

HYPERGRAPH OPERATIONS PRESERVING *sc*-GREEDINESS

PIOTR BOROWIECKI

Institute of Control and Computation Engineering
University of Zielona Góra

e-mail: p.borowiecki@issi.uz.zgora.pl

EWA DRGAS-BURCHARDT

AND

ELŻBIETA SIDOROWICZ

Institute of Mathematics
University of Zielona Góra

e-mail: e.drgas-burchardt@wmie.uz.zgora.pl
e.sidorowicz@wmie.uz.zgora.pl

Abstract

Given a hypergraph \mathcal{H} and a function $f : V(\mathcal{H}) \rightarrow \mathbb{N}$, we say that \mathcal{H} is f -choosable if there exists a proper vertex colouring ϕ of \mathcal{H} such that $\phi(v) \in L(v)$ for all $v \in V(\mathcal{H})$, where $L : V(\mathcal{H}) \rightarrow 2^{\mathbb{N}}$ is an assignment of $f(v)$ colours to a vertex v . The sum-choice-number $\chi_{sc}(\mathcal{H})$ of \mathcal{H} is a minimum $\sum_{v \in V(\mathcal{H})} f(v)$ taken over all functions f such that \mathcal{H} is f -choosable. The hypergraphs for which $\chi_{sc}(\mathcal{H}) = |V(\mathcal{H})| + |\mathcal{E}(\mathcal{H})|$ are called *sc*-greedy. The class of *sc*-greedy hypergraphs is closed under the union of hypergraphs having at most one vertex in common. In this paper we consider *sc*-greediness of the union of hypergraphs having two vertices in common. We investigate this operation when one of the arguments is an arbitrary *sc*-greedy hypergraph while the second one is a hyperpath. Our research is motivated by the possibility of obtaining improved bounds on the sum-choice-number of graphs and new applications to the resource allocation problems in computer systems.

Keywords: hypergraph, list colouring, generalized colouring, minimum sum, resource allocation, greedy algorithm.

2020 Mathematics Subject Classification: 05C15, 68R10, 05C90, 68M20.

1. INTRODUCTION

In this paper we consider the problem of sum-list choosability of hypergraphs in the classical as well as the generalized variant in which the colour classes do not have to be edgeless. Our main concern is the class Γ_{sc} of *sc-greedy* hypergraphs. In this context we define and analyze new operations with respect to which Γ_{sc} is closed. In particular we are interested in preserving the property of being *sc-greedy* by the union of hypergraphs having two vertices in common.

Our motivation is twofold. On the one hand we are interested in *sc-greedy* hypergraphs, since these are exactly the hypergraphs attaining an upper bound on the hypergraph sum-choice number—a property that we further use to obtain bounds on the sum-choice number of various graphs, both in the classical and generalized variants of the problem. In this context, any operations that result in *sc-greedy* hypergraphs are of primary interest. On the other hand we identify new areas of applications of the generalized variant of sum-list choosability that turns out to be a natural complement for one of the resource allocation models by allowing the so-called resource preferences.

Since every *sc-greedy* hypergraph is linear, the above problems we consider for simple, finite and linear hypergraphs \mathcal{H} with the non-empty vertex set $V(\mathcal{H})$ and edge set $\mathcal{E}(\mathcal{H})$, where each edge in $\mathcal{E}(\mathcal{H})$ is a subset of $V(\mathcal{H})$ and contains at least two vertices (a hypergraph is *simple* if no edge is contained in the other, and it is *linear* if any two distinct edges have at most one vertex in common).

1.1. Sum-list choosability

A *proper colouring* of a hypergraph \mathcal{H} is a mapping $\phi : V(\mathcal{H}) \rightarrow \mathbb{N}$ such that every edge of \mathcal{H} contains at least two vertices v_1, v_2 coloured with distinct colours, i.e., $\phi(v_1) \neq \phi(v_2)$ (in other words no edge is allowed to be monochromatic). In the most classical setting there are no other restrictions on colour assignment and we simply aim at finding a colouring with the smallest number of colours. A well-known way of imposing colour restrictions is a *list assignment* $L : V(\mathcal{H}) \rightarrow 2^{\mathbb{N}}$ under which an *L-colouring* (*list colouring*) of \mathcal{H} is defined as a colouring ϕ such that for every vertex $v \in V(\mathcal{H})$ the colour $\phi(v)$ belongs to $L(v)$. In the problem of choosability we ask for smallest ℓ such that for every list assignment L with $|L(v)| \geq \ell$ for all $v \in V(\mathcal{H})$, there exists a proper *L-colouring* of \mathcal{H} .

A common approach is to consider list colouring with the lists of equal size. However, if we allow lists of varying sizes, it is known that the average size can get smaller (see, e.g. [12, 19]). Therefore, let us consider a positive integer valued function f on the vertex set of a hypergraph \mathcal{H} . We call f a *size function* and use $\text{size}(f)$ to denote $\sum_{v \in V(\mathcal{H})} f(v)$. An *f-assignment* for \mathcal{H} is a list assignment L such that $|L(v)| = f(v)$ for every vertex $v \in V(\mathcal{H})$. The size function f is a *choice function* for \mathcal{H} if for every *f-assignment* L there is a proper *L-colouring*

of \mathcal{H} . A hypergraph \mathcal{H} is *f-choosable* if f is a choice function for \mathcal{H} . Finally the *sum-choice-number* $\chi_{sc}(\mathcal{H})$ of \mathcal{H} is a minimum $\text{size}(f)$ taken over all choice functions f for \mathcal{H} , i.e.,

$$\chi_{sc}(\mathcal{H}) = \min_f \{ \text{size}(f) \mid \mathcal{H} \text{ is } f\text{-choosable} \}.$$

The sum-list colouring and sum-list choosability were first introduced for graphs. Erdős, Rubin and Taylor [12] analyzed the size functions, and then Isaak [15] was the first to consider the minimum sum of the list sizes. Many of the above concepts and results related to the earlier work on graphs have been recently generalized to hypergraphs (see, e.g. [10, 11]). Several generalizations, e.g. sum-list colourings in which colour classes need not be edgeless were investigated in [8, 9, 16]. Effectively computable upper bound on the minimum sum of the list sizes for graphs we find in [14]. For a similar bound in terms of β -degrees in hypergraphs see [9, 10].

1.2. Choice functions and *sc*-greedy hypergraphs

Let U be a subset of the vertex set of a hypergraph \mathcal{H} . By the *subhypergraph of \mathcal{H} induced by U* , denoted by $\mathcal{H}[U]$, we mean a hypergraph \mathcal{H}' for which $V(\mathcal{H}') = U$ and $\mathcal{E}(\mathcal{H}') = \{E \in \mathcal{E}(\mathcal{H}) \mid E \subseteq U\}$, whereas a *subhypergraph of \mathcal{H}* is an arbitrary hypergraph \mathcal{H}' satisfying $V(\mathcal{H}') \subseteq V(\mathcal{H})$ and $\mathcal{E}(\mathcal{H}') \subseteq \mathcal{E}(\mathcal{H})$. In this context, considering a choice function f for a hypergraph \mathcal{H} and its subhypergraphs \mathcal{H}' , observe that $f|_{V(\mathcal{H}')}$ is a choice function for \mathcal{H}' . Therefore $\chi_{sc}(\mathcal{H}) \geq \chi_{sc}(\mathcal{H}')$ for every subhypergraph \mathcal{H}' of a hypergraph \mathcal{H} .

A little bit more insightful analysis shows that

$$(1) \quad \chi_{sc}(\mathcal{H}) \leq |V(\mathcal{H})| + |\mathcal{E}(\mathcal{H})|.$$

In order to see that (1) holds for every hypergraph \mathcal{H} , take an arbitrary ordering v_1, \dots, v_n of the vertices of \mathcal{H} and let \mathcal{H}_i stand for a subhypergraph of \mathcal{H} induced by $\{v_1, \dots, v_i\}$. Now, consider a special size function f^* defined as follows: $f^*(v_i) = \deg_{\mathcal{H}_i}(v_i) + 1$ for $i \in \{1, \dots, |V(\mathcal{H})|\}$. It is not hard to argue that f^* is a choice function for \mathcal{H} . Indeed, if vertices are coloured according to the above ordering, then for each vertex v_i there are at most $\deg_{\mathcal{H}_i}(v_i)$ colour choices resulting in a monochromatic edge containing v_i . Consequently, there always remains at least one colour in $L(v_i)$ that can be used for v_i in any proper list colouring of \mathcal{H}_i . Hence $\chi_{sc}(\mathcal{H}) \leq \text{size}(f^*)$. Now, it remains to observe that $\text{size}(f^*) = |V(\mathcal{H})| + |\mathcal{E}(\mathcal{H})|$.

Note that f^* is a choice function for \mathcal{H} , even if the colour of each vertex is chosen greedily. Following [10, 18] the hypergraphs \mathcal{H} satisfying $\chi_{sc}(\mathcal{H}) = |V(\mathcal{H})| + |\mathcal{E}(\mathcal{H})|$ are called *sc-greedy*. A fundamental property of this class is that it is closed with respect to taking induced subhypergraphs but it is also known

that it is not closed with respect to taking subhypergraphs [10, 18], where the latter is easily seen by taking an *sc*-greedy graph obtained from $K_{2,3}$ by joining the vertices of degree 3 and observing that $K_{2,3}$ is not *sc*-greedy. Several other properties of *sc*-greedy hypergraphs are already known, where one of the most important is their linearity [10]; we state it explicitly for a further reference.

Property 1. *Every sc-greedy hypergraph is linear.*

It is also known that hypertrees, hypercycles as well as graphs obtained recursively by specific operation of identification of appropriate vertices of some hypergraphs are *sc*-greedy (for more details see [10, 11] and Section 2).

Since every *sc*-greedy hypergraph achieves the upper bound (1), one of the approaches to determine the sum-choice-number of an arbitrary hypergraph \mathcal{H} is to focus on its *sc*-greedy subhypergraphs with relatively large sum of the order and size, thus finding a lower bound on $\chi_{sc}(\mathcal{H})$. As we will see later this can be further used to bound the sum-choice-number of graphs in proper and generalized variants of colouring. With this as one of our aims in mind, we define and analyze new operations on *sc*-greedy hypergraphs that preserve their *sc*-greediness. On the way, we naturally narrow down to linear hypergraphs.

1.3. Related work and our results

The idea of investigating the operations on *sc*-greedy hypergraphs is partially inspired by the result for graphs stating that the operation of adding a handle, starting with an appropriate cycle, is sufficient to generate any 2-connected graph (see, e.g. Diestel [7]). In papers [6, 13, 18] we find several results on preserving *sc*-greediness by the operation of adding a handle to cycles and some special *graphs*. In this context, similar but more general operations that we consider for *hypergraphs*, also use elementary structures, such as hyperpaths or hypercycles, as the ‘building blocks’ of structurally more complex hypergraphs. At this point, however, it is necessary to remark that providing a set of basic operations whose recursive application allows generation of *all* 2-connected hypergraphs seems challenging.

