-coloring whose largest color class has cardinality 3, so L admits a proper 3-bounded coloring.

In Case 2, list $L(v_3)$ must contain a color which is neither b nor g. Define a new coloring c' in which $c'(v_3) \notin \{b, g\}$ and $c' = c$ for all other vertices. This new coloring c' is a proper L-coloring whose largest color class has cardinality 3, so L admits a proper 3-bounded coloring.

In Cases 3 and 4, we will attempt to change the color of vertex u_2. If there is a color in $L(u_2)$ which is not b, g, or r, we can obtain a proper 3-bounded L-coloring by changing u_2 to that color. If not, then $L(u_2) = \{b, g, r\}$. Fortunately, list $L(u_1)$ must contain a color that is not blue or red. Define a new coloring c' in which $c'(u_2) = r$, $c'(u_1) \notin \{b, r\}$, and $c' = c$ for all other vertices. This new coloring c' is a proper L-coloring whose largest color class has cardinality 3. Thus, in Cases 3 and 4, L always admits a proper 3-bounded coloring.

In Case 5, we will attempt to change the color of vertex u_2. If there is a color in $L(u_2)$ which is not b, r, g, or y, we can obtain a proper 3-bounded L-coloring by changing u_2 to that color. If not, then $L(u_2)$ must contain b and exactly two of the colors r, g, and y. Without a loss of generality, suppose $L(u_2)$ contains y. Fortunately, list $L(u_1)$ must contain a color that is not b or y. Define a new coloring c' in which $c'(u_2) = y$, $c'(u_1) \notin \{b, y\}$, and $c' = c$ for all other vertices. This new coloring c' is a proper L-coloring whose largest color class has cardinality 3. Thus, in Case 5, L always admits a proper 3-bounded coloring.

In every case, we are able to obtain a proper 3-bounded L-coloring. This means that the arbitrary 3-list-assignment L always admits a proper 3-bounded coloring, so by definition Π_4 is equitably 3-choosable. ■

3. Proof of Main Result

In this section we prove that Π_n, $n \geq 6$, is equitably 3-choosable. Together with Lemmas 9 and 10, this constitutes a proof of our main result, Theorem 8. For the remainder of the paper, let L be an arbitrary 3-list-assignment of Π_n. We know that a proper L-coloring of Π_n exists by Observation 3, so we will use a minimum counterexample technique and a discharging argument to prove that L admits a proper $\lceil 2n/3 \rceil$-bounded coloring.

Let c be a proper L-coloring of Π_n, and let $C_1, C_2, \ldots, C_r$ be the color classes of c. Further, let $|C_i| = n_i$ for each color class and, without a loss of generality, suppose $n_1 \geq n_2 \geq \cdots \geq n_r$. The *color word* of c is the length-r list $w_c = n_1 n_2 \cdots n_r$. We say c is a *lex-min coloring* of Π_n if its color word w_c is lexicographically minimum. That is, if c' is another proper L-coloring of Π_n with color word $w_{c'} = m_1 m_2 \cdots m_s$, then either $w_c = w_{c'}$ or there exists an index k $(1 \leq k \leq s)$, such that $n_i = m_i$ for $1 \leq i \leq k - 1$ and $n_k < m_k$.

The following two lemmas describe some useful properties of a lex-min L-coloring of Π_n, $n \geq 6$.

Lemma 11. *Let c be a proper lex-min L-coloring of Π_n, $n \geq 6$. If the lists in L are all the same, then c is $\lceil 2n/3 \rceil$-bounded.*

Proof. Suppose c is a proper lex-min L-coloring of Π_n. As L is a 3-list-assignment of Π_n, if all lists are the same, then finding a proper L-coloring of Π_n is equivalent to finding a proper 3-coloring of Π_n. According to Theorem 6, Π_n is equitably 3-colorable. That is, there exists a proper 3-coloring of Π_n in which each color class has cardinality equal to $\lceil 2n/3 \rceil$ or $\lfloor 2n/3 \rfloor$. In the lex-min coloring c, the largest color class may be even smaller than $\lceil 2n/3 \rceil$. Therefore, c is $\lceil 2n/3 \rceil$-bounded. ■

Lemma 12. *Let c be a proper lex-min L-coloring of Π_n, $n \geq 6$. If c is not $\lceil 2n/3 \rceil$-bounded, then the range of c includes at least 4 colors.*

Proof. By Lemma 11, we know that the lists used to define c are not all the same, so at least 4 colors appear in the union of all the color lists. Suppose c uses fewer than 4 colors. Then there exists a vertex v such that $L(v)$ contains a color *not* used by c, say c_0. Define a coloring c' in which $c'(v) = c_0$ and $c' = c$ for all other vertices. Since color c_0 was not used by coloring c, we may conclude that $w_{c'}$ is lexicographically less than w_c. This contradicts the assumption that c is lex-min, so it must be that c uses at least 4 colors. ■

The remainder of this section is organized as follows. We start by assuming that c is a lex-min L-coloring of Π_n, $n \geq 6$, which is not $\lceil 2n/3 \rceil$-bounded. In Section 3.1, we use reducible configurations and a counting argument to prove that the two largest color classes of c must differ in cardinality by at least 2. Then in Section 3.2, we use additional reducible configurations and a discharging argument to prove that c must be $\lceil 2n/3 \rceil$-bounded, and so Π_n, $n \geq 6$, is equitably 3-choosable.

3.1. Two large color classes

In this subsection we prove that if c is lex-min and not $\lceil 2n/3 \rceil$-bounded, then its two largest color classes must differ in size by at least 2. To simplify the notation, let us say that Blue is the most common color and Red is the second most common color. We assume that $|\text{Blue}| \geq \lceil 2n/3 \rceil + 1$ and either $|\text{Red}| = |\text{Blue}|$ or $|\text{Red}| = |\text{Blue}| - 1$. First, we determine how much larger $|\text{Red}|$ and $|\text{Blue}|$ are than the other color classes.

