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RESTRAINED DIFFERENTIAL OF A GRAPH

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Abstract

Given a graph $G = (V(G), E(G))$ and a vertex $v \in V(G)$, the open neighbourhood of v is defined to be $N(v) = \{u \in V(G) : uv \in E(G)\}$. The external neighbourhood of a set $S \subseteq V(G)$ is defined as $S_e = \left(\bigcup_{v\in S} N(v)\right) \setminus S$, while the *restrained external neighbourhood* of S is defined as $S_r = \{v \in S_e : N(v) \cap S_e \neq \emptyset\}$. The restrained differential of a graph G is defined as $\partial_r(G) = \max\{|S_r| - |S| : S \subseteq V(G)\}$. In this paper, we introduce the study of the restrained differential of a graph. We show that this novel parameter is perfectly integrated into the theory of domination in graphs. We prove a Gallai-type theorem which shows that the theory of restrained differentials can be applied to develop the theory of restrained Roman domination, and we also show that the problem of finding the restrained differential of a graph is NP-hard. The relationships between the restrained differential of a graph and other types of differentials are also studied. Finally, we obtain several bounds on the restrained differential of a graph and we discuss the tightness of these bounds.

Keywords: differentials in graphs, restrained differential, restrained Roman domination.

2020 Mathematics Subject Classification: 05C69.

1. Introduction

Given a graph $G = (V(G), E(G))$ and a vertex $v \in V(G)$, the open neighbourhood of v is defined to be $N(v) = \{u \in V(G) : uv \in E(G)\}$. The open neighbourhood of a set $S \subseteq V(G)$ is defined as $N(S) = \bigcup_{v\in S} N(v)$, while the external neighbourhood of S, or boundary of S, is defined as $S_e = N(S) \setminus S$. The differential of a set $S \subseteq V(G)$ is defined as $\partial(S) = |S_e| - |S|$, while the differential of a graph G is defined to be

$$\partial(G) = \max\{\partial(S) : S \subseteq V(G)\}.$$

As described in [14], the definition of $\partial(G)$ was given by Hedetniemi about 25 years ago in an unpublished paper, and was also considered by Goddard and Henning [10]. After that, the differential of a graph has been studied by several authors, including [2–4, 16]. Currently, the study of differentials in graphs and their variants is of great interest because it has been observed that the study of different types of domination can be approached through a variant of the differential which is related to them. Specifically, we are referring to domination parameters that are necessarily defined through the use of functions, such as Roman domination, perfect Roman domination, Italian domination and unique response Roman domination. In each case, the main result linking the domination parameter to the corresponding differential is a Gallai-type theorem, which allows us to study these domination parameters without the use of functions. For instance, the differential ∂ is related to the Roman domination number γ_R [3], the perfect differential ∂_p is related to the perfect Roman domination number γ_R^p [5], the strong differential ∂_s is related to the Italian domination number γ_I [6], while the 2-packing differential $\partial_{2\rho}$ is related to the unique response Roman domination number μ_R [7]. We will omit here the definition and properties of most of the above-mentioned differentials, referring the reader to the corresponding papers for details. Next we introduce the study of the restrained differential.

Given a set $S \subseteq V(G)$ we define the *restrained external neighbourhood* of S as

$$S_r = \{v \in S_e : N(v) \cap S_e \neq \emptyset\}.$$

The *restrained differential of a set* $S \subseteq V(G)$ is defined as

$$\partial_r(S) = |S_r| - |S|.$$

In this paper we study the *restrained differential* of a graph G, which is defined as

$$\partial_r(G) = \max\{\partial_r(S) : S \subseteq V(G)\}.$$

Two examples are shown in Figure 1.

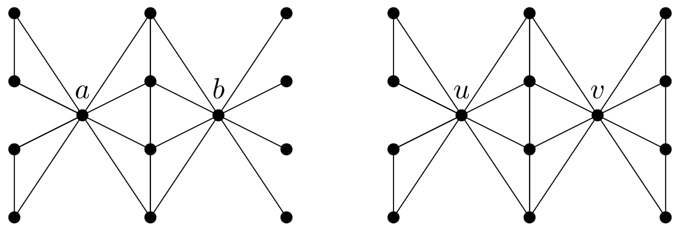

Figure 1. For the left hand side graph we have $\partial(G) = \partial(\{a, b\}) = 10 > 7 = \partial_r(\{a\}) = \partial_r(G)$, while for the right hand side graph we have $\partial(G) = \partial(\{u, v\}) = \partial_r(\{u, v\}) = \partial_r(G) = 10$.

From now on, for simplicity we say that a set $S \subseteq V(G)$ is a $\partial_r(G)$-set if $\partial_r(S) = \partial_r(G)$.

We will show that the restrained differential is perfectly integrated into the theory of domination. The paper is organised as follows. In Section 2 we prove a Gallai-type theorem which shows that the theory of restrained differentials can be applied to develop the theory of restrained Roman domination and we also show that the problem of finding the restrained differential of a graph is NP-hard. Section 3 is devoted to the relationships between restrained differential and other types of differentials of graphs. In Sections 4 and 5 we obtain several bounds on the restrained differential of a graph and we discuss the tightness of these bounds. Specifically, in Section 4 we obtain results for arbitrary graphs while Section 5 is devoted to trees.

2. A Gallai-Type Theorem

Let $f : V(G) \longrightarrow \{0, 1, 2\}$ be a function on a graph G and let $V_i = \{x \in V(G) : f(x) = i\}$ for $i \in \{0, 1, 2\}$. Since f is defined from the sets V_0, V_1 and V_2, and vice versa, the function f will be denoted by $f(V_0, V_1, V_2)$. The *weight* of $f(V_0, V_1, V_2)$ is defined to be

$$\omega(f) = \sum_{v \in V(G)} f(v) = |V_1| + 2|V_2|.$$

Cockayne *et al.* [8] defined a *Roman dominating function* (RDF) on a graph G as a function $f(V_0, V_1, V_2)$ such that $N(v) \cap V_2 \neq \emptyset$ for every vertex $v \in V_0$. The *Roman domination number* of G, denoted by $\gamma_R(G)$, is the minimum weight among all RDFs on G.

A *restrained Roman dominating function* (RRDF) on a graph G is a RDF $f(V_0, V_1, V_2)$ such that the subgraph of G induced by V_0 has no isolated vertex. The *restrained Roman domination number* of G, denoted by $\gamma_{rR}(G)$, is the minimum weight among all RRDFs on G.

A Gallai-type theorem has the form $a(G) + b(G) = n$, where n denotes the order of G, while $a(G)$ and $b(G)$ are other parameters defined on G. In [2, 5–7] we can find Gallai-type results for $a \in \{\partial, \partial_p, \partial_s, \partial_{2\rho}\}$ and $b \in \{\gamma_R, \gamma_R^p, \gamma_I, \mu_R\}$, respectively. Here we present the similar result for $a = \partial_r$ and $b = \gamma_{rR}$.

