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3-NEIGHBOR BOOTSTRAP PERCOLATION ON GRIDS

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Abstract

Given a graph G and assuming that some vertices of G are infected, the r-neighbor bootstrap percolation rule makes an uninfected vertex v infected if v has at least r infected neighbors. The r-percolation number, $m(G,r)$, of G is the minimum cardinality of a set of initially infected vertices in G such that after continuously performing the r-neighbor bootstrap percolation rule each vertex of G eventually becomes infected. In this paper, we consider the 3-bootstrap percolation number of grids with fixed widths. If G is the Cartesian product $P_3 \,\square\, P_m$ of two paths of orders 3 and m, we prove that $m(G,3) = \frac{3}{2}(m+1) - 1$, when m is odd, and $m(G,3) = \frac{3}{2}m + 1$, when m is even. Moreover, if G is the Cartesian product $P_5 \,\square\, P_m$, we prove that $m(G,3) = 2m+2$, when m is odd, and $m(G,3) = 2m+3$, when m is even. If G is the Cartesian product $P_4 \,\square\, P_m$, we prove that $m(G,3)$ takes on one of two possible values, namely $m(G,3) = \left\lfloor \frac{5(m+1)}{3} \right\rfloor + 1$ or $m(G,3) = \left\lfloor \frac{5(m+1)}{3} \right\rfloor + 2$.

Keywords: bootstrap percolation, 3-percolation number, grids.

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1. Introduction

For notation and graph theory terminology, we in general follow [21, 22]. Specifically, let G be a graph with vertex set $V(G)$ and edge set $E(G)$, and of order $n(G) = |V(G)|$ and size $m(G) = |E(G)|$. A *neighbor* of a vertex v in G is a vertex u that is adjacent to v, that is, $uv \in E(G)$. The *open neighborhood* $N_G(v)$ of a vertex v in G is the set of neighbors of v, while the *closed neighborhood* of v is the set $N_G[v] = \{v\} \cup N_G(v)$. For a set $S \subseteq V(G)$, its *open neighborhood* is the set $N_G(S) = \bigcup_{v\in S} N_G(v)$, and its *closed neighborhood* is the set $N_G[S] = N_G(S) \cup S$.

We denote the *degree* of a vertex v in G by $\deg_G(v)$, or simply by $\deg(v)$ if the graph G is clear from the context, and so $\deg(v) = |N_G(v)|$. If $X \subseteq V(G)$ and $v \in V(G)$, then $\deg_X(v)$ is the number of neighbors of the vertex v in G that belong to the set X, that is, $\deg_X(v) = |N_G(v) \cap X|$. In particular, if $X = V(G)$, then $\deg_X(v) = \deg_G(v)$.

We denote a cycle and a path on n vertices by C_n and P_n, respectively. For a nonempty set of vertices $S \subseteq V(G)$, the subgraph induced by S is denoted by $G[S]$. Thus, $G[S]$ is the graph having vertex set S and whose edge set consists of all those edges of G incident with two vertices in S. Moreover, we denote the graph obtained from G by deleting all vertices in the set S by $G - S$, that is, $G-S = G[V(G) \setminus S]$. A subgraph H of G is an *induced subgraph of* G if $H = G[S]$ for some subset S of $V(G)$.

For any integer $r \geq 2$, the r-neighbor bootstrap percolation process is an update rule for the states of vertices in a given graph G. At any given time a vertex can either be *infected* or *uninfected*. From an initial set of infected vertices, the following updates occur simultaneously and in discrete intervals: any uninfected vertex with at least r infected neighbors becomes infected, while infected vertices never change their state.

More formally, let $A_0 \subseteq V(G)$ be an initial set of infected vertices and for every $t \geq 1$ define

$$A_t = A_{t-1} \cup \{v \in V(G) \colon |N_G(v) \cap A_{t-1}| \geq r\}.$$

The set $A_t \setminus A_{t-1}$ is referred to as the *set of vertices infected at time* t. A vertex v is infected before vertex u if $v \in A_t$ and $u \notin A_t$ for some $t \geq 0$. We say that the set A_0 is an *r-percolating set*, or simply *r-percolates*, in the graph G if

$$\bigcup_{t=0}^{\infty} A_t = V(G).$$

A natural extremal problem is finding a smallest r-percolating set A_0 in a given graph G. For a given graph G and integer $r \geq 2$, the *r-percolation number* of G, denoted $m(G, r)$, is the minimum cardinality of an r-percolating set in G, that is,

$$m(G,r) = \min\{|A_0|\colon\ A_0 \subseteq V(G),\ A_0 \text{ is an } r\text{-percolating set in } G\}\,.$$

A *minimum r-percolating set* in G is an r-percolating set S of G satisfying $m(G,r) = |S|$. Bootstrap percolation is very well studied in graphs, see, for example, [1–17, 23–27].

The *Cartesian product* $G \,\square\, H$ of two graphs G and H is the graph whose vertex set is $V(G) \times V(H)$, and where two vertices (g_1, h_1) and (g_2, h_2) are adjacent in $G \,\square\, H$ if either $g_1 = g_2$ and $h_1h_2 \in E(H)$, or $h_1 = h_2$ and $g_1g_2 \in E(G)$. For a vertex $g \in V(G)$, the subgraph of $G \,\square\, H$ induced by the set $\{(g,h) : h \in V(H)\}$ is called a *H-fiber* and is denoted by ${}^{g}H$. Similarly, for $h \in V(H)$, the *G-fiber*, G^h, is the subgraph induced by $\{(g,h) : g \in V(G)\}$. We note that all G-fibers are isomorphic to G and all H-fibers are isomorphic to H. A *fiber* in $G \,\square\, H$ is a *G-fiber* or an *H-fiber*.

If $G = P_n \,\square\, P_m$ is the Cartesian product of two paths P_n and P_m for some $n, m \geq 2$ (such a graph is called a *grid*), then a vertex $v \in V(G)$ is called a *boundary vertex* or a *vertex on the boundary of* G if $\deg_G(v) \leq 3$.

In 2006, Bollobás [7] presented a problem of disease spreading on an $n \times n$ grid where the infection spreads if an uninfected vertex has at least two infected neighbors, also providing an elegant proof of the problem. While grids are in fact Cartesian products of paths, not much was established for the product of arbitrary graphs. Coelho *et al.* [12] determined the 2-bootstrap percolation numbers of the strong and lexicographic products of two graphs, while the Cartesian product of two graphs proved to be more complex.

Special cases of the Cartesian product were studied by Balogh in [1] with 3-bootstrap percolation in the hypercube, and Brešar and Valencia-Pabon [8] in the case of 2-bootstrap percolation in Hamming graphs. Grid-like graphs arise in applications from computer networks and integrated circuit designs to city street layouts, and the study of domination related parameters in grids (the Cartesian product of two paths) are very well studied (see, for example, [19–21]).

Further research on 2-dimensional grids was done by Benevides *et al.* in [4] where they studied $n \times n$ grids under r-bootstrap percolation for $r = 3$ and $r = 4$. For grids in higher dimensions, Przykucki and Shelton [25] established the r-bootstrap percolation number of an r-dimensional square grid. Most of the research to date focused on r-bootstrap percolation in square $n \times n$ grids, while in 2023 Dukes, Noel, and Romer [15] studied the so-called perfect lethal sets (sets which attain the well known general upper bound for r-bootstrap percolation in grids) under 3-bootstrap percolation in rectangular grids of dimensions 2 or 3. However the cases where the upper bound is attained proved to be sparse. The problem to determine closed formulas for the 3-bootstrap percolation number of an $n \times m$ grid for general n and m was therefore left open.

2. Main Results

Our aim in this paper is to study 3-neighbor bootstrap percolation on grids. We determine closed formulas for the 3-percolation number of a $3 \times m$ grid for all $m \geq 3$ and the 3-percolation number of a $5 \times m$ grid for all $m \geq 5$. Moreover, we show that the 3-percolation number of a $4 \times m$ grid for all $m \geq 4$ takes on one of two possible values. We shall prove the following results.

Theorem 1. *For $m \geq 3$, if $G = P_3 \,\Box\, P_m$, then*

$$m(G,3) = \begin{cases} \frac{3}{2}(m+1) - 1; & m \text{ odd;} \\ \frac{3}{2}m + 1; & m \text{ even.} \end{cases}$$

Theorem 2. *For $m \geq 5$, if $G = P_5 \,\Box\, P_m$, then*

$$m(G,3) = \begin{cases} 2m + 2; & m \text{ odd;} \\ 2m + 3; & m \text{ even.} \end{cases}$$

Theorem 3. *For $m \geq 4$, if $G = P_4 \,\Box\, P_m$, then*

$$m(G,3) = \left\lfloor \frac{5(m+1)}{3} \right\rfloor + \Phi_m(G),$$

where $\Phi_m(G) \in \{1,2\}$. Moreover, $\Phi_m(G) = 1$ if $m \in \{5,7,11\}$.

3. Preliminary Results

In this section, we present some preliminary lemmas that we will need when proving our main results in Section 2. We remark that Lemma 4 is already known in the literature, but for completeness we provide short proofs of the elementary results we present in this section since we use them frequently when proving our main results.

Lemma 4. *For $r \geq 2$ if H is a subgraph of a graph G such that every vertex in H has strictly less than r neighbors in G that belong to $V(G) \setminus V(H)$, then every r-percolating set of G contains at least one vertex of H.*

Proof. For $r \geq 2$ let H be an induced subgraph of a graph G and let $X = V(G) \setminus V(H)$. Suppose that every vertex in H has strictly less than r neighbors in the graph G that belong to the set X, that is, $\deg_X(v) < r$ for every vertex $v \in V(H)$. In this case, even if every vertex in X is infected, no vertex in H becomes infected since every vertex in H has strictly less than r infected neighbors. Therefore, every r-percolating set of G contains at least one vertex of H. ■

We call the subgraph H in the statement of Lemma 4 an *r-forbidden subgraph of G*. We describe next some structural properties of 3-forbidden subgraphs in grids. As an immediate consequence of Lemma 4, we infer the following 3-forbidden subgraphs in grids.

Corollary 5. *Let $G = P_n \,\square\, P_m$ for some $m, n \in \mathbb{N}$ and let S be a minimum 3-percolating set of G. If H is a subgraph of G satisfying* (a) *or* (b), *then H is a 3-forbidden subgraph of G.*

(a) *H is a path joining two boundary vertices in G;*

(b) *H is a cycle in G.*

We note that if $G = P_n \,\square\, P_m$, then two adjacent boundary vertices in G form a path joining two boundary vertices in G. Moreover, every P_n-fiber and P_m-fiber in G is a path joining two boundary vertices in G. We also note that every 4-cycle in G is a 3-forbidden subgraph. Hence as special cases of Corollary 5, we have the following 3-forbidden subgraphs in a grid.