Presumably, the problem does not get easier even if following the results in [10] we further restrict our analysis to linear 2-connected hypergraphs; starting with a hypercycle and repeating the operation of adding a handle cannot guarantee that every linear 2-connected hypergraph is obtained. However, adding a handle preserves linearity and 2-connectivity, when the handle has at least two edges. The research on adding hypergraph handles was initiated in papers [10, 11] where necessary and sufficient conditions for preserving *sc*-greediness were given in the case of handles added to a *hypercycle*. In this paper we continue this line of research by considering the operation of adding handles to *arbitrary sc-greedy hypergraphs*.

The remainder of this article is organized as follows. In Section 2 we present previously known hypergraph operations that preserve sc -greediness. In Section 3.1 we prove several new properties of sc -greedy hypergraphs and use them in Section 3.2 to prove our main result stating that the class of sc -greedy hypergraphs is closed with respect to two new variants of adding hypergraph handles. In Section 4, on applications, we provide examples that demonstrate the use of our new operations in bounding the sum-choice numbers of graphs, and describe new areas of applications of the sum-list choosability to the resource allocation problems in computer systems.

For any notions not defined in this paper the reader is referred to Berge [1] and Diestel [7].

2. BASIC OPERATIONS PRESERVING sc -GREEDINESS

This section is based mainly on papers [10, 11] and collects several results in the area of the hypergraph operations that preserve sc -greediness, and is intended to provide the necessary background for new operations on sc -greedy hypergraphs. We start with a union of hypergraphs.

Definition 1. Let \mathcal{H}_1 and \mathcal{H}_2 be two not necessarily vertex disjoint hypergraphs. The *union* of hypergraphs \mathcal{H}_1 and \mathcal{H}_2 , denoted $\mathcal{H}_1 \cup \mathcal{H}_2$, is the hypergraph with the vertex set $V(\mathcal{H}_1 \cup \mathcal{H}_2) = V(\mathcal{H}_1) \cup V(\mathcal{H}_2)$ and the edge set $\mathcal{E}(\mathcal{H}_1 \cup \mathcal{H}_2) = \mathcal{E}(\mathcal{H}_1) \cup \mathcal{E}(\mathcal{H}_2)$.

The following properties of hypergraph union were given in [10].

Property 2. If $\mathcal{H}_1, \mathcal{H}_2$ are vertex disjoint hypergraphs, then

$$\chi_{sc}(\mathcal{H}_1 \cup \mathcal{H}_2) = \chi_{sc}(\mathcal{H}_1) + \chi_{sc}(\mathcal{H}_2).$$

Property 3. If $\mathcal{H}_1, \mathcal{H}_2$ are two hypergraphs that have exactly one vertex in common, then

$$\chi_{sc}(\mathcal{H}_1 \cup \mathcal{H}_2) = \chi_{sc}(\mathcal{H}_1) + \chi_{sc}(\mathcal{H}_2) - 1.$$

From Properties 2 and 3 we draw the following easy but important conclusion.

Corollary 1. If $\mathcal{H}_1, \mathcal{H}_2$ are two hypergraphs that have at most one vertex in common, then $\mathcal{H}_1 \cup \mathcal{H}_2$ is sc -greedy if and only if both \mathcal{H}_1 and \mathcal{H}_2 are sc -greedy.

In order to keep our considerations precise, we need to define hyperpaths, hypercycles and hypertrees. For the former two we also need a specific notation.

Definition 2. By a *hyperpath* in a given hypergraph \mathcal{H} we mean a sequence $(v_1, E_1, v_2, E_2, \dots, E_{q-1}, v_q)$ with $q \geq 2$, where v_1, \dots, v_q are distinct vertices of \mathcal{H} and E_1, \dots, E_{q-1} are distinct edges of \mathcal{H} . Moreover, for all $i \in \{1, \dots, q-2\}$ it holds $E_i \cap E_{i+1} = \{v_{i+1}\}$, while for all edges $E_i \cap E_j = \emptyset$ whenever $|i - j| > 1$.

Definition 3. By a *hypercycle* in a given hypergraph \mathcal{H} we mean a sequence $(v_1, E_1, v_2, E_2, \dots, E_{q-1}, v_q, E_q, v_1)$ with $q \geq 3$, where v_1, \dots, v_q are distinct vertices of \mathcal{H} and E_1, \dots, E_q are distinct edges of \mathcal{H} . Moreover, $E_1 \cap E_q = \{v_1\}$ and for all $i \in \{1, \dots, q-1\}$ it holds $E_i \cap E_{i+1} = \{v_{i+1}\}$, while for all edges $E_i \cap E_j = \emptyset$ whenever $|i - j| > 1$.

The hyperpath $(v_1, E_1, v_2, E_2, \dots, E_{q-1}, v_q)$ is called (v_1, v_q) -*hyperpath*. For a hyperpath \mathcal{P} we also use $\text{Int}(\mathcal{P})$ to denote the set $\bigcup_{i=1}^{q-1} E_i \setminus \{v_1, v_q\}$ consisting of vertices called *internal*. By the *length* of a hyperpath (a hypercycle) we simply mean the number of its edges. Naturally, both a hyperpath \mathcal{P} and a hypercycle \mathcal{C} in a hypergraph \mathcal{H} can be viewed as its subhypergraphs, with the vertex and edge sets defined as $V(\mathcal{P}) = \bigcup_{i=1}^{q-1} E_i$ and $\mathcal{E}(\mathcal{P}) = \{E_1, \dots, E_{q-1}\}$ and $V(\mathcal{C}) = \bigcup_{i=1}^q E_i$ and $\mathcal{E}(\mathcal{C}) = \{E_1, \dots, E_q\}$, respectively (clearly they are linear subhypergraphs). Also, both can be treated as standalone hypergraphs, commonly called a *hyperpath* and a *hypercycle* if their vertex and edge sets coincide with those of the hypergraph \mathcal{H} .

Definition 4. A *hypertree* is either a single-edge hypergraph or any hypergraph that is the union of two hypertrees having exactly one vertex in common.

It is intuitive and easily proven that hypertrees and hypercycles are *sc-greedy*.

Property 4. *Hypertrees and hypercycles are sc-greedy.*

In the light of Properties 2–4 we conclude that further considerations on hypergraph *sc-greediness* could be carried out in the class of linear 2-connected hypergraphs, i.e., linear hypergraphs that cannot be obtained as a result of the union of two hypergraphs having at most one vertex in common.

With reference to the idea of constructing 2-connected graphs and the question of describing hypergraph operations that allow recursive construction of preferably all 2-connected *sc-greedy* hypergraphs (see [10] for more details) we need to recall the operation of adding a handle.

Definition 5 (Adding a handle). An operation of *adding a handle to a hypergraph* \mathcal{H} consists in taking the union of \mathcal{H} and (v_1, v_q) -hyperpath \mathcal{P} such that v_1, v_q are the only common vertices of \mathcal{H} and \mathcal{P} . The vertices v_1 and v_q are called the *connectors*, while the resulting hypergraph \mathcal{H}' is called a *hypergraph with (v_1, v_q) -handle* or simply a *hypergraph with handle*.

The following results on adding handles to hypercycles we find in [11] (see Figure 1 for an illustration).

Theorem 1. *Let \mathcal{H} be a hypercycle with handle and let the degrees of all vertices of \mathcal{H} be less than 3. The hypergraph \mathcal{H} is sc-greedy if and only if at least one of the following conditions holds:*

- (a) \mathcal{H} does not contain a hypercycle of length 3,
 (b) \mathcal{H} has an edge containing four vertices of degree 2.

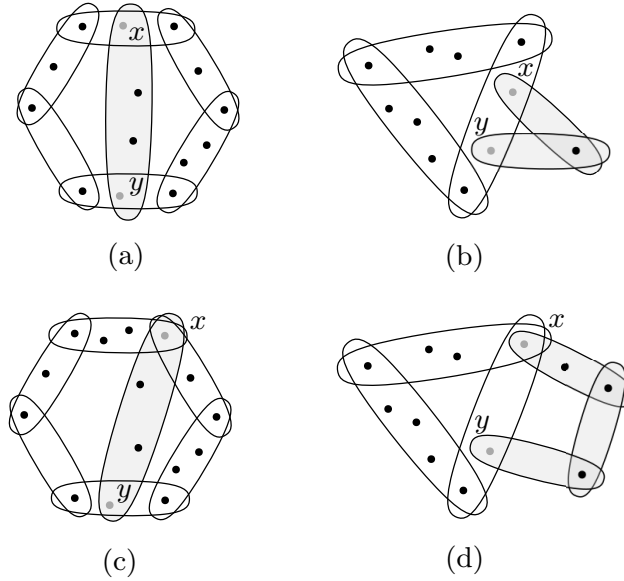


Figure 1. Examples of *sc*-greedy hypergraphs resulting from Theorem 1 (see, (a) and (b) in the upper row) and Theorem 2 (see, (c) and (d) in the bottom row). Connectors and handles are marked grey.

Theorem 2. Let \mathcal{H} be a hypercycle with handle and let the degrees of all vertices of \mathcal{H} be less than 3 except the distinguished vertex x of degree 3. The hypergraph \mathcal{H} is *sc*-greedy if and only if at least one of the following conditions holds:

- (a) \mathcal{H} does not contain a hypercycle of length 3,
 (b) x belongs to an edge containing three vertices of degree at least 2.

In a similar way the following theorem for θ -hypergraphs (see [10]) can be interpreted as a result on adding a handle to a hypercycle, when both connectors are of degree 2 (a θ -hypergraph, denoted by θ_{k_1, k_2, k_3}^h , is a hypergraph consisting of two vertices of degree 3 connected by three internally disjoint hyperpaths of lengths k_1, k_2, k_3). Clearly, θ -hypergraphs need not to be linear (see, e.g. $\theta_{1,1,1}^h$).

Theorem 3. Let k_1, k_2, k_3 be positive integers. A hypergraph θ_{k_1, k_2, k_3}^h is *sc*-greedy if and only if one of its distinguished hyperpaths, say the hyperpath of length k_2 , has only edges of size 2, and one of the following conditions holds:

- (a) both $k_1 + k_2$ and $k_2 + k_3$ are odd numbers, and $k_1 \geq 2$ or $k_3 \geq 2$,

- (b) $k_1 + k_2$ is an odd number and $k_2 + k_3$ is an even number and $k_3 \geq 3$,
- (c) both $k_1 + k_2$ and $k_2 + k_3$ are even numbers and $k_1 \geq 3$ and $k_3 \geq 3$.

Although by starting from a hypercycle and repeating the operation of adding a handle we are not able to generate all linear 2-connected hypergraphs, under certain conditions the operation preserves *sc-greediness*. Theorems 1, 2 and 3 describe such conditions when handles are added to hypercycles. In the next section we continue this topic by revealing *sc-greediness* preserving variants of adding handles to *arbitrary sc-greedy* hypergraphs.

3. ADDING HANDLES TO *sc*-GREEDY HYPERGRAPHS

In this section we define two new variants of adding handles to *sc-greedy* hypergraphs. Similarly as in Theorems 1, 2 and 3, the necessary conditions for adding a handle in both variants guarantee the resulting hypergraphs to be *sc-greedy*. The key concept of the first variant is that of an almost generated hyperpath of sufficient length. This allows adding handles of minimum length 3 and gives much freedom of choice of the connectors provided they are far enough apart. The second variant relies mainly on the closeness of the connectors and allows short handles.

Let us start with appropriate definitions.