Theorem 1 (Gallai-type theorem)**.** *For any graph G,*

$$\gamma_{rR}(G) + \partial_r(G) = n.$$

Proof. Let D be a $\partial_r(G)$-set. Notice that the function $g(W_0, W_1, W_2)$, defined from $W_2 = D$ and $W_0 = D_r$, is a RRDF on G, which implies that

$$\begin{aligned}\gamma_{rR}(G) \leq \omega(g) &= 2|D| + |V(G) \setminus (D_r \cup D)| = 2|D| + n - |D_r| - |D| \\ &= n - (|D_r| - |D|) = n - \partial_r(G).\end{aligned}$$

We proceed to show that $\gamma_{rR}(G) \geq n - \partial_r(G)$. Let $f(V_0, V_1, V_2)$ be a $\gamma_{rR}(G)$-function. By definition of f and the fact that $V_0 \subseteq (V_2)_r$, we obtain that

$$\begin{aligned}\partial_r(G) \geq \partial_r(V_2) &= |(V_2)_r| - |V_2| \geq |V_0| - |V_2| \\ &= n - 2|V_2| - |V_1| = n - \gamma_{rR}(G).\end{aligned}$$

Therefore, the result follows. ■

According to the result above we can see the theory of restrained differential in graphs as a new approach to the theory of restrained Roman domination. One of the advantages of this approach is that it allows us to study the restrained Roman domination number of a graph without the use of functions.

By Theorem 1 we can also show the relation between the $\gamma_{rR}(G)$-functions and the $\partial_r(G)$-sets.

Proposition 2. *If $f(V_0, V_1, V_2)$ is a $\gamma_{rR}(G)$-function, then V_2 is a $\partial_r(G)$-set. Conversely, for any $\partial_r(G)$-set S, there exists a $\gamma_{rR}(G)$-function $f(V_0, V_1, V_2)$ such that $V_2 = S$ and $V_1 = V(G) \setminus (S_r \cup S)$.*

Proof. Let $f(V_0, V_1, V_2)$ be a $\gamma_{rR}(G)$-function. Since $V_0 = (V_2)_r$,

$$\begin{aligned}\gamma_{rR}(G) &= 2|V_2| + |V_1| = n + |V_2| - |V_0| \\ &= n - (|(V_2)_r| - |V_2|) = n - \partial_r(V_2).\end{aligned}$$

Thus $\gamma_{rR}(G) + \partial_r(V_2) = n$, and so Theorem 1 leads to $\partial_r(V_2) = \partial_r(G)$, which implies that V_2 is a $\partial_r(G)$-set.

Conversely, let S be a $\partial_r(G)$-set, and define a function $g\colon V(G) \to \{0, 1, 2\}$ by $g(v) = 2$ whenever $v \in S$ and $g(v) = 1$ for every $v \in V(G) \setminus (S_r \cup S)$. Notice that g is an RRDF of weight $w(g) = n - \partial_r(S) = n - \partial_r(G)$ (since S is a $\partial_r(G)$-set). Thus, from Theorem 1 we conclude that g is a $\gamma_{rR}(G)$-function. Therefore, the result follows. ■

Now we analyse the case of the computational complexity. Given a positive integer k and a graph G, the problem of deciding if G has a restrained Roman dominating function f of weight $w(f) \leq k$ is NP-complete [1]. Hence, the problem of computing the restrained Roman domination number of a graph is NP-hard. Therefore, by Theorem 1 we immediately obtain the analogous result for the restrained differential.

Theorem 3. *The problem of computing the restrained differential of a graph is NP-hard.*

3. Relations Between Restrained Differential and Other Types of Differentials

It is known that for any graph G,

$$\gamma_I(G) \leq \gamma_R(G) \leq \gamma_R^p(G) \leq \mu_R(G),$$

where γ_I is the Italian domination number, γ_R is the Roman domination number, γ_R^p is the perfect Roman domination number and $\mu_R(G)$ is the unique response Roman domination number. Hence, the correspondent differentials are related by the following inequality chain.

Remark 4. For any connected graph G, $\partial_s(G) \geq \partial(G) \geq \partial_p(G) \geq \partial_{2\rho}(G)$.

Now, by the definitions of $\partial(G)$ and $\partial_r(G)$ we derive the following remark.

Remark 5. For any graph G,

$$\partial(G) \geq \partial_r(G).$$

In Remark 6 we will show that there are graphs with $\partial(G) = \partial_r(G)$, and also there are graphs where the difference between $\partial(G)$ and $\partial_r(G)$ can be arbitrarily large. Moreover, we will show that $\partial_r(G)$ and $\partial_{2\rho}(G)$ are incomparable; the same for $\partial_r(G)$ and $\partial_p(G)$. To this end, we consider the following families of graphs.

Let $\mathcal{G}_1$ be a family of graphs obtained from two different copies of a complete graphs K_ℓ joined by the edge for $\ell \geq 3$. A tree $S_{k,\ell}$ of order $\ell + k + 2$ containing exactly two support vertices u, v such that u has degree $k + 1$ and v has degree $\ell + 1$, is called a double-star. Let $\mathcal{G}_2$ be a family of double stars $S_{\ell,\ell}$ for $\ell \geq 2$. Let $\mathcal{G}_3$ be a family of generalized lexicographic product graphs $P_5 \circ \{P_\ell, P_1, P_\ell, P_1, P_\ell\}$ for $\ell \geq 3$. Figure 2 on the left shows a graph belonging to $\mathcal{G}_3$ for $\ell = 4$, while the same figure on the right shows the graph $S_{4,4} \in \mathcal{G}_2$.

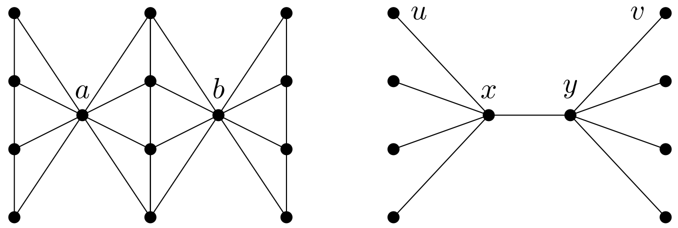

Figure 2. For the left hand side graph we have $\partial(G) = \partial(\{a, b\}) = \partial_r(\{a, b\}) = \partial_r(G) = 10$, while for the right hand side graph we have $\partial(G) = \partial(\{x, y\}) = 6$ and $\partial_r(G) = \partial_r(\{u, v\}) = 0$.

Remark 6.

- If $G \in \mathcal{G}_1$, then $\partial_s(G) = \partial(G) = \partial_r(G) = \partial_p(G) = \partial_{2\rho}(G) = 2\ell - 4$.
- If $G \in \mathcal{G}_2$, then $\partial_r(G) = 0$, $\partial_s(G) = \partial(G) = \partial_p(G) = 2\ell - 2$ and $\partial_{2\rho}(G) = \ell$.
- If $G \in \mathcal{G}_3$, then $\partial_p(G) = \partial_{2\rho}(G) = 2\ell + 1$, $\partial_s(G) = \partial(G) = \partial_r(G) = 3\ell - 2$.