Corollary 6. *Let $G = P_n \,\square\, P_m$ for some $m, n \in \mathbb{N}$ and let S be a minimum 3-percolating set of G. If H is an induced subgraph of G satisfying* (a), (b) *or* (c), *then H is a 3-forbidden subgraph of G.*

(a) *$H = P_2$, where the two vertices in H are adjacent boundary vertices in G;*

(b) *H is a fiber in G;*

(c) *$H = C_4$.*

Lemma 7. *If $G = P_n \,\square\, P_m$ for some $m, n \in \mathbb{N}$, then there exists a minimum 3-percolating set of G that does not contain three consecutive boundary vertices of G.*

Proof. Let $G = P_n \,\square\, P_m$ and let u, v, z be three consecutive boundary vertices of G where v is adjacent to both u and z. Let S be a minimum 3-percolating set of G that contains as few vertices from the set $\{u, v, z\}$ as possible. Suppose, to the contrary, that $\{u, v, z\} \subseteq S$. Let x be the third neighbor of v different from u and z. If $x \in S$, then $S \setminus \{v\}$ is also a percolating set, since v is adjacent to three infected vertices. However this contradicts the minimality of the set S. Therefore, $x \notin S$. We now consider the set $S' = (S \setminus \{v\}) \cup \{x\}$. We note that $|S'| = |S|$. The vertex v becomes immediately infected in the 3-neighbor bootstrap percolation process since it has three infected neighbors in the set S'. Since the resulting set of infected vertices contains the 3-percolating set S of G as a subset, we infer that the set S' is a 3-percolating set of G, implying that S' is a minimum 3-percolating set of G. However since the set S' contains fewer vertices that belong to the set $\{u, v, z\}$ than does the set S, this contradicts our choice of the set S. Hence, $|\{u, v, z\} \cap S| \leq 2$, that is, there exists a minimum 3-percolating set of G that does not contain three consecutive boundary vertices of G. ∎

3.1. 3-Bootstrap percolation in $3 \times m$ grids

In this section we present a proof of Theorem 1. Recall its statement.

Theorem 1. *For $m \ge 3$, if $G = P_3 \,\Box\, P_m$, then*

$$m(G,3) = \begin{cases} \frac{3}{2}(m+1) - 1; & m \text{ odd;} \\ \frac{3}{2}m + 1; & m \text{ even.} \end{cases}$$

Proof. For $m \ge 3$, let G be the grid $P_3 \,\Box\, P_m$ with

$$V(G) = \bigcup_{i=1}^{m} \{a_i, b_i, c_i\},$$

where the path $a_i b_i c_i$ is a P_3-fiber in G for $i \in [m]$, and where the paths $a_1 a_2 \cdots a_m$, $b_1 b_2 \cdots b_m$, and $c_1 c_2 \cdots c_m$ are P_m-fibers in G. For example, when $m = 5$ the grid $G = P_3 \,\Box\, P_m$ is illustrated in Figure 1.

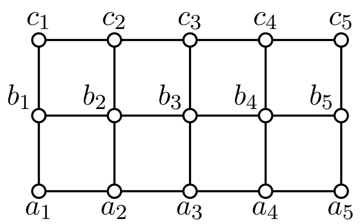


Figure 1. The graph $G = P_3 \,\Box\, P_5$.

For $i \in [m]$, let $V_i = \{a_i, b_i, c_i\}$ and let

$$V_{\le i} = \bigcup_{j=1}^{i} V_i \quad \text{and} \quad V_{\ge i} = \bigcup_{j=i}^{m} V_i.$$

Thus, $V(G) = V_{\le m} = V_{\ge 1}$. Let $A = \{a_1, a_2, \ldots, a_m\}$, $B = \{b_1, b_2, \ldots, b_m\}$, and $C = \{c_1, c_2, \ldots, c_m\}$. Further let

$$A_{\rm odd} = \bigcup_{i=1}^{\lceil \frac{m}{2} \rceil} \{a_{2i-1}\}, \quad B_{\rm even} = \bigcup_{i=1}^{\lfloor \frac{m}{2} \rfloor} \{b_{2i}\}, \quad \text{and} \quad C_{\rm odd} = \bigcup_{i=1}^{\lceil \frac{m}{2} \rceil} \{c_{2i-1}\}.$$

By Lemma 7, there exists a minimum 3-percolating set of G that does not contain three consecutive boundary vertices of G. Among all minimum 3-percolating set of G, let S be chosen so that

(1) S does not contain three consecutive boundary vertices of G,

(2) subject to (1), $|S \cap B_{\text{even}}|$ is a maximum, and

(3) subject to (2), $|S \cap (A_{\text{odd}} \cup C_{\text{odd}})|$ is a maximum.

Since each vertex in the set $X = \{a_1, c_1, a_m, c_m\}$ has degree 2 in G, the set X is necessarily a subset of the 3-percolating set S. Thus since the set S does not contain three consecutive boundary vertices of G, we note that $b_1 \notin S$ and $b_m \notin S$.

Suppose that $m = 3$. In this case, $X = \{a_1, c_1, a_3, c_3\}$. However the set X is not a 3-percolating set of G, implying that S contains at least one additional vertex that does not belong to the set X. Since $S \cap \{b_1, a_2, c_2, b_3\} = \emptyset$ by Lemma 7, we infer that $S = X \cup \{b_2\}$, and so $m(G, 3) = |S| = 5 = \frac{3}{2}(m+1) - 1$. Hence, we may assume that $m \geq 4$, for otherwise the desired value of $m(G, 3)$ holds.

Claim 8. $b_2 \in S$.

Proof. Suppose, to the contrary, that $b_2 \notin S$. Since vertex b_1 only gets infected after vertex b_2 is infected, the three neighbors a_2, c_2 and b_3 of b_2 must all be infected in order to infect b_2. Thus, vertex b_2 only gets infected after the vertices a_2, c_2 and b_3 are all infected. However if $a_2 \notin S$, then vertex a_2 only gets infected after vertex b_2 is infected, a contradiction. Hence, $a_2 \in S$. Analogously, $c_2 \in S$. By Lemma 7, we infer that $a_3 \notin S$ and $c_3 \notin S$. If $b_3 \notin S$, then it would not be possible to infect b_3 since at most one of its neighbors gets infected. Hence, $b_3 \in S$, and so $S \cap V_{\leq 3} = \{a_1, c_1, a_2, c_2, b_3\}$, as illustrated in Figure 2(a). In this case, we note that the set

$$S' = (S \setminus \{a_2, c_2, b_3\}) \cup \{b_2, a_3, c_3\}$$

is also a minimum 3-percolating set of G, as illustrated in Figure 2(b). Thus, $S' \cap V_{\leq 3} = \{a_1, c_1, b_2, a_3, c_3\}$ and $S \cap V_{\geq 4} = S' \cap V_{\geq 4}$.

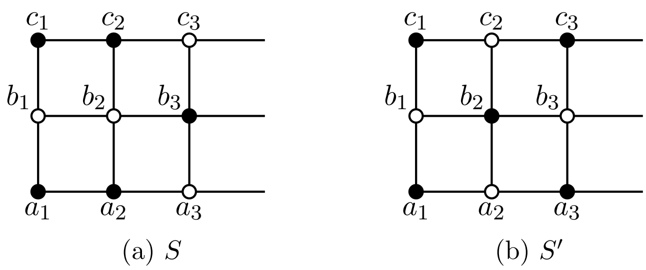


Figure 2. The sets S and S' in the proof of Claim 8.

By construction, $|S' \cap B_{\text{even}}| > |S \cap B_{\text{even}}|$. Hence if S' satisfies (1), then we contradict our choice of the set S. Therefore, S' does not satisfy (1), and so S'

contains three consecutive boundary vertices. Since the set S does not contain three consecutive boundary vertices, we infer that $\{a_4, a_5\} \subset S$ or $\{c_4, c_5\} \subset S$. If $\{a_4, a_5\} \subset S$ and $\{c_4, c_5\} \subset S$, then the set $S'' = (S' \setminus \{a_4, c_4\}) \cup \{b_4\}$ is a 3-percolating set of G, contradicting the minimality of S'. Hence, exactly one of $\{a_4, a_5\} \subset S$ or $\{c_4, c_5\} \subset S$ holds. By symmetry, we may assume that $\{a_4, a_5\} \subset S$, and so a_3, a_4, a_5 are three consecutive boundary vertices that belong to S'. The set $S'' = (S' \setminus \{a_4\}) \cup \{b_4\}$ is a minimum 3-percolating set of G that satisfies (1). However, $|S'' \cap B_{\rm even}| > |S \cap B_{\rm even}|$, contradicting our choice of the set S. Therefore, $b_2 \in S$. □

By Claim 8, we have $b_2 \in S$.

Claim 9. $S \cap \{a_2, c_2\} \neq \emptyset$.

Proof. Suppose that at least one of a_2 and c_2 belongs to the set S. By symmetry, we may assume that $a_2 \in S$. In this case, we consider the set $S' = (S \setminus \{a_2\}) \cup \{a_3\}$. Necessarily, S' is a minimum 3-percolating set of G. We note that $|S' \cap B_{\rm even}| = |S \cap B_{\rm even}|$ and $|S' \cap (A_{\rm odd} \cup C_{\rm odd})| > |S \cap (A_{\rm odd} \cup C_{\rm odd})|$. If S' satisfies (1), then we contradict our choice of the set S. Hence, S' does not satisfy (1), implying that a_3, a_4, a_5 are three consecutive boundary vertices that belong to S'. If $b_4 \in S'$, then $S' \setminus \{a_4\}$ is a 3-percolating set of G, contradicting the minimality of S'. Hence, $b_4 \notin S'$ and the set $S'' = (S' \setminus \{a_4\}) \cup \{b_4\}$ is a minimum 3-percolating set of G that satisfies (1) and $|S'' \cap B_{\rm even}| > |S \cap B_{\rm even}|$, contradicting our choice of the set S. □

By Claim 9, neither a_2 nor c_2 belongs to the set S. Hence, a_2 and c_2 only get infected after a_3 and c_3, respectively, are infected. Since a_3 and c_3 are boundary vertices, this implies that $a_3 \in S$ and $c_3 \in S$. If $b_3 \in S$, then $S \setminus \{b_3\}$ is a 3-percolating set of G, contradicting the minimality of S. Hence, $b_3 \notin S$. Thus, $S \cap V_{\leq 3} = \{a_1, c_1, b_2, a_3, c_3\}$, as illustrated in Figure 3.