Definition 6. Let \mathcal{H} be a hypergraph. We say that a hyperpath \mathcal{P} in \mathcal{H} is *generated* if $\deg_{\mathcal{P}}(v) = \deg_{\mathcal{H}}(v)$ for all $v \in \text{Int}(\mathcal{P})$. Similarly, a hyperpath \mathcal{P} in \mathcal{H} is called *almost generated* if for every $v \in \bigcup_{i=2}^{q-2} E_i$ it holds $\deg_{\mathcal{P}}(v) = \deg_{\mathcal{H}}(v)$, where $\mathcal{P} = (v_1, E_1, v_2, E_2, \dots, E_{q-1}, v_q)$.

Now, the main results of this paper can be stated as follows.

Theorem 4. Let \mathcal{H}' be an *sc-greedy* hypergraph and $u, w \in V(\mathcal{H}')$ be two vertices of degree one. If u, w are joined by an almost generated hyperpath of length at least 5 containing at least three edges, each with more than two vertices, then a hypergraph \mathcal{H} obtained by adding to \mathcal{H}' a (u, w) -handle of length at least 3 is *sc-greedy*.

Theorem 5. If \mathcal{H}' is an *sc-greedy* hypergraph and u, w are adjacent vertices of degree one, then a hypergraph \mathcal{H} obtained by adding to \mathcal{H}' a (u, w) -handle of length at least 2 is *sc-greedy*.

In the next section we give several new properties of *sc-greedy* hypergraphs, then we use them in Section 3.2 to prove Theorems 4 and 5.

3.1. New properties related to *sc*-greediness

In what follows, for convenience we write $\mathcal{H} - U$ instead of $\mathcal{H}[V(\mathcal{H}) \setminus U]$ and $\mathcal{H} - v$ in place of $\mathcal{H} - \{v\}$. Similarly we use the symbol $\mathcal{H} - E$ to denote a hypergraph obtained from a hypergraph \mathcal{H} by deleting a fixed edge. Also, for $A \subseteq V(\mathcal{H})$, by $\mathcal{E}(A)$ we mean $\bigcup_{v \in A} \mathcal{E}(v)$, where $\mathcal{E}(v) = \{E \in \mathcal{E}(\mathcal{H}) \mid v \in E\}$.

Lemma 6. *Let f be a size function for \mathcal{H} such that $\text{size}(f) < |V(\mathcal{H})| + |\mathcal{E}(\mathcal{H})|$. If there exists $A \subseteq V(\mathcal{H})$ such that $\mathcal{H} - A$ is *sc*-greedy and $\text{size}(f|_A) \geq |A| + |\mathcal{E}(A)|$, then \mathcal{H} is not f -choosable.*

Proof. Suppose to the contrary that \mathcal{H} is f -choosable. Let $\mathcal{H}' = \mathcal{H} - A$. Clearly,

$$\begin{aligned} \text{size}(f|_{V(\mathcal{H}')}) &= \text{size}(f) - \text{size}(f|_A) < |V(\mathcal{H})| + |\mathcal{E}(\mathcal{H})| - \text{size}(f|_A) \\ &\leq |V(\mathcal{H})| + |\mathcal{E}(\mathcal{H})| - (|A| + |\mathcal{E}(A)|) = |V(\mathcal{H}')| + |\mathcal{E}(\mathcal{H}')|. \end{aligned}$$

Since $\text{size}(f|_{V(\mathcal{H}')}) < |V(\mathcal{H}')| + |\mathcal{E}(\mathcal{H}')|$ and \mathcal{H}' is *sc*-greedy, we infer that \mathcal{H}' is not $f|_{V(\mathcal{H}')}$ -choosable. Thus \mathcal{H} is not f -choosable, a contradiction. ■

It follows from Lemma 6 that if \mathcal{H} is not *sc*-greedy, f is a choice function for \mathcal{H} such that $\text{size}(f) = \chi_{sc}(\mathcal{H})$, and $\mathcal{H} - A$ is *sc*-greedy, then $\text{size}(f|_A) \leq |A| + |\mathcal{E}(A)| - 1$. Thus, for a singleton A we have the following.

Corollary 2. *Let \mathcal{H} be a hypergraph that is not *sc*-greedy and let f be a choice function for \mathcal{H} such that $\text{size}(f) = \chi_{sc}(\mathcal{H})$. If $\mathcal{H} - v$ is *sc*-greedy, then $f(v) \leq \deg_{\mathcal{H}}(v)$.*

Lemma 7. *If $(v_1, E_1, v_2, \dots, v_q, E_q, v_1)$ is a hypercycle in \mathcal{H} , then*

$$\sum_{i \in \{1, \dots, q\}} \deg_{\mathcal{H}}(v_i) \geq |\mathcal{E}(\{v_1, \dots, v_q\})| + q.$$

Proof. Let $\mathcal{H}_i = \mathcal{H} - \{v_{i+1}, \dots, v_q\}$ for $i \in \{1, \dots, q-1\}$ and let $\mathcal{H}_q = \mathcal{H}$. Clearly, $\deg_{\mathcal{H}}(v_i) \geq \deg_{\mathcal{H}_i}(v_i) + 1$ for $i \in \{2, \dots, q-1\}$ and $\deg_{\mathcal{H}}(v_1) \geq \deg_{\mathcal{H}_1}(v_1) + 2$. Thus,

$$\sum_{i \in \{1, \dots, q\}} \deg_{\mathcal{H}}(v_i) \geq \sum_{i \in \{1, \dots, q\}} \deg_{\mathcal{H}_i}(v_i) + q = |\mathcal{E}(\{v_1, \dots, v_q\})| + q. \quad \blacksquare$$

Lemmas 6 and 7 now lead to the following.

Corollary 3. *Let f be a size function for \mathcal{H} such that $\text{size}(f) < |V(\mathcal{H})| + |\mathcal{E}(\mathcal{H})|$. If \mathcal{H} contains a hypercycle $(v_1, E_1, v_2, \dots, v_q, E_q, v_1)$ such that $f(v_i) = \deg_{\mathcal{H}}(v_i)$ for $i \in \{1, \dots, q\}$ and $\mathcal{H} - \{v_1, \dots, v_q\}$ is *sc*-greedy, then \mathcal{H} is not f -choosable.*

In the next lemma we present a useful property of a hypergraph for which there is a choice function with the size smaller than the sum of the numbers of vertices and edges.

Lemma 8. *Suppose that hypergraph \mathcal{H} is not sc -greedy and let $v \in V(\mathcal{H})$ be such that $\mathcal{H} - E$ is sc -greedy for every $E \in \mathcal{E}(v)$. If f is a choice function for \mathcal{H} such that $\text{size}(f) = \chi_{sc}(\mathcal{H})$, then $\deg_{\mathcal{H}}(v) \geq 2$ implies $f(v) \geq 2$ provided that each edge in $\mathcal{E}(v)$ contains at most two vertices of degree greater than one.*

Proof. Let $\deg_{\mathcal{H}}(v) \geq 2$ and let v satisfy all assumptions (in particular all those concerning the edges in $\mathcal{E}(v)$). Suppose also that $f(v) = 1$. The assumption that for every $E \in \mathcal{E}(v)$ the hypergraph $\mathcal{H} - E$ is sc -greedy implies that $\mathcal{H} - w$ is sc -greedy for every neighbour w of v . Indeed, let $E' \in \mathcal{E}(v)$ be an edge that contains w , thus $\mathcal{H} - E'$ is sc -greedy, however $\mathcal{H} - w$ is the induced subhypergraph of $\mathcal{H} - E'$, so it also has to be sc -greedy. Let $\mathcal{E}(v) = \{E_1, \dots, E_k\}$. Observe that by Corollary 2, we have $f(w) = 1$ for $w \in E_1 \cup \dots \cup E_k$ with $\deg_{\mathcal{H}}(w) = 1$.

If there is an edge $E \in \{E_1, \dots, E_k\}$ that contains only one vertex of degree greater than one (only v), then for each vertex in E the value of f is equal to one, and consequently \mathcal{H} is not f -choosable, a contradiction. Thus for each $i \in \{1, \dots, k\}$ there is exactly one vertex v_i different from v such that $v_i \in E_i$ and $\deg_{\mathcal{H}}(v_i) \geq 2$. Let $\mathcal{H}' = \mathcal{H}[(V(\mathcal{H}) \setminus \bigcup_{i=1}^k E_i) \cup \{v_1, \dots, v_k\}]$, i.e., to obtain \mathcal{H}' we delete from \mathcal{H} the vertex v and some vertices of degree one, say we delete $t - 1$ vertices of degree one. Thus, $|\mathcal{E}(\mathcal{H}')| = |\mathcal{E}(\mathcal{H})| - k$ and $|V(\mathcal{H}')| = |V(\mathcal{H})| - t$. Now, we define the following size function f' for \mathcal{H}' :

- (a) $f'(u) = f(u)$ if $u \notin \{v_1, \dots, v_k\}$,
- (b) $f'(v_i) = f(v_i) - 1$ for $i \in \{1, \dots, k\}$.

Thus $\sum_{u \in V(\mathcal{H}')} f'(u) = \sum_{u \in V(\mathcal{H})} f(u) - k - t$. By our assumption on \mathcal{H} it holds $\sum_{u \in V(\mathcal{H})} f(u) \leq |V(\mathcal{H})| + |\mathcal{E}(\mathcal{H})| - 1$, and consequently $\sum_{u \in V(\mathcal{H}')} f'(u) \leq |V(\mathcal{H})| + |\mathcal{E}(\mathcal{H})| - 1 - k - t = |V(\mathcal{H}')| + |\mathcal{E}(\mathcal{H}')| - 1$. Thus \mathcal{H}' is not f' -choosable. Let L' be an f' -assignment for \mathcal{H}' such that there is no proper L' -colouring of \mathcal{H}' and let $a \notin \bigcup_{u \in V(\mathcal{H}')} L'(u)$. We define an f -assignment L for \mathcal{H} in the following way: $L(u) = L'(u)$ for $u \in V(\mathcal{H}') \setminus \{v_1, \dots, v_k\}$, next $L(v_i) = L'(v_i) \cup \{a\}$ for $i \in \{1, \dots, k\}$ and $L(u) = \{a\}$ for $u \in V(\mathcal{H}) \setminus V(\mathcal{H}')$. It is not hard to see that there is no proper L -colouring of \mathcal{H} , which means that \mathcal{H} is not f -choosable and contradicts the assumption. Hence $f(v) \geq 2$, which completes the proof. ■

In the proofs of the main theorems we often use arguments related to extendability of certain partial L -colourings. The following few lemmas are devoted to extendability for hyperpaths.

Definition 7. Given an f -assignment L for \mathcal{H} , $\{v_1, \dots, v_p\} \subseteq V(\mathcal{H})$ and given $(c_1, \dots, c_p) \subseteq L(v_1) \times \dots \times L(v_p)$, we say that a p -tuple (c_1, \dots, c_p) is $(L, v_1, \dots,$

v_p)-extendable for \mathcal{H} if there is a proper L -colouring ϕ of \mathcal{H} such that $\phi(v_i) = c_i$ for $i \in \{1, \dots, p\}$.