Lemma 13. *If c is lex-min and not $\lceil 2n/3 \rceil$-bounded, and $|Red| = |Blue|$ or $|Red| = |Blue| - 1$, then $|Blue|$ and $|Red|$ are at least 2 larger than every color class.*

Proof. Recall that by Lemma 12, c has at least 4 nonempty color classes. Suppose $|\text{Red}| = |\text{Blue}| - 1$, and let C denote an arbitrary third color class. Then because Π_n has $2n$ vertices and $|\text{Blue}| \geq \lceil 2n/3 \rceil + 1$, we may conclude that

$$|C| \leq 2n - |\text{Blue}| - |\text{Red}| - 1 \leq 2n - 2\lceil 2n/3 \rceil - 2 \leq 2n/3 - 2 \leq \lceil 2n/3 \rceil - 2.$$

Note that $\lceil 2n/3 \rceil - 2 \leq |\text{Blue}| - 3$ and $\lceil 2n/3 \rceil - 2 \leq |\text{Red}| - 2$. So when $|\text{Red}| = |\text{Blue}| - 1$, Blue contains at least 3 more vertices and Red contains at least 2 more vertices than every other color class.

A similar bounding argument shows that when $|\text{Red}| = |\text{Blue}|$, Red and Blue both contain at least 4 more vertices than every other color class. ■

Next, we identify four color configurations that cannot occur in c. Note that in these diagrams, there are vertices that have not been labeled with a color. We put no restriction on which colors c assigns to these unlabeled vertices.

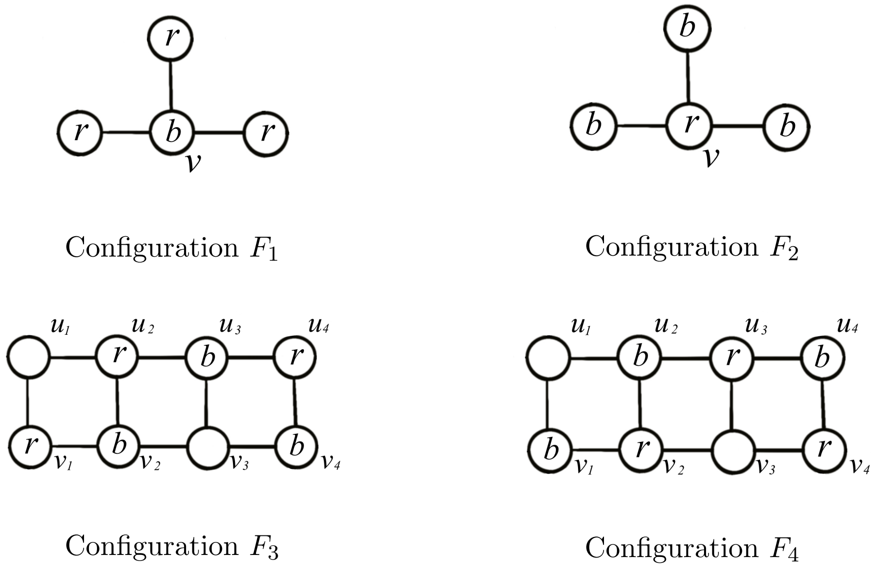


Figure 2. Reducible configurations for Lemma 14.

Lemma 14. *If c is lex-min and not $\lceil 2n/3 \rceil$-bounded, and $|Red| = |Blue|$ or $|Red| = |Blue| - 1$, then the color configurations in Figure 2 do not occur in c.*

Proof. In configurations F_1 and F_2, notice that $L(v)$ contains a color which is neither r nor b. Define a new coloring c' in which $c'(v) \notin \{b, r\}$ and $c' = c$ for all other vertices. Because either $|\text{Red}|$ or $|\text{Blue}|$ decreases by 1 and a third color class increases by 1, $w_{c'}$ is lexicographically less than w_c. This is a contradiction, so configurations F_1 and F_2 cannot occur in the lex-min coloring c.

In configurations F_3 and F_4, we attempt to recolor one of vertices u_2, u_3, and v_2. If one of these vertices' lists contains a color other than r, b, and the blank neighbor's color, then we may recolor that vertex and obtain a lexicographically smaller color word. This is a contradiction, so it must be that $L(u_2) = \{b, r, c(u_1)\}$, and $L(u_3) = L(v_2) = \{b, r, c(v_3)\}$. Note that this implies $c(v_3)$ is neither r nor b.

In F_3, there must be a color c_0 in $L(v_3)$ such that $c_0 \notin \{b, c(v_3)\}$. Define a new coloring c' in which $c'(v_3) = c_0$, $c'(u_3) = c'(v_2) = c(v_3)$, $c'(u_2) = b$, and $c' = c$ for all other vertices. If $c_0 = r$, then |Red| stays the same, |Blue| decreases by 1, and the color class corresponding to $c(v_3)$ increases by 1. If $c_0 \neq r$, then |Red| and |Blue| both decrease by 1, and the color classes corresponding to $c(v_3)$ and $c'(v_3)$ both increase by 1. Either way, $w_{c'}$ is lexicographically less than w_c. This contradicts the assumption that c was a lex-min coloring, so F_3 cannot occur in c.