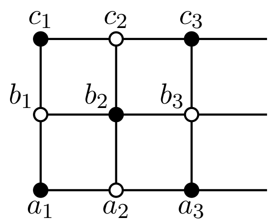


Figure 3. The set $S \cap V_{\leq 3}$.

The set S infects vertices b_1, a_2, c_2 and b_3. If $m = 4$, then by our earlier observations, $\{a_4, c_4\} \subset S$ and $b_4 \notin S$, implying that $S = \{a_1, c_1, b_2, a_3, c_3, a_4, c_4\}$, and so $m(G, 3) = |S| = 7 = \frac{3}{2}m + 1$. Hence, we may assume that $m \geq 5$, for otherwise the desired value of $m(G, 3)$ holds.

Suppose that $m = 5$. By our earlier observations, $\{a_5, c_5\} \subset S$ and $b_5 \notin S$. In order for b_5 to be infected, the vertex b_4 must be infected first. If $b_4 \notin S$, then in order for b_4 to be infected before b_5, both boundary vertices a_4 and c_4 must belong to the set S. But then $(S \setminus \{a_4, c_4\}) \cup \{b_4\}$ is a 3-percolating set of G, contradicting the minimality of S. Therefore, $b_4 \in S$, implying that $S = \{a_1, c_1, b_2, a_3, c_3, b_4, a_5, c_5\}$, and so $m(G, 3) = |S| = 8 = \frac{3}{2}(m+1) - 1$. Hence, we may assume that $m \geq 6$.

Claim 10. $b_4 \in S$.

Proof. Suppose, to the contrary, that $b_4 \notin S$. For b_4 to be infected, it needs two more infected neighbors in addition to the vertex b_3, implying that at least one of a_4 or c_4 must be infected before b_4. By symmetry, we may assume that a_4 is infected before b_4, implying that the boundary vertex a_4 belongs to the set S (and to the set S). Since S satisfies (1) and $\{a_3, a_4\} \subset S$, we note that $a_5 \notin S$.

Suppose that $c_4 \notin S$. By Corollary 6(a), the adjacent boundary vertex c_5 of c_4 therefore belongs to S. In order for c_4 to be infected, the vertex b_4 must be infected first. However in order for b_4 to be infected before c_4, the vertex b_5 must be infected before b_4. If $b_5 \notin S$, then both vertices a_5 and b_6 must be infected before b_5, implying that the boundary vertex a_5 belongs to the set S, a contradiction. Hence, $b_5 \in S$. We now consider the set

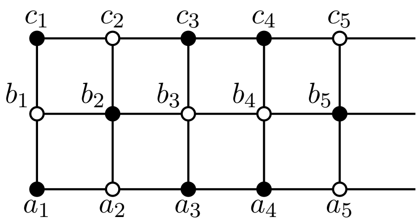
$$S' = (S \setminus \{a_4, b_5\}) \cup \{b_4, a_5\}.$$

Since S is a 3-percolating set of G, so too is the set S'. Thus since $|S'| = |S|$, the set S' is a minimum 3-percolating set of G. We note that $|S' \cap B_{\rm even}| > |S \cap B_{\rm even}|$, and so if S' satisfies (1), then we contradict our choice of the set S. Hence, S' does not satisfy (1), implying that a_5, a_6, a_7 are three consecutive boundary vertices that belong to S'. If $b_6 \in S'$, then $S' \setminus \{a_6\}$ is a 3-percolating set of G, contradicting the minimality of S'. Hence, $b_6 \notin S'$ and the set $S'' = (S' \setminus \{a_6\}) \cup \{b_6\}$ is a minimum 3-percolating set of G that satisfies (1) and $|S'' \cap B_{\rm even}| > |S \cap B_{\rm even}|$, contradicting our choice of the set S. Hence, $c_4 \in S$. Since $\{c_3, c_4\} \subset S$ and S satisfies (1), we note that $c_5 \notin S$.

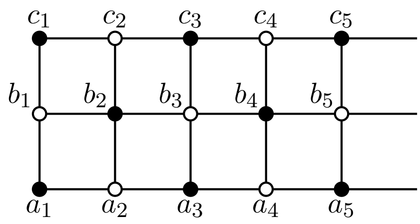
If $b_5 \notin S$, then b_5 must be infected before the boundary vertices a_5 and c_5. However, this would not be possible since then b_5 would have at most two infected neighbors at any stage of the percolation process. Thus, $b_5 \in S$, and so $S \cap V_{\leq 5} = \{a_1, c_1, b_2, a_3, c_3, a_4, c_4, b_5\}$, as illustrated in Figure 4(a). In this case, we note that the set

$$S'' = (S \setminus \{a_4, c_4, b_5\}) \cup \{b_4, a_5, c_5\}$$

is also a minimum 3-percolating set of G, as illustrated in Figure 4(b). Thus, $S' \cap V_{\leq 5} = \{a_1, c_1, b_2, a_3, c_3, b_4, a_5, c_5\}$ and $S \cap V_{\geq 6} = S' \cap V_{\geq 6}$.



(b) S



(b) S''

Figure 4. The sets S and S'' in the proof of Claim 10.

By construction, $|S'' \cap B_{\text{even}}| > |S \cap B_{\text{even}}|$. Hence if S'' satisfies (1), then we contradict our choice of the set S. Therefore, S'' does not satisfy (1), and so S'' contains three consecutive boundary vertices. Since the set S does not contain three consecutive boundary vertices, we infer that $\{a_6, a_7\} \subset S$ or $\{c_6, c_7\} \subset S$. If $\{a_6, a_7\} \subset S$ and $\{c_6, c_7\} \subset S$, then the set $(S'' \setminus \{a_6, c_6\}) \cup \{b_6\}$ is a 3-percolating set of G, contradicting the minimality of S''. Hence, exactly one of $\{a_6, a_7\} \subset S$ or $\{c_6, c_7\} \subset S$ holds. By symmetry, we may assume that $\{a_6, a_7\} \subset S$, and so a_5, a_6, a_7 are three consecutive boundary vertices that belong to S''. The set $S^* = (S'' \setminus \{a_6\}) \cup \{b_6\}$ is a minimum 3-percolating set of G that satisfies (1). However, $|S^* \cap B_{\text{even}}| > |S \cap B_{\text{even}}|$, contradicting our choice of the set S. Therefore, $b_4 \in S$. $\square$

By Claim 10, we have $b_4 \in S$.

Claim 11. $S \cap \{a_4, c_4\} \neq \emptyset$.

Proof. Suppose that at least one of a_4 and c_4 belongs to the set S. By symmetry, we may assume that $a_4 \in S$. In this case, we consider the set $S' = (S \setminus \{a_4\}) \cup \{a_5\}$. Necessarily, S' is a minimum 3-percolating set of G. We note that $|S' \cap B_{\text{even}}| = |S \cap B_{\text{even}}|$ and $|S' \cap (A_{\text{odd}} \cup C_{\text{odd}})| > |S \cap (A_{\text{odd}} \cup C_{\text{odd}})|$. If S' satisfies (1), then we contradict our choice of the set S. Hence, S' does not satisfy (1), implying that a_5, a_6, a_7 are three consecutive boundary vertices that belong to S'. If $b_6 \in S'$, then $S' \setminus \{a_6\}$ is a 3-percolating set of G, contradicting the minimality of S'. Hence, $b_6 \notin S'$ and the set $S'' = (S' \setminus \{a_6\}) \cup \{b_6\}$ is a minimum 3-percolating set of G that satisfies (1) and $|S'' \cap B_{\text{even}}| > |S \cap B_{\text{even}}|$, contradicting our choice of the set S. $\square$

By Claim 11, neither a_4 nor c_4 belongs to the set S. Hence, a_4 and c_4 only get infected after a_5 and c_5, respectively, are infected. Since a_5 and c_5 are boundary vertices, this implies that $a_5 \in S$ and $c_5 \in S$. If $b_5 \in S$, then $S \setminus \{b_5\}$ is a 3-percolating set of G, contradicting the minimality of S. Hence, $b_5 \notin S$. Thus, $S \cap V_{\leq 5} = \{a_1, c_1, b_2, a_3, c_3, b_4, a_5, c_5\}$, as illustrated in Figure 5.

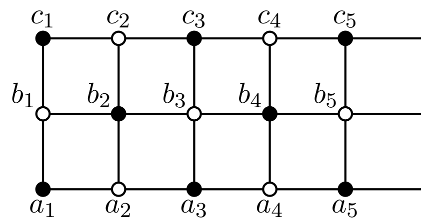


Figure 5. The set $S \cap V_{\leq 5}$.

By our earlier assumption, $m \geq 6$. Continuing the above process, this pattern concludes naturally if m is odd and yields the set

$$S = A_{\text{odd}} \cup B_{\text{even}} \cup C_{\text{odd}},$$

implying that in this case when m is odd, we have

$$\begin{aligned} m(G,3) = |S| &= |A_{\text{odd}}| + |B_{\text{even}}| + |C_{\text{odd}}| \\ &= \tfrac{1}{2}(m+1) + \tfrac{1}{2}(m-1) + \tfrac{1}{2}(m+1) = \tfrac{3}{2}(m+1) - 1. \end{aligned}$$

If m is even, then recalling that $\{a_m, c_m\} \subset S$ and $b_m \notin S$, this yields the set

$$S = (A_{\text{odd}} \cup \{a_m\}) \cup (B_{\text{even}} \setminus \{b_m\}) \cup (C_{\text{odd}} \cup \{C_m\}),$$

implying that in this case when m is even, we have

$$\begin{aligned} m(G,3) = |S| &= \big(|A_{\text{odd}}| + 1\big) + \big(|B_{\text{even}}| - 1\big) + \big(|C_{\text{odd}}| + 1\big) \\ &= \big(\tfrac{1}{2}m + 1\big) + \big(\tfrac{1}{2}m - 1\big) + \big(\tfrac{1}{2}m + 1\big) = \tfrac{3}{2}m + 1. \end{aligned}$$

This completes the proof of Theorem 1. ■

3.2. 3-Bootstrap percolation in $5 \times m$ grids

We present in this section a proof of Theorem 2. Recall its statement.