Lemma 9 [10]. *Let \mathcal{P} be a (v_1, v_q) -hyperpath with at least one edge, and let f be a size function for \mathcal{P} such that $f(v_1) = f(v_q) = 2$ and $f(v) = \deg_{\mathcal{P}}(v)$ for $v \notin \{v_1, v_q\}$. If L is an f -assignment for \mathcal{P} , then at most two pairs in $L(v_1) \times L(v_q)$ are not (L, v_1, v_q) -extendable for \mathcal{P} . Moreover,*

- (a) *there are exactly two pairs that are not (L, v_1, v_q) -extendable for \mathcal{P} if and only if \mathcal{P} contains only 2-edges and $L(v_1) = \dots = L(v_q)$, and*
- (b) *if \mathcal{P} is of even length and there are exactly two pairs that are not (L, v_1, v_q) -extendable for \mathcal{P} , then (a, a) and (b, b) are (L, v_1, v_q) -extendable for \mathcal{P} , where a, b are colours in the list of v_1 , and*
- (c) *if \mathcal{P} is of odd length and there are exactly two pairs that are not (L, v_1, v_q) -extendable for \mathcal{P} , then (a, b) and (b, a) are (L, v_1, v_q) -extendable for \mathcal{P} , where a, b are colours in the list of v_1 .*

Lemma 9 implies the following.

Corollary 4. *Let \mathcal{P} be a (v_1, v_q) -hyperpath that has at least one edge with more than two vertices. If f is a size function for \mathcal{P} such that $f(v_1) = f(v_q) = 2$ and $f(v) = \deg_{\mathcal{P}}(v)$ for $v \notin \{v_1, v_q\}$, then for every f -assignment L for \mathcal{P} there are at least three distinct pairs $(a, b) \in L(v_1) \times L(v_q)$ that are (L, v_1, v_q) -extendable for \mathcal{P} .*

Lemma 10. *Let \mathcal{P} be a hyperpath $(v_1, E_1, \dots, E_{q-1}, v_q)$ that has at least three edges. If f is a size function for \mathcal{P} such that $f(v) = \deg_{\mathcal{P}}(v)$ for all $v \in V(\mathcal{P})$, then for every pair (a, b) there is an f -assignment L such that $a \in L(v_1), b \in L(v_q)$ and (a, b) is not (L, v_1, v_q) -extendable for \mathcal{P} .*

Proof. It is enough to define appropriate f -assignments L for \mathcal{P} in the following two cases.

Case 1. ($a \neq b$)

If length of \mathcal{P} is even, we let

- $L(v_1) = \{a\}$, $L(v_q) = \{b\}$ and $L(v_i) = \{a, b\}$ for $i \in \{2, \dots, q-1\}$,
- $L(v) = \{a\}$ for $v \in E_i \setminus \{v_i, v_{i+1}\}$ when i is odd,
- $L(v) = \{b\}$ for $v \in E_i \setminus \{v_i, v_{i+1}\}$ when i is even.

If length of \mathcal{P} is odd, we assume $c \neq a$ and $c \neq b$, and set

- $L(v_2) = \{a, c\}$, $L(v_3) = \{b, c\}$, $L(v_q) = \{b\}$ and $L(v_i) = \{a, b\}$ for $i \in \{4, \dots, q-1\}$,
- $L(v) = \{c\}$ for $v \in E_2 \setminus \{v_2, v_3\}$,

- $L(v) = \{a\}$ for $v \in E_1 \setminus \{v_2\}$, and for $v \in E_i \setminus \{v_i, v_{i+1}\}$ with even i in $\{4, \dots, q-1\}$,
- $L(v) = \{b\}$ for $v \in E_i \setminus \{v_i, v_{i+1}\}$ with odd $i \in \{3, \dots, q-1\}$.

Case 2. ($a = b$)

If length of \mathcal{P} is even, we assume $c \neq a, c \neq d$ and $a \neq d$, and set

- $L(v_2) = \{a, c\}$, $L(v_3) = \{c, d\}$, $L(v_i) = \{a, d\}$ for $i \in \{4, \dots, q-1\}$,
- $L(v) = \{a\}$ for $v \in \{v_q\} \cup (E_1 \setminus \{v_2\})$, and for $v \in E_i \setminus \{v_i, v_{i+1}\}$ with even i in $\{4, \dots, q-1\}$,
- $L(v) = \{c\}$ for $v \in E_2 \setminus \{v_2, v_3\}$,
- $L(v) = \{d\}$ for $v \in E_i \setminus \{v_i, v_{i+1}\}$ with odd i in $\{3, \dots, q-1\}$.

If length of \mathcal{P} is odd, we assume $a \neq c$ and set

- $L(v_i) = \{a, c\}$ for $i \in \{2, \dots, q-1\}$,
- $L(v) = \{a\}$ for $v \in \{v_q\} \cup (E_1 \setminus \{v_2\})$, and for $v \in E_i \setminus \{v_i, v_{i+1}\}$ with odd i in $\{3, \dots, q-1\}$,
- $L(v) = \{c\}$ for $v \in E_i \setminus \{v_i, v_{i+1}\}$ with even i in $\{2, \dots, q-1\}$. ■

Naturally, if \mathcal{P} is a hyperpath with two edges, i.e., $\mathcal{P} = (v_1, E_1, v_2, E_2, v_3)$, and f is a size function for \mathcal{P} such that $f(v) = \deg_{\mathcal{P}}(v)$ for every vertex v of \mathcal{P} , then for every pair (a, b) of distinct integers there exists an f -assignment L such that (a, b) is not (L, v_1, v_3) -extendable for \mathcal{P} . Thus by Lemma 10 we get the following.

Lemma 11. *Let \mathcal{P} be a (v_1, v_q) -hyperpath with at least two edges. If f is a size function for \mathcal{P} such that $f(v) = \deg_{\mathcal{P}}(v)$ for $v \in V(\mathcal{P})$, then for every pair (a, b) such that $a \neq b$ there exists an f -assignment L such that $a \in L(v_1)$, $b \in L(v_q)$ and (a, b) is not (L, v_1, v_q) -extendable for \mathcal{P} .*

This leads to the following statement.

Corollary 5. *If \mathcal{P} is a hyperpath and f is a size function for \mathcal{P} such that $f(v) = \deg_{\mathcal{P}}(v)$ for every vertex v of \mathcal{P} , then \mathcal{P} is not f -choosable.*

In the last lemma of this section the generated paths come into play.

Lemma 12. *Let \mathcal{H} be a hypergraph that is not sc -greedy and let $v_1, v_q \in V(\mathcal{H})$ be two vertices of degree two in \mathcal{H} that are joined by a generated hyperpath \mathcal{P} of length at least 3. If $\mathcal{H} - \text{Int}(\mathcal{P})$ is sc -greedy, then $f(v) = \deg_{\mathcal{H}}(v)$ for every $v \in V(\mathcal{P})$ and for every choice function f for \mathcal{H} with $\text{size}(f) = \chi_{sc}(\mathcal{H})$.*

Proof. The argument that $\mathcal{H} - \text{Int}(\mathcal{P})$ is sc -greedy implies that also $\mathcal{H} - v$ is sc -greedy for every vertex $v \in V(\mathcal{P})$. Indeed, $\mathcal{H} - v$ is a hypergraph \mathcal{H}'' with some additional isolated vertices, where

- (a) if $v \in \text{Int}(\mathcal{P})$, then \mathcal{H}'' is the union of $\mathcal{H} - \text{Int}(\mathcal{P})$ and two disjoint hyperpaths $\mathcal{P}_1, \mathcal{P}_2$ such that v_1 is the only common vertex of $\mathcal{H} - \text{Int}(\mathcal{P})$ and \mathcal{P}_1 , v_q is the only common vertex of $\mathcal{H} - \text{Int}(\mathcal{P})$ and \mathcal{P}_2 ,
- (b) if $v = v_1$ (or $v = v_q$), then \mathcal{H}'' is the union of $\mathcal{H} - (V(\mathcal{P}) \setminus \{v_q\})$ (or $\mathcal{H} - (V(\mathcal{P}) \setminus \{v_1\})$) and some hyperpath that have one vertex v_q (or v_1) in common.

Clearly $\mathcal{H} - (V(\mathcal{P}) \setminus \{v_1\})$ and $\mathcal{H} - (V(\mathcal{P}) \setminus \{v_q\})$ are *sc*-greedy, because they are induced subhypergraphs of the *sc*-greedy hypergraph. Thus, by Corollary 1, $\mathcal{H} - v$ is *sc*-greedy for $v \in V(\mathcal{P})$. Similarly, we can see that for every $E \in \mathcal{E}(\mathcal{P})$, the hypergraph $\mathcal{H} - E$ is *sc*-greedy.

Let f be a choice function for \mathcal{H} with $\text{size}(f) = \chi_{sc}(\mathcal{H})$. Thus by Corollary 2, $f(v) \leq \deg_{\mathcal{H}}(v)$ for $v \in V(\mathcal{P})$, and consequently $f(v) = \deg_{\mathcal{H}}(v)$ for vertices of \mathcal{P} of degree one. By Lemma 8, every internal vertex of \mathcal{P} with degree two has the list size equal to two. Thus, it remains to prove that also $f(v_1) = 2$ and $f(v_q) = 2$. Suppose first that $f(v_1) = 1$ and $f(v_q) = 1$. By Corollary 5, \mathcal{P} is not $f|_{V(\mathcal{P})}$ -choosable, so \mathcal{H} is not f -choosable. Thus, at least one of v_1, v_q has to have a list size equal to two. Let $f(v_1) = 1$, $f(v_q) = 2$ and suppose that $\mathcal{P} = (v_1, E_1, v_2, \dots, v_{q-1}, E_{q-1}, v_q)$, where $q \geq 4$ by the assumptions (see Figure 2).

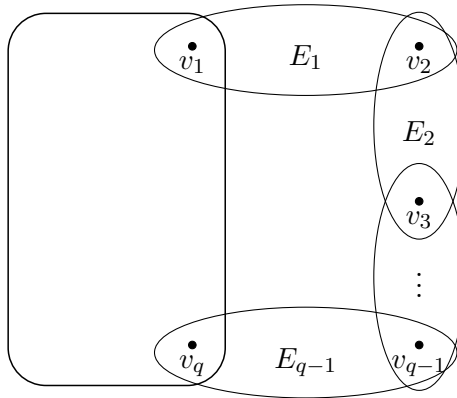


Figure 2. The hypergraph \mathcal{H} that illustrates Lemma 12.

Let $\mathcal{H}' = \mathcal{H} - v_2$ and g be the size function for \mathcal{H}' such that $g(v) = f(v)$ for $v \in V(\mathcal{H}') \setminus \{v_3\}$ and $g(v_3) = f(v_3) - 1$. Observe that $\text{size}(g) < |V(\mathcal{H}')| + |\mathcal{E}(\mathcal{H}')|$ and so \mathcal{H}' is not g -choosable, because \mathcal{H}' is *sc*-greedy. So there is a g -assignment $L_{\mathcal{H}'}$ such that \mathcal{H}' is not properly $L_{\mathcal{H}'}$ -colourable. Let $L_{\mathcal{H}'}(v_1) = \{a\}$ and $L_{\mathcal{H}'}(v_3) = \{b\}$ (it may be that $a = b$), thus (a, b) is not $(L_{\mathcal{H}'}, v_1, v_3)$ -extendable for \mathcal{H}' . We define an f -assignment $L_{\mathcal{H}}$ for \mathcal{H} in the following way:

- $L_{\mathcal{H}}(v) = L_{\mathcal{H}'}(v)$ for $v \in V(\mathcal{H}') \setminus (E_1 \cup E_2)$,
- $L_{\mathcal{H}}(v_2) = \{a, c\}$, where $c \neq a$,
- $L_{\mathcal{H}}(v) = \{a\}$ for every $v \in E_1 \setminus \{v_2\}$,
- $L_{\mathcal{H}}(v) = \{c\}$ for every $v \in E_2 \setminus \{v_2, v_3\}$,
- $L_{\mathcal{H}}(v_3) = \{b, c\}$.