The argument for F_4 is similar. We let c_0 denote a color in $L(v_3)$ such that $c_0 \notin \{r, c(v_3)\}$, then we define c' in the same way. To determine that $w_{c'}$ is lexicographically less than w_c, we consider cases $c_0 = b$ and $c_0 \neq b$. In both cases, we contradict the assumption that c was lex-min, so F_4 cannot occur in c. ■

Lemma 15. *If c is lex-min and not $\lceil 2n/3 \rceil$-bounded, then there does not exist a subgraph of 6 consecutive rungs of Π_n, $n \geq 6$, in which 9 or more of the vertices are colored red or blue.*

Proof. Suppose there is a subgraph H of 6 consecutive rungs of Π_n, $n \geq 6$, in which 9 or more of the vertices are colored red or blue. H cannot contain 6 blue vertices and 3 or more red vertices because then reducible configuration F_2 would occur. Similarly, if H contained 6 red vertices and at least 3 blue vertices, reducible configuration F_1 would occur. So the only possible counts for red and blue vertices are: 5 blue, 4 red; 5 blue, 5 red; and 4 blue, 5 red.

Suppose H contains 5 blue vertices. Then H looks like one of the diagrams in Figure 3. Note that the blank vertices cannot be blue and must be properly colored.

In Case 1, the blue vertices are arranged in 5 consecutive rungs. If 4 of the blank vertices are colored red, and the coloring is proper, then at least one of u_2, u_4, and v_3 would be forced to be red. However, this creates a copy of the reducible configuration F_2, so this arrangement of blue vertices cannot occur.

In Case 2, the blue vertices are arranged in 4 consecutive rungs with the fifth blue vertex being v_6. If u_2 or v_3 is red, then reducible configuration F_2 occurs, which is a contradiction. So if 4 of the blank vertices are colored red, it must be that v_1, u_4, v_5, and u_6 are red. However, this creates a copy of the reducible configuration F_4, so this arrangement of blue vertices cannot occur.

In Case 3, the blue vertices are arranged in 4 consecutive rungs with the fifth blue vertex being u_6. If 4 of the blank vertices are colored red, and the coloring is proper, then at least one of u_2 and v_3 would be forced to be red. However, this creates a copy of F_2, so this arrangement of blue vertices cannot occur.

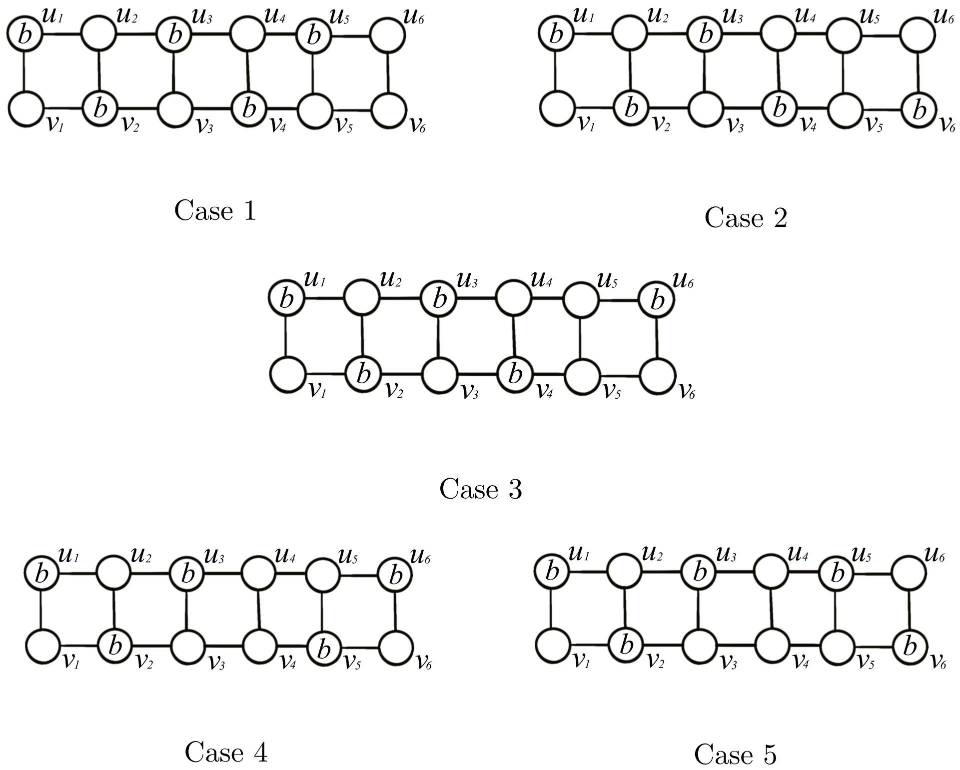


Figure 3. Potential colorings of 6 consecutive rungs.

In Case 4, the blue vertices are arranged in 3 consecutive rungs, with the fourth and fifth blue vertices being v_5 and u_6. To avoid F_2, u_2 cannot be red. The largest independent set of vertices which are neither blue nor u_2 is 4, so there cannot be 5 red vertices in this case. If v_1, v_3, u_4, and v_6 are red, a copy of F_3 is created, which is a contradiction. If v_1, v_4, u_5, and v_6 are red, a copy of F_1 is created, which is a contradiction. So it must be that v_1, v_3, u_5, and v_6 are the only red vertices in this case. Notice that u_4 and v_4 are neither red nor blue, and $c(u_4) \neq c(v_4)$. An updated diagram for Case 4 is shown in Figure 4. For simplicity, we will say $c(u_4) = g$ and $c(v_4) = y$.

If we were able to change any blue or red vertex to a color other than red and blue, we would contradict the assumption that c is lex-min. Therefore, $L(u_5) = \{b, g, r\}$ and $L(v_3) = L(v_5) = \{b, r, y\}$. It must be that $L(v_4) \subseteq \{b, g, r, y\}$, because otherwise we could define a new coloring c' in which $c'(v_5) = y$, $c'(v_4) \notin \{b, g, r, y\}$, and $c' = c$ for all other vertices. The color word of c' is lexicographically less than w_c, which is a contradiction. For similar reasons, $L(u_4) \subseteq \{b, g, r, y\}$.