Theorem 2 *For $m \geq 5$, if $G = P_5 \,\Box\, P_m$, then*

$$m(G,3) = \begin{cases} 2m+2; & m \textit{ odd}; \\ 2m+3; & m \textit{ even}. \end{cases}$$

Proof. For $m \geq 5$, let G be the grid $P_5 \,\Box\, P_m$ with

$$V(G) = \bigcup_{i=1}^{m} \{a_i, b_i, c_i, d_i, e_i\},$$

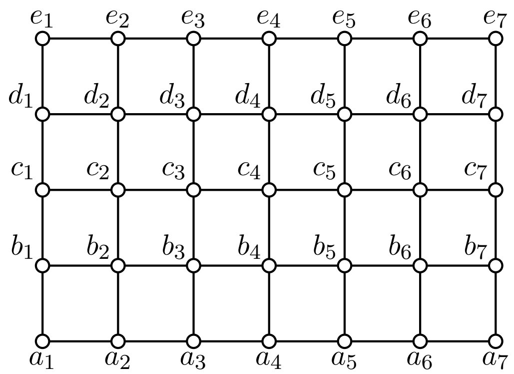


Figure 6. The graph $G = P_5 \,\square\, P_7$.

where the path $a_ib_ic_id_ie_i$ is a P_5-fiber in G for $i \in [m]$, and where $a_1a_2\cdots a_m$, $b_1b_2\cdots b_m$, $c_1c_2\cdots c_m$, $d_1d_2\cdots d_m$, and $e_1e_2\cdots e_m$ are P_m-fibers in G. For example, when $m = 7$ the grid $G = P_5 \,\square\, P_m$ is illustrated in Figure 6.

For $i \in [m]$, let $V_i = \{a_i, b_i, c_i, d_i, e_i\}$ and let

$$V_{\le i} = \bigcup_{j=1}^{i} V_i \quad \text{and} \quad V_{\ge i} = \bigcup_{j=i}^{m} V_i.$$

Thus, $V(G) = V_{\le m} = V_{\ge 1}$. Let $A = \{a_1, a_2, \ldots, a_m\}$, $B = \{b_1, b_2, \ldots, b_m\}$, $C = \{c_1, c_2, \ldots, c_m\}$, $D = \{d_1, d_2, \ldots, d_m\}$, and $E = \{e_1, e_2, \ldots, e_m\}$. In what follows, let S be a minimum 3-percolating set of G that does not contain three consecutive boundary vertices of G. We note that such a set S exists by Lemma 7.

Claim 12. *The following properties hold.*

(a) $|S \cap V_1| \ge 3$ *and* $|S \cap (V_1 \cup V_2)| \ge 5$*;*

(b) $|S \cap V_m| \ge 3$ *and* $|S \cap (V_{m-1} \cup V_m)| \ge 5$.

Proof. Since the vertices a_1 and e_1 both have degree 2 in G, we note that $\{a_1, e_1\} \subset S$. Suppose that $c_1 \notin S$. By Corollary 6(a), this implies that $\{b_1, d_1\} \subset S$. By Corollary 6(b), $|S \cap V_2| \ge 1$. Hence in this case when $c_1 \notin S$, we have $|S \cap V_1| \ge 4$ and $|S \cap (V_1 \cup V_2)| \ge 5$. Hence we may assume that $c_1 \in S$, for otherwise the desired lower bounds hold. Since S does not contain three consecutive boundary vertices of G, we infer that $S \cap V_1 = \{a_1, c_1, e_1\}$. Let $H_1 = G[\{a_2, b_1, b_2\}]$ and let $H_2 = G[\{d_1, d_2, e_2\}]$. Each of H_1 and H_2 is a path joining two boundary vertices of G, and is therefore a 3-forbidden subgraph of G by Corollary 5(a). Thus, S contains at least one vertex from each of H_1 and H_2, implying that $|S \cap \{a_2, b_2\}| \ge 1$ and $|S \cap \{d_2, e_2\}| \ge 1$. Thus, $|S \cap V_1| = 3$ and $|S \cap V_2| \ge 2$, and so $|S \cap (V_1 \cup V_2)| \ge 5$. This proves part (a). By symmetry, part (b) holds. $\square$

Claim 13. $|S \cap (V_i \cup V_{i+1})| \geq 4$ *for all* i *where* $2 \leq i \leq m-1$.

Proof. Consider the set $V_i \cup V_{i+1}$ for some i where $2 \leq i \leq m-1$. We show that $|S \cap (V_i \cup V_{i+1})| \geq 4$. By Corollary 6(a), $|S \cap \{a_i, a_{i+1}\}| \geq 1$ and $|S \cap \{e_i, e_{i+1}\}| \geq 1$. By Corollary 6(b), $|S \cap V_i| \geq 1$ and $|S \cap V_{i+1}| \geq 1$.

Suppose firstly that $\{a_i, e_i\} \subset S$. Let $H_1 = G[\{b_i, b_{i+1}, c_i, c_{i+1}\}]$ and let $H_2 = G[\{c_i, c_{i+1}, d_i, d_{i+1}\}]$. Since $H_1 = C_4$ and $H_2 = C_4$, by Corollary 6(c) both H_1 and H_2 are 3-forbidden subgraph of G. Let $H_3 = G[V_{i+1}]$. Since H_3 is a fibre in G, by Corollary 6(b), the fibre H_3 is a 3-forbidden subgraph of G. Let $H_4 = G[\{a_{i+1}, b_{i+1}, b_i, c_i, d_i, d_{i+1}, e_{i+1}\}]$. Since H_4 is a path joining two boundary vertices, H_4 is a 3-forbidden subgraph of G. Hence, the set S must contain at least one vertex from each of the 3-forbidden subgraphs H_1, H_2, H_3 and H_4. This is only possible if S contains at least two vertices in $V_i \cup V_{i+1}$ different from a_i and e_i, implying that $|S \cap (V_i \cup V_{i+1})| \geq 4$.

Hence we may assume that at most one of a_i and e_i belong to the set S, for otherwise the desired result holds. By analogous arguments, we may assume that at most one of a_{i+1} and e_{i+1} belong to the set S. As observed earlier, at least one a_i and a_{i+1} belongs to the set S and at least one e_i and e_{i+1} belongs to the set S. By symmetry, we may therefore assume that $a_{i+1} \in S$ and $e_i \in S$, and so $a_i \notin S$ and $e_{i+1} \notin S$.

Let $F_1 = G[\{a_i, b_i, b_{i+1}, c_{i+1}, d_{i+1}, e_{i+1}\}]$, $F_2 = G[\{a_i, b_i, c_i, c_{i+1}, d_{i+1}, e_{i+1}\}]$ and $F_3 = G[\{a_i, b_i, c_i, d_i, d_{i+1}, e_{i+1}\}]$. Each of F_1, F_2 and F_3 is a path joining two boundary vertices of G, and is therefore a 3-forbidden subgraph of G by Corollary 5(a). Moreover if $F_4 = G[\{b_i, c_i, b_{i+1}, c_{i+1}\}]$ and $F_5 = G[\{c_i, d_i, c_{i+1}, d_{i+1}\}]$, then $F_4 = C_4$ and $F_5 = C_4$, and so by Corollary 6 both F_4 and F_5 are 3-forbidden subgraph of G. Hence, the set S must contain at least one vertex from each of the subgraphs F_1, F_2, F_3, F_4 and F_5. This implies that S contains at least two vertices in $V_i \cup V_{i+1}$ different from a_{i+1} and e_i, implying once again that $|S \cap (V_i \cup V_{i+1})| \geq 4$. □

Claim 14. $m(G,3) \geq 2m+2$.

Proof. Suppose firstly that m is even. By Claim 12(a) and Claim 13, we have

$$\begin{aligned} m(G,3) = |S| &= \sum_{i=1}^{m} |S \cap V_i| = |S \cap V_1| + |S \cap V_m| + \sum_{i=2}^{m-1} |S \cap V_i| \\ &= |S \cap V_1| + |S \cap V_m| + \sum_{i=1}^{\frac{m}{2}-1} |S \cap (V_{2i} \cup V_{2i+1})| \\ &\geq 3 + 3 + 4\Big(\frac{m}{2} - 1\Big) \geq 2m + 2. \end{aligned}$$

Suppose secondly that m is odd. By Claim 12(a) and Claim 13, we have

$$\begin{aligned} m(G,3) = |S| &= \sum_{i=1}^{m} |S \cap V_i| = |S \cap (V_1 \cup V_2)| + |S \cap V_m| + \sum_{i=3}^{m-1} |S \cap V_i| \\ &= |S \cap (V_1 \cup V_2)| + |S \cap V_m| + \sum_{i=1}^{\frac{m-3}{2}} |S \cap (V_{2i+1} \cup V_{2i+2})| \\ &\geq 5 + 3 + 4\Big(\frac{m-3}{2}\Big) \geq 2m + 2. \end{aligned}$$

In both cases, we have $m(G,3) \geq 2m+2$. $\square$

In order to establish upper bounds on the 3-percolation number $m(G,3)$ of G, let

$$A_{\text{odd}} = \bigcup_{i=1}^{\lceil \frac{m}{2} \rceil} \{a_{2i-1}\}, \quad B_{\text{even}} = \bigcup_{i=1}^{\lfloor \frac{m}{2} \rfloor} \{b_{2i}\}, \quad D_{\text{even}} = \bigcup_{i=1}^{\lfloor \frac{m}{2} \rfloor} \{d_{2i}\}, \quad \text{and} \quad E_{\text{odd}} = \bigcup_{i=1}^{\lceil \frac{m}{2} \rceil} \{e_{2i-1}\}.$$

Claim 15. *If m is odd, then $m(G,3) = 2m+2$.*

Proof. Suppose that m is odd. Let

$$S_{\text{odd}} = A_{\text{odd}} \cup B_{\text{even}} \cup \{c_1, c_m\} \cup D_{\text{even}} \cup E_{\text{odd}}.$$

For example, when $m = 7$ the set S_{odd} is illustrated in Figure 7.

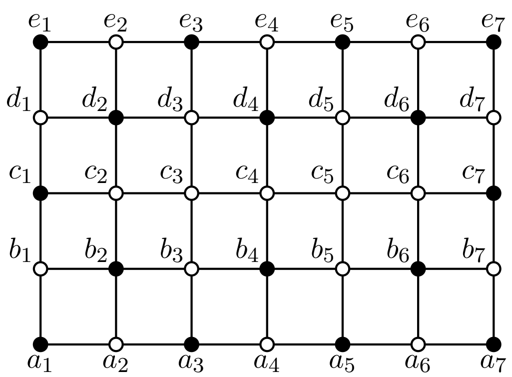


Figure 7. The set S_{odd} in the graph $G = P_5 \,\square\, P_7$.