Now, since all vertices of E_1 except v_2 have exactly one colour on the lists (the colour a), in any proper $L_{\mathcal{H}}$ -colouring v_2 must receive c . Next, since E_2 cannot be monochromatic, v_3 must receive b . However, as we observed before, (a, b) is not $(L_{\mathcal{H}'}, v_1, v_3)$ -extendable for \mathcal{H}' , so \mathcal{H} is not properly $L_{\mathcal{H}}$ -colourable. Hence $f(v_1) = 2 = \deg_{\mathcal{H}}(v_1)$ and $f(v_q) = 2 = \deg_{\mathcal{H}}(v_2)$, which completes the proof. ■

3.2. New variants of the operation of adding a handle

3.2.1. Variant 1 — proof of Theorem 4

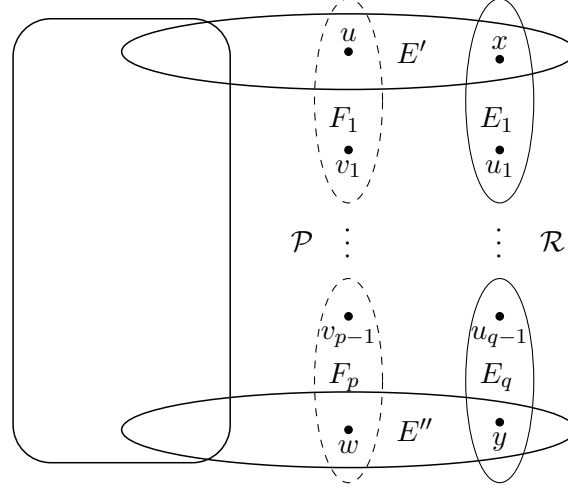
Theorem 4. *Let \mathcal{H}' be an sc -greedy hypergraph and $u, w \in V(\mathcal{H}')$ be two vertices of degree one. If u, w are joined by an almost generated hyperpath of length at least 5 containing at least three edges, each with more than two vertices, then a hypergraph \mathcal{H} obtained by adding to \mathcal{H}' a (u, w) -handle of length at least 3 is sc -greedy.*

Proof. By Corollary 1 every component of an sc -greedy hypergraph is sc -greedy. Hence, we assume that \mathcal{H}' is connected, for otherwise we can consider the component of \mathcal{H}' containing u and w . Let $(u, E', x, E_1, u_1, \dots, u_{q-1}, E_q, y, E'', w)$ be the assumed almost generated (u, w) -hyperpath in \mathcal{H}' and let \mathcal{P} be the (u, w) -handle, where $\mathcal{P} = (u, F_1, v_1, F_2, v_2, \dots, v_{p-1}, F_p, w)$. Clearly $\mathcal{H} = \mathcal{H}' \cup \mathcal{P}$. Moreover, let $\mathcal{R} = (x, E_1, u_1, \dots, u_{q-1}, E_q, y)$. For an illustration see Figure 3.

Claim 1. $\mathcal{H} - \text{Int}(\mathcal{P})$ is sc -greedy.

Proof. Suppose, on the contrary, that \mathcal{H} is not sc -greedy and observe that $\mathcal{H} - v$ is sc -greedy for every $v \in \text{Int}(\mathcal{P})$. Indeed, $\mathcal{H} - v$ can be seen as a hypergraph \mathcal{H}'' with some additional isolated vertices, where \mathcal{H}'' is either a union of \mathcal{H}' and a hyperpath having one vertex in common or \mathcal{H}'' is a union of \mathcal{H}' and two disjoint hyperpaths $\mathcal{P}_1, \mathcal{P}_2$, where u is the common vertex of \mathcal{H}' and \mathcal{P}_1 , and w is the common vertex of \mathcal{H}' and \mathcal{P}_2 . Thus, by Corollary 1, $\mathcal{H} - v$ is sc -greedy for all $v \in \text{Int}(\mathcal{P})$ and hence also $\mathcal{H} - \text{Int}(\mathcal{P})$ is sc -greedy. □

Similarly, we can show that $\mathcal{H} - v$ is sc -greedy for $v \in E' \cup E''$. We state this fact explicitly for a further reference.

Figure 3. The hypergraph \mathcal{H} in Theorem 4.

Claim 2. *If $v \in E' \cup E''$, then $\mathcal{H} - v$ is sc -greedy.*

In order to finish the proof we need to argue that $\mathcal{H} - \text{Int}(\mathcal{R})$ is sc -greedy (see Claim 3). Assuming Claims 1 and 3 hold, the proof proceeds as follows.

Let g be a choice function for \mathcal{H} such that $\text{size}(g) = \chi_{sc}(\mathcal{H})$. Since by the assumption \mathcal{H} is not sc -greedy, $\text{size}(g) < |V(\mathcal{H})| + |\mathcal{E}(\mathcal{H})|$. We know that $\mathcal{H} - \text{Int}(\mathcal{P})$ is sc -greedy (Claim 1) and hence by Lemma 12 we have $g(v) = \deg_{\mathcal{H}}(v)$ for every $v \in V(\mathcal{P})$. The argument that $\mathcal{H} - \text{Int}(\mathcal{R})$ is sc -greedy (Claim 3), again by Lemma 12, implies that $g(v) = \deg_{\mathcal{H}}(v)$ for every $v \in V(\mathcal{R})$. Since $\text{size}(g|_{V(\mathcal{P}) \cup V(\mathcal{R})}) = \sum_{v \in V(\mathcal{P}) \cup V(\mathcal{R})} \deg_{\mathcal{H}}(v) = |V(\mathcal{P}) \cup V(\mathcal{R})| + |\mathcal{E}(\mathcal{P}) \cup \mathcal{E}(\mathcal{R})|$, from Lemma 6 it follows that \mathcal{H} is not g -choosable, which is a contradiction.

In the remaining part of the proof we need additional notation. Namely, let S_1, \dots, S_p be a family of sets and \mathcal{G} be a hypergraph (we allow $S_i \cap V(\mathcal{G}) \neq \emptyset$). By $\mathcal{G} + \{S_1, \dots, S_p\}$ we denote a hypergraph with the vertex set $V(\mathcal{G}) \cup S_1 \cup \dots \cup S_p$ and the edge set $\mathcal{E}(\mathcal{G}) \cup \{S_1\} \cup \dots \cup \{S_p\}$.

Claim 3. $\mathcal{H} - \text{Int}(\mathcal{R})$ is sc -greedy.

Proof. Let $\mathcal{H}_{\mathcal{R}} = \mathcal{H} - \text{Int}(\mathcal{R})$. First, observe that if each of both E' and E'' contains exactly two vertices of degree at least two (say u, x and w, y , respectively), then \mathcal{H} is a hypercycle and hence sc -greedy, a contradiction. On the other hand, if either E' or E'' contains exactly two vertices of degree at least two, then $\mathcal{H}_{\mathcal{R}}$ is a union of two sc -greedy hypergraphs having exactly one vertex in common. Indeed, suppose that E'' is the edge that has exactly two vertices of degree at least two in \mathcal{H} . Now, by our assumption w and y are the only vertices of degree at least two in E'' and they have degree exactly two in \mathcal{H} . Thus, the common

vertex of these two *sc*-greedy hypergraphs is u and one of them is the hyperpath $\mathcal{P} + E''$. From Corollary 1 it follows that $\mathcal{H}_{\mathcal{R}}$ is *sc*-greedy in this case. Thus, we may assume that each of E', E'' has at least three vertices of degree at least two (E', E'' may also contain vertices of degree one). Let $E' = \{x_1, \dots, x_t, x, u\}$ and $E'' = \{y_1, \dots, y_s, y, w\}$.

Suppose, contrary to our claim, that $\mathcal{H}_{\mathcal{R}}$ is not *sc*-greedy. The idea of the remaining part of the proof is as follows. First, we will prove that every choice function f for $\mathcal{H}_{\mathcal{R}}$ with the size at most $|V(\mathcal{H}_{\mathcal{R}})| + |\mathcal{E}(\mathcal{H}_{\mathcal{R}})| - 1$ has special properties: if restricted to $V(\mathcal{H}_{\mathcal{P}, \mathcal{R}})$, where $\mathcal{H}_{\mathcal{P}, \mathcal{R}} = \mathcal{H} - (V(\mathcal{P}) \cup V(\mathcal{R}))$, then it has the size equal to $|V(\mathcal{H}_{\mathcal{P}, \mathcal{R}})| + |\mathcal{E}(\mathcal{H}_{\mathcal{P}, \mathcal{R}})|$ (i.e., $\text{size}(f|_{V(\mathcal{H}_{\mathcal{P}, \mathcal{R}})}) = |V(\mathcal{H}_{\mathcal{P}, \mathcal{R}})| + |\mathcal{E}(\mathcal{H}_{\mathcal{P}, \mathcal{R}})|$) and it has the properties that we will denote by (1) and (2). Next, we will construct the choice function g for \mathcal{H}' (for vertices of $\mathcal{H}_{\mathcal{P}, \mathcal{R}}$ we will assign the values of f , where f is any choice function for $\mathcal{H}_{\mathcal{R}}$ with the size at most $|V(\mathcal{H}_{\mathcal{R}})| + |\mathcal{E}(\mathcal{H}_{\mathcal{R}})| - 1$). Since the size of g will be equal to $|V(\mathcal{H}')| + |\mathcal{E}(\mathcal{H}')| - 1$, this will contradict the *sc*-greediness of \mathcal{H}' . Let f be a choice function for $\mathcal{H}_{\mathcal{R}}$ such that $\text{size}(f) = \chi_{sc}(\mathcal{H}_{\mathcal{R}}) \leq |V(\mathcal{H}_{\mathcal{R}})| + |\mathcal{E}(\mathcal{H}_{\mathcal{R}})| - 1$. Since $\mathcal{H} - (\text{Int}(\mathcal{P}) \cup \text{Int}(\mathcal{R}))$ is *sc*-greedy as an induced subhypergraph of \mathcal{H}' , Lemma 12 implies that $f(v) = \deg_{\mathcal{H}_{\mathcal{R}}}(v)$ for $v \in V(\mathcal{P})$. Furthermore, by Claim 2, for each $v \in E' \cup E''$, $\mathcal{H}_{\mathcal{R}} - v$ is *sc*-greedy and hence by Lemma 2, $f(v) = 1$ when $\deg_{\mathcal{H}}(v) = 1$ (in particular $f(x) = 1$ and $f(y) = 1$). Recall that $\mathcal{H}_{\mathcal{P}, \mathcal{R}} = \mathcal{H} - (V(\mathcal{P}) \cup V(\mathcal{R}))$. Since \mathcal{H}' is *sc*-greedy, $\mathcal{H}_{\mathcal{P}, \mathcal{R}}$ is also *sc*-greedy. Considering $f|_{V(\mathcal{H}_{\mathcal{P}, \mathcal{R}})}$, our remarks about list sizes of the vertices of \mathcal{P} imply