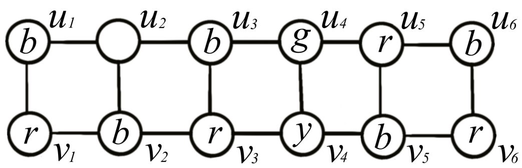


Figure 4. Updated coloring for Case 4.

Now, notice that $L(u_4) = \{b, g, y\}$. If not, then $r \in L(u_4)$, so we could define a new coloring c' in which $c'(u_5) = g$, $c'(u_4) = r$, and $c' = c$ for all other vertices. The color word of c' is equal to w_c, so they are both lex-min colorings. However, c' contains a copy of F_3, which is a contradiction. Further, notice that $L(v_4) = \{b, r, y\}$. If not, then $g \in L(v_4)$, and we can define a new coloring c'' such that $c''(u_5) = g$, $c''(u_4) = y$, $c''(v_4) = g$, and $c'' = c$ for all other vertices. The color word of c'' is less than w_c, which is a contradiction.

Now that we know the color lists for u_5, u_4, v_4, and v_3, we can define a new coloring c' in the following way. Let $c'(u_5) = g$, $c'(u_4) = y$, $c'(v_4) = r$, $c'(v_3) = y$, and $c' = c$ for all other vertices. The color word of c' is less than w_c, which is a contradiction, so the Case 4 color arrangement cannot occur in coloring c.

In Case 5, the blue vertices are arranged in 3 consecutive rungs, with the fourth and fifth blue vertices being u_5 and v_6. If u_2 is red, then F_2 occurs, which is a contradiction. And if u_4, u_6, and v_5 are all red, then F_1 occurs, which is a contradiction. So there cannot be 5 red vertices in this case. Further, to avoid F_1 and F_2 when there are 4 red vertices, it must be that v_1, v_3, and exactly 2 of u_4, u_6, and v_5 are red. If u_4 is red, a copy of F_3 is created, which is a contradiction. So it must be that v_1, v_3, v_5, and u_6 are the only red vertices in this case. Notice that u_4 and v_4 are neither red nor blue, and $c(u_4) \neq c(v_4)$. An updated diagram for Case 5 is shown in Figure 5. For simplicity, we will say $c(u_4) = g$ and $c(v_4) = y$.

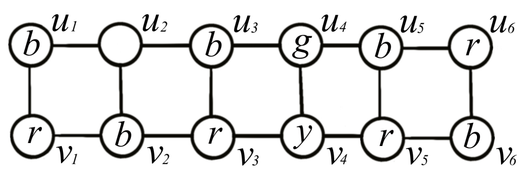


Figure 5. Updated coloring for Case 5.

By an argument similar to the one from Case 4, we can determine that $L(u_5) = \{b, g, r\}$, $L(v_3) = L(v_5) = \{b, r, y\}$, $L(v_4) \subseteq \{b, g, r, y\}$, and $L(u_4) \subseteq$

$\{b, g, r, y\}$. Now, notice that $L(v_4) = \{b, g, y\}$. If not, then $r \in L(v_4)$, so we could define a new coloring c' in which $c'(v_4) = r$, $c'(v_3) = c'(v_5) = y$, and $c' = c$ for all other vertices. The color word of c' is lexicographically less than w_c, which is a contradiction.

We know there must exist some $c_o \in L(u_4)$ such that $c_0 \notin \{b, g\}$. In particular, $c_0 \in \{r, y\}$. So define a new coloring c' in the following way: $c'(u_5) = g$; $c'(u_4) = c_0$; $c'(v_4) = b$; $c'(v_5) = y$; if $c_0 = r$, then $c'(v_3) = y$, otherwise $c'(v_3) = r$; and for all other vertices, $c' = c$. When switching from c to c', the blue and green color classes stay the same size, the red color class decreases by 1, and the yellow color class increases by 1, so $w_{c'}$ is lexicographically less than w_c. This is a contradiction, so the Case 5 color arrangement cannot occur in coloring c.

Similar arguments hold when H contains 5 red vertices and 4 blue vertices. Simply swap the colors red and blue, swap the reducible configurations F_1 and F_2, and swap the configurations F_3 and F_4. Therefore, when c is lex-min, every subgraph of 6 consecutive rungs of Π_n can contain at most 8 vertices that are blue or red. ■

Lemma 16. *If c is lex-min and not $\lceil 2n/3 \rceil$-bounded, then $|Blue| \geq |Red| + 2$.*

Proof. Suppose $|\text{Blue}| < |\text{Red}| + 2$. That is, suppose $|\text{Red}| = |\text{Blue}|$ or $|\text{Red}| = |\text{Blue}| - 1$.

By Lemma 15, we know that every 6 consecutive rungs must contain at most 8 vertices that are blue or red. There are n distinct subgraphs of 6 consecutive rungs, and each red or blue vertex is in exactly 6 of these subgraphs. So we can now bound the number of red and blue vertices in Π_n, $n \geq 6$, in the following way:

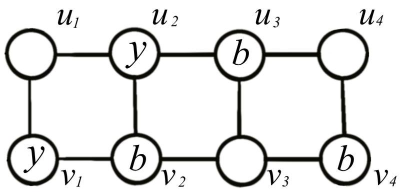
$$|\text{Blue}| + |\text{Red}| \leq \frac{8n}{6} \leq 2 \lceil 2n/3 \rceil = (\lceil 2n/3 \rceil + 1) + \lceil 2n/3 \rceil - 1 \leq |\text{Blue}| + |\text{Red}| - 1.$$

This is a contradiction, so if $|\text{Blue}| \geq \lceil 2n/3 \rceil + 1$, we must have $|\text{Red}| \leq \lceil 2n/3 \rceil - 1 \leq |\text{Blue}| - 2$. ■

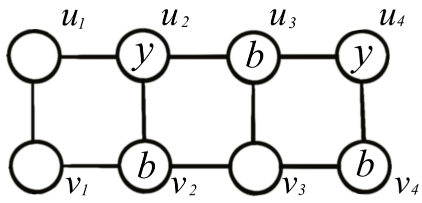
3.2. One large color class

In Section 3.1, we determined that the largest color class of a lex-min, non-$\lceil 2n/3 \rceil$-bounded coloring of Π_n, $n \geq 6$, must contain at least 2 more vertices than every other color class. In this subsection, we use this fact to finish proving our main result. To simplify notation, we continue to say Blue is the largest color class.