We show that S_{odd} is a 3-percolating set of G. The vertices b_1, d_1, b_m and d_m all have three infected neighbors, and so become infected during the percolation process starting with the initial set S_{odd}. Hence, all vertices in the first column

V_1 and in the last column V_m are infected. The vertices a_{2i} and e_{2i} for i where $1 \le i \le \frac{1}{2}(m-1)$ all have three infected neighbors, and the vertices b_{2i+1} and d_{2i+1} for i where $1 \le i \le \frac{1}{2}(m-3)$ all have three infected neighbors. Therefore, these vertices all become infected during the percolation process. Hence, all vertices in $A \cup B \cup D \cup E$ are infected. Thereafter, all vertices in C become infected by considering the vertices $c_2, c_3, \ldots, c_{m-1}$ sequentially and noting that the vertex c_i becomes infected from its three infected neighbors c_{i-1}, b_i and d_i for i where $2 \le i \le m-1$. Hence, all vertices in $V(G)$ become infected, implying that

$$\begin{aligned} m(G,3) \le |S_{\rm odd}| &= |A_{\rm odd}| + |B_{\rm even}| + |\{c_1, c_m\}| + |D_{\rm even}| + |E_{\rm odd}| \\ &= \tfrac{m+1}{2} + \tfrac{m-1}{2} + 2 + \tfrac{m-1}{2} + \tfrac{m+1}{2} = 2m + 2. \end{aligned}$$

Hence, $m(G,3) \le 2m+2$. By Claim 14, $m(G,3) \ge 2m+2$. Consequently, $m(G,3) = 2m+2$. □

Claim 16. *If m is even, then $m(G,3) \ge 2m+3$.*

Proof. Suppose that m is even. By Claim 14, we know that $m(G,3) \ge 2m+2$. Suppose, to the contrary, that $m(G,3) = 2m+2$. Hence we must have equality throughout the inequality chain the first paragraph of the proof of Claim 14, implying that $|S \cap V_1| = |S \cap V_m| = 3$ and $|S \cap (V_{2i} \cup V_{2i+1})| = 4$ for all i where $1 \le i \le \frac{m}{2} - 1$. As shown in the proofs of Claim 12 and 14, since $|S \cap V_1| = 3$ we infer that $S \cap V_1 = \{a_1, c_1, e_1\}$ and $|S \cap V_2| \ge 2$. Analogously, since $|S \cap V_m| = 3$ we infer that $S \cap V_m = \{a_m, c_m, e_m\}$ and $|S \cap V_{m-1}| \ge 2$.

Claim 16.1. *$|S \cap V_i| = 2$ for all i where $2 \le i \le m-1$.*

Proof. If $|S \cap V_2| \ge 3$, then by our earlier observations we have

$$\begin{aligned} m(G,3) = |S| &= |S \cap V_1| + |S \cap V_2| + |S \cap V_{m-1}| + |S \cap V_m| \\ &+ \sum_{i=1}^{\frac{m}{2}-2} |S \cap (V_{2i+1} \cup V_{2i+2})| + |S \cap V_{m-1}| + |S \cap V_m| \\ &\ge 3 + 3 + 2 + 3 + 4\Big(\frac{m}{2} - 2\Big) = 2m + 3, \end{aligned}$$

a contradiction. Hence, $|S \cap V_2| = 2$. By symmetry, $|S \cap V_{m-1}| = 2$. We show next that $|S \cap V_i| = 2$ for all i where $3 \le i \le m-3$. Let i be the smallest such integer such that $|S \cap V_i| \ne 2$. By Claim 13, $|S \cap (V_{i-1} \cup V_i)| \ge 4$. By our choice of the integer i, we have $|S \cap V_{i-1}| = 2$, implying that $|S \cap V_i| \ge 3$.

If i is odd, then $|S \cap (V_{i-1} \cup V_i)| = |S \cap V_{i-1}| + |S \cap V_i| \ge 2 + 3 = 5$, contradicting our earlier observation that $|S \cap (V_{2i} \cup V_{2i+1})| = 4$ for all i where $1 \le i \le \frac{m}{2} - 1$. Hence, i is even. Thus, $4 = |S \cap (V_i \cup V_{i+1})| = |S \cap V_i| + |S \cap V_{i+1}| \ge 3 + |S \cap V_{i+1}|$, implying that $|S \cap V_{i+1}| = 1$. By Claim 13, $|S \cap (V_{i+1} \cup V_{i+2})| \ge 4$, implying that $|S \cap V_{i+2}| = 3$, which in turn implies that $|S \cap V_{i+3}| = 1$. Continuing in this manner, we have $|S \cap V_j| = 3$ for all j even where $i \le j \le m-1$ and $|S \cap V_j| = 1$ for all j odd where $i+1 \le j \le m-2$. In particular, $|S \cap V_{m-1}| = 1$, contradicting our earlier observation that $|S \cap V_{m-1}| = 2$. □

By Claim 16.1, $|S \cap V_i| = 2$ for all i where $2 \le i \le m-1$. By our earlier observation, $S \cap V_1 = \{a_1, c_1, e_1\}$ and $S \cap V_m = \{a_m, c_m, e_m\}$. As shown in the proof of Claim 12 we infer that $|S \cap \{a_2, b_2\}| = 1$ and $|S \cap \{d_2, e_2\}| = 1$. Since $G[\{b_1, b_2, c_2, d_2, d_1\}]$ is a path joining two boundary vertices of G, this subgraph is a 3-forbidden subgraph of G, implying that S must contain at least one vertex from the set $\{b_2, d_2\}$. By symmetry, we may assume that $b_2 \in S$, and so $a_2 \notin S$. Now either $d_2 \in S$ or $e_2 \in S$. We show firstly that the case $e_2 \in S$ cannot occur.

Claim 16.2. *If $e_2 \in S$, then we obtain a contradiction.*

Proof. Suppose that $e_2 \in S$. Thus, $S \cap V_2 = \{b_2, e_2\}$. Since $a_2 \notin S$, the boundary vertex $a_3 \in S$. Let $Q_1 = G[\{c_2, d_2, c_3, d_3\}]$ and let $Q_2 = G[\{d_1, d_2, d_3, e_3\}]$. Since $Q_1 = C_4$ and since Q_2 is a path joining two boundary vertices of G, both Q_1 and Q_2 are 3-forbidden subgraphs of G, implying that S must contain at least one vertex from each of Q_1 and Q_2. Since $|S \cap V_3| = 2$ and $a_3 \in S$, we infer that $d_3 \in S$. Thus, $S \cap V_3 = \{a_3, d_3\}$.

Since $e_3 \notin S$, the vertex $e_4 \in S$ by Corollary 6. Let $Q_3 = G[\{b_3, c_3, b_4, c_4\}]$ and let $Q_2 = G[\{d_1, d_2, c_2, c_3, b_2, b_4, a_4\}]$. Since $Q_3 = C_4$ and since Q_4 is a path joining two boundary vertices of G, both Q_3 and Q_4 are 3-forbidden subgraphs of G, implying that S must contain at least one vertex from each of Q_3 and Q_4. Since $|S \cap V_4| = 2$ and $e_4 \in S$, we infer that $b_4 \in S$. Thus, $S \cap V_4 = \{b_4, e_4\}$.

Since $a_4 \notin S$, the boundary vertex $a_5 \in S$. Let $Q_5 = G[\{c_4, d_4, c_5, d_5\}]$ and let $Q_6 = G[\{d_1, d_2, c_2, c_3, c_4, d_4, d_5, e_5\}]$. Since $Q_5 = C_4$ and since Q_6 is a path joining two boundary vertices of G, both Q_5 and Q_6 are 3-forbidden subgraphs of G, implying that S must contain at least one vertex from each of Q_5 and Q_6. Since $|S \cap V_5| = 2$ and $a_5 \in S$, we infer that $d_5 \in S$. Thus, $S \cap V_5 = \{a_5, d_5\}$.

Continuing in this way, the above pattern repeats itself, that is, $S \cap V_i = \{b_i, e_i\}$ for i even and $2 \le i \le m-2$ and $S \cap V_i = \{a_i, d_i\}$ for i odd and $3 \le i \le m-1$. The set S is now fully determined. For example, when $m = 8$ the set S is illustrated in Figure 8. However, the subgraph $G[\{\{d_1, d_m\} \cup (V(C) \setminus \{c_1, c_m\}\}]$ is a path joining two boundary vertices of G and is therefore a 3-forbidden subgraph of G. However, this subgraph contains no vertex of S, a contradiction. □

By Claim 16.2, $e_2 \notin S$. By our earlier observations, $|S \cap \{d_2, e_2\}| = 1$, implying that $d_2 \in S$. Thus, $S \cap V_2 = \{b_2, d_2\}$. Since $a_2 \notin S$, this forces $a_3 \in S$, and since $e_2 \notin S$, this forces $e_3 \in S$. Thus, $S \cap V_3 = \{a_3, e_3\}$.

Let $R_1 = G[\{b_3, b_4, c_3, c_4\}]$ and let $R_2 = G[\{c_3, c_4, d_3, d_4\}]$. Since $R_1 = C_4$ and $R_1 = C_4$, both R_1 and R_2 are 3-forbidden subgraphs of G, implying that S

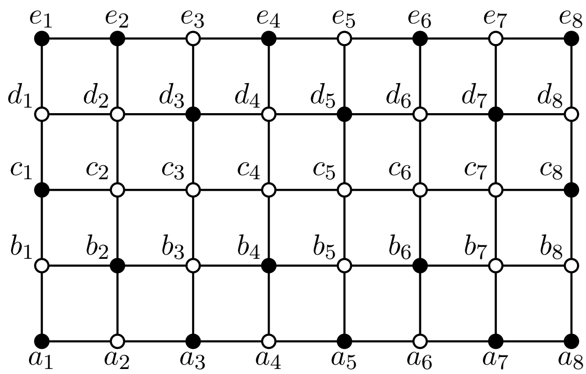


Figure 8. The set S in the graph $G = P_5 \,\square\, P_8$ in the proof of Claim 16.2.

must contain at least one vertex from each of R_1 and R_2. This implies that at most one of a_4 and e_4 belong to the set S. By symmetry, we may assume that $e_4 \notin S$, implying that $e_5 \in S$. If $a_4 \in S$, then this forces $c_4 \in S$ in order for the set S to contain a vertex from each of R_1 and R_2. We note that the case $S \cap V_4 = \{b_4, c_4\}$ is symmetric to the case $S \cap V_4 = \{c_4, d_4\}$. Hence by symmetry, there are three possibilities for the set $S \cap V_4$, namely $S \cap V_4 = \{b_4, c_4\}$, $S \cap V_4 = \{b_4, d_4\}$, or $S \cap V_4 = \{a_4, c_4\}$.

We show next that the cases $S \cap V_4 = \{a_4, c_4\}$ and $S \cap V_4 = \{b_4, c_4\}$ cannot occur.