These examples show that $\partial_r(G)$ and $\partial_{2\rho}(G)$ ($\partial_p(G)$, respectively) are incomparable. If $G \in \mathcal{G}_3$, then $\partial_r(G) - \partial_{2\rho}(G) = \partial_r(G) - \partial_p(G) = \ell - 3$, while if $G \in \mathcal{G}_2$, then $\partial_{2\rho}(G) - \partial_r(G) = \ell$ and $\partial_p(G) - \partial_r(G) = 2\ell - 2$.

The following result shows another family of graphs with $\partial_r(G) = \partial(G)$.

Proposition 7. *If G is a claw-free cubic graph, then $\partial_r(G) = \partial(G)$.*

Proof. Let D be a $\partial(G)$-set such that $|D_r|$ is maximum among all $\partial(G)$-sets. We suppose that $D_e \setminus D_r \neq \emptyset$. Let $v \in D_e \setminus D_r$, $u \in N(v) \cap D$ and $N(v) \setminus \{u\} = \{u_1, u_2\}$. As $u_1, u_2 \in V(G) \setminus D_r$, we obtain that either $\{u_1, u_2\} \cap D \neq \emptyset$ or $u_1, u_2 \in V(G) \setminus (D \cup D_r)$. Next, we analyse these two cases.

Case 1. $u_1, u_2 \in V(G) \setminus (D \cup D_r)$. In this case, it is easy to see that $u_1u_2 \in E(G)$, because G is claw free. Now, we consider $D' = D \cup \{u_1\}$. Observe that $\partial(D') = |D'_e| - |D'| \geq |D_e| + 1 - |D| - 1 = \partial(D) = \partial(G)$. Moreover, $D_r \cup \{u_2\} \subseteq D'_r$. Hence, D' is a $\partial(G)$-set with $|D'_r| > |D_r|$, which is a contradiction.

Case 2. $\{u_1, u_2\} \cap D \neq \emptyset$. In this case, we consider that $u_1 \in D$. Recall that $u_2 \in V(G) \setminus D_r$ and that $N(x) \cap D = \emptyset$ for every vertex $x \in V(G) \setminus (D_e \cup D)$. From the previous facts and as G is claw-free, we can assume, without loss of generality, that $uu_1 \in E(G)$. Now, if we consider the set $D'' = D \setminus \{u_1\}$, we deduce that $\partial(D'') = |D''_e| - |D''| \geq |D_e| - |D| + 1 > \partial(D) = \partial(G)$, which is a contradiction.

Therefore, from the two cases above we deduce that $D_e = D_r$, and as a consequence,

$$\partial_r(G) \leq \partial(G) = \partial(D) = |D_e| - |D| = |D_r| - |D| = \partial_r(D) \leq \partial_r(G),$$

which implies that $\partial_r(G) = \partial(G)$. ∎

4. General Bounds

In this section we present lower and upper bounds of the restrained differential of a graph. To begin with, we need to state some notation and terminology. A set of vertices S is a *dominating set* if $S \cup N(S) = V(G)$; or equivalently, every vertex in $V(G) \setminus S$ is adjacent to at least one vertex in S. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set in G. For detailed information on the theory of domination in graphs we suggest the books [12, 13].

A set $S \subseteq V(G)$ is a *restrained dominating set* if every vertex not in S is adjacent to a vertex in S and to a vertex in $V(G) \setminus S$. The *restrained domination number* of G, denoted by $\gamma_r(G)$, is the smallest cardinality among all restrained dominating sets of G [9]. As a consequence of Theorem 1 and the bounds $\gamma_r(G) \leq \gamma_{rR}(G) \leq 2\gamma_r(G)$ given in [15], we deduce the following result.

Proposition 8. *For any graph G of order n,*

$$n - 2\gamma_r(G) \leq \partial_r(G) \leq n - \gamma_r(G).$$

Next we discuss the case where the equalities hold.

Proposition 9. *For any graph G of order n, the following statements hold.*

(i) $\partial_r(G) = n - \gamma_r(G)$ *if and only if* $\gamma_r(G) = n$.

(ii) $\partial_r(G) = n - 2\gamma_r(G)$ *if and only if every* $\gamma_r(G)$*-set is a* $\partial_r(G)$*-set.*

Proof. If $\gamma_r(G) = n$, then Proposition 8 leads to $\partial_r(G) = 0$ and so $\partial_r(G) = n - \gamma_r(G)$.

Conversely, assume $\partial_r(G) = n - \gamma_r(G)$. Let D be a $\partial_r(G)$-set and let $D' = V(G) \setminus (D \cup D_r)$. Notice that $D \cup D'$ is a restrained dominating set of G, and so $\gamma_r(G) \leq |D'| + |D|$. Hence,

$$n - \gamma_r(G) = \partial_r(G) = |D_r| - |D| = n - |D'| - 2|D|,$$

which implies that $|D'| + 2|D| = \gamma_r(G) \leq |D'| + |D|$. Therefore, $D = \emptyset$ and $\gamma_r(G) = |D'| = n$, concluding the proof of (i).

Now, if $\partial_r(G) = n - 2\gamma_r(G)$, then for any $\gamma_r(G)$-set S we have $n - 2\gamma_r(G) = \partial_r(G) \geq \partial_r(S) = |S_r| - |S| = n - 2\gamma_r(G)$, and so S is a $\partial_r(G)$-set.

Conversely, if every $\gamma_r(G)$-set F is a $\partial_r(G)$-set, then $\partial_r(G) = \partial_r(F) = n - 2|F| = n - 2\gamma_r(G)$, which concludes the proof of (ii). ∎

Corollary 10. *If G is a graph with at least two adjacent vertices of degree at least two, then $\partial_r(G) \leq n - \gamma_r(G) - 1$.*

As an example of graph G with $\partial_r(G) = n - \gamma_r(G) - 1$ we can take a graph of order $n \geq 3$ having at least one vertex of degree $n-1$ and exactly $k \in \{0, \ldots, n-3\}$ vertices of degree one. In such a case, $\gamma_r(G) = k + 1$ and $\partial_r(G) = n - k - 2$.

Using Theorem 1 and some results obtained in [15, 17] for the restrained Roman domination number we immediately obtain the following.

Proposition 11. *For any connected graph G of order $n \geq 2$,*

(a) $0 \leq \partial_r(G) \leq n - 2$.

(b) $\partial_r(G) = n - 2$ *if and only if* $n = 2$ *or* G *has maximum degree* $\Delta = n - 1$ *and minimum degree* $\delta \geq 2$.

(c) $\partial_r(G) = 0$ *if and only if* G *is a tree of diameter at most* 5 *or* $G \in \{C_4, C_5, G_1, G_2\}$, *where* G_1, G_2 *are the graphs shown in Figure* 3.