To begin, we show that the color configurations in Figure 6 do not occur in a lex-min, non-$\lceil 2n/3 \rceil$-bounded coloring. In these diagrams, the blue vertices must come from the largest color class, but the yellow vertices do not need to come from the second largest color class. A blank vertex could be any color which is not already in use by its neighbors.



Configuration F_5



Configuration F_6

Figure 6. Reducible configurations for Lemma 17.

Lemma 17. *If c is lex-min and not $\lceil 2n/3 \rceil$-bounded, then the color configurations in Figure* 6 *do not occur.*

Proof. In configuration F_5, we attempt to recolor v_2. If $L(v_2)$ contains a color other than b, y, and $c(v_3)$, then we may recolor it and obtain a lexicographically smaller color word. This is a contradiction, so it must be that $L(v_2) = \{b, y, c(v_3)\}$. Note that this implies $c(v_3)$ is neither b nor y. Define a new coloring c' in which $c'(v_2) = c(v_3)$ and $c'(v_3) \notin \{b, c(v_3)\}$. Since $|\text{Blue}|$ decreases by 1, $w_{c'}$ is lexicographically less than w_c, which is a contradiction. So F_5 cannot occur in the lex-min coloring c.

In configuration F_6, we attempt to recolor u_3. By an argument similar to the F_5 case, $L(u_3) = \{b, y, c(v_3)\}$ and $c(v_3) \notin \{b, y\}$. Define a new coloring c' in which $c'(u_3) = c(v_3)$ and $c'(v_3) \notin \{b, c(v_3)\}$. Then $w_{c'}$ is lexicographically less than w_c, which is a contradiction. So F_6 cannot occur in the lex-min coloring c. ■

Next, we will show that there cannot be 4 consecutive rungs which contain blue vertices.

Lemma 18. *If c is lex-min and not $\lceil 2n/3 \rceil$-bounded, then there does not exist a subgraph of* 4 *consecutive rungs of Π_n, $n \geq 6$, in which* 4 *vertices are colored blue.*

Proof. Suppose Π_n contains a subgraph H of four consecutive rungs with blue vertices. A diagram of this subgraph is shown in Figure 7. Note that the blank vertices cannot be blue because the graph must be properly colored.

To avoid configuration F_5, $c(v_1) \neq c(u_2)$ and $c(u_4) \neq c(v_3)$. To avoid F_6, $c(u_2) \neq c(u_4)$ and $c(v_1) \neq c(v_3)$. Therefore, without a loss of generality, there are four possible ways to color v_1, u_2, v_3, and v_4, which can be seen in Figure 8.

In all cases, the color list of every blue vertex must only contain blue and colors utilized by neighbors. If not, we could recolor one blue vertex to obtain

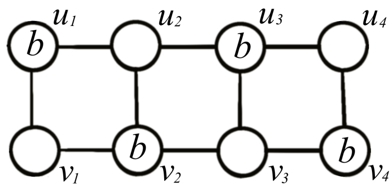


Figure 7. Diagram of subgraph H.

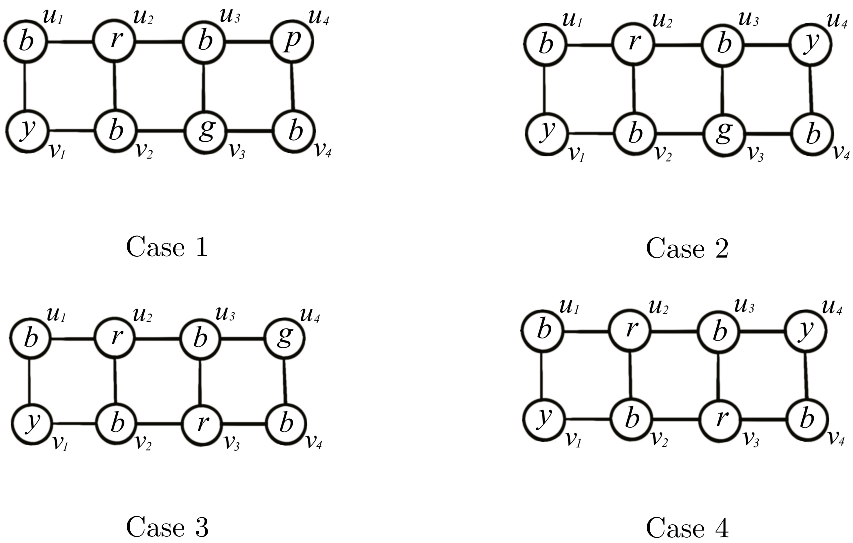


Figure 8. Possible colorings of H.

a new coloring whose color word is lexicographically less than w_c, which is a contradiction.

In Cases 1 and 2, we know $L(v_2) \subseteq \{b, g, r, y\}$. If $r \in L(v_2)$, define a new coloring c' in which $c'(v_2) = r$, $c'(u_2) \notin \{b, r\}$, and $c' = c$ for all other vertices. If $g \in L(v_2)$, define a new coloring c'' in which $c''(v_2) = g$, $c''(v_3) \notin \{b, g\}$, and $c'' = c$ for all other vertices. Both $w_{c'}$ and $w_{c''}$ are lexicographically less than w_c, so we obtain a contradiction and Cases 1 and 2 cannot occur.