Claim 16.3. *If $S \cap V_4 = \{a_4, c_4\}$, then we obtain a contradiction.*

Proof. Suppose that $S \cap V_4 = \{a_4, c_4\}$. Since $\{a_3, a_4\} \subset S$, we know that $a_5 \notin S$. Let $L_1 = G[\{e_4, d_4, d_3, c_3, b_3, b_4, b_5, a_5\}]$. Since L_1 is a path joining two boundary vertices of G, the subgraph L_1 is a 3-forbidden subgraph of G, and so S must contain at least one vertex from L_1, implying that $b_5 \in S$. Thus, $S \cap V_5 = \{b_5, e_5\}$.

Since $a_5 \notin S$, this forces $a_6 \in S$. Let $L_2 = G[\{e_4, d_4, d_5, d_6, e_6\}]$ and let $L_3 = G[\{c_5, c_6, d_5, d_6\}]$. Since L_2 is a path joining two boundary vertices of G and since $L_3 = C_4$, the subgraphs L_2 and L_3 are 3-forbidden subgraphs of G, and so S must contain at least one vertex from each of L_2 and L_3, implying that $d_6 \in S$. Thus, $S \cap V_6 = \{a_6, d_6\}$.

Since $e_6 \notin S$, this forces $e_7 \in S$. Let $L_4 = G[\{e_4, d_4, d_5, c_5, c_6, b_6, b_7, a_7\}]$ and let $L_5 = G[\{b_6, b_7, c_6, c_7\}]$. Since L_4 is a path joining two boundary vertices of G and since $L_5 = C_4$, the subgraphs L_4 and L_5 are 3-forbidden subgraphs of G, and so S must contain at least one vertex from each of L_4 and L_5, implying that $b_7 \in S$. Thus, $S \cap V_7 = \{b_7, e_7\}$.

Continuing in this way, the above pattern repeats itself, that is, $S \cap V_i = \{b_i, e_i\}$ for i odd and $5 \le i \le m - 1$ and $S \cap V_i = \{a_i, d_i\}$ for i even and $6 \le i \le m - 2$. The set S is now fully determined. However, the subgraph

$G[\{e_4, d_4, d_5, d_{m-1}, d_m\} \cup (V(C) \setminus \{c_1, c_2, c_3, c_4, c_m\})]$ is a path joining two boundary vertices of G and is therefore a 3-forbidden subgraph of G. However, this subgraph contains no vertex of S, a contradiction. $\square$

Claim 16.4. *If $S \cap V_4 = \{b_4, c_4\}$, then we obtain a contradiction.*

Proof. Suppose that $S \cap V_4 = \{b_4, c_4\}$. Since $a_4 \notin S$, this forces $a_5 \in S$. Recall that $e_5 \in S$, and so $S \cap V_5 = \{a_5, e_5\}$. Let $Z = \{b_m, b_{m-1}, c_{m-1}, d_{m-1}, d_m\}$.

If $m = 6$, then the set S is fully determined. In this case, the subgraph $G[Z]$ is a path joining two boundary vertices of G and is therefore a 3-forbidden subgraph of G. However, this subgraph contains no vertex of S, a contradiction. Hence, $m \geq 8$. Let $T_1 = G[\{e_4, d_4, d_5, d_6, e_6\}]$, $T_2 = G[\{c_5, c_6, d_5, d_6\}]$, $T_3 = G[\{e_4, d_4, d_5, c_5, b_5, b_6, a_6\}]$, and $T_4 = G[\{b_5, b_6, c_5, c_6\}]$. Since T_1 and T_3 are paths joining two boundary vertices of G and since $T_2 = C_4$ and $T_4 = C_4$, the subgraphs T_1, T_2, T_3 and T_4 are 3-forbidden subgraphs of G, and so S must contain at least one vertex from each of T_1, T_2, T_3 and T_4, implying that $S \cap V_6 = \{b_6, d_6\}$. Since $a_6 \notin S$, this forces $a_7 \in S$, and since $e_6 \notin S$, this forces $e_7 \in S$, and so $S \cap V_7 = \{a_6, e_6\}$.

Continuing in this way, the above pattern repeats itself, that is, $S \cap V_i = \{a_i, e_i\}$ for i odd and $5 \leq i \leq m-1$ and $S \cap V_i = \{b_i, d_i\}$ for i even and $6 \leq i \leq m-2$. The set S is now fully determined. However, as before the subgraph $G[Z]$ is a path joining two boundary vertices of G and is therefore a 3-forbidden subgraph of G. However, this subgraph contains no vertex of S, a contradiction. $\square$

By Claim 16.3, the case $S \cap V_4 = \{a_4, c_4\}$ cannot occur. By Claim 16.4, the case $S \cap V_4 = \{b_4, c_4\}$ cannot occur. Hence by our earlier assumptions, $S \cap V_4 = \{b_4, d_4\}$. Since $a_4 \notin S$, this forces $a_5 \in S$, and since $e_4 \notin S$, this forces $e_5 \in S$, and so $S \cap V_5 = \{a_5, e_5\}$. Let $Z = \{b_m, b_{m-1}, c_{m-1}, d_{m-1}, d_m\}$.

If $m = 6$, then the set S is fully determined. In this case, the subgraph $G[Z]$ is a path joining two boundary vertices of G and is therefore a 3-forbidden subgraph of G. However, this subgraph contains no vertex of S, a contradiction. Hence, $m \geq 8$.

If $S \cap V_6 = \{a_6, c_6\}$, then proceeding analogously as in the proof of Claim 16.3 we obtain a contradiction. If $S \cap V_6 = \{b_6, c_6\}$, then proceeding analogously as in the proof of Claim 16.4 we obtain a contradiction. Hence, $S \cap V_6 = \{b_6, d_6\}$.

Continuing in this way, the above pattern repeats itself, that is, $S \cap V_i = \{b_i, d_i\}$ for i even and $2 \leq i \leq m-2$ and $S \cap V_i = \{a_i, e_i\}$ for i odd and $3 \leq i \leq m-1$. The set S is now fully determined. For example, when $m = 8$ the set S is illustrated in Figure 9. However, as before the subgraph $G[Z]$ is a path joining two boundary vertices of G and is therefore a 3-forbidden subgraph of G. However, this subgraph contains no vertex of S, a contradiction. We deduce, therefore, that our supposition that $m(G,3) = 2m+2$ is incorrect. Hence, $m(G,3) \geq 2m+3$. This completes the proof of Claim 16. □

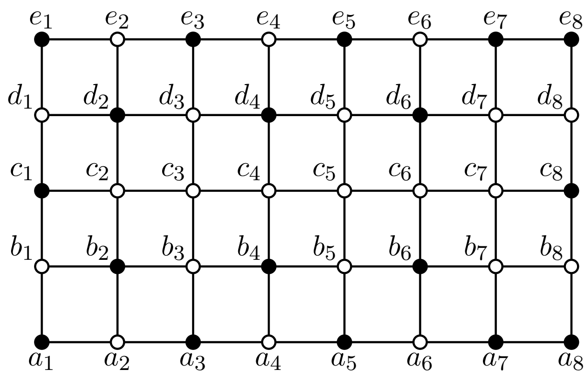


Figure 9. The set S in the graph $G = P_5 \,\square\, P_8$ in the proof of Claim 16.

Claim 17. *If m is even, then $m(G,3) = 2m+3$.*

Proof. Suppose that m is even. Let

$$\begin{aligned} S_{\text{even}} = {} & (A_{\text{odd}} \cup \{a_m\}) \cup (B_{\text{even}} \setminus \{b_m\}) \cup \{c_1, c_{m-1}, c_m\} \\ & \cup (D_{\text{even}} \setminus \{d_m\}) \cup (E_{\text{odd}} \cup \{a_m\}). \end{aligned}$$

For example, when $m = 8$ the set S_{even} is illustrated in Figure 10.

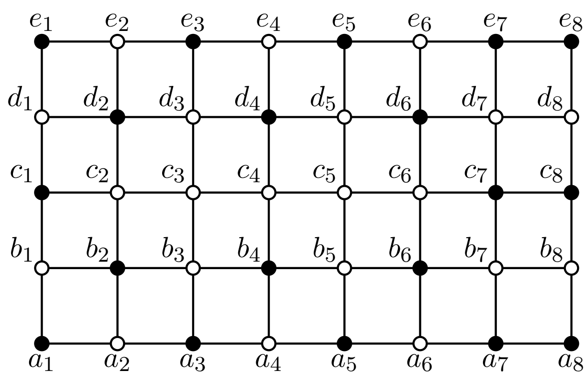


Figure 10. The set S_{even} in the graph $G = P_5 \,\square\, P_8$.

We show that S_{even} is a 3-percolating set of G. The vertices b_1 and d_1 both have three infected neighbors, and so become infected during the percolation process starting with the initial set S_{even}. Hence, all vertices in the first column V_1

are infected. Every vertex in $A \cup E$ is in the set $S_{\rm even}$ or has three infected neighbors, and so become infected during the percolation process. Hence, all vertices in $A \cup E$ are infected. Thereafter, all vertices in B become infected by considering the vertices sequentially (that is, the vertex b_1 is first infected, followed by $b_3, b_5, \ldots, b_{m-1}$, and finally b_m is infected). Identical argument show that all vertices in D become infected. Hence all vertices in $B \cup D$ are infected. Thereafter, all vertices in C become infected by considering the vertices sequentially $c_2, c_2, \ldots, c_{m-2}$. Thus, all vertices in $V(G)$ become infected, implying that

$$\begin{aligned} m(G,3) \le |S_{\rm even}| &= (|A_{\rm odd}| + 1) + (|B_{\rm even}| - 1) + |\{c_1, c_{m-1}, c_m\}| \\ &+ (|D_{\rm even}| - 1) + (|E_{\rm odd}| + 1) \\ &= |A_{\rm odd}| + |B_{\rm even}| + 3 + |D_{\rm even}| + |E_{\rm odd}| \\ &= \tfrac{m}{2} + \tfrac{m}{2} + 3 + \tfrac{m}{2} + \tfrac{m}{2} = 2m + 3. \end{aligned}$$

Hence, $m(G,3) \le 2m + 3$. By Claim 16, $m(G,3) \ge 2m + 3$. Consequently, $m(G,3) = 2m + 3$. □

The proof of Theorem 2 now follows from Claim 15 and 17. ■

3.3. 3-Bootstrap percolation in $4 \times m$ grids

In this section, we show that the 3-percolation number of a $4 \times m$ grid for all $m \ge 4$ takes on one of two possible values. We first prove a lower bound on the 3-percolation number of a $4 \times m$ grid.