$$\begin{aligned}
 \text{size}(f|_{V(\mathcal{H}_{\mathcal{P}, \mathcal{R}})}) &\leq \text{size}(f) - \sum_{v \in V(\mathcal{P})} f(v) - 2 \\
 &\leq |V(\mathcal{H}_{\mathcal{R}})| + |\mathcal{E}(\mathcal{H}_{\mathcal{R}})| - 1 - \sum_{v \in V(\mathcal{P})} f(v) - 2 \\
 &= |V(\mathcal{H}_{\mathcal{R}})| + |\mathcal{E}(\mathcal{H}_{\mathcal{R}})| - 3 - \sum_{v \in V(\mathcal{P})} \deg_{\mathcal{H}}(v) \\
 &= |V(\mathcal{H}_{\mathcal{R}})| + |\mathcal{E}(\mathcal{H}_{\mathcal{R}})| - 3 - (|V(\mathcal{P})| + |\mathcal{E}(\mathcal{P})| + 1) \\
 &= |V(\mathcal{H}_{\mathcal{R}})| - |V(\mathcal{P})| - 2 + |\mathcal{E}(\mathcal{H}_{\mathcal{R}})| - |\mathcal{E}(\mathcal{P})| - 2 \\
 &= |V(\mathcal{H}_{\mathcal{P}, \mathcal{R}})| + |\mathcal{E}(\mathcal{H}_{\mathcal{P}, \mathcal{R}})|.
 \end{aligned}$$

Since $\mathcal{H}_{\mathcal{P}, \mathcal{R}}$ is *sc*-greedy and $f|_{V(\mathcal{H}_{\mathcal{P}, \mathcal{R}})}$ is a choice function for $\mathcal{H}_{\mathcal{P}, \mathcal{R}}$, we have $\text{size}(f|_{V(\mathcal{H}_{\mathcal{P}, \mathcal{R}})}) \geq |V(\mathcal{H}_{\mathcal{P}, \mathcal{R}})| + |\mathcal{E}(\mathcal{H}_{\mathcal{P}, \mathcal{R}})|$. Thus $\text{size}(f|_{V(\mathcal{H}_{\mathcal{P}, \mathcal{R}})}) = |V(\mathcal{H}_{\mathcal{P}, \mathcal{R}})| + |\mathcal{E}(\mathcal{H}_{\mathcal{P}, \mathcal{R}})|$.

Now, suppose that there exists an $f|_{V(\mathcal{H}_{\mathcal{P}, \mathcal{R}})}$ -assignment $L_{\mathcal{H}_{\mathcal{P}, \mathcal{R}}}$ such that $(c_1, \dots, c_t, c'_1, \dots, c'_s)$ is $(L_{\mathcal{H}_{\mathcal{P}, \mathcal{R}}}, x_1, \dots, x_t, y_1, \dots, y_s)$ -extendable only if $c_1 = \dots = c_t = a$ and $c'_1 = \dots = c'_s = c$ (it may happen that $a = c$), where $a \in L_{\mathcal{H}_{\mathcal{P}, \mathcal{R}}}(x_1) \cap \dots \cap L_{\mathcal{H}_{\mathcal{P}, \mathcal{R}}}(x_t)$ and $c \in L_{\mathcal{H}_{\mathcal{P}, \mathcal{R}}}(y_1) \cap \dots \cap L_{\mathcal{H}_{\mathcal{P}, \mathcal{R}}}(y_s)$. Let $b \neq a$ and

$d \neq c$, by Lemma 10 there exists an $f|_{V(\mathcal{P})}$ -assignment $L_{\mathcal{P}}$ such that $L_{\mathcal{P}}(u) = \{a, b\}$, $L_{\mathcal{P}}(w) = \{c, d\}$ and the pair (b, d) is not $(L_{\mathcal{P}}, u, w)$ -extendable for \mathcal{P} . The f -assignment $L_{\mathcal{H}_{\mathcal{R}}}$ for $\mathcal{H}_{\mathcal{R}}$ we define in the following way:

- $L_{\mathcal{H}_{\mathcal{R}}}(v) = L_{\mathcal{H}_{\mathcal{P}, \mathcal{R}}}(v)$ for $v \in V(\mathcal{H}_{\mathcal{P}, \mathcal{R}})$,
- $L_{\mathcal{H}_{\mathcal{R}}}(v) = L_{\mathcal{P}}(v)$ for $v \in V(\mathcal{P})$,
- $L_{\mathcal{H}_{\mathcal{R}}}(x) = \{a\}$ and $L_{\mathcal{H}_{\mathcal{R}}}(y) = \{c\}$.

Since in every proper $L_{\mathcal{H}_{\mathcal{R}}}$ -colouring of the vertices of $V(\mathcal{H}_{\mathcal{P}, \mathcal{R}})$ the vertices in $E_1 \setminus \{u\}$ have to receive colour a and the vertices in $E_2 \setminus \{w\}$ have to receive colour c , we have to choose colour b for u and d for w . However, the pair (b, d) is not $(L_{\mathcal{P}}, u, w)$ -extendable for \mathcal{P} . Thus for every $f|_{V(\mathcal{H}_{\mathcal{P}, \mathcal{R}})}$ -assignment $L_{\mathcal{H}_{\mathcal{P}, \mathcal{R}}}$ for $\mathcal{H}_{\mathcal{P}, \mathcal{R}}$ we have the following properties. Either

- (1) there is a $(t+s)$ -tuple of colours $(c_1, \dots, c_t, c'_1, \dots, c'_s)$ that is $(L_{\mathcal{H}_{\mathcal{P}, \mathcal{R}}}, x_1, \dots, x_t, y_1, \dots, y_s)$ -extendable and (c_1, \dots, c_t) or (c'_1, \dots, c'_s) contains at least two distinct colours—say this is (c_1, \dots, c_t) , or
- (2) all $(t+s)$ -tuples $(c_1, \dots, c_t, c'_1, \dots, c'_s)$ that are $(L_{\mathcal{H}_{\mathcal{P}, \mathcal{R}}}, x_1, \dots, x_t, y_1, \dots, y_s)$ -extendable satisfy $c_1 = c_2 = \dots = c_t$ and $c'_1 = c'_2 = \dots = c'_s$, but there are at least two $(t+s)$ -tuples of colours that are $(L_{\mathcal{H}_{\mathcal{P}, \mathcal{R}}}, x_1, \dots, x_t, y_1, \dots, y_s)$ -extendable, i.e.,
 - (a) $(a, \dots, a, c, \dots, c)$ and $(a, \dots, a, d, \dots, d)$, or
 - (b) $(a, \dots, a, c, \dots, c)$ and $(b, \dots, b, d, \dots, d)$
 are $(L_{\mathcal{H}_{\mathcal{P}, \mathcal{R}}}, x_1, \dots, x_t, y_1, \dots, y_s)$ -extendable.

Note that we can omit the cases $(a, \dots, a, c, \dots, c)$ and $(b, \dots, b, c, \dots, c)$ because of the symmetry in (2a).

Now, we construct a function g on $V(\mathcal{H}')$ and next we show that g is a choice function for \mathcal{H}' . Let g be defined as follows:

- $g(v) = f(v)$ for $v \in V(\mathcal{H}_{\mathcal{P}, \mathcal{R}})$,
- $g(u) = g(w) = 1$,
- $g(v) = \deg_{\mathcal{H}}(v)$ for $v \in V(\mathcal{R})$.

Recall that f is an arbitrary choice function for $\mathcal{H}_{\mathcal{R}}$ with the size less than or equal to $\chi_{sc}(\mathcal{H}_{\mathcal{R}}) \leq |V(\mathcal{H}_{\mathcal{R}})| + |\mathcal{E}(\mathcal{H}_{\mathcal{R}})| - 1$. Observe that $\text{size}(g) = |V(\mathcal{H}')| + |\mathcal{E}(\mathcal{H}')| - 1$. Now, we claim that g is really a choice function for \mathcal{H}' . For this purpose, let $L_{\mathcal{H}'}$ be any g -assignment for \mathcal{H}' . Clearly, $L_{\mathcal{H}'}|_{\mathcal{H}_{\mathcal{P}, \mathcal{R}}}$ is the $f|_{V(\mathcal{H}_{\mathcal{P}, \mathcal{R}})}$ -assignment, so it has property (1) or (2). If (1) holds, then to make E'' not monochromatic at most one colour in $L_{\mathcal{H}'}(y)$ is forbidden. However, since (c_1, \dots, c_t) contains at least two different colours, x can be coloured with

any colour from its list and E' will not be monochromatic. Thus at most two pairs of colours in $L_{\mathcal{H}'}(x) \times L_{\mathcal{H}'}(y)$ are forbidden in any proper $L_{\mathcal{H}'}$ -colouring of $\mathcal{H}_{\mathcal{P},\mathcal{R}} + \{E', E''\}$. Recall that by our assumption there is at least one edge in \mathcal{R} that has more than two vertices. Consequently, by Corollary 4, at least three pairs of colours in $L_{\mathcal{H}'}(x) \times L_{\mathcal{H}'}(y)$ are $(L_{\mathcal{H}'}|_{V(\mathcal{R})}, x, y)$ -extendable for \mathcal{R} and hence we have at least one available pair of colours for vertices (x, y) to extend the proper $L_{\mathcal{H}'}$ -colouring of $\mathcal{H}_{\mathcal{P},\mathcal{R}} + \{E', E''\}$ to a proper $L_{\mathcal{H}'}$ -colouring of \mathcal{H}' . It remains to consider $f|_{V(\mathcal{H}_{\mathcal{P},\mathcal{R}})}$ -assignment with property (2). Similarly as above, we can see that at most two pairs of colours in $L_{\mathcal{H}'}(x) \times L_{\mathcal{H}'}(y)$ are forbidden in any proper $L_{\mathcal{H}'}$ -colouring of $\mathcal{H}_{\mathcal{P},\mathcal{R}} + \{E', E''\}$. Since by Corollary 4 at least three pairs of colours in $L_{\mathcal{H}'}(x) \times L_{\mathcal{H}'}(y)$ are $(L_{\mathcal{H}'}|_{V(\mathcal{R})}, x, y)$ -extendable for \mathcal{R} , there is at least one pair of colours for the pair (x, y) that can be used to extend the proper $L_{\mathcal{H}'}$ -colouring of $\mathcal{H}_{\mathcal{P},\mathcal{R}} + \{E', E''\}$ to a proper $L_{\mathcal{H}'}$ -colouring of \mathcal{H}' . Thus, we have proved that there is a choice function for \mathcal{H}' with the size at most $|V(\mathcal{H}_{\mathcal{R}})| + |\mathcal{E}(\mathcal{H}_{\mathcal{R}})| - 1$, which contradicts the sc -greediness of \mathcal{H}' . It follows that $\mathcal{H}_{\mathcal{R}}$ must be sc -greedy. \square

Finally, Claims 1, 3 and Lemma 12 yield the sc -greediness of \mathcal{H} , as explained after Claim 2. \blacksquare

We have just proved that if an sc -greedy hypergraph contains appropriate almost generated hyperpath of sufficient length, then it is possible to add a handle of length at least 3 such that the resulting hypergraph is sc -greedy. In the next theorem we consider a variant in which the existence of an almost generated hyperpath is not required. Instead, we require the end-vertices of a handle to be adjacent.