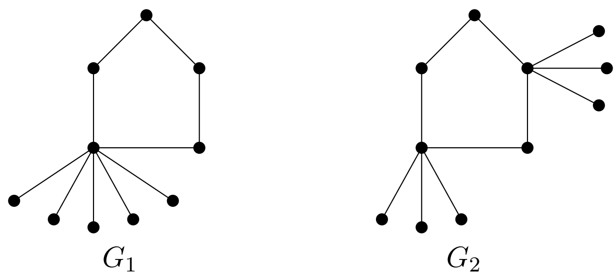

Figure 3. Graphs G_1 and G_2.

Proof. To prove that $\partial_r(G) \geq 0$, it is enough to show that $\partial_r(\emptyset) = 0$. Obviously, $\emptyset_r = \emptyset$, what implies $\partial_r(\emptyset) = |\emptyset_r| - |\emptyset| = 0$. Concerning the upper bound, since $\gamma_{rR}(G) \geq 2$, Theorem 1 leads to $\partial_r(G) \leq n-2$.

It was shown in [17] that $\gamma_{rR} = 2$ if and only if $n = 2$ or $\Delta = n-1$ and $\delta \geq 2$. Hence, by Theorem 1 we conclude that (b) holds. It is known from [15] that $\gamma_{rR}(G) = n$ if and only if G is a tree with $diam(G) \leq 5$ or $G \in \{C_4, C_5, G_1, G_2\}$, where G_1, G_2 are graphs from the Figure 3, so again by Theorem 1 we conclude that (c) holds. ■

Remark 12. Characterization of trees and graphs G with small girth 3, 4 or 5 with $\partial_r(G) = 1$ can be obtained from Propositions 15-20 from [17] and by Theorem 1. Moreover, characterization of graphs G with $\partial_r(G) = n-3$ is a consequence of Proposition 5 from [17] and by Theorem 1.

Lemma 13. *Let G be a graph of minimum degree $\delta \geq 2$. If D is a $\partial_r(G)$-set of maximum cardinality among all $\partial_r(G)$-sets, then either $D' = V(G) \setminus (D \cup D_r) = \emptyset$ or the subgraph induced by D' is empty.*

Proof. Suppose, to the contrary, that there exist two adjacent vertices $u, v \in D'$. If $N(u) \cap D_r \neq \emptyset$, then $\partial_r(D \cup \{v\}) \geq \partial_r(D)$, which is a contradiction. Hence, $D_r \cap (N(u) \cup N(v)) = \emptyset$. Now, if $N(u) \cap D \neq \emptyset$, then there exists $v' \in N(v) \cap D' \setminus \{u\}$ and so $\partial_r(D \cup \{v'\}) \geq \partial_r(D)$, which is a contradiction. Hence, $(D \cup D_r) \cap (N(u) \cup N(v)) = \emptyset$. Obviously, $N(u) \cap N(v) \cap D' = \emptyset$, which implies that there exist two different vertices $u_1 \in N(u) \cap D' \setminus \{v\}$ and $v_1 \in N(v) \cap D' \setminus \{u\}$, and so $\partial_r(D \cup \{u_1, v_1\}) \geq \partial_r(D)$, which is a contradiction again. Therefore, the result follows. ■

Lemma 14. *For any graph G and any $\partial_r(G)$-set D,*

$$|D| \leq \gamma_r(G).$$

Proof. Suppose, to the contrary, that D is a $\partial_r(G)$-set with $|D| > \gamma_r(G)$. In such a case, $|D_r| \leq n - |D| < n - \gamma_r(G)$, and so

$$\partial_r(G) = |D_r| - |D| < n - \gamma_r(G) - |D| < n - 2\gamma_r(G),$$

which contradicts Proposition 8. Therefore, $|D| \leq \gamma_r(G)$, as required. ■

Lemma 15. *For any graph G and any subgraph H of G,*

$$\partial_r(G) \geq \partial_r(H).$$

Proof. Given a set $D \subseteq V(H)$, we use the notation $D_r(H) = D_r$, just to emphasize that $D_r(H)$ is defined on H, while $D_r(G)$ is the analogous one, but defined on G. Thus, $D_r(H) \subseteq D_r(G)$, and so if D is a $\partial_r(H)$-set, then $\partial_r(H) = |D_r(H)| - |D| \leq |D_r(G)| - |D| \leq \partial_r(G)$. ■

Recall that the largest cardinality of a set of vertices of G, no two of which are adjacent, is called the *independence number* of G and it is denoted by $\alpha(G)$. A set $S \subseteq V(G)$ is a 2-*packing* if the distance between any two vertices belonging to S is at least three. The 2-*packing number* of G, denoted by $\rho(G)$, is the maximum cardinality among all a 2-packings of G. The following result improves the upper bound given in Proposition 8 for the graphs with minimum degree at least two.

Proposition 16. *For any graph G of order n and minimum degree $\delta \geq 2$,*

$$\partial_r(G) \leq n - \gamma_r(G) - \max\left\{\gamma_r(G) - \alpha(G), \left\lceil \frac{\gamma_r(G) + n(\delta - 1)}{\delta(\Delta + 1)} \right\rceil\right\}.$$

Proof. Let D be a $\partial_r(G)$-set of maximum cardinality among all $\partial_r(G)$-sets. By Lemma 13, either $D' = V(G) \setminus (D \cup D_r) = \emptyset$ or the subgraph induced by D' is empty.

Now, since $D \cup D'$ is a restrained dominating set of G and $|D'| \leq \alpha(G)$, we deduce the following.

$$\partial_r(G) = |D_r| - |D| = n - 2(|D| + |D'|) + |D'| \leq n - 2\gamma_r(G) + \alpha(G).$$

Finally, we proceed to prove the bound $\partial_r(G) \leq n - \gamma_r(G) - \frac{\gamma_r(G)+n(\delta-1)}{\delta(\Delta+1)}$. Let $D'' = \{x \in D' : N(x) \subseteq D_r\}$. We now claim that if $D'' \neq \emptyset$, then D'' is a 2-packing of G. Suppose that there exist two vertices $v, v' \in D''$ such that $u \in N(v) \cap N(v')$. Since $u \in D_r$ and $\delta \geq 2$, it follows that $\partial_r(D \cup \{u\}) \geq \partial_r(D)$, which is a contradiction. Therefore, if $D'' \neq \emptyset$, then D'' is a 2-packing of G, as required. This implies that $|D''| \leq \rho(G)$, and as a consequence,

$$|D''| \leq \frac{|D_r|}{\delta} \quad \text{and} \quad |D' \setminus D''| \leq \Delta|D| - |D_r|.$$

Hence,

$$|D| + |D'| \leq (\Delta + 1)|D| - \frac{(\delta - 1)|D_r|}{\delta} = (\Delta + 1)|D| - \frac{(\delta - 1)(n - (|D| + |D'|)}{\delta},$$

which implies that

$$|D| \geq \frac{|D| + |D'| + n(\delta - 1)}{\delta(\Delta + 1)} \geq \frac{\gamma_r(G) + n(\delta - 1)}{\delta(\Delta + 1)}.$$

Since $|D|$ is an integer,

$$|D| \geq \left\lceil \frac{\gamma_r(G) + n(\delta - 1)}{\delta(\Delta + 1)} \right\rceil.$$

Therefore,

$$\partial_r(G) = |D_r| - |D| = n - (|D| + |D'|) - |D| \leq n - \gamma_r(G) - \left\lceil \frac{\gamma_r(G) + n(\delta - 1)}{\delta(\Delta + 1)} \right\rceil,$$

which completes the proof. ■

To see that the bound $\partial_r(G) \leq n-\gamma_r(G)-\left\lceil \frac{\gamma_r(G)+n(\delta-1)}{\delta(\Delta+1)} \right\rceil$ is tight we consider the graph shown in Figure 4. In this case, $\partial_r(G) = 4$ and $\gamma_r(G) = 2$. We would like to emphasize that we are not sure about the tightness of the bound $\partial_r(G) \leq n - 2\gamma_r(G) + \alpha(G)$.