Theorem 18. *For $m \ge 4$, if $G = P_4 \,\square\, P_m$, then $m(G,3) \ge \left\lfloor \frac{5(m+1)}{3} \right\rfloor + 1$.*

Proof. For $m \ge 4$, let G be the grid $P_4 \,\square\, P_m$ with

$$V(G) = \bigcup_{i=1}^{m} \{a_i, b_i, c_i, d_i\},$$

where the path $a_i b_i c_i d_i$ is a P_5-fiber in G for $i \in [m]$, and where $a_1 a_2 \cdots a_m$, $b_1 b_2 \cdots b_m$, $c_1 c_2 \cdots c_m$, and $d_1 d_2 \cdots d_m$ are P_m-fibers in G. For example, when $m = 6$ the grid $G = P_4 \,\square\, P_m$ is illustrated in Figure 11.

For $i \in [m]$, let $V_i = \{a_i, b_i, c_i, d_i\}$ and let

$$V_{\le i} = \bigcup_{j=1}^{i} V_i \quad \text{and} \quad V_{\ge i} = \bigcup_{j=i}^{m} V_i.$$

Thus, $V(G) = V_{\le m} = V_{\ge 1}$. Let $A = \{a_1, a_2, \ldots, a_m\}$, $B = \{b_1, b_2, \ldots, b_m\}$, $C = \{c_1, c_2, \ldots, c_m\}$, and $D = \{d_1, d_2, \ldots, d_m\}$. In what follows, let S be a minimum 3-percolating set of G that does not contain three consecutive boundary vertices of G. We note that such a set S exists by Lemma 7.

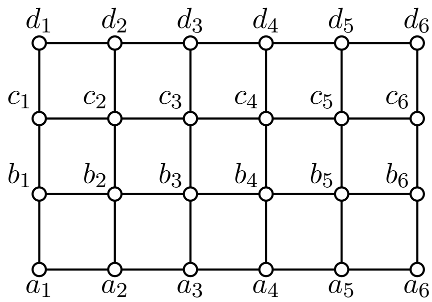


Figure 11. The graph $G = P_4 \,\square\, P_6$.

Claim 19. *The following properties hold.*

(a) $|S \cap V_1| \geq 3$ *and* $|S \cap V_m| \geq 3$.

(b) $|S \cap V_i| \geq 2$ *for all* i *where* $2 \leq i \leq m-1$.

(c) $|S \cap (V_i \cup V_{i+1})| \geq 3$ *for all* i *where* $2 \leq i \leq m-1$.

(d) $|S \cap (V_i \cup V_{i+1} \cup V_{i+2})| \geq 5$ *for all* i *where* $2 \leq i \leq m-2$.

Proof. Since the vertices a_1 and d_1 both have degree 2 in G, we note that $\{a_1, d_1\} \subset S$. Since the set S contains at least one of every two adjacent boundary vertices, we note that $|S \cap \{b_1, c_1\}| \geq 1$, implying that $|S \cap V_1| \geq 3$. By symmetry, $|S \cap V_m| \geq 3$. Thus, property (a) holds. Property (b) follows from Corollary 6(b).

To prove property (c), consider the set $S \cap (V_i \cup V_{i+1})$ for some i where $2 \leq i \leq m-1$. By Corollary 6(a), $|S \cap \{a_i, a_{i+1}\}| \geq 1$ and $|S \cap \{d_i, d_{i+1}\}| \geq 1$. If $Q_1 = G[\{b_i, b_{i+1}, c_i, c_{i+1}\}]$, then $Q_1 = C_4$, and so the subgraph Q_1 is a 3-forbidden subgraph of G, and so S must contain at least one vertex from of Q_1. These observations imply that $|S \cap (V_i \cup V_{i+1})| \geq 3$, and so property (c) holds.

To prove property (d), consider the set $S \cap (V_i \cup V_{i+1} \cup V_{i+2})$ for some i where $2 \leq i \leq m-2$. For notation convenience, we may assume that $i = 2$, that is, we consider the set $S \cap (V_2 \cup V_3 \cup V_4)$. If S contains at least three boundary vertices in $V_2 \cup V_3 \cup V_4$, then the desired result is immediate. Hence, we may assume that $|S \cap \{a_2, a_3, a_4, d_2, d_3, d_4\}| \leq 4$, for otherwise the desired lower bound holds. Since S contains no three consecutive boundaries, we note that $|S \cap \{a_2, a_3, a_4\}| \leq 2$ and $|S \cap \{d_2, d_3, d_4\}| \leq 2$.

Suppose that S contains four boundary vertices in $V_2 \cup V_3 \cup V_4$, implying that $\{a_2, a_4, d_2, d_4\} \subset S$. By property (b) we have $|S \cap V_3| \geq 1$, and we infer in this case that $|S \cap (V_2 \cup V_2 \cup V_4)| \geq 5$. Hence, we may assume that S contains at most three boundary vertices in $V_2 \cup V_3 \cup V_4$, for otherwise the desired lower bound holds.

Suppose that S contains exactly three boundary vertices in $V_2 \cup V_3 \cup V_4$. By symmetry, we may assume that $\{a_2, a_4, d_2\} \subset S$ or $\{a_2, a_4, d_3\} \subset S$. Suppose that $\{a_2, a_4, d_2\} \subset S$. Let $Q_2 = G[\{b_2, b_3, c_2, c_3\}]$, $Q_3 = G[\{c_3, c_4, d_3, d_4\}]$, and $Q_4 = G[\{a_3, b_3, b_4, c_4, d_4\}]$. Since $Q_2 = Q_3 = C_4$ and since Q_3 is a path joining two boundary vertices of G, the subgraphs Q_2, Q_3 and Q_4 are all 3-forbidden subgraphs of G, and so S must contain at least one vertex from each of Q_2, Q_3 and Q_4. At least two vertices in S are needed for this purpose, implying that $|S \cap (V_2 \cup V_3 \cup V_4)| \geq 5$, as desired. Suppose next that $\{a_2, a_4, d_3\} \subset S$. Let $Q_5 = G[\{d_2, c_2, c_3, c_4, d_4\}]$, $Q_6 = G[\{d_2, c_2, b_2, b_3, a_3\}]$, and $Q_7 = G[\{b_3, b_4, c_3, c_4\}]$. Since Q_5 and Q_6 are paths joining two boundary vertices of G and since $Q_7 = C_4$, the subgraphs Q_5, Q_6 and Q_7 are all 3-forbidden subgraphs of G, and so S must contain at least one vertex from each of Q_5, Q_6 and Q_7. At least two vertices in S are needed for this purpose, implying once again that $|S \cap (V_2 \cup V_3 \cup V_4)| \geq 5$, as desired.

Hence, we may assume that S contains at most two boundary vertices in $V_2 \cup V_3 \cup V_4$, for otherwise the desired lower bound holds. Since S contains at least one vertex among every two adjacent boundary vertices, this implies that a_3 and d_3 are the two boundary vertices in S. We note that $|S \cap V_2| \geq 1$ and $|S \cap V_4| \geq 1$. Suppose that $|S \cap V_2| = 1$ and $|S \cap V_4| = 1$, implying by our earlier assumptions that $|S \cap \{b_2, c_2\}| = 1$ and $|S \cap \{b_4, c_4\}| = 1$. By symmetry, we may assume that $\{b_2, c_4\} \subset S$ or $\{b_2, b_4\} \subset S$. If $\{b_2, c_4\} \subset S$, then $G[\{d_2, c_2, c_3, b_3, b_4, a_4\}]$ is a path joining two boundary vertices that contains no vertex of S, and if $\{b_2, b_4\} \subset S$, then $G[\{d_2, c_2, c_3, c_4, d_4\}]$ is a path joining two boundary vertices that contains no vertex of S. Both cases produce a contradiction. We deduce, therefore, that $|S \cap V_2| \geq 2$ or $|S \cap V_4| \geq 2$, implying that $|S \cap (V_2 \cup V_3 \cup V_4)| \geq 5$, as desired. This completes the proof of property (d). □

We now return to the proof of Theorem 18 and calculate the lower bound on $m(G, 3)$.

Claim 20. *If $m \equiv 0 \pmod 3$, then $m(G, 3) \geq \left\lfloor \frac{5(m+1)}{3} \right\rfloor + 1$.*

Proof. Suppose that $m \equiv 0 \pmod 3$. By Claim 19 we have

$$\begin{aligned} m(G,3) = |S| &= |S \cap V_1| + |S \cap V_2| + |S \cap V_m| + \sum_{i=3}^{m-1} |S \cap V_i| \\ &\geq 3 + 1 + 3 + \sum_{i=1}^{\frac{m-3}{3}} |S \cap (V_{3i} \cup V_{3i+1} \cup V_{3i+2})| \\ &\geq 3 + 1 + 3 + \tfrac{m-3}{3} \times 5 = \tfrac{5}{3}(m+1) + \tfrac{1}{3} = \left\lfloor \frac{5(m+1)}{3} \right\rfloor + 1, \end{aligned}$$

noting that in this case $m \equiv 0 \pmod 3$. □

Claim 21. *If $m \equiv 1 \pmod 3$, then $m(G, 3) \geq \left\lfloor \frac{5(m+1)}{3} \right\rfloor + 1$.*

Proof. Suppose that $m \equiv 1 \pmod 3$. By Claim 19 we have

$$\begin{aligned} m(G,3) = |S| &= |S \cap V_1| + |S \cap (V_2 \cup V_3)| + |S \cap V_m| + \sum_{i=4}^{m-1} |S \cap V_i| \\ &\geq 3 + 3 + 3 + \sum_{i=1}^{\frac{m-4}{3}} |S \cap (V_{3i+1} \cup V_{3i+2} \cup V_{3i+3})| \\ &\geq 3 + 3 + 3 + \tfrac{m-4}{3} \times 5 = \tfrac{5}{3}(m+1) + \tfrac{2}{3} = \left\lfloor \tfrac{5(m+1)}{3} \right\rfloor + 1, \end{aligned}$$

noting that in this case $m \equiv 1 \pmod 3$. □

Claim 22. *If $m \equiv 2 \pmod 3$, then $m(G,3) \geq \left\lfloor \frac{5(m+1)}{3} \right\rfloor + 1$.*

Proof. Suppose that $m \equiv 2 \pmod 3$. By Claim 19 we have

$$\begin{aligned} m(G,3) = |S| &= |S \cap V_1| + |S \cap V_m| + \sum_{i=2}^{m-1} |S \cap V_i| \\ &\geq 3 + 3 + \sum_{i=1}^{\frac{m-2}{3}} |S \cap (V_{3i-1} \cup V_{3i} \cup V_{3i+1})| \\ &\geq 3 + 3 + \tfrac{m-2}{3} \times 5 = \tfrac{5}{3}(m+1) + 1 = \left\lfloor \tfrac{5(m+1)}{3} \right\rfloor + 1, \end{aligned}$$

noting that in this case $m \equiv 2 \pmod 3$. □

By Claims 20, 21, and 22, we have $m(G,3) \geq \left\lfloor \frac{5(m+1)}{3} \right\rfloor + 1$. This completes the proof of Theorem 18. ■

We establish next upper bounds on the 3-percolation number of $4 \times m$ grids for all $m \geq 4$.