In Cases 3 and 4, we know $L(v_2) = \{b, r, y\}$ and $L(u_3) = \{b, r, c(u_4)\}$. Define a new coloring c' in which $c'(v_2) = c'(u_3) = r$, $c'(u_2), c'(v_3) \notin \{b, r\}$, and $c' = c$ for all other vertices. If $c'(u_2) \neq c'(v_3)$, then the blue color class has decreased by 2 and two other color classes each increase by 1. Since Blue is at least 2 larger than every other color classes, we know that $w_{c'}$ is lexicographically less than w_c, which is a contradiction. So it must be that $c'(u_2) = c'(v_3)$. If we are forced to make $c'(u_2) = c'(v_3)$, let's say that the new color of u_2 and v_3 in c' is pink. If pink occurred at least 3 times fewer than blue in c, then $w_{c'}$ is still lexicographically less than w_c. So $c'(u_2) = c'(v_3)$, pink was the second most common color class of c, pink occurred exactly 2 times fewer than Blue in c, and we cannot avoid utilizing pink when defining c'. This causes us to obtain $w_{c'} = w_c$, so we must define c' in a different way to obtain a contradiction.

Since $|\text{Pink}| = |\text{Blue}| - 2$ in c and c uses at least 4 colors, it must be that $|\text{Blue}| = \lceil 2n/3 \rceil + 1$ and $|\text{Pink}| = \lceil 2n/3 \rceil - 1$ in c. Further, since we could not avoid coloring u_2 and v_3 pink, it must be that $L(u_2) = L(v_3) = \{b, p, r\}$. Recall that $L(v_2) = \{b, r, y\}$ and $L(u_3) = \{b, r, c(u_4)\}$. We will consider the rung to the left of u_1 and v_1 as pictured in Figure 9.

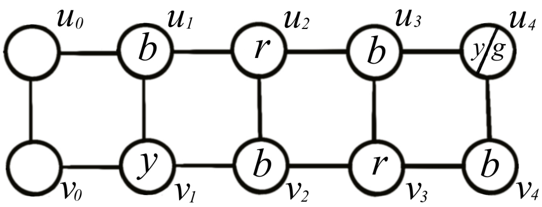


Figure 9. Diagram for Cases 3 and 4.

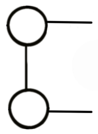
Notice that $c(u_0) \neq b$ because c is a proper coloring. Also, $c(v_0) \neq b$, other wise we could define a new coloring c' in which $c'(v_2) = y$, $c'(v_1) \notin \{b, y\}$, and $c' = c$ for all other vertices. Since $w_{c'}$ is less than w_c, this gives us a contradiction. Additionally, notice that $L(v_1) = \{b, y, c(v_0)\}$. If not, we could obtain a contradiction by defining c' so $c'(v_2) = y$, $c'(v_1) \notin \{b, y, c(v_0)\}$, and $c' = c$ for all other vertices. And $L(u_1) \subseteq \{b, r, y, c(u_0)\}$, otherwise we could obtain a contradiction by recoloring u_1 with something other than those 4 colors.

If $c(u_0) \in \{r, y\}$, then $L(u_1) = \{b, r, y\}$. If $c(u_0) = y$, obtain a contradiction by defining a new coloring c' in which $c'(u_1) = r$, $c'(u_2) = p$, and $c' = c$ for all other vertices. If $c(u_0) = r$, obtain a contradiction by defining a new coloring c'' in which $c''(u_1) = y$, $c''(v_1) = b$, $c''(v_2) = p$, and $c'' = c$ for all other vertices.

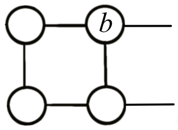
If $c(u_0) \notin \{r, y\}$, there are 3 possible lists for u_1. If $L(u_1) = \{b, r, y\}$ or $L(u_1) = \{b, y, c(u_0)\}$, obtain a contradiction by defining a new coloring c' in which $c'(u_1) = y$, $c'(v_1) = b$, $c'(v_2) = p$, and $c' = c$ for all other vertices. If $L(u_1) = \{b, r, c(u_0)\}$, obtain a contradiction by defining a new coloring c'' in which $c''(u_1) = r$, $c''(u_2) = p$, and $c'' = c$ for all other vertices.

In every case and sub-case we are able to obtain a contradiction. Thus, it must be that c does not contain four blue vertices in four consecutive rungs. ■

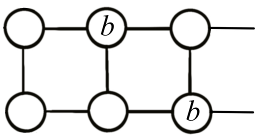
A consequence of Lemma 18 is that in the lex-min coloring c, the blue vertices of Π_n, $n \geq 6$, must be arranged into blocks of 1, 2, or 3 consecutive rungs. That is, we can cover the vertices and edges of Π_n with disjoint copies of the blocks shown in Figure 10, which may be reflected vertically. In these diagrams, the vertically-oriented edges correspond to rungs of Π_n and the blank vertices must *not* be blue.



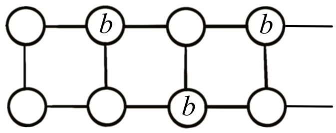
Block B_0



Block B_1



Block B_2



Block B_3

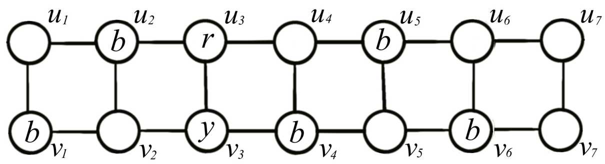
Figure 10. Blocks in the decomposition of Π_n, $n \geq 6$.

The block decomposition of Π_n, $n \geq 6$ will play an important role in the proof of Theorem 20. But first, in Lemma 19, we show that a B_3 block cannot be adjacent to either a B_2 block or another B_3 block.