3.2.2. Variant 2 — proof of Theorem 5

Theorem 5. *If \mathcal{H}' is an sc -greedy hypergraph and u, w are its adjacent vertices of degree one, then hypergraph \mathcal{H} obtained by adding to \mathcal{H}' a (u, w) -handle of length at least 2 is sc -greedy.*

Proof. Suppose, for a contradiction, that \mathcal{H} is not sc -greedy. Let \mathcal{P} be an (u, w) -handle and let f be a choice function for \mathcal{H} such that $\text{size}(f) = \chi_{sc}(\mathcal{H}) < |V(\mathcal{H})| + |\mathcal{E}(\mathcal{H})|$. It is not hard to see that $\mathcal{H} - u$ and $\mathcal{H} - w$ are sc -greedy. Hence, by Corollary 2 it holds $f(u) \leq 2$ and $f(w) \leq 2$. If $f(u) = 1$ and $f(w) = 1$, then \mathcal{H} is not f -choosable, since $f|_{V(\mathcal{P})}$ is not a choice function for \mathcal{P} . Thus $f(u) = 2$ or $f(w) = 2$. Furthermore, for every $E \in \mathcal{E}(\mathcal{P})$, the hypergraph $\mathcal{H} - E$ is sc -greedy. Hence, by Lemma 8 for every internal vertex v of \mathcal{P} it holds $\deg_{\mathcal{H}}(v) = f(v)$. Now, assume that $f(u) = 2$ and $f(w) = 1$. Consider the size function g for \mathcal{H}' such that $g(v) = f(v)$ for $v \in V(\mathcal{H}') \setminus \{u\}$ and $g(u) = f(u) - 1$. Observe that

$$\begin{aligned}
\text{size}(g) &= \text{size}(f|_{V(\mathcal{H}')}) - 1 = \text{size}(f) - 1 - \sum_{v \in V(\mathcal{P}) \setminus \{u, w\}} \deg(v) \\
&< |V(\mathcal{H})| + |\mathcal{E}(\mathcal{H})| - 1 - \sum_{v \in V(\mathcal{P}) \setminus \{u, w\}} \deg(v) \\
&= |V(\mathcal{H})| + |\mathcal{E}(\mathcal{H})| - 1 - (|V(\mathcal{P})| - 2 + |\mathcal{E}(\mathcal{P})| - 1) \\
&= |V(\mathcal{H}')| + |\mathcal{E}(\mathcal{H}')|.
\end{aligned}$$

Since \mathcal{H}' is *sc*-greedy, \mathcal{H}' is not *g*-choosable. So there is a *g*-assignment $L_{\mathcal{H}'}$ such that \mathcal{H}' is not properly $L_{\mathcal{H}'}$ -colourable. Let $L_{\mathcal{H}'}(u) = \{a\}$ and $L_{\mathcal{H}'}(w) = \{b\}$. Clearly, the pair (a, b) is not $(L_{\mathcal{H}'}, u, w)$ -extendable. The vertices u and w are contained in the common edge and both have degree one in \mathcal{H}' . This fact implies that if $a \neq b$, then we may assume that also (a, a) is not $(L_{\mathcal{H}'}, u, w)$ -extendable. Let $c \neq a$. By Lemma 11 there is an $f|_{V(\mathcal{P})}$ -assignment $L_{\mathcal{P}}$ such that (a, c) is not $(L_{\mathcal{P}}, u, w)$ -extendable for \mathcal{P} . We define an f -assignment $L_{\mathcal{H}}$ in the following way:

- $L_{\mathcal{H}}(v) = L_{\mathcal{H}'}(v)$ for $v \in V(\mathcal{H}') \setminus \{u\}$,
- $L_{\mathcal{H}}(u) = \{a, c\}$,
- $L_{\mathcal{H}}(v) = L_{\mathcal{P}}(v)$ for $v \in V(\mathcal{P}) \setminus \{u, w\}$.

Since (a, a) is not $(L_{\mathcal{H}'}, u, w)$ -extendable for \mathcal{H}' and (a, c) is not $(L_{\mathcal{P}}, u, w)$ -extendable for \mathcal{P} , the hypergraph \mathcal{H} is not properly $L_{\mathcal{H}}$ -colourable. Thus, we may assume that $f(u) = 2$ and $f(w) = 2$ and hence $f(v) = \deg_{\mathcal{H}}(v)$ for every $v \in V(\mathcal{P})$. Clearly $\sum_{v \in V(\mathcal{P})} \deg_{\mathcal{H}}(v) = |V(\mathcal{P})| + |\mathcal{E}(V(\mathcal{P}))|$. Consequently, $\text{size}(f|_{V(\mathcal{P})}) = |V(\mathcal{P})| + |\mathcal{E}(V(\mathcal{P}))|$. Thus, by Lemma 6 the hypergraph \mathcal{H} is not *f*-choosable and hence there is no choice function for \mathcal{H} with the size less than $|V(\mathcal{H})| + |\mathcal{E}(\mathcal{H})|$. So \mathcal{H} is *sc*-greedy. ■

4. APPLICATIONS

In this section we focus mainly on giving examples of applications of our new theorems, but we will also indicate new area of applications of sum-list colouring and choosability in theoretical and practical variants of optimisation problems in the field of resource allocation and chromatic scheduling. We believe that proper as well as generalized variants of sum-list colouring will be interesting and worth investigating not only in the above fields but also across the entire range of other applications.

4.1. New bounds

As we already know every *sc*-greedy hypergraph achieves effectively computable bound (1) and hence one of the approaches to determine a lower bound on the

sum-choice-number of a given hypergraph \mathcal{H} is to find its *sc*-greedy subhypergraph with large sum of the order and size. We will also use this approach to bound the sum-choice-number of graphs in proper and generalized variants of list-colouring.

Given a graph G and an \leq -hereditary class \mathcal{R} (i.e., a class closed with respect to taking induced subgraphs) we consider the hypergraph $\mathcal{H}(G, \mathcal{R})$ with the vertex set $V(G)$ and the edge set $\mathcal{E}(\mathcal{H}(G, \mathcal{R})) = \{E \mid E \subseteq V(G) \text{ and } G[E] \in \mathcal{C}(\mathcal{R})\}$, where $\mathcal{C}(\mathcal{R})$ is a set of minimal forbidden graphs characterizing the class \mathcal{R} . We use $\mathcal{H}(G, \mathcal{R})$ in conjunction with \mathcal{R} -colouring of the graph G which is a partition of $V(G)$ such that each set of the partition induces in G a subgraph belonging to \mathcal{R} . Naturally, an assignment ϕ of colours to the vertices of G is an \mathcal{R} -colouring of G if and only if ϕ is a proper colouring of $\mathcal{H}(G, \mathcal{R})$. This clearly holds also for sum-list \mathcal{R} -colouring, the colouring variant introduced in [9] and investigated, e.g. in [8, 16]. The \mathcal{R} -sum-choice-number $\chi_{sc}^{\mathcal{R}}(G)$ of a graph G is simply the minimum $\text{size}(f)$ over all f -assignments L for which G admits an \mathcal{R} -colouring with the colour of each vertex v belonging to $L(v)$.

Property 5. *For every \leq -hereditary class \mathcal{R} and every graph G it holds*

$$(2) \quad \chi_{sc}(G) \geq \chi_{sc}^{\mathcal{R}}(G) = \chi_{sc}(\mathcal{H}(G, \mathcal{R})) \geq \chi_{sc}(\mathcal{H}'),$$

where \mathcal{H}' is an arbitrary subhypergraph of $\mathcal{H}(G, \mathcal{R})$.

The main advantage of the operations defined in Theorems 4 and 5 over those in Theorems 1, 2 and 3 is that they can be used recursively. Despite specific cases, our new theorems allow recursive construction of connected subhypergraphs of $\mathcal{H}(G, \mathcal{R})$ with the size larger than before; thus overcoming the main weakness of taking unions which is the small number of ‘common’ vertices.

In the example that follows we consider \mathcal{R} -sum choosability of grids and grids with holes, where by the *grid* $G_{n,m}$ we mean the Cartesian product $P_n \square P_m$ of the paths P_n and P_m while *grid with holes* is an arbitrary induced subgraph of $G_{m,n}$. The example is intended to present selected cases that reveal differences between operations and show when Theorems 4 and 5 can be used to obtain improved bounds compared to those in [8, 11].

Example 1. Let $\mathcal{R}_1, \mathcal{R}_2$ be the classes of P_3 -free and subcubic graphs (the graphs G for which $\Delta(G) \leq 3$), respectively (recall that \mathcal{R}_1 -colouring is also known as P_3 -free colouring or *subcolouring*). Also note that the set of minimum forbidden graphs defining \mathcal{R}_2 consists only of the graphs of order 5, each of which contains a vertex of degree 4 and hence, on grids the problem of \mathcal{R}_2 -colouring can be seen as $K_{1,4}$ -free colouring. Let us briefly analyze the applications of individual operations to establishing bounds on grids $G_1 = G_{6,22}$, $G_2 = G_{11,11}$ and the

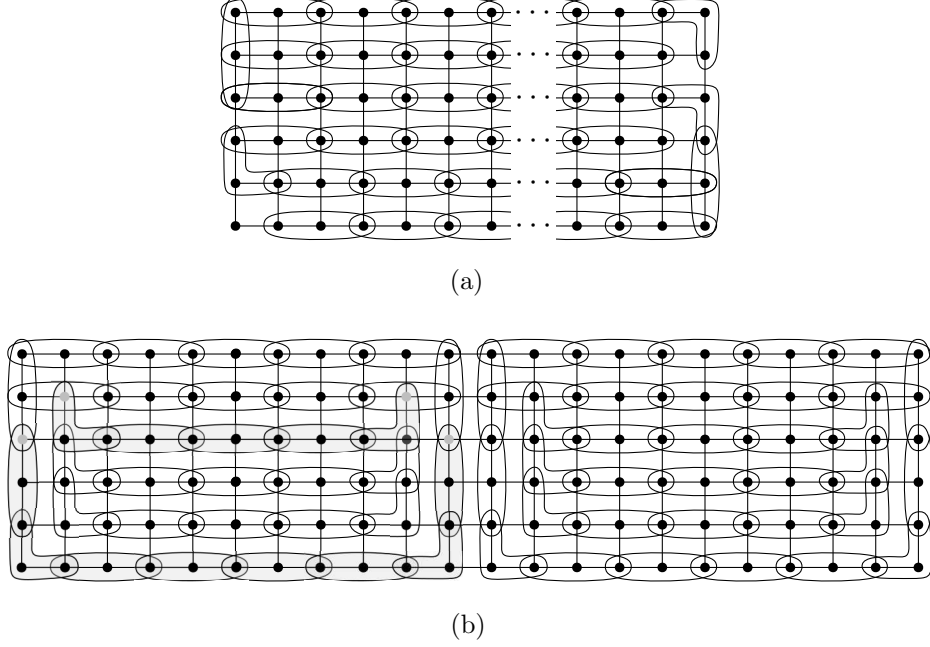


Figure 4. Grid G_1 and appropriate subhypergraphs for \mathcal{R}_1 : (a) a hypertree, (b) a union of hypergraphs obtained by the recursive application of Theorem 4 (selected connectors and handles are marked grey).

grid with holes G_3 that is an induced subgraph of $G_{11,11}$ (see Figures 4 and 5, respectively).