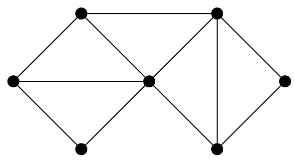

Figure 4. A graph with $\partial_r(G) = n - \gamma_r(G) - \left\lceil \frac{\gamma_r(G)+n(\delta-1)}{\delta(\Delta+1)} \right\rceil$.

Proposition 17. *If G is a graph of diameter three and minimum degree δ, then*

$$\partial_r(G) \geq \rho(G)(\delta - 1).$$

Proof. Let D be a $\rho(G)$-set. Notice that $|D| \geq 2$, as $diam(G) = 3$. Let $u \in D$ and $u' \in N(u)$. If $N(u') \cap N(v) = \emptyset$ for some $v \in D \setminus \{u\}$, then the eccentricity of u is greater than three, which is a contradiction. Hence, $D_e = D_r$, and so

$$\partial_r(G) \geq \partial_r(D) = |D_r| - |D| = \sum_{u \in D} \deg(u) - \rho(G) \geq \rho(G)(\delta - 1).$$

■

As the next result shows, the bound above is tight.

Proposition 18. *If G is a regular graph of order n and degree $\delta \geq 2$ with $\gamma(G) = \rho(G)$, then*

$$\partial_r(G) = n - 2\rho(G) = \rho(G)(\delta - 1) = \frac{n(\delta - 1)}{\delta + 1}.$$

Proof. Let S_1 be a $\gamma(G)$-set and let S_2 be a $\rho(G)$-set. Since $|S_1| = \gamma(G) = \rho(G) = |S_2|$,

$$n - |S_2| = n - \gamma(G) = |(S_1)_e| \leq \delta|S_1| = \delta|S_2|,$$

which implies that S_2 is a $\gamma(G)$-set, and therefore a $\gamma_r(G)$-set. Hence,

$$\partial_r(S_2) = n - 2\gamma(G) = n - 2\rho(G).$$

Notice also that $\partial_r(S_2) = \rho(G)(\delta - 1)$, as S_2 is a $\gamma(G)$-set and a $\rho(G)$-set.

Now, let S' be a $\partial_r(G)$-set. If $|S'| > \rho(G)$, then $|S'_r| \leq n - |S'| < n - \rho(G)$, and so $\partial_r(G) = \partial_r(S') < n - \rho(G) - |S'| < n - 2\rho(G) = \partial_r(S_2)$, which is a contradiction. Hence, $|S'| \leq \rho(G)$ and, as a consequence,

$$\partial_r(G) = \partial_r(S') \leq |S'|\delta - |S'| = |S'|(\delta - 1) \leq \rho(G)(\delta - 1) = \partial_r(S_2) \leq \partial_r(G).$$

Therefore, $\partial_r(G) = \partial_r(S_2) = n - 2\rho(G) = \rho(G)(\delta - 1) = \frac{n(\delta-1)}{\delta+1}$. ■

As an example of application of the result above, we consider the 3-cube graph, where $n = 8$, $\delta = 3$, $\rho(G) = 2$ and $\partial_r(G) = 4$.

Proposition 19. *If G is a claw-free graph of minimum degree $\delta \geq 2$, then*

$$\partial_r(G) \geq \rho(G)(\delta - 2).$$

Proof. If $\delta = 2$, then the result follows. Hence, we assume that $\delta \geq 3$. Let D be a $\rho(G)$-set. Notice that every vertex $v \in D$ has at most one vertex $u \in N(v)$ such that $N(u) \cap (N(v) \setminus \{u\}) = \emptyset$, as G is claw-free. From this fact, we deduce that $|D_r| \geq \sum_{x \in D}(\deg(x) - 1) \geq \rho(G)(\delta - 1)$. Hence,

$$\partial_r(G) \geq \partial_r(D) = |D_r| - |D| \geq \rho(G)(\delta - 1) - \rho(G) = \rho(G)(\delta - 2),$$

which completes the proof. ■

The bound above is tight. For instance, it is achieved for the graphs C_4 and C_5.

By Proposition 8, $n - 2\gamma_r(G) \leq \partial_r(G) \leq n - \gamma_r(G)$ for any nontrivial graph G of order n. The following result provides some new upper bounds for the restrained differential of G.

Proposition 20. *Let G be a graph of order $n \geq 2$ and maximum degree Δ. For any integer k such that $0 \leq k \leq \gamma_r(G)$,*

$$\partial_r(G) \leq \max\left\{n - 2\gamma_r(G) + k, \frac{(\Delta - 1)(n - k - 1)}{\Delta + 1}\right\}.$$

Proof. Let k be an integer such that $0 \leq k \leq \gamma_r(G)$ and let D be a $\partial_r(G)$-set. By definition, $V(G) \setminus D_r$ is a restrained dominating set of G, which implies that $|D_r| \leq n - \gamma_r(G)$. Now, we first suppose that $|V(G) \setminus (D \cup D_r)| \leq k$, that is $|D| \geq n - |D_r| - k$. Hence,

$$\begin{aligned}\partial_r(G) = \partial_r(D) = |D_r| - |D| &\leq |D_r| - (n - |D_r| - k) \leq 2(n - \gamma_r(G)) - n + k \\ &= n - 2\gamma_r(G) + k.\end{aligned}$$

From now on, we assume that $|V(G) \setminus (D \cup D_r)| \geq k + 1$. In addition, notice that $|D_r| \leq \Delta|D|$ and that $2|D| = n - \partial_r(G) - |V(G) \setminus (D \cup D_r)| \leq n - \partial_r(G) - k - 1$. Hence, from the previous inequalities we deduce the following.

$$\partial_r(G) = |D_r| - |D| \leq (\Delta - 1)|D| \leq \frac{(\Delta - 1)(n - \partial_r(G) - k - 1)}{2}.$$

Hence, $(\Delta+1)\partial_r(G) \leq (\Delta-1)(n-k-1)$, which implies that $\partial_r(G) \leq \frac{(\Delta-1)(n-k-1)}{\Delta+1}$.