Theorem 23. *For $m \geq 4$, if $G = P_4 \,\square\, P_m$, then*

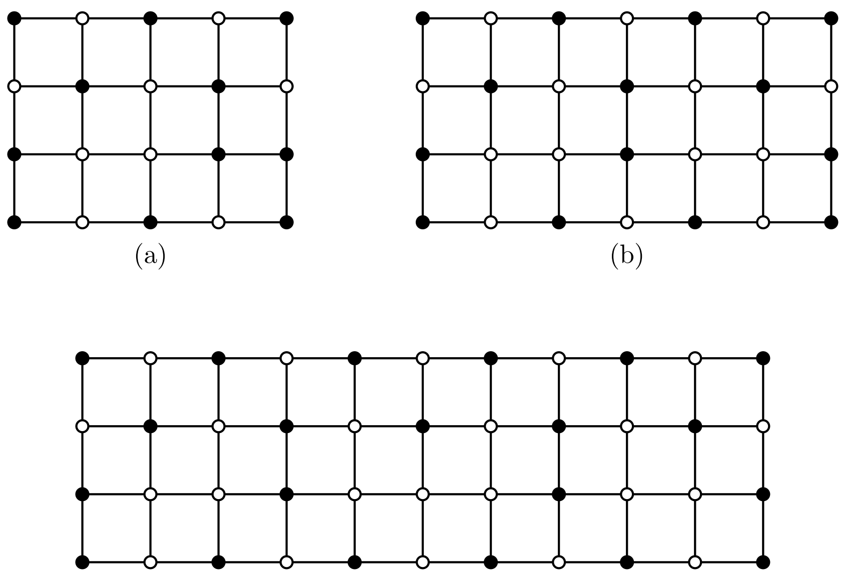
$$m(G,3) \leq \begin{cases} \left\lfloor \frac{5(m+1)}{3} \right\rfloor + 1; & m \in \{5,7,11\}; \\ \left\lfloor \frac{5(m+1)}{3} \right\rfloor + 2; & \textit{otherwise.} \end{cases}$$

Proof. For $m \geq 4$, let G_m be the grid $P_4 \,\square\, P_m$ where we follow the notation in the proof of Theorem 18. The sets shown in Figure 12(a), 12(b), and 12(c) are 3-percolating sets of G_5, G_7, and G_{11}, respectively, of cardinalities 11, 14 and 21, respectively, implying that

$$m(G_m,3) \leq \left\lfloor \frac{5(m+1)}{3} \right\rfloor + 1$$

for $m \in \{5, 7, 11\}$. Hence in what follows, we may assume that $m \notin \{5, 7, 11\}$, for otherwise the desired upper bound holds. For $i \in \{2, \ldots, m-3\}$, let

$$X_i = \{a_i, c_i, b_{i+1}, d_{i+1}, a_{i+2}\} \quad \text{and} \quad Y_i = \{b_i, d_i, a_{i+1}, c_{i+1}, d_{i+2}\}.$$



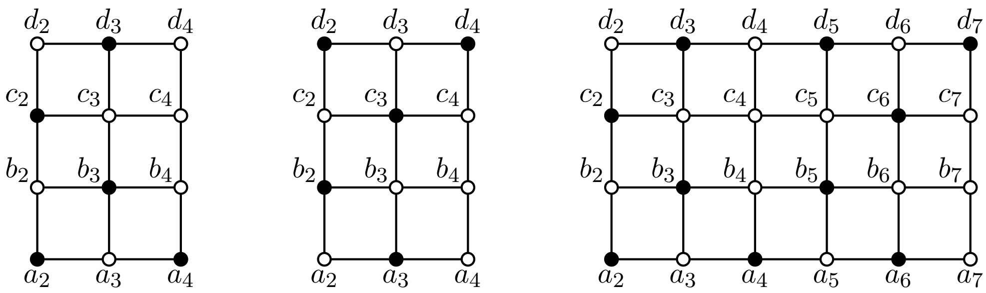
(a) (b) (c)

Figure 12. 3-percolating sets for $P_4 \,\square\, P_5$, $P_4 \,\square\, P_7$, and $P_4 \,\square\, P_{11}$.

For $i \in \{2, \ldots, m-5\}$, we denote by X_iY_{i+3} the set $X_i \cup Y_{i+3}$, and we denote by Y_iX_{i+3} the set $Y_i \cup X_{i+3}$. The sets X_2, Y_2 and X_2Y_5, for example, are illustrated in Figure 13. We note that all vertices in V_{i+1}, V_{i+2}, V_{i+3} and V_{i+4} are infected by the set X_iY_{i+3} (respectively, by the set Y_iX_{i+3}) in the 4×6 grid induced by the sets $V_i \cup V_{i+1} \cup \cdots \cup V_{i+5}$. For notational simplicity, if the subscripts are clear from the context, we simply write X and Y rather than X_i and Y_i, respectively, and we write XY and YX rather than X_iY_{i+3} and Y_iX_{i+3}, respectively. We also extend our notation to include multiple copies of X and Y. For example, we denote by $X_iY_{i+3}X_{i+6}$ the set $X_i \cup Y_{i+3} \cup X_{i+6}$ and simply denote this by the sequence XYX. Using the sequence of sets $XYXY \cdots$, we obtain grids of size $4 \times 3k$ where every internal column from 2 to $3k-1$ becomes infected. We now construct a percolating set S as follows.

Claim 24. *If $m \equiv 2 \pmod 3$, then $m(G_m, 3) \leq \left\lfloor \frac{5(m+1)}{3} \right\rfloor + 2$.*

Proof. Suppose that $m \equiv 2 \pmod 3$. Thus, $m = 3k+2$ for some $k \geq 1$. Let S consist of vertices in $(V_1 \setminus \{c_1\}) \cup V_m$ and from the set $V(G) \setminus (V_1 \cup V_m)$, we add to S



(a) X_2 (b) Y_2 (c) X_2Y_5

Figure 13. Vertex sets X_2, Y_2 and X_2Y_5.

the vertices given by the alternating sequence, $XYXY\cdots X$ or $XYXY\cdots Y$, of sets X and Y starting with the set X. From our earlier observations, we infer that all vertices in $V_3 \cup V_{m-2}$ become infected. Moreover, the vertices of S infect all vertices in $V_1 \cup V_2$ and infect all vertices in $V_{m-1} \cup V_m$, implying that the set S is a 3-percolating set. Thus, $m(G_m, 3) \le |S| = 7+5k = 7+5\times\frac{m-2}{3} = \frac{5}{3}(m+1)+2$. $\square$

Claim 25. *If* $m \equiv 1 \pmod 3$, *then* $m(G_m, 3) \le \left\lfloor \frac{5(m+1)}{3} \right\rfloor + 2$.

Proof. Suppose that $m \equiv 1 \pmod 3$. Thus, $m = 3k+1$ for some $k \ge 1$. We now construct the set S as follows. Let $S \cap V_1 = \{a_1, b_1, d_1\}$.

If $m \equiv 1 \pmod 6$, then we let $S \cap (V_2 \cup V_3 \cup \cdots \cup V_{m-3})$ consist of the alternating sequence $XYXY\cdots X$ that starts and ends with the set X, and we let $S \cap (V_{m-2} \cup V_{m-1} \cup V_m) = \{b_{m-2}, d_{m-2}, a_{m-1}, c_{m-1}, a_m, b_m, d_m\}$.

If $m \equiv 4 \pmod 6$, then we let $S \cap (V_2 \cup V_3 \cup \cdots \cup V_{m-3})$ consist of the alternating sequence $XYXY\cdots Y$ that starts with the set X and ends with the set Y, and we let $S \cap (V_{m-2} \cup V_{m-1} \cup V_m) = \{a_{m-2}, c_{m-2}, b_{m-1}, d_{m-1}, a_m, b_m, d_m\}$.

In both cases, the resulting set S is a 3-percolating set of G_m. Thus, $m(G_m, 3) \le |S| = 10 + 5(k-1) = 10 + 5\times\frac{m-4}{3} = \frac{5}{3}(m+1) + \frac{5}{3}$. $\square$

Claim 26. *If* $m \equiv 0 \pmod 3$, *then* $m(G_m, 3) \le \left\lfloor \frac{5(m+1)}{3} \right\rfloor + 2$.

Proof. Suppose that $m \equiv 0 \pmod 3$. Thus, $m = 3k$ for some $k \ge 2$. We now construct the set S as follows. Let $S \cap V_1 = \{a_1, b_1, d_1\}$.

If $m \equiv 0 \pmod 6$, then we let $S \cap (V_2 \cup V_3 \cup \cdots \cup V_{m-2})$ consist of the alternating sequence $XYXY\cdots X$ that starts and ends with the set X, and we let $S \cap (V_{m-1} \cup V_m) = \{b_{m-1}, d_{m-1}, a_m, c_m, d_m\}$.

If $m \equiv 3 \pmod 6$, then we let $S \cap (V_2 \cup V_3 \cup \cdots \cup V_{m-2})$ consist of the alternating sequence $XYXY\cdots Y$ that starts with the set X and ends with the set Y, and we let $S \cap (V_{m-1} \cup V_m) = \{a_{m-1}, c_{m-1}, a_m, b_m, d_m\}$.

In both cases, the resulting set S is a 3-percolating set of G_m. Thus, $m(G_m, 3) \leq 8 + 5(k-1) = 8 + 5 \times \frac{m-3}{3} + \frac{4}{3}$. $\square$

By Claims 24, 25, and 26, we have $m(G, 3) \leq \left\lfloor \frac{5(m+1)}{3} \right\rfloor + 2$. This completes the proof of Theorem 23. $\blacksquare$

Theorem 3 follows as an immediate consequence of Theorems 18 and 23.

4. Open Problems

As shown in Theorem 3, for $m \geq 4$ if $G = P_4 \,\square\, P_m$, then $m(G, 3)$ takes on one of two possible values, namely $m(G, 3) = \left\lfloor \frac{5(m+1)}{3} \right\rfloor + 1$ or $m(G, 3) = \left\lfloor \frac{5(m+1)}{3} \right\rfloor + 2$. It would be interesting to determine the exact value of $m(G, 3)$ in this case for all $m \geq 4$. More generally, it would be interesting to determine the exact value of $m(G, 3)$ when $G = P_n \,\square\, P_m$ for all $n \geq m \geq 6$.

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