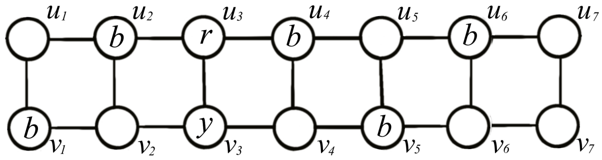
Lemma 19. *If c is lex-min and not $\lceil 2n/3 \rceil$-bounded, then a B_3 block in Π_n, $n \geq 6$ cannot be adjacent to a B_2 block or another B_3 block.*

Proof. Suppose not. Then Π_n contains a subgraph of 7 consecutive rungs which looks like one of configurations F_7 and F_8 in Figure 11. Vertices u_3 and v_3 must be non-blue and different colors, so without a loss of generality, we suppose they have colors r and y, respectively. Note further that the blue vertices must be blue and each blank vertex could be any non-blue color which is not already in use by its neighbors.

In configuration F_7, we will first determine $L(v_4)$. Notice that $L(v_4) \subseteq \{b, y, c(u_4), c(v_5)\}$. If not, we could recolor v_4 and obtain a lexicographically smaller color word. Further, notice that $c(u_4) \neq y$. If it were, then $L(v_4) = \{b, y, c(v_5)\}$ so we could define a lexicographically smaller coloring c' in which $c'(v_4) = c(v_5)$, $c'(v_5) \notin \{b, c(v_5)\}$, and $c' = c$ for all other vertices. For simplicity, say $c(u_4) = g$. Then $L(v_4) = \{b, g, y\}$. If not, then there exists a color c_0 in



Configuration F_7



Configuration F_8

Figure 11. Reducible configurations for Lemma 19.

$L(v_4) - \{b, g, y\}$. So we could define a lexicographically smaller coloring c'' in which $c''(v_4) = c_0$, $c''(v_5) \notin \{b, c_0\}$, and $c'' = c$ for all other vertices.

Now we will determine $L(u_4)$, $L(u_3)$, and $L(v_3)$. Notice that $L(u_4) = \{b, g, r\}$. If not, we could obtain a lexicographically smaller coloring c' in which $c'(v_4) = g$, $c'(u_4) \notin \{b, g, r\}$, $c'(v_5) \notin \{b, g\}$, and $c' = c$ for all other vertices. Next, notice that $L(u_3) = \{b, r, y\}$. If not, we could obtain a lexicographically smaller coloring c'' in which $c''(v_4) = g$, $c''(u_4) = r$, $c''(u_3) \notin \{b, r, y\}$, $c''(v_5) \notin \{b, g\}$, and $c'' = c$ for all other vertices. Finally, notice that $L(v_3) = \{b, y, c(v_2)\}$. If not, we could obtain a lexicographically smaller coloring c''' in which $c'''(v_4) = g$, $c'''(u_4) = r$, $c'''(u_3) = y$, $c'''(v_3) \notin \{b, y, c(v_2)\}$, $c'''(v_5) \notin \{b, g\}$, and $c''' = c$ for all other vertices.

Since we now know the color lists for u_3, v_3, u_4, and v_4, we will use them to obtain a lexicographically smaller coloring c'. If $c(v_2) \neq r$, define c' so that $c'(v_4) = y$, $c'(v_3) = c(v_2)$, $c'(v_2) \notin \{b, c(v_2)\}$, and $c' = c$ for all other vertices. If $c(v_2) = r$, define c' so that $c'(v_4) = g$, $c'(u_4) = r$, $c'(u_3) = y$, $c'(v_3) = r$, $c'(v_2) \notin \{b, r\}$, and $c' = c$ for all other vertices. In both cases, we contradict the assumption that c is a lex-min coloring. Thus, configuration F_7 cannot occur in coloring c.

In configuration F_8, we will first determine $L(u_4)$ and $L(u_2)$. Notice that $L(u_4) \subseteq \{b, r, c(v_4), c(u_5)\}$. If not, we could recolor u_4 and obtain a lexicographically smaller color word. Futher, notice that $c(v_4) \neq r$. If it were, then $L(u_4) = \{b, r, c(u_5)\}$ so we could define a lexicographically smaller coloring c' in which $c'(u_4) = c(u_5)$, $c'(u_5) \notin \{b, c(u_5)\}$, and $c' = c$ for all other vertices. For simplicity, say $c(v_4) = g$. Then $L(u_4) = \{b, g, r\}$. If not, then $L(u_4) = \{b, r, c(u_5)\}$ with $c(u_5) \neq g$. So we could define a lexicographically smaller coloring c'' in which $c''(u_4) = c(u_5)$, $c''(u_5) \notin \{b, c(u_5)\}$, and $c'' = c$ for all other vertices.

We will now consider three cases: $c(u_5) = r$, $c(u_5) = g$, and $c(u_5) \notin \{g, r\}$.

Case 1. Suppose that $c(u_5) = r$. We will determine $L(v_4)$, $L(v_3)$, and $L(u_3)$. Notice that $L(v_4) = \{b, g, y\}$. If not, we could obtain a lexicographically smaller coloring c' in which $c'(u_4) = g$, $c'(v_4) \notin \{b, g, y\}$, and $c' = c$ for all other vertices. Next, notice that $L(v_3) = \{r, y, c(v_2)\}$. If not, we could obtain a lexicographically smaller coloring c'' in which $c''(u_4) = g$, $c''(v_4) = y$, $c''(v_3) \notin \{r, y, c(v_2)\}$, and $c'' = c$ for all other vertices. Finally, notice that $L(u_3) = \{b, g, r\}$. If not, we could obtain a lexicographically smaller coloring c''' in which $c'''(u_4) = g$, $c'''(v_4) = y$, $c'''(v_3) = r$, $c'''(u_3) \notin \{b, g, r\}$, and $c''' = c$ for all other vertices.