Therefore, in any case, we have that $\partial_r(G) \leq \max\left\{n-2\gamma_r(G)+k, \frac{(\Delta-1)(n-k-1)}{\Delta+1}\right\}$, which completes the proof. ■

By Propositions 8 and 20 we deduce the following result.

Proposition 21. *Let G be a graph of order $n \geq 2$ and maximum degree Δ. If $\gamma_r(G) \leq \frac{n}{\Delta+1} + \frac{\Delta-1}{2(\Delta+1)}$, then $\partial_r(G) = n - 2\gamma_r(G)$.*

It is known [4] that $\partial_r(G) \leq \partial(G) \leq \frac{n(\Delta-1)}{\Delta+1}$ for any graph G of order n. We next provide an interesting equivalence.

Proposition 22. *Given a graph G of order n and maximum degree Δ, the following statements are equivalent.*

- $\partial_r(G) = \frac{n(\Delta-1)}{\Delta+1}$.
- $\gamma_r(G) = \frac{n}{\Delta+1}$.

Proof. If $\gamma_r(G) = \frac{n}{\Delta+1}$, then by Proposition 21 we have $\partial_r(G) = n - 2\gamma_r(G) = \frac{n(\Delta-1)}{\Delta+1}$.

Conversely, assume $\partial_r(G) = \frac{n(\Delta-1)}{\Delta+1}$. For any $\partial_r(G)$-set D,

$$n \geq (|D_r| - |D|) + 2|D| = \frac{n(\Delta-1)}{\Delta+1} + 2|D| = n - 2\left(\frac{n}{\Delta+1}\right) + 2|D|,$$

which implies that $|D| \leq \frac{n}{\Delta+1}$. In addition, by using the fact that $|D_r| \leq \Delta|D|$, we obtain that

$$\frac{n(\Delta-1)}{\Delta+1} = \partial_r(G) = |D_r| - |D| \leq (\Delta-1)|D| \leq \frac{n(\Delta-1)}{\Delta+1}.$$

Thus, $|D| = \frac{n}{\Delta+1}$ and $|D_r| = \frac{n\Delta}{\Delta+1} = n - \frac{n}{\Delta+1} = n - |D|$, which implies that D is a restrained dominating set of G. Hence, $\frac{n}{\Delta+1} \leq \gamma(G) \leq \gamma_r(G) \leq |D| = \frac{n}{\Delta+1}$. Therefore, $\gamma_r(G) = \frac{n}{\Delta+1}$, which completes the proof. ■

5. Restrained Differential of Trees

It was shown in [1,11] that the restrained Roman domination number of any tree of diameter at least three, order n with l leaves and s support vertices is bounded by $\gamma_{rR}(T) \geq \frac{2n+l-s+4}{3}$. From this bound and Theorem 1 we deduce the following result.

Proposition 23. *If T is a tree of diameter at least three and order n, with l leaves and s support vertices, then $\partial_r(T) \leq \frac{n-l+s-4}{3}$.*

A description for the trees achieving the equality was given in [11], but this description was not correct, since they use induction and the equality $\gamma_{rR}(T) =$

$(2n+l-s+4)/3$ does not hold for the base graphs P_5 and P_6 used in the proof. Here we describe the family of extremal trees for which the equality holds. To this end, we need to introduce some additional notation and terminology. Given a tree T, let $S(T) = S_1(T) \cup S_2(T)$ be the set of supports of T, where $S_1(T)$ and $S_2(T)$ are the sets of weak and strong support of T, respectively. Let $\Omega(T)$ be the set of leaves of T.

We introduce the family $\mathcal{R}$ of trees $T = T_i$ which can be obtained in the following way: Let $T_1 = P_4$. If i is a positive integer, then T_{i+1} can be obtained recursively from T_i by one of two operations.

Operation $\mathcal{O}_1$. If $v \in S(T_i)$, then we add a new vertex x and the edge xv.

Operation $\mathcal{O}_2$. If $v \in V(T_i)$ is such that $d(v, \Omega(T_i)) \equiv 0 \pmod 3$ and $N_{T_i}(v) \cap S_2(T_i) = \emptyset$, then we add a path $P_3 = (x, y, z)$ and the edge xv.

An example of the tree belonging to the family $\mathcal{R}$ we can see in Figure 5.

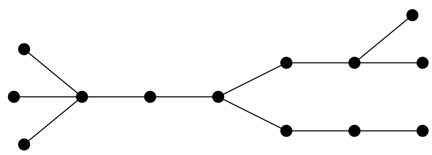

Figure 5. A tree $T \in \mathcal{R}$.

If T is a tree, then we define T^* as a tree obtained from T by removing all but one leaves for every strong support vertex of T. Obviously, if $S_2(T) = \emptyset$, then $T = T^*$.

Observation 24. *If $T \in \mathcal{R}$, then*

(a) $|V(T^*)| \equiv 1 \pmod 3$*;*

(b) $d_{T^*}(u, v) \equiv 0 \pmod 3$ *for any two vertices $u, v \in \Omega(T^*)$;*

(c) *the set $\{x \colon d_{T^*}(x, \Omega(T^*)) \equiv 0 \pmod 3\}$ is a maximum 2-packing of T^* and also a $\partial_r(T^*)$-set;*

(d) $|V(T)| - |\Omega(T)| = |V(T^*)| - |\Omega(T^*)|$ *and* $|S(T)| = |S(T^*)|$.

Lemma 25. *If $x, y \in V(G)$ are two vertices of degree one which are at distance two, then $\partial_r(G) = \partial_r(G - x)$.*

Proof. Let $x, y, z \in V(G)$ be three vertices of G such that x and y have degree one and both are neighbours of z.

Let D be a $\partial_r(G)$-set. It is readily seen that $|\{x, y, z\} \cap D| \le 1$. Thus, if $\{x, y\} \cap D = \emptyset$, then $\{x, y\} \cap D_r = \emptyset$, and so $\partial_r(G) = \partial_r(D) = \partial_r(D \setminus \{x\}) \le \partial_r(G - x)$. Now, if $|\{x, y\} \cap D| = 1$, say $y \in D$, then $z \in D_r$ and $x \notin D \cup D_r$, which implies that $\partial_r(G) = \partial_r(D) = \partial_r(D \setminus \{x\}) \le \partial_r(G - x)$.

On the other side, let S be a $\partial_r(G-x)$-set. If $y \in S$, then $z \in S_r$, and so in G we have that $x \notin S \cup S_r$, which implies that $\partial_r(G-x) = \partial_r(S) = \partial_r(S \setminus \{x\}) \leq \partial_r(G)$. Now, if $y \notin S$, then $y \notin S_r$, and in G we have that $\{x, y\} \cap (S \cup S_r) = \emptyset$, which leads to $\partial_r(G-x) = \partial_r(S) = \partial_r(S \setminus \{x\}) \leq \partial_r(G)$.