First, we consider the class \mathcal{R}_1 . A hypertree \mathcal{H}' presented in Figure 4(a) is sc -greedy by Corollary 1. From the definition of sc -greediness and formula (2) it follows that $\chi_{sc}(G_1) \geq \chi_{sc}^{\mathcal{R}_1}(G_1) \geq \chi_{sc}(\mathcal{H}') = 132 + 65 = 197$. However, using Theorem 4 it is possible to obtain an improved bound $\chi_{sc}(G_1) \geq \chi_{sc}^{\mathcal{R}_1}(G_1) \geq \chi_{sc}(\mathcal{H}'') = 132 + 70 = 202$, where the construction of each of the two components of \mathcal{H}'' starts at a hypercycle of length 12 with handle of length 5, and then consists in adding subsequent handles; see Figure 4(b).

Now, consider the class \mathcal{R}_2 . Applying the union operation and Theorem 4 we construct a subhypergraph consisting of a hypercycle of length 12 with handle of length 7 and two disjoint hyperpaths of length 2; see Figure 5(a). It is worth mentioning that for \mathcal{R}_2 the operation given in Theorem 4 seems to be more appropriate in the case of grids with holes. On the other hand, however, the pattern of holes can vary considerably, which allows the construction of grids with holes for which none of the above-mentioned operations gives the result better than the operation given in Theorem 5. Namely, several adjacent connectors can be used to add subsequent handles of length 3 to the hypercycle of length 8; see

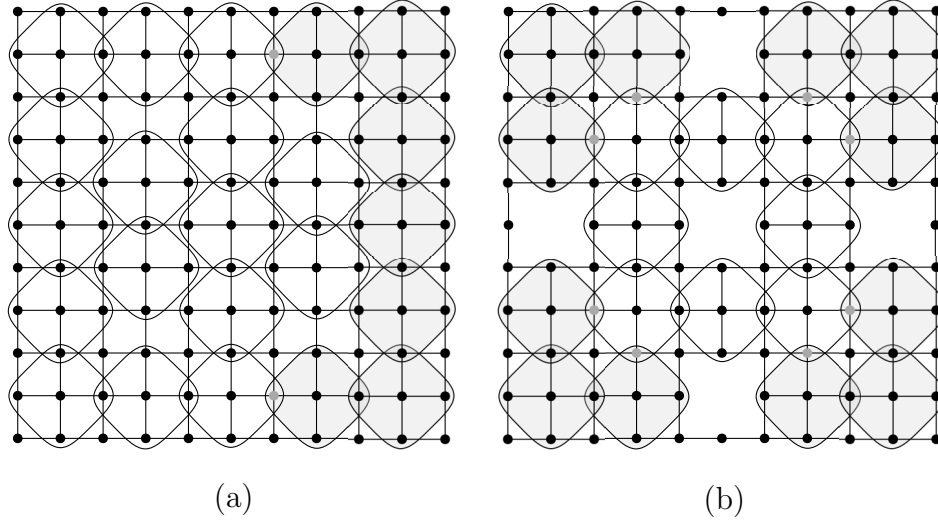


Figure 5. Grids G_2 and G_3 with appropriate subhypergraphs for \mathcal{R}_2 : (a) a union of hyperpaths and a hypercycle with handle, (b) the result of recursive application of Theorem 5 (connectors and handles are marked grey).

Figure 5(b). This results in sc -greedy hypergraph \mathcal{H}'' for which the following bound holds $\chi_{sc}(G_3) \geq \chi_{sc}^{\mathcal{R}_2}(G_3) \geq \chi_{sc}(\mathcal{H}'') = 116 + 20 = 136$.

The above example is just the tip of the iceberg, intended to demonstrate basic relationships and motivate further investigation.

4.2. New areas of applications

The whole range of problems related to the classical list colouring and choosability, e.g. the variant with the minimum sum criterion or generalization to \mathcal{R} -colouring have interesting applications in system modeling and optimization.

Restricting the colour choice by setting the list at each vertex is a natural mechanism desired in many chromatic optimization models, e.g. in chromatic scheduling such restriction is often termed an *availability constraint* and relates to the resources such as time slots, machines (processors), storage cells or communication channels, usually represented as colours (see, e.g. [17]). Appropriate resources (colours) are assigned to jobs represented by the vertices of the so called *resource conflict graph* thus resolving the resource sharing problem under *mutual exclusion* constraint. Clearly, the list $L(v)$ restricts the resources available to vertex v , but on the other hand the same list can be viewed as *job's preference* specifying the resources that job prefers because of costs, quality or security reasons. This complements well with the criterion of minimizing the sum of the list lengths that is closer to the classical criteria representing the viewpoint of the

system owner more than that of individual jobs or system users. The system owner policy is therefore a kind of a trade-off between short lists (more profit) and having to guarantee resource choosability.

All of the above-mentioned aspects become apparent in the context of the problem of partitioning (\mathcal{R} -colouring) for scheduling with resource conflicts (see [3]), where the system is faced with a sequence of jobs competing for unsharable resources. As the result of partitioning one gets distinct subsets of jobs that form separate instances of the resource-constrained scheduling problem—in fact, each instance is assigned to a distinct *subsystem* that holds its own replica of the resources. The objective is to generate a partition into the smallest number of such instances or equivalently, to minimize the number of subsystems (resource replicas) such that the response time of each job in each instance (subsystem) is bounded by a given constant. In [3] each job is allowed to have its own response time requirement, and the above-mentioned partitions guarantee meeting jobs quality of service criteria. This can be further extended by allowing each job to have its own list of preferred subsystems (preferred colours) with their choice guaranteed by appropriate f -assignment. Though originally stated as an online problem [3], it seems equally interesting when partitioning as well as scheduling are offline or dynamic (see [2, 4, 5]). At this point we mention that due to the generality of the problem statement the scheduling part can be seamlessly replaced by other optimization problems. For a detailed description of the above model and applications to communication channel assignment and scheduling multiprocessor jobs on dedicated processors the reader is referred to [3].

5. OPEN PROBLEM

As we already know, adding a handle of length at least two is a hypergraph operation preserving hypergraph linearity and 2-connectivity, the properties that play an important role in the analysis of *sc*-greediness. In this paper we introduced new variants of adding handles and proved that the class of *sc*-greedy hypergraphs is closed with respect to these operations. Despite the development of new operations, the main question originating in [10] remains unanswered.

Problem 1. *Under which conditions an *sc*-greedy hypergraph can be obtained by adding a handle to an *sc*-greedy hypergraph?*

REFERENCES

- [1] C. Berge, *Hypergraphs: Combinatorics of Finite Sets* (North-Holland, Amsterdam, 1989).
- [2] P. Borowiecki, *Computational aspects of greedy partitioning of graphs*, J. Comb. Optim. **35** (2018) 641–665.
<https://doi.org/10.1007/s10878-017-0185-2>

- [3] P. Borowiecki, *On-line partitioning for on-line scheduling with resource conflicts*, in: Proc. 7th Int. Conf. on Parallel Processing and Applied Mathematics (PPAM'07), Lecture Notes in Comput. Sci. **4967**, R. Wyrzykowski, J. Dongarra, K. Karczewski and J. Wasniewski (Ed(s)), (Springer, Berlin, Heidelberg, 2007) 981–990.
https://doi.org/10.1007/978-3-540-68111-3_104
- [4] P. Borowiecki and E. Sidorowicz, *Dynamic F -free coloring of graphs*, Graphs Combin. **34** (2018) 457–475.
<https://doi.org/10.1007/s00373-018-1886-8>
- [5] P. Borowiecki and E. Sidorowicz, *Dynamic coloring of graphs*, Fund. Inform. **114** (2012) 105–128.
<https://doi.org/10.3233/FI-2012-620>
- [6] C. Brause, A. Kemnitz, M. Marangio, A. Pruchnewski and M. Voigt, *Sum choice number of generalized θ -graphs*, Discrete Math. **340** (2017) 2633–2640.
<https://doi.org/10.1016/j.disc.2016.11.028>
- [7] R. Diestel, Graph Theory, Grad. Texts in Math. **173** (Springer, Berlin, Heidelberg, 2017).
<https://doi.org/10.1007/978-3-662-53622-3>
- [8] E. Drgas-Burchardt and A. Drzystek, *Acyclic sum-list-coloring of grids and other classes of graphs*, Opuscula Math. **37** (2017) 535–556.
<https://doi.org/10.7494/OpMath.2017.37.4.535>
- [9] E. Drgas-Burchardt and A. Drzystek, *General and acyclic sum-list-coloring of graphs*, Appl. Anal. Discrete Math. **10** (2016) 479–500.
<https://doi.org/10.2298/AADM161011026D>
- [10] E. Drgas-Burchardt, A. Drzystek and E. Sidorowicz, *Sum-list-coloring of θ -hypergraphs*, Ars Math. Contemp. **22** (2022) #P1.05.
<https://doi.org/10.26493/1855-3974.2083.e80>
- [11] E. Drgas-Burchardt and E. Sidorowicz, *Sum-list colouring of unions of a hypercycle and a path with at most two vertices in common*, Discuss. Math. Graph Theory **40** (2020) 893–917.
<https://doi.org/10.7151/dmgt.2312>
- [12] P. Erdős, A.L. Rubin and H. Taylor, *Choosability in graphs*, in: Proc. West Coast Conf. on Combinatorics, Graph Theory and Computing, Congr. Numer. **26** (1979) 125–157.
- [13] B. Heinold, Sum List Coloring and Choosability, PhD Thesis (Lehigh University, 2006).
- [14] G. Isaak, *Sum list coloring block graphs*, Graphs Combin. **20** (2004) 499–506.
<https://doi.org/10.1007/s00373-004-0564-1>
- [15] G. Isaak, *Sum list coloring $2 \times n$ arrays*, Electron. J. Combin. **9** (2002) #N8.
<https://doi.org/10.37236/1669>

- [16] A. Kemnitz, M. Marangio and M. Voigt, *Generalized sum list colorings of graphs*, Discuss. Math. Graph Theory **39** (2019) 689–703.
<https://doi.org/10.7151/dmgt.2174>
- [17] M. Kubale, Graph Colorings, Contemp. Math. **352** (American Mathematical Society, 2004).
<https://doi.org/10.1090/conm/352>
- [18] M. Lastrina, List-Coloring and Sum-List-Coloring Problems on Graphs, PhD Thesis (Iowa State University, 2012).
<https://doi.org/10.31274/etd-180810-1195>
- [19] C. Thomassen, *Every planar graph is 5-choosable*, J. Combin. Theory Ser. B **62** (1994) 180–181.
<https://doi.org/10.1006/jctb.1994.1062>

Received 22 August 2023

Revised 30 November 2023

Accepted 30 November 2023

Available online 16 December 2023