Since we now know the color lists for u_4, v_4, and v_3, we will use them to obtain a lexicographically smaller coloring c'. Let $c'(u_4) = g$, $c'(v_4) = y$, $c'(v_3) = c(v_2)$, $c'(v_2) \notin \{b, c(v_2)\}$, and $c' = c$ for all other vertices. This contradicts the assumption that c is a lex-min coloring.

Case 2. Suppose $c(u_5) = g$. We will determine $L(u_3)$ and $L(v_3)$. Notice that $L(u_3) = \{b, r, y\}$. If not, we could obtain a lexicographically smaller coloring c' in which $c'(u_4) = r$, $c'(u_3) \notin \{b, r, y\}$, and $c' = c$ for all other vertices. Next, notice that $L(v_3) = \{g, y, c(v_2)\}$. If not, we could obtain a lexicographically smaller coloring c'' in which $c''(u_4) = r$, $c''(u_3) = y$, $c''(v_3) \notin \{g, y, c(v_2)\}$, and $c'' = c$ for all other vertices.

Since we now know the color lists for u_4, u_3, and v_3, we will use them to obtain a lexicographically smaller coloring c'. Let $c'(u_4) = r$, $c'(u_3) = y$, $c'(v_3) = c(v_2)$, $c'(v_2) \notin \{b, c(v_2)\}$, and $c' = c$ for all other vertices. This contradicts the assumption that c is a lex-min coloring.

Case 3. Suppose $c(u_5) \notin \{g, r\}$. We can apply the same argument as Case 2 to obtain a coloring that is lexicographically smaller than c.

In each of the three cases, we contradict the assumption that c is a lex-min coloring. Thus, configuration F_8 cannot occur in coloring c. ■

We are now ready to finish proving our main result. Theorem 20 argues that every lex-min 3-list-coloring of Π_n, $n \geq 6$ is $\lceil 2n/3 \rceil$-bounded. Together with Observation 3 and Lemmas 9 and 10, this final proof confirms that our main result, Theorem 8, is true, and Π_n, $n \geq 3$, is equitably 3-choosable.

Theorem 20. *If c is lex-min L-coloring of Π_n, $n \geq 6$, then c is $\lceil 2n/3 \rceil$-bounded.*

Proof. Suppose not. Define a charge function *chg* on the vertices and faces of a planar embedding of Π_n, $n \geq 6$, as follows.

For every vertex v, $chg(v) = 1 - |B(v)|$ where $B(v)$ is the set of neighbors of v in Π_n which are colored blue under c.

For every 4-face f, $chg(f) = \frac{4}{3} - |B(f)|$ where $B(f)$ is the set of vertices on face f which are colored blue under c.

For every n-face f, $chg(f) = 0$.

Let V denote the set of vertices of Π_n and F denote the set of faces of Π_n. Note that F consists of two n-faces and n 4-faces. Since $|\text{Blue}| > \lceil 2n/3 \rceil$, the total charge of Π_n is

$$\begin{aligned} chg(\Pi_n) &= \sum_{f \in F} chg(f) + \sum_{v \in V} chg(v) = 2(0) + \frac{4n}{3} - 2|\text{Blue}| + 2n - 3|\text{Blue}| \\ &= 10n/3 - 5|\text{Blue}| \le 10n/3 - 5\left(\lceil 2n/3 \rceil + 1\right) < 0. \end{aligned}$$

That is, when c is not $\lceil 2n/3 \rceil$-bounded, the total charge of Π_n is negative.

Recall that we can cover Π_n with disjoint copies of the blocks B_0, B_1, B_2, and B_3 pictured in Figure 10. As suggested by the figure, we introduce the convention that each block includes the 4-face to its right, and compute the charge of each block by summing the charges of its vertices and faces. To begin, the blocks have the following charges.

$chg(B_0) = 10/3$ if the block to its left is B_0 and $chg(B_0) = 7/3$ otherwise.

$chg(B_1) = 8/3$ if the block to its left is B_0 and $chg(B_1) = 5/3$ otherwise.

$chg(B_2) = 1$ if the block to its left is B_0 and $chg(B_2) = 0$ otherwise.

$chg(B_3) = -2/3$ if the block to its left is B_0 and $chg(B_3) = -5/3$ otherwise.

We introduce the following discharging rules.

Rule 1. Every B_0 steals $+1$ charge from the block to its right.

Rule 2. Every B_3 steals $+5/3$ charge from the block to its right.

After applying Rule 1, blocks of each type all have the same charge. In particular, $chg(B_0) = 10/3$, $chg(B_1) = 5/3$, $chg(B_2) = 0$, and $chg(B_3) = -5/3$. After applying Rule 2, $chg(B_3) = 0$ for every B_3 block. Further, by Lemma 19, we know that the block to the right of any B_3 cannot be a B_2 or a B_3. Thus, we end with the following charges.

$chg(B_0) = 5/3$ if the block to its left is B_3 and $chg(B_0) = 10/3$ otherwise.

$chg(B_1) = 0$ if the block to its left is B_3 and $chg(B_1) = 5/3$ otherwise.

$chg(B_2) = chg(B_3) = 0$ for all B_2 and B_3 blocks.

Since $chg(B_i) \geq 0$ for all $i \in \{0, 1, 2, 3\}$, and $chg(\Pi_n)$ is equal to the sum of the charges of the blocks, we may conclude that $chg(\Pi_n) \geq 0$. This contradicts our earlier conclusion that $chg(\Pi_n) < 0$, so it must be that our assumption about c is wrong. Instead, every lex-min coloring c of Π_n, $n \geq 6$, must be $\lceil 2n/3 \rceil$-bounded. ■

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