Therefore, $\partial_r(G) = \partial_r(G-x)$. ■

Lemma 26. *If $T \in \mathcal{R}$, then $\partial_r(T) = (n-l+s-4)/3$.*

Proof. Let $T \in \mathcal{R}$. By Lemma 25 and Observation 24(d) it is enough to prove that $\partial_r(T^*) = (n(T^*)-l(T^*)+s(T^*)-4)/3$. From Observation 24(b), $d_{T^*}(u,v) = 0 \pmod 3$ for any $u, v \in \Omega(T^*)$. Let $D_1 = \{x \in V(T^*) \setminus \Omega(T^*) \colon d(x, \Omega(T^*) = 0 \pmod 3)\}$ and $D_2 = \Omega(T^*)$. Hence $D_3 = V(T^*) \setminus (D_1 \cup D_2)$ and $T^*[D_3] = (|D_1 \cup D_2| - 1)K_2$. Notice that $D_1 \cup D_2$ is a maximum 2-packing of T^*, hence by Observation 24(c), we obtain that $\partial_r(T^*) = |D_3| - (|D_1| + |D_2|) = 2(|D_1| + |D_2| - 1) - (|D_1| + |D_2|) = |D_1| + |D_2| - 2$. Hence, $n(T^*) = |D_1| + |D_2| + |D_3|$, $l(T^*) = |D_2|$, $s(T^*) = |D_2|$. Therefore, $(n(T^*) - l(T^*) + s(T^*) - 4)/3 = (|D_1| + |D_2| + 2(|D_1| + |D_2| - 1) - |D_2| + |D_2| - 4)/3 = (3|D_1| + 3|D_2| - 6)/3 = |D_1| + |D_2| - 2 = \partial_r(T^*)$ and the result holds. ■

Lemma 27. *If $\partial_r(T) = (n-l+s-4)/3$, then $T \in \mathcal{R}$.*

Proof. If $diam(T) = 3$, then $T \in \mathcal{R}$; so assume $diam(T) \geq 4$; thus $n \geq 5$. We use an induction on n, the order of a tree. Assume the result holds for any tree T' of order $n' < n$ with s' supports and l' leaves. Let T be a tree such that $\partial_r(T) = (n-l+s-4)/3$. Assume first that $S_2(T) \neq \emptyset$ and let $x \in S_2(T)$ and let y, z be two leaves adjacent to x. Let D be a $\partial_r(T)$-set. Since D is maximum, $|D \cap \{y, z\}| \leq 1$ and, without loss of generality, let $z \notin D$. Consider $T' = T \setminus \{z\}$; then obviously $D \setminus \{z\}$ is a restrained differential set of $T.'$ Combining this fact with $n' = n-1, l' = l-1$ and $s' = s$ we obtain

$$\partial_r(T') \geq \partial_r(D \setminus \{z\}) = \partial_r(T) = \frac{n-l+s-4}{3} = \frac{n'-l'+s'-4}{3}.$$

By Proposition 23, $\partial_r(T') = (n'-l'+s'-4)/3$. Thus, by induction hypothesis $T' \in \mathcal{R}$ and it is easy to observe that T can be obtained from T' by Operation $\mathcal{O}_1$. Hence $T \in \mathcal{R}$. From now we can assume that T has no strong support vertex.

Claim*. *If D is a $\partial_r(T)$-set, then $\Omega(T) \subseteq D$.*

Now we proceed to prove the claim. To the contrary suppose that D^* is a $\partial_r(T)$-set such that $x \in \Omega(T) \setminus D^*$. Consider $T' = T - \{x\}$ and notice that $\partial_r(T') \geq \partial_r(D^*) = \partial_r(T)$, $n' = n-1, l' = l$ and $s' \in \{s, s-1\}$, so $s \geq s'$. Thus

$$\partial_r(T') \geq \partial_r(T) = \frac{n-l+s-4}{3} \geq \frac{n'+1-l'+s'-4}{3} = \frac{n'-l'+s'-4}{3} + \frac{1}{3},$$

a contradiction with Proposition 23. Therefore, the claim follows. □

Let $P = (v_0, \ldots, v_k)$ be a longest path of T; $k \geq 4$. Let D be a $\partial_r(T)$-set; from Claim*, $v_0 \in D$ and since $S_2(T) = \emptyset$, $d_T(v_1) = 2$.

Suppose $d_T(v_2) > 2$. First consider the case when a path P_2 different from (v_0, v_1) is attached to v_2. Then, since D is $\partial_r(T)$-set and $\Omega(T) \subseteq D$, $v_2 \notin D$ and $v_3 \in D$. Thus $D \setminus \{v_0\}$ is a restrained differential set of T of cardinality $\partial_r(T)$, a contradiction with the claim above. Thus the only path attached to v_2 is (v_0, v_1) and v_2 is a support vertex. Let us denote by x a leaf adjacent to v_2. Notice that $v_3 \notin D$ (otherwise, $D \setminus \{v_0, x\}$ would be a restrained differential set of T of cardinality $\partial_r(T)$, a contradiction with Claim*). Moreover, v_3 has a neighbour in D (if not, $(D \setminus \{x\}) \cup \{v_3\}$ would be a restrained differential set of T of cardinality $\partial_r(T)$, again a contradiction with Claim*). These two fact imply that $(D \setminus \{v_0, x\}) \cup \{v_1\}$ would be a restrained differential set of T of cardinality $\partial_r(T)$. This final contradiction with Claim* shows that $d_T(v_2) = 2$. Since D is a $\partial_r(T)$-set, $v_3 \in D$ and using similar arguments like before we can prove that v_3 is neither a support vertex nor a neighbour of a support vertex of T. Thus $d_T(v_3, \Omega(T)) \equiv 0 \pmod 3$. Now let $T' = T \setminus \{v_0, v_1, v_2\}$. Notice that v_3 is not a neighbour of a strong support of T' and $d'_T(v_3, \Omega(T')) \equiv 0 \pmod 3$. Finally, T can be obtained from T' by Operation $\mathcal{O}_2$, what completes the proof. ■

Corollary 28. *For a tree T is $\partial_r(T) = (n - l + s - 4)/3$ if and only if $T \in \mathcal{R}$.*

The following result shows that $\partial_r(T) \leq n - \alpha(T) - 1$ for any tree T of order $n \geq 2$. Notice that this bound does not hold for arbitrary graphs. For instance, if we take the join graph $G = K_1 + C_4$, then $\partial_r(G) = 3 > 2 = n - \alpha(G) - 1$. In order to prove the result we need to introduce the following terminology. A *rooted tree* T is a tree with a distinguished special vertex r, called the root. For each vertex $v \neq r$ of T, the parent of v is the neighbour of v on the unique $r - v$ path. A descendant of v is a vertex $u \neq v$ such that the unique $r - u$ path contains v. The set of descendants of v is denoted by $D(v)$. The maximal subtree at v, denoted by T_v, is the subtree of T induced by $D(v) \cup \{v\}$. Obviously if v is a leaf, then $T_v = T$.

Proposition 29. *If $T \ncong K_{1,k}$ is a tree of order $n \geq 4$, then*

$$\partial_r(T) \leq n - \alpha(T) - 2.$$

Proof. In order to avoid confusion, in this proof the order of a tree T will be denoted by $n(T)$. We proceed by induction on the order of T. If $n(T) \in \{4, 5\}$, then it is easy to check that $\partial_r(T) \leq n(T) - \alpha(T) - 2$, as $T \ncong K_{1,k}$. These particular cases establish the base cases.

From now on, we consider that $T \ncong K_{1,k}$ has order at least six and that every tree T^*, different from a star graph, with $5 \leq n(T^*) < n(T)$ satisfies that

$\partial_r(T^*) \le n(T^*) - \alpha(T^*) - 2$. It is readily seen that if T has a support vertex which is adjacent to two leaves x and x', then the subtree T' of T, obtained by removing the leaf x, satisfies $\alpha(T) = \alpha(T') + 1$. Now, by Lemma 25, $\partial_r(T) = \partial_r(T')$. Hence,

$$\partial_r(T) = \partial_r(T') \le n(T') - \alpha(T') - 2 = (n(T)-1) - (\alpha(T)-1) - 2 = n(T) - \alpha(T) - 2.$$

Therefore, we can assume that every support vertex of T is adjacent to exactly one leaf.

Now, let $z, h \in V(T)$ be two antipodal vertices, i.e., there exists a diametral path $z - h$ of T. So, z and h are leaves of T. Also, let s be the parent of h in the rooted tree T_z. Given a subtree T' of T and a set $D \subseteq V(T')$, we use the notation $D_r(T') = D_r$, just to emphasize that $D_r(T')$ is defined on T', while $D_r(T)$ is the analogous one, but defined on T. We next analyse the following three cases, considering that D is a $\partial_r(T)$-set.

Case 1. $|N(s)| \ge 3$. Since h is an antipodal vertex, $|N(s) \cap \Omega(T)| \ge 2$, which is a contradiction, as we are assuming that every support vertex of T is adjacent to exactly one leaf. Hence, this case does not occur.

Case 2. $N(s) = \{h, v\}$ and $|N(v)| \ge 3$. Let u be the parent of v in the rooted tree T_z. Now, and without loss of generality, we assume that $|D \cap \{h, s, v, u\}|$ is maximum. We consider the next subcases.

Subcasee 2.1. $v \notin D_r(T)$. In this subcase, we have that $h, s \notin D \cup D_r(T)$. Let $T' = T - \{h, s\}$. Hence, $\partial_r(T) = |D_r(T)| - |D| = |D_r(T')| - |D| = \partial_r(D) \le \partial_r(T')$. Moreover, we have that $\alpha(T) \le \alpha(T') + 1$. Hence, by inequalities above and the induction hypothesis we obtain that

$$\partial_r(T) \le \partial_r(T') \le n(T') - \alpha(T') - 2 \le (n(T)-2) - (\alpha(T)-1) - 2 < n(T) - \alpha(T) - 2,$$

as required.

Subcase 2.2. $v \in D_r(T)$ and $h, s \notin D \cup D_r(T)$. In this subcase, the procedure is analogous to Subcase 2.1, obtaining that $\partial_r(T) \le n(T) - \alpha(T) - 2$, as required.

Subcase 2.3. $v \in D_r(T)$ and $s \in D$. In this subcase, we have that $h \notin D \cup D_r(T)$. Let $T'' = T - \{h\}$. Hence, $\partial_r(T) = |D_r(T)| - |D| = |D_r(T'')| - |D| = \partial_r(D) \le \partial_r(T'')$. Moreover, we have that $\alpha(T) \le \alpha(T'') + 1$. Hence, by inequalities above and the induction hypothesis we obtain that

$$\partial_r(T) \le \partial_r(T'') \le n(T'') - \alpha(T'') - 2 \le (n(T)-1) - (\alpha(T)-1) - 2 = n(T) - \alpha(T) - 2,$$

as required.

Subcase 2.4. $v \in D_r(T)$ and $s \in D_r(T)$. In this subcase, we have that $h \in D$. We first assume that $u \in D_r(T)$. Let $T' = T - \{h, s\}$ and $D' = D \setminus \{h\}$. Hence,

$\partial_r(T) = |D_r(T)| - |D| = |D'_r(T')| - |D'| = \partial_r(D') \leq \partial_r(T')$. Moreover, we have that $\alpha(T) \leq \alpha(T')+1$. Hence, by inequalities above and the induction hypothesis we obtain that

$$\partial_r(T) \leq \partial_r(T') \leq n(T')-\alpha(T')-2 \leq (n(T)-2)-(\alpha(T)-1)-2 < n(T)-\alpha(T)-2,$$

as required. Finally, we assume that $u \notin D_r(T)$. Let $s' \in N(v) \setminus \{s, u\}$. Notice that $T_{s'} \in \{P_1, P_2\}$, which implies that $u \in D$ by the maximality of $|D \cap \{h, s, v, u\}|$. Now, if $T_{s'} = P_1$, then we consider the subtree $T'' = T - \{s'\}$ and proceed in a manner analogous to Subcase 2.3, obtaining that $\partial_r(T) \leq n(T) - \alpha(T) - 2$, as required. Otherwise $T_{s'} = P_2$, and without loss of generality we can assume that $(D \cup D_r(T)) \cap V(T_{s'}) = \emptyset$. So, proceeding in a manner analogous to Subcase 2.1 (assuming that $T' = T - V(T_{s'})$), we obtain that $\partial_r(T) \leq n(T) - \alpha(T) - 2$, as required.

Case 3. $|N(s)| = |N(v)| = 2$. Let u be the parent of v in the rooted tree T_z. Notice that $u \neq z$ since $n(T) \geq 6$. If $v \notin D_r(T)$, then the procedure is analogous to Subcase 2.1. Now, if $v \in D_r(T)$ and $s \in D$, then the procedure is analogous to Subcase 2.3. In both cases we have $\partial_r(T) \leq n(T) - \alpha(T) - 2$, as required.

Finally, we consider the case $v \in D_r(T)$ and $s \in D_r(T)$. In this case, we have that $h, u \in D$. Let $T' = T - \{h, s, v\}$ and $D' = D \setminus \{h\}$. Hence, $\partial_r(T) = |D_r(T)| - |D| = |D'_r(T')| - |D'| + 1 = \partial_r(D') + 1 \leq \partial_r(T') + 1$. Moreover, we have that $\alpha(T) \leq \alpha(T') + 2$. Notice that if $n(T) = 6$, then $T \cong P_6$, and we are done. Now, if $n(T) \geq 7$, by inequalities above and the induction hypothesis we obtain that

$$\partial_r(T) \leq \partial_r(T')+1 \leq n(T')-\alpha(T')-1 \leq (n(T)-3)-(\alpha(T)-2)-1 = n(T)-\alpha(T)-2,$$

as required.

Therefore, the proof is complete. ■

To see that the bound above is achieved we can consider the tree T obtained from any star graph $K_{1,k}$ by subdividing twice all edges. In such a case, $\partial_r(T) = k - 1$, $n(T) = 3k + 1$ and $\alpha(T) = 2k$